A New Solution to the Swing up Control Problem for the Acrobot

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Abstract: This paper studies the swing up control for the Acrobot, i.e., to move the Acrobot from its stable downward position to its unstable inverted position and balance it about the vertical. A new control scheme which combines of the partial linearization control (related to the first link) for the swing up phase introduced in Spong (1995) and robust control for the capture and balance phase is proposed in this paper. The key idea is to treat the speed of the second link when the Acrobot rotates across the vertical as an uncertainty, and to design a robust controller to cope with such uncertainty by using the LMI solution to the quadratic stabilization problem. Hence, the basin of attraction of the robust controller is bigger than that of the LQR controller which is designed based on the linearized model of the Acrobot around the unstable equilibrium. The simulation results have verified the effectiveness of the proposed control scheme.

Keywords: Acrobot, Swing up Control, Robust Control, Quadratic Stability, LMI.

1 Introduction

In recent years a great deal of research activity has been received in the study of underactuated mechanical systems, which possess fewer actuators than degrees of freedom. The design of mechanisms that can perform complex tasks with fewer actuators allows to reduce cost and weight. However, complex nonlinear dynamics, nonholonomic behavior, and lack of linearizability are often exhibited by the underactuated mechanical systems. These make the control of such systems attractive and challenging, see e.g., [3], [9], [1], [7], [5].

The Acrobot as shown in Figure 1, is a two-degree-of-freedom planar robot with single actuator at the joint of two links. The first link attached to a passive joint can rotate freely. Note that a similar mechanism was constructed to investigate so-called brachiation motions in [5]. The Acrobot has been studied as a typical example of underactuated mechanical systems, see e.g., [9], [1], [7], [2], [11].

In this paper we study the swing up control for the Acrobot, i.e. to move the Acrobot from its stable downward position to its unstable inverted position and balance it about the vertical. Owing to the large range of motion, the swing up control problem is highly nonlinear and challenging.

In [9], an interesting approach of the combination of the partial linearization control and LQR method has been proposed for solving such problem. Under the condition of strong inertial coupling, Spong [9] shows that the first link of the Acrobot can be moved to the vertical by nonlinear controller which achieves the linearization of the dynamics of the link. The typical steady state behavior is for the first link to converge exponentially to the vertical and for the second link either to swing up around the downward position or to rotate. Under the circumstance that the second link rotates, an LQR controller is switched when the Acrobot moves close to the vertical in [9]. Since the LQR controller is designed based on the linearized model of the Acrobot around the unstable equilibrium, the speed of the second link when the Acrobot crosses the vertical should be made small to guarantee the success of the capture and balance phase. However, as indicated in [9], the basin of attraction of the LQR controller is very small, and it is difficult to tune the control parameters to accomplish the capture and balance phase successfully.

Motivated by such feature of the LQR controller, first we treat the non-zero speed of the second link when the Acrobot crosses the vertical as an uncertainty. Next, we design a robust controller based on the quadratic stabilization method to cope with such uncertainty. In this way, the capture and balance phase can be accomplished easier by the robust controller than by the LQR controller.

2 Preliminaries

We recall the control method in [9] for describing our proposed control scheme presented in the next section.

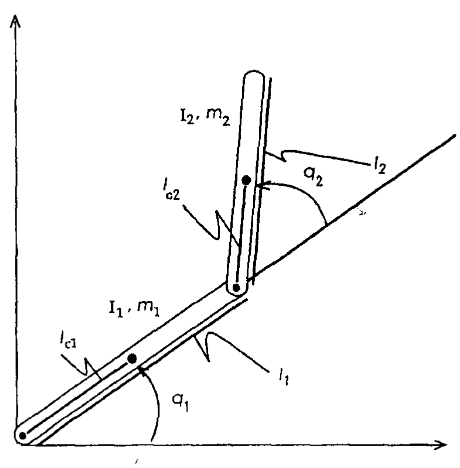

Figure 1: The Acrobot.

With the notation and conventions shown in Figure 1, from [8] and [9], the equations of motion of the Acrobot are:

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \tag{1}$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau \tag{2}$$

where τ is control input

$$\begin{aligned}
d_{11} &= c_1 + c_2 + 2c_3\cos(q_2) \\
d_{12} &= c_2 + c_3\cos(q_2) \\
d_{21} &= d_{12} \\
d_{22} &= c_2 \\
h_1 &= -c_3\sin(q_2)\dot{q}_2{}^2 - 2c_3\sin(q_2)\dot{q}_2\dot{q}_1 \\
h_2 &= c_3\sin(q_2)\dot{q}_1{}^2 \\
\phi_1 &= k_1\cos(q_1) + k_2\cos(q_1+q_2) \\
\phi_2 &= k_2\cos(q_1+q_2)
\end{aligned}$$

with

$$\begin{aligned}
c_1 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\
c_2 &= m_2 l_{c2}^2 + I_2 \\
c_3 &= m_2 l_1 l_{c2} \\
k_1 &= (m_1 l_{c1} + m_2 l_1)g \\
k_2 &= m_2 l_{c2} g
\end{aligned}$$

Denote

$$x = \begin{bmatrix} q_1 - \pi/2 & q_2 & \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T, \quad u = \tau$$

From (1), (2), the state space equations of the Acrobot can be described as

$$\dot{x}_1 = x_3 \tag{3}$$

$$\dot{x}_2 = x_4 \tag{4}$$

$$\dot{x}_3 = -\frac{d_{22}}{\Xi}(h_1+\phi_1) + \frac{d_{12}}{\Xi}(h_2+\phi_2) - \frac{d_{12}}{\Xi}u \tag{5}$$

$$\dot{x}_4 = \frac{d_{12}}{\Xi}(h_1+\phi_1) - \frac{d_{11}}{\Xi}(h_2+\phi_2) + \frac{d_{11}}{\Xi}u \tag{6}$$

where $\Xi = d_{11}d_{22} - d_{12}^2 \neq 0$ for all q_2.

Let ν_1 be the term of the right side of (5), i.e.,

$$\nu_1 = -\frac{d_{22}}{\Xi}(h_1+\phi_1) + \frac{d_{12}}{\Xi}(h_2+\phi_2) - \frac{d_{12}}{\Xi}u \tag{7}$$

Through this paper, we assume $d_{12} \neq 0$ holds for all q_2, i.e., the *strong inertial coupling condition* holds. Under such assumption, $\tau = u$ can be determined from (7) if ν_1 is provided.

The objective of swing up control is to drive the Acrobot from $x(0) = \begin{bmatrix} -\pi & 0 & 0 & 0 \end{bmatrix}^T$ to $x^d = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$. Here, $x_1 = 0$ and $x_2 = 0$ hold in the meaning of modulo 2π.

Spong [9] uses

$$\nu_1 = -k_p x_1 - k_d x_3, \quad k_p > 0,\ k_d > 0 \tag{8}$$

Under such control law, the typical steady state behavior is for the first link to converge exponentially to

$$x_1 = 0, \quad x_3 = 0 \tag{9}$$

and for the second link either to swing up around the downward position or to rotate.

To accomplish the objective of the swing up control, one has to tune k_p and k_d carefully such that not only the second link rotates, but also x_4 (the angular velocity of the second link) should be small when the Acrobot rotates across the vertical. Only in such situation, the LQR controller for linearized model of (3)-(6) around $x = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ can be switched to capture and balance the Acrobot about the vertical successfully.

Suppose that the Acrobot can enter the vicinity of $x = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$. Then linearizing the equations (3)-(6) around this point, we obtain

$$\dot{x} = Ax + Bu \tag{10}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_1 & (a_{11} + a_{12})k_2 & 0 & 0 \\ \alpha_2 & (a_{12} + a_{22})k_2 & 0 & 0 \end{bmatrix} \tag{11}$$

$$B = \begin{bmatrix} 0 & 0 & a_{12} & a_{22} \end{bmatrix}^T \tag{12}$$

where

$$\alpha_1 = a_{11}(k_1 + k_2) + a_{12}k_2$$

$$\alpha_2 = a_{12}(k_1 + k_2) + a_{22}k_2$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + 2c_3 & c_2 + c_3 \\ c_2 + c_3 & c_2 \end{bmatrix}^{-1}$$

For (10), we can find a state feedback stabilizing controller by solving the following LQR problem with performance index

$$J = \int_0^\infty (x^T Q x + R u^2) dt \tag{13}$$

being minimized, where $Q \geq 0$ and $R > 0$. Then

$$u = -Kx, \quad K = R^{-1} B^T P \tag{14}$$

stabilizes (10), where $P = P^T > 0$ is the solution of the following Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{15}$$

It is pointed out in [9] that tuning of k_p and k_d such that the Acrobot can enter the vicinity of $x = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ is critical and is not easy.

3 Quadratic Stabilization Based Balancing Controller

We want to switch the swing up phase to the capture and balance phase under the situation that the first link has moved to the vertical and the second link is approaching close to the vertical. It follows that x_1, x_2 and x_3 are very small. However, in such situation, x_4 is usually not zero which often makes the capturing and balancing the Acrobot about the vertical difficult.

To achieve a successful capture and balance phase, we have to take x_4 into consideration. To this end, we linearize the equations (3)-(6) around $x = \begin{bmatrix} 0 & 0 & 0 & x_4 \end{bmatrix}^T$ rather than $x = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ to obtain

$$\dot{x} = (A + D x_4^2 E)x + Bu \tag{16}$$

where A and B are described as in (11) and (12), respectively, and

$$D = \begin{bmatrix} 0 \\ 0 \\ a_{11} \\ a_{12} \end{bmatrix}, \quad E = \begin{bmatrix} 0 & c_3 & 0 & 0 \end{bmatrix} \tag{17}$$

In deriving (16), we have used the approximation

$$\sin(x_2)x_4^2 \approx x_2 x_4^2$$

and

$$\sin(x_2)x_3x_4 \approx x_2x_3x_4 \approx 0$$

where the latter has been obtained owing to the fact that both x_2 and x_3 are very small.

Now, we treat x_4 as an uncertainty and apply the quadratic stabilization method to construct a state feedback stabilizing controller for (16). To this end, we recall the notion of quadratic stability from [4] as follows: Consider an autonomous uncertain continuous-time system of the form

$$\dot{x}(t) = (A + D\Delta(t)E)x(t), \quad x(0) = x_0 \quad (18)$$

where x is the state, and $\Delta(t)$ is uncertainty, and A, D, E are constant matrices. If there exist appropriate positive matrix $P > 0$ and scalar $\alpha > 0$ such the time derivative of

$$V(x) = x^T Px \quad (19)$$

along the trajectory of (18) satisfies

$$\dot{V}(x) = x^T[(A+D\Delta(t)E)^T P+P(A+D\Delta(t)E)]x$$

$$\leq -\alpha||x||^2 \quad (20)$$

then system (18) is called *quadratically stable.*

Now consider the following uncertain system

$$\dot{x} = (A + D\Delta(t)E)x + Bu, \quad ||\Delta(t)|| \leq 1 \quad (21)$$

the quadratic stabilization problem for (21) is to find control law u such that the close loop is quadratically stable. In [4], the ARE (Algebraic Riccati Equation) solution to the quadratic stabilization problem is proposed. Since the ARE in [4] is parametrized by a positive scalar parameter, one has to search such parameter appropriately to determine whether the ARE has a positive solution. In order to avoid such searching, this paper uses the result of LMI (Linear Matrix Inequality) based solution to the quadratic stabilization problem in [10] rather than the ARE based one in [4].

Lemma 1 *[10] There exists a state feedback controller which quadratically stabilizes system (21) if and only if there exist matrices $Y = Y^T > 0$, W and scalar $\eta > 0$ satisfies the following LMI*

$$\begin{bmatrix} YA^T + AY - BW - W^TB^T + \eta DD^T & YE^T \\ EY & -\eta I \end{bmatrix}$$

$$< 0 \quad (22)$$

In this case, the quadratically stabilizing controller is given as $u = -Kx$ with

$$K = WP, \quad P = Y^{-1} \quad (23)$$

In what follows, we apply Lemma 1 to system (16). When the Acrobot rotates across the vertical, we assume that the corresponding angular velocity x_4 of the second link satisfies

$$|x_4| < \delta \quad (24)$$

where $\delta > 0$ is a given constant. Then we can rewrite (16) as

$$\dot{x} = (A + \bar{D}\Delta(t)\bar{E})x + Bu \quad (25)$$

where

$$\bar{D} = \delta D, \quad \bar{E} = \delta E \quad (26)$$

$$\Delta(t) = x_4^2/\delta^2, \quad |\Delta(t)| < 1 \quad (27)$$

By using Lemma 1 to system (25), we can obtain the quadratically stabilizing controller.

If δ is sufficiently small, then $\bar{D}$ and $\bar{E}$ in (26) are sufficiently small, and system (25) can be reduced to system (10). Therefore, designing the LQR controller for (10) becomes a special case of designing the robust controller designed via the quadratic stabilization method for (25).

Hence, for the gains k_p and k_d in the swing up phase, if the successful capture and balance phase can be achieved by using the LQR controller, it can also be achieved by using the above robust controller. Furthermore, even if the successful capture and balance phase can not be achieved by using the LQR controller for some gains k_p and k_d, it can still be achieved by using the above robust controller when δ in a certain range. We can expect that the quadratic stabilization based balancing controller can achieve better performance of the capturing and balancing of the Acrobot than the LQR controller.

4 Simulation Results

We simulate the Acrobot using the same parameters as those in [9], i.e.,

$$m_1 = 1[\text{kg}],\ m_2 = 1[\text{kg}],\ l_1 = 1[\text{m}],\ l_2 = 2[\text{m}]$$

$$l_{c1} = 0.5[\text{m}],\ l_{c2} = 1[\text{m}],\ I_1 = 0.083[\text{kg}{\cdot}\text{m}^2]$$

$$I_2 = 0.33[\text{kg}{\cdot}\text{m}^2],\ g = 9.8[\text{m/s}^2]$$

As mentioned in Section 3, for the gains k_p and k_d in the swing up phase, if the successful capture and balance phase can be achieved by using the LQR controller, it can also be achieved by using the above robust controller. Such fact has been verified through simulations.

Furthermore, even if the successful capture and balance phase can not be achieved by using the LQR controller for some gains k_p and k_d, it can be achieved by using the above robust controller. Such statement has also been verified through simulations and some corresponding results are presented below.

For

$$k_p = 55,\ k_d = 5 \tag{28}$$

the first link converges exponentially to $x_1 = 0$ and the second link can rotate more than 2π.

Now, we consider the capture and balance phase. Since the second link may have rotated several times when the balancing controller is switched, the value of the angle $x_2 = q_2$ corresponding to the vertical may be not zero. Therefore, in the balance phase, x_2 in $u = -Kx$ should be replaced by x_{2m} defined as

$$x_{2m} = \text{mod}(x_2 + \pi, 2\pi) - \pi \tag{29}$$

In this way, when the Acrobot is at the vertical position, $x_{2m} = 0$ holds.

Taking

$$Q = \text{diag}(1,1,1,1),\ R = 1 \tag{30}$$

we obtain from (14) that the state feedback gain of the LQR controller is

$$K = \begin{bmatrix} -246.4813 & -98.6903 & -106.4640 & -50.1381 \end{bmatrix} \tag{31}$$

However, the above LQR controller fails to capture and balance the Acrobot about the vertical, see Fig. 2.

Next, under the same k_p and k_d in (28), by adjusting Q and R in (30), we attempt to achieve a successful capture and balance of the Acrobot about the vertical. To begin, with the same Q, we have varied R from 0.1 to 2, however, we still fail to obtain a successful capture and balance phase. Then with $R = 1$, the similar failure occurs when we change Q around the value given in (30).

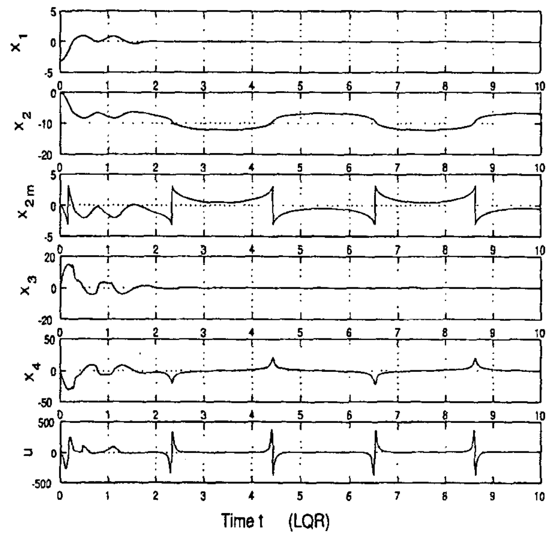

Figure 2: LQR based capturing and balancing.

With the same k_p and k_d as in (28), we investigate the control effect of quadratically stabilizing controller. Through simulations, we have found that the capturing and balancing the Acrobot is successful for $\delta \in [0.8\ \ 1.2]$. For $\delta = 1$, the state feedback matrix of the quadratically stabilizing controller is

$$K = \begin{bmatrix} -1.0444 & -0.4607 & -0.4548 & -0.2227 \end{bmatrix} \times 10^3 \tag{32}$$

See Fig. 3 for a successful capture and balance phase with the above K.

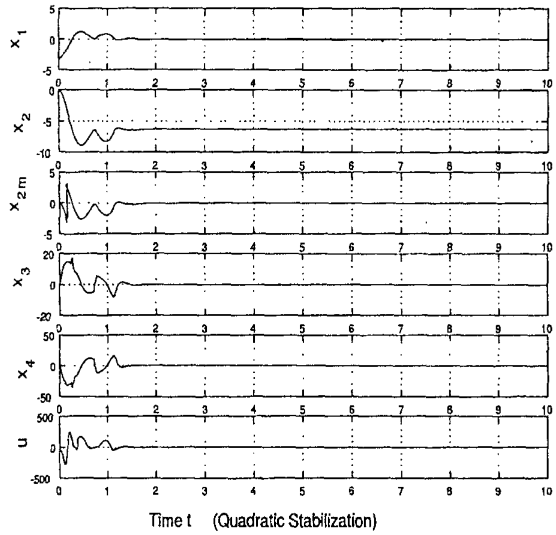

Figure 3: Robust controller based capturing and balancing.

In summary, it has been verified that the designed robust controller is superior to the LQR controller for the capture and balance phase when δ is in a certain range, and the basin of attraction of the robust controller is bigger than that of the LQR controller.

5 Conclusions

This paper has studied the swing up control problem for the Acrobot. The control scheme which combines the partial linearization control for the swing up phase introduced in [9] and the robust control for the capture and balance phase has been proposed in this paper. It has been shown that the basin of attraction of the robust controller designed in this paper is bigger than that of the LQR controller which is used in [9]. Hence, the capture and balance phase can be accomplished easier by the robust controller than by the LQR controller. The simulation results have verified the effectiveness of the proposed control scheme.

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References

[1] M. D. Berkemeier and R. S. Fearing: Tracking fast inverted trajectories of the underactuated Acrobot, *IEEE Transactions on Robotics and Automation*, vol. 15, no. 4, pp. 740-750 (1999).

[2] S. C. Brown and K. M. Passino: Intelligent control for an acrobot, *Journal of Intelligent & Robotic Systems*, vol. 18, no. 3, pp 209-248 (1997).

[3] I. Kolmanovsky and N. H. McClamroch: Developments in nonholonomic control problems, *IEEE Control Systems Magazine*, vol. 15, no. 6, pp. 20-36 (1995).

[4] I. R. Petersen and C. V. Hollot: A Riccati equation approach to the stabilization of uncertain linear system. Automatica, vol. 22, no. 4, pp. 397-411 (1986).

[5] J. Nakanishi, T. Fukuda and D. E. Koditschek: A brachiating robot controller, *IEEE Transactions on Robotics and Automation*, vol. 16, no. 2, pp. 109-123 (1999).

[6] H. K. Khalil, *Nonlinear Systems*, second edition, Prentice-Hall, New Jersey (1996).

[7] R. Olfati-Saber and A. Megretski: Controller design for a class of underactuated nonlinear systems, *Proceedings of 37th IEEE Conference of Decision and Control*, pp. 4182-4187 (1998).

[8] M. W. Spong and M. Vidyasagar: *Robot Dynamics and Control*, New York, Wiley (1989).

[9] M. W. Spong: The swing up control problem for the Acrobot, *IEEE Control Systems Magazine*, vol. 15, no. 1, pp. 49-55 (1995).

[10] L. Xie: Output feedback H_∞ control of systems with parametric uncertainty, *International Journal of Control*, vol. 63, no. 4, pp. 741-750 (1996).

[11] E. Zergeroglu, W. E. Dixon, D. M. Dawson and A. Behal: Lyapunov-based set point control of the acrobot, *International Journal of Robotics & Automation*, vol. 14, no. 4, pp. 161-170 (1999).