***Introduction to probability* cheat sheet**

1. Classical probability

- Variations with repetitions: n^k
- Variations without repetition: $\frac{n!}{(n-k)!}$
- Permutations: $n!$
- Combinations: $\binom{n}{k}$

2. Axiomatic def. of probability

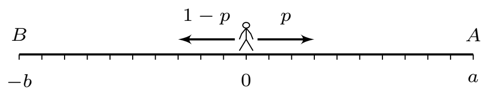
- Definition of σ-algebra $\mathcal{F} \subseteq 2^{\Omega}$:
  1. $\Omega \in \mathcal{F}$
  2. If $A \in \mathcal{F}$ then $A' \in \mathcal{F}$
  3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $A_1 \cup A_2 \cup \ldots \in \mathcal{F}$
- Properties of σ-algebra: $\emptyset \in \mathcal{F}$; if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, $A \setminus B \in \mathcal{F}$
- Properties of probability:
  - $P(\emptyset) = 0$, $P(A') = 1 - P(A)$
  - If $A \subseteq B$ then $P(B \setminus A) = P(B) - P(A)$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - $P(A_1 \cup \ldots \cup A_n) \leqslant P(A_1) + \ldots + P(A_n)$, equality for disjoint events ($A_i \cap A_j = \emptyset$ for $i \neq j$)

3. Conditional probability

- Definition: $P(A|B) = \frac{P(A \cap B)}{P(B)}$ for $P(B) > 0$
- $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
- Chain rule: $P(A_1 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1) \cdot \ldots \cdot P(A_n|A_1 \cap \ldots \cap A_{n-1})$
- Partition $A_1, \ldots, A_n$: $A_i \cap A_j = \emptyset$ for $i \neq j$, and $A_1 \cup \ldots \cup A_n = \Omega$
- Total probability: if $A_1, \ldots, A_n$ – partition: $P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$
- Bayes' rule: if $A_1, \ldots, A_n$ – partition: $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$

4. Independence

- Definition $P(A \cap B) = P(A)P(B)$. More generally: $A_1, \ldots, A_n$ independent if for each $S \subseteq \{1, 2, \ldots, n\}$: $P\left(\cap_{i \in S} A_i\right) = \Pi_{i \in S} P(A_i)$
- If $A \perp B$ then $A \perp B'$, $A' \perp B$, $A' \perp B'$
- If $A_1, \ldots, A_n$ – independent then $P(A_1 \cup \ldots \cup A_n) = 1 - P(A_1') \cdot \ldots \cdot P(A_n')$
- Conditional independence (given C): $P(A \cap B|C) = P(A|C)P(B|C)$
- Random walk:



  - Prob. of reaching A:
    $$P(A) = \begin{cases} \frac{b}{a+b} & (p = \frac{1}{2}) \\ \frac{\left(\frac{p}{1-p}\right)^a - \left(\frac{p}{1-p}\right)^{a+b}}{1 - \left(\frac{p}{1-p}\right)^{a+b}} & (p \neq \frac{1}{2}) \end{cases}$$
  - Prob. of reaching B:
    $$P(B) = 1 - P(A)$$

5. Random variables

- Definition: measurable function $X : \Omega \to \mathbb{R}$

- Distribution of random variable: measure P_X over $\mathbb{R}$ with Borel σ-algebra, such that $P_X(A) = P(X \in A) = P(X^{-1}(A))$
- C.d.f.: $F_X(x) = P(X \leqslant x)$
- Properties of F_X: nondecreasing; $F(\infty) = 1$, $F(-\infty) = 0$; $P(a < X \leqslant b) = F(b) - F(a)$
- Degenerate distribution: $P(X = c) = 1$
- Uniform distribution: $X \in \{x_1, \ldots, x_n\}$, $P(X = x_i) = \frac{1}{n}$
- Bernoulli distribution $B(p)$: $X \in \{0, 1\}$, $P(X = 1) = p$, $P(X = 0) = 1 - p$
- Binomial distribution $B(n, p)$: $X \in \{0, 1, \ldots, n\}$, $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- Geometric distribution $G_1(p)$: $X \in \{1, 2, \ldots\}$, $P(X = k) = (1-p)^{k-1} p$
- Geometric distribution $G_0(p)$: $X \in \{0, 1, \ldots\}$, $P(X = k) = (1-p)^k p$
- For $X \sim G_1(p)$: $P(X > k) = (1-p)^k$
- Memorylessness $X \sim G_1(p)$: $P(X > k + \ell | X > k) = P(X > \ell)$
- Negative binomial distribution $NB(r, p)$: $P(X = k) = \binom{r+k-1}{r-1} (1-p)^r p^k$
- Poisson distribution $\mathrm{Pois}(\lambda)$: $X \in \{0, 1, \ldots\}$, $P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$
- $B(n, p) \to \mathrm{Pois}(\lambda)$ for $n \to \infty$ and $\lambda = np$

6. Moments of random variables

- For $X \in \{0, 1, \ldots\}$: $EX = \Sigma_{k=1}^{\infty} P(X \geqslant k)$
- For $Y = f(X)$: $EY = \Sigma_x f(x) P(X = x)$
- Linearity: $E(aX + b) = aEX + b$
- $D^2(X) = E\big((X - EX)^2\big) = E(X^2) - (EX)^2$
- $D^2(aX + b) = a^2 D^2(X)$
- $D^2(X) \geqslant 0$ and $D^2(X) = 0 \iff X$ has degenerate distr.
- Expected value and variance

Distribution of X	EX	$D^2(X)$
$B(p)$	p	$p(1-p)$
$B(n, p)$	np	$np(1-p)$
$G_1(p)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$NB(r, p)$	$\frac{rp}{1-p}$	$\frac{rp}{(1-p)^2}$
$\mathrm{Pois}(\lambda)$	λ	λ

- k-th order moment: $m_k = E(X^k)$
- k-th order central moment: $\mu_k = E\big((X - EX)^k\big)$
- Markov's inequality: for nonnegative X and $a > 0$: $P(X \geqslant a) \leqslant \frac{EX}{a}$
- Chebyshev's inequality: $P(|X - EX| \geqslant \epsilon) \leqslant \frac{D^2(X)}{\epsilon^2}$
- For $X \sim B(n, p)$ the most probable value is: (a) $\lfloor (n+1)p \rfloor$ if $(n+1)p$ is non-integer; (b) $(n+1)p$ and $(n+1)p - 1$ (two values) if $(n+1)p$ is integer