USING DEA MODELS TO MEASURE EFFICIENCY[1]

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It is one of the key activities for any firm to monitor its efficiency. In our modern society, there is a number of methods based either on the traditional approach or using IT. Efficiency measurement methods can be divided into three main categories: ratio indicators, parametric and nonparametric methods. In selecting indicators to gauge efficiency we focus primarily on a firm's inputs and outputs.

Ratios rank among the most simple methods. Their drawback is that they evaluate just a handful of indicators and cannot influence overall corporate efficiency. As an example, we may cite income per unit of costs.

Nonparametric methods include *Data Envelopment Analyses (DEA)* and the *Free Disposal Hull (FDH)*. We use them to measure technical (technological) efficiency. Technical efficiency looks at the level of inputs or outputs. Being technically efficient means to minimise inputs at a given level of outputs, or maximise outputs at a given level of inputs.

Parametric methods of efficiency measurement include the *Stochastic Frontier Approach (SFA), Thick Frontier Approach (TFA)* and *Distribution Free Approach (DFA)*. These methods measure economic efficiency. Economic efficiency is a broader term than technical efficiency. It covers an optimal choice of the level and structure of inputs and outputs based on reactions to market prices. Being economically efficient means to choose a certain volume and structure of inputs and outputs in order to minimise cost or maximise profit. Economic efficiency requires both technical efficiency and efficient allocation. While technical efficiency only requires input and output data, economic efficiency requires price data as well.

DEA is a nonparametric method. It is a linear programming model, assuming no random mistakes, used to measure technical efficiency. Efficient firms are those that produce a certain amount of or more outputs while spending a given amount of inputs, or use the same amount of or less inputs to produce a given amount of outputs, as compared with other firms in the test group.

FDH is another nonparametric and nonstochastic method, which can be seen as a generalised DEA model with variable returns to scale. This particular model does not require the estimated efficiency boundary to have a convex shape.

The SFA econometric model presents a method assuming two error elements. In this approach, inefficiency is assumed to have asymmetrical distribution, usually a half normal distribution and random error is expected to have standard symmetrical distribution. SFA deals with the problem that not all deviations from criteria are due to a lack of efficiency. They may also occur as a result of misfortune (fortune) or measurement errors.

TFA compares the average efficiency of a group of firms, rather than trying to estimate efficiency thresholds.

DFA relies on average variations of a cost function estimated on a data set to construct a cost efficiency threshold. This method requires no specific form of distribution or average efficiency of each firm.

The objective of this paper is to suggest possible ways for productive units to measure their efficiency. To that end, a variety of analysis, synthesis and comparison methods were used. The paper focuses on efficiency measurement by means DEA models, which can be broken up into certain subcategories. The DEA methodology gives us a tool to estimate "relative" efficiency of a chosen entity in a given group of units and criteria. The theory is demonstrated on a simple numerical and graphical example.

1. Productive unit efficiency

The measurement of efficiency in production units and the identification of sources of their inefficiency is a precondition to improve the performance of any productive unit in a competitive environment. Generally speaking, the term productive unit refers to a unit producing certain outputs by spending certain inputs.

Banks, or bank branches, can be treated as produc-

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tion units too. In general, they are homogeneous units performing the same or similar activities. All inputs and outputs have an impact on efficient operation of such units, even though some are considered more or less important.

The most frequent method used to measure efficiency is based on ratios. Their handicap is that they reflect only a few of the factors having an impact on the overall efficiency of a productive unit.

Say we have a population of n productive units $DMU_1, DMU_2, \ldots, DMU_n$. Each unit produces s outputs while consuming m inputs. Let us write an input matrix $X = [xij,\ i = 1, 2, \ldots, m,\ j= 1, 2, \ldots, n]$ and an output matrix $Y = [yij,\ i = 1, 2, \ldots, s,\ j= 1, 2, \ldots, n]$. The q-th line – i.e. X_q and Y_q – of these matrixes thus shows quantified inputs/outputs of unit DMU_q. The efficiency rate of such a unit can then be generally expressed as:

$$\frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}} = \frac{\sum_{i=1}^{s} u_i y_{iq}}{\sum_{j=1}^{m} v_j x_{jq}}$$

where:
$v_j,\ j = 1, 2, \ldots, m$, are weights assigned to j-th input,
$u_i,\ i = 1, 2, \ldots, s$, are weights assigned to i-th output.

There are several ways to estimate the efficiency rate as defined above, namely multicriterial decision methods and data envelopment analyses (DEA). These approaches differ in how they obtain input and output weights.

• Multicriterial decision methods usually expect the user to define the weights vj and ui upfront, i.e. the user determines the significance of individual inputs and outputs in the analysis. Such an analysis yields the rate of utility of given units. It reflects the relative importance of inputs and outputs represented by their respective weights. Based on this analysis units can be ranked from the worst to the best performer.

• DEA models derive input and output weights by means of an optimising calculation. Based on that, units can be classified into efficient and inefficient. In inefficient units, they tell us target values of inputs and outputs which would lead to efficiency.

2. Basic DEA models

2.1 CCR and BCC models

In DEA models, we evaluate n productive units, DMU_S, where each DMU takes m different inputs to produce s different outputs. The essence of DEA models in measuring the efficiency of productive unit DMU_q lies in maximising its efficiency rate. However, subject to the condition that the efficiency rate of any other units in the population must not be greater than 1. The models must include all characteristics considered, i.e. the weights of all inputs and outputs must be greater than zero. Such a model is defined as a linear divisive programming model:

$$\begin{aligned} &\text{maximize} && \frac{\Sigma_i u_i y_{iq}}{\Sigma_j v_j x_{jq}} && (1) \\ &\text{subject to} && \frac{\Sigma_i u_i y_{ik}}{\Sigma_j v_j x_{jk}} \le 1 \quad k = 1, 2, \ldots, n \\ &&& u_i \ge \in \quad i = 1, 2, \ldots, s \\ &&& v_j \le \in \quad j = 1, 2, \ldots, m \end{aligned}$$

This model can be converted into a linear programming model[2] and transformed into a matrix:

$$\begin{aligned} &\text{maximize} && z = u^T Y_q && (2) \\ &\text{subject to} && v^T X_q = 1 \\ &&& u^T Y - v^T X \le 0 \\ &&& u \ge \in \\ &&& v \le \in \end{aligned}$$

Model (2) is often called primary CCR model (*Charnes, Cooper, Rhodes*). The dual model to this can be stated as follows:

$$\begin{aligned} &\text{minimize} && f = \theta - \in (e^T s^+ + e^T s^-) && (3) \\ &\text{subject to} && Y\lambda - s^+ = Y_q \\ &&& X\lambda + s^- = \theta X_q \\ &&& \lambda, s^+, s^- \ge 0 \end{aligned}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda \ge 0$ is a vector assigned to individual productive units, s^+ and s^- are vectors of addition input and output variables, $e^T = (1, 1, \ldots, 1)$ and $\in$ is a constant[3] greater than zero, which is normally pitched at 10^{-6} or 10^{-8}. In evaluating the efficiency of unit DMU_q, model (3) seeks a virtual unit characterised by inputs $X\lambda$ and outputs $Y\lambda$, which are a linear combination of inputs and outputs of other units of the population and which are better that the inputs and outputs of unit DMU_q which is being evaluated. For inputs of the virtual unit $X\lambda \le X_q$ and for outputs $Y\lambda \ge Y_q$. Unit DMU_q is rated efficient if no virtual unit with requested traits

[2] The term linear programming consists of two words explaining the substance of this particular branch of operational research. Programming is a synonym for predicting future development. The word linear means that all equations and inequalities used in the model are linear (Jablonský, 2002).

[3] Economic reasoning: for any amount of output we must use at least a minimum quantity of every input. If any of the inputs equals zero, the total output is zero as well.

exists or if the virtual unit is identical with the unit evaluated, i.e. $X\lambda = X_q$ and $Y\lambda = Y_q$.

If unit DMU is CCR efficient, then:

– the value of variable θ is zero,

– the values of all additional variables s^+ and s^- equal zero.

Consequently, unit DMU_q is CCR efficient if the optimum value of the model (3) objective function equals one. Otherwise, the unit is inefficient. The optimum value of the objective function f^* marks the efficiency rate of the unit concerned. The lower the rate, the less efficient the unit is compared to the rest of the population. In inefficient units θ is less than one. This value shows the need for a proportional reduction of inputs for unit DMU_q to become efficient. The advantage of the DEA model is that it advises how the unit evaluated should mend its behaviour to reach efficiency.

Models (2) and (3) are input-oriented – they try to find out how to improve the input characteristics of the unit concerned for it to become efficient. There are output-oriented models as well. Such a model could be written as follows:

$$\begin{aligned} &\text{maximize} && g = \Phi + \in(e^T s^+ + e^T s^-) \qquad (4)\\ &\text{subject to} && Y\lambda - s^+ = \Phi Y_q\\ &&& X\lambda + s^- = X_q\\ &&& \lambda,\ s^+,\ s^- \geq 0 \end{aligned}$$

This model can be interpreted as follows: unit DMU_q is CCR efficient if the optimal value of the objective function in model (4) equals one, $g^* = 1$. If the value of the function is greater than one, the unit is inefficient. The variable Φ indicates the need for increased output to achieve efficiency. For the optimal solution to the CCR model, the values of objective functions should be inverted, i.e. $f^* = 1/g^*$.

Models (2), (3) and (4) assume constant returns to scale.[4] However, in efficiency analysis, variable returns to scale can also be considered. In that case, models (3) and (4) need to be rewritten to include a condition of convexity $e^T\lambda = 1$. Afterwards, they are referred to as BCC (*Banker, Charnes, Cooper*) models.

The aim of DEA analysis is not only to determine the efficiency rate of the units reviewed, but in particular to find target values for inputs X'_q and outputs Y'_q for an inefficient unit. After reaching these values, the unit would arrive at the threshold of efficiency. Target values are calculated:

1. by means of productive unit vectors:

$$X'_q = X\lambda^*$$
$$Y'_q = Y\lambda^*$$

where λ^* is the vector of optimal variable values.

2. by means of the efficiency rate and values of additional variables s^- and s^+:

input-oriented CCR model:

$$X'_q = \theta X_q - s^- \qquad Y'_q = Y_q + s^+$$

output-oriented CCR model:

$$X'_q = X_q - s^- \qquad Y'_q = \Phi Y_q + s^+$$

where θ is the efficiency rate in the input-oriented model and Φ is the efficiency rate in the output-oriented model.

2.2 SBM and super SBM models

Besides the basic models, certain modifications exist. One of them, labelled the SBM model, was designed by Tonen (2001). This model serves as the basis for the definition of superefficiency. Efficiency is measured only by additional variables s^+ and s^-. The model formula, provided constant returns to scale, is:

$$\text{minimize} \qquad p = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}(s_i^- / x_{iq})}{1 + \frac{1}{r}\sum_{i=1}^{r}(s_i^+ / y_{iq})} \qquad (5)$$

$$\begin{aligned} &\text{subject to} && \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{iq} \qquad i = 1, 2, ..., m\\ &&& \sum_{j=1}^{n} y_{ij}\lambda_j - s_i^+ = y_{iq} \qquad i = 1, 2, ..., r\\ &&& \lambda_j,\ s_i^+,\ s_i^- \geq 0 \end{aligned}$$

The variables s^+ and s^- measure the distance of inputs $X\lambda$ and outputs $Y\lambda$ of a virtual unit from those of the unit evaluated (X_q). The numerator and the denominator of the objective function of model (5) measures the average distance of inputs and outputs, respectively, from the efficiency threshold.

For variable returns to scale, condition $e^T\lambda = 1$ only needs to be appended to the formula. We can demonstrate that the SBM efficiency rate is always lower or equal to that of the input-oriented CCR model. This means that a unit rated as SBM efficient is CCR efficient at the same time.

In order to review efficient units, we can use superefficiency models. Unlike the CCR (BCC) model, they can also evaluate the efficiency rate of efficient units.

The super SBM model is based on the SBM model. After removing from the population unit DMU_q which is being evaluated, it seeks to find a virtual unit DMU^* with inputs X^* and outputs Y^* which will be SBM efficient after the removal. Apparently, inputs into unit DMU^* will

[4] For instance, a double increase in inputs leading to a double increase in outputs.

be higher or equal to those into the evaluated unit DMU_q and all outputs will be lower or equal to those of DMU_q. The superefficiency rate is defined as the distance between the inputs and outputs of both units – DMU^* and DMU_q. The distance is shown in variable δ. The super SBM model can then be written as follows:

$$\text{minimize } \delta = \frac{\frac{1}{m}\sum_{i=1}^{m}(x_i^* / x_{iq})}{\frac{1}{r}\sum_{i=1}^{r}(y_i^* / y_{iq})} \qquad (6)$$

subject to

$$\sum_{j=1,\neq q}^{n} x_{ij}\lambda_j + s_i^- = x_{iq} \qquad i = 1, 2, ..., m$$

$$\sum_{j=1,\neq q}^{n} y_{ij}\lambda_j - s_i^+ = y_{iq} \qquad i = 1, 2, ..., r$$

$$x_i^* \geq x_{iq} \qquad i = 1, 2, ..., m$$

$$y_i^* \leq y_{iq} \qquad i = 1, 2, ..., r$$

$$\lambda_j,\ s_i^+,\ s_i^-\ y_i^* \geq 0$$

The input-oriented super SBM model is derived from model (6) with the denominator set to 1. The super SBM model returns a value of the objective function which is greater or equal to one. The higher the value, the more efficient the unit.

2.3 Graphical and numerical example

In this section, we will apply theoretical knowledge to simulate a graphical and numerical example. We will attempt to determine the efficiency of five virtual productive units described by a single input and a single output. Efficiency is measured by DEA models and their variations.

Assume we have 5 productive units which spend one input to produce one output. The input matrix is written X = (1, 2, 3, 4, 6), the output matrix is Y = (1, 4, 6, 5, 7). Efficiency rates were calculated using input- (3) and output-oriented (4) CCR models and an input-oriented BCC model. Based on the input- and output-oriented CCR model, productive units DMU(2) and DMU(3) were found to have the objective function value equal to one, i.e. they are efficient (Table 1). According to the BCC model, units DMU(1, 2, 3, 5) are efficient. The curves connecting the efficient productive units delineate the boundaries of efficiency. Other productive units lie below the curves, i.e. they are inefficient within the population of units being reviewed.

As Table 1 shows, the models differ in their efficiency rating. Consequently, they will also differ in the way they shift inefficient units to the efficiency threshold.

Both the input- and the output-oriented CCR models singled out productive units DMU(2) and DMU(3) as efficient. In this case, the efficiency boundary is a straight line cutting through DMU(2) and DMU(3). All other units are inefficient, i.e. they fall short of the efficiency curve. Note that the DEA model allows us to determine how a productive unit should change its behaviour to become efficient and rise to the efficiency curve. The input-oriented CCR model suggests that for a unit to become efficient it must lower its inputs. In our case, units DMU(1, 4, 5) will be efficient if they lower the level of their respective inputs to points *P1', P4'* and *P5'* on the efficiency curve (Figure 1). In order to calculate the efficient input values, we can use the production unit vectors λ or efficiency rates and additional variables s_1^-. Then we can compute the efficient input value of DMU(1) with data from Table 2.

The efficient input value for DMU(1) will thus be determined as a combination of inputs into DMU(2, 3) as follows: λ_2 = 0 and λ_3 = 0.166667 (Table 2), where λ_2 and λ_3 are vectors of efficient productive units DMU. Subsequently, the target input value X will be:

$$0.166667 \,.\, X(3) = 0.166667 \,.\, 3 = 0.5$$

Table 1 Productive unit efficiency rate, CCR and BCC models

DMU	Efficiency rate CCR input-oriented	 CCR output-oriented	 BCC input-oriented
1	0.5	2.0	1.0
2	1.0	1.0	1.0
3	1.0	1.0	1.0
4	0.625	1.6	0.625
5	0.5833	1.7143	1.0

Source: own calculations.

Figure 1 Graphical illustration of productive units

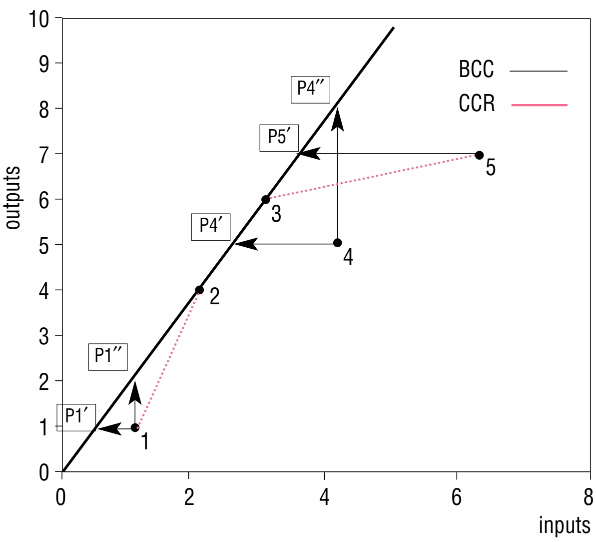

Table 2 Productive unit efficiency rate – input-oriented CCR model

DMU	Efficiency rate	Ranking	λ_1	λ_2	λ_3	λ_4	λ_5	s_1^-	s_1^+	X	X*	Y	Y*
1	0,5	5	0	0	0.166667	0	0	0	0	1	0.5	1	1[1)]
2	1	1	0	1	0	0	0	0	0	2	2	4	4
3	1	1	0	0	1	0	0	0	0	3	3	6	6
4	0.625	3	0	0	0.83	0	0	0	0	4	2.49	5	5
5	0.5833	4	0	0	1.16667	0	0	0	0	6	3.5	7	7

Source: own calculations.

1) $Y^*(1) = Y(1) + s^+ = 1 + 0 = 1$

Table 3 Productive unit efficiency rate – output-oriented CCR model

DMU	Efficiency rate	Ranking	λ_1	λ_2	λ_3	λ_4	λ_5	s_1^-	s_1^+	X	X*	Y	Y*
1	2	5	0	0	0.333333	0	0	0	0	1	1[1)]	1	2[2)]
2	1	1	0	1	0	0	0	0	0	2	2	4	4
3	1	1	0	0	1	0	0	0	0	3	3	6	6
4	1,6	3	0	0	1.333333	0	0	0	0	4	4	5	8
5	1.71429	4	0	0	2	0	0	0	0	6	6	7	12

Source: own calculations.

1) $X^*(1) = X(1) - s^- = 1 - 0 = 1$

2) $Y^*(1) = \Phi Y(1) + s^+ = 2 \,.\, 1 + 0 = 2$ or $Y^*(1) = \lambda 3 \,.\, Y(3) = 0{,}333333 \,.\, 6 = 2$

Another way of projecting an inefficient point to the efficiency boundary is to use the efficiency rate and the values of additional variables s_1^- (Table 2). Then:

$$X(1) \,.\, \text{efficiency rate}(1) - s_1^- = 1 \,.\, 0.5 - 0 = 0.5$$

So DMU(1, 4, 5) would be efficient if they reduced inputs to X^* (Table 2), with outputs left unchanged ($Y = Y^*$).

The output-oriented CCR model says that in order to attain efficiency, productive units must increase their outputs. In case of DMU(1, 4, 5) it would therefore be necessary for these productive units to raise the level of their output to reach points *P1˝, P4˝* and *P5˝* on the efficiency curve (Figure 1). Efficient output values can be computed similarly as efficient inputs, either by means of productive unit vectors of efficient DMUs or by using the efficiency rate and the values of additional variables s_1^+ (Table 3). The BCC model efficiency threshold runs through efficient units DMU(1, 2, 3, 5). Due to variable returns to scale, the BCC efficiency threshold is a convex curve, rather than a straight line.

Conclusion

This paper did not set out to find the line between an efficient and inefficient unit, but rather seeks to outline possible ways of measuring efficiency and interpreting the outcome of analyses.

It must be noted that the DEA is not completely flawless. It does facilitate an estimate of "relative" efficiency of a selected unit within a group, but stops short of estimating absolute efficiency. In other words, it tells us how well a unit performs within a given group based on chosen criteria. Another shortcoming is that the DEA method is based on extreme points and compares each unit to the best performers. This particular feature makes the DEA analysis more sensitive to data noise and measurement errors.

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