ZPC-22

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Instructions

Please read the following instructions carefully before proceeding further:

1. The test is of 2 hours. It will end *sharp* at **8:00 pm**.
2. It is a closed web and closed book test (Except the relevant reading material provided of course).
3. The answers to all the questions are subjective. Please make a pdf (handwritten, word, or LaTeX) of your solutions and email them to evariste@sc.iiitd.ac.in. Please include your *name, roll number, branch, year* in the email.
4. Any submission received after **8:10 pm** will not be evaluated.
5. Relevant reading material for all the questions has been provided in the document itself.
6. If you are stuck on a question that you cannot figure out, move on. Nothing good ever comes out of being hung up on something.
7. Please feel free to reach out to us on the Google Meet in case of any queries.
8. We will provide hints to questions in the last hour, depending on the participation and responses.
9. All the best. GL HF :)

POLYNOMIALS

1 Functions

A relation between two sets is a collection of ordered pairs containing one object from each set. If the object x is from the first set and the object y is from the second set, then the objects are said to be related if the ordered pair (x, y) is in the relation.
A function is a specific type of relation between two sets:
A relation between two sets A and B is said to be a function if it assigns exactly one element of B to every element of A, or equivalently, $\forall a \in A$ there uniquely exists $b \in B$ such that (a, b) is part of the relation.

A function, named f, from A to B is often denoted as $f : A \to B$

2 Polynomials

Polynomials are a special type of functions that are widely used.

Defintion: A function is said to be a (real) polynomial if there exists finite $n \geq 0$ and real numbers $a_n, a_{n-1}, a_{n-2}, \ldots\ldots a_2, a_1, a_0$, such that

$$f(x) \equiv a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots\ldots a_2x^2 + a_1x + a_0$$

Here, $a_n, a_{n-1}, \ldots\ldots a_1, a_0$ are called coefficients of the polynomial.
a_n is the leading coefficient, a_0 is called the constant term.

Examples: $f(x) = 3x^2 + 2x + 1$, $g(x) = 5$ and $h(x) = 2x^3 + 3$ are polynomial functions

3 Degree of Polynomials

Definition: In a polynomial $f(x) \equiv a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots a_2x^2 + a_1x + a_0$, the degree is defined as the largest possible power of x that is multiplied with some non-zero constant in the polynomial.
if $a_n \neq 0$, the degree is n

Example: $f(x) = x^4 + 2x^2 + 1$ is a polynomial with degree 4

A polynomial with degree 0 always has a non-zero constant value ($f(x) = kx^0$).
If the constant is 0, $f(x) = 0$, is called the **zero polynomial**, and we shall consider its degree to be -1

4 Operations on polynomials

Let $\deg(f)$ denote the degree of the polynomial function f(x)

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ...a_1 x + a_0$
and $g(x) = b_m x^m + b_{m-1}x^{m-1} + ...b_1 x + b_0$
where both $a_n, b_m \neq 0$

It is evident that $\deg(f) = n$ and $\deg(g) = m$

Sum: $(f + g)(x) := f(x) + g(x)$

$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$,
Moreover, if $n \neq m$, $\deg(f + g) = \max\{\deg(f), \deg(g)\}$

Polynomial Product: $(fg)(x) := f(x) \times g(x)$

If neither f nor g are the zero polynomials,
$\deg(fg) = \deg(f) + \deg(g)$ 1

Both sum and products of polynomials can easily be extended for k different polynomials. Also if $f(x) = g(x)$ then $(fg)(x) = f(x) \times g(x) = (f(x))^2$. So, we can also extend the notion of product of polynomials to power of polynomials

Scalar Product: $(cf)(x) := c \times f(x)$ and $\deg(cf) = \deg(f)$, where c is a non-zero scalar.

5 An Interesting Technique

Many a times, We have polynomials which we have to find given the value of the polynomial for some numbers. Here, a useful technique is to define a new polynomial and then think in terms of that polynomial. We use the following result in these problems:
$f(x) = g(x)$ for $x = x_1,\ x_2, \ldots, x_n$ then $q(x) = f(x) - g(x)$ has roots $x_1,\ x_2, \ldots, x_n$ and can be written as $q(x) = t(x)(x - x_1)(x - x_2)\ldots(x - x_n)$ where $t(x)$ is a polynomial. So, $f(x) = t(x)(x - x_1)(x - x_2)\ldots(x - x_n) + g(x)$.

Example : Let $f(x)$ be a cubic polynomial such that $f(1) = 1,\ f(2) = 2,\ f(3) = 3$ and $f(4) = 28$. Calculate $f(5)$.

Solution: One way to solve this problem is to assume $f(x) = ax^3 + bx^2 + cx + d$, and then solve 4 equations in four variables but that is a pretty inefficient approach and requires a lot of work. We can solve this problem using this approach :

Consider the polynomial $q(x) = f(x) - x$, Now $q(x)$ is a cubic polynomial as $f(x)$ is cubic and 1, 2, 3 are the roots of $q(x)$ from the given conditions ($f(1) = 1,\ f(2) = 2,\ f(3) = 3$). So, now $q(x)$ is of the form $q(x) = a(x-1)(x-2)(x-3)$ where a is some non zero constant. So, $f(x) - x = a(x-1)(x-2)(x-3) \Rightarrow f(x) = a(x-1)(x-2)(x-3) + x$. Now, substitute $x = 4$ to get $a = 4$.
So, we finally get $f(x) = 4(x-1)(x-2)(x-3) + x \Rightarrow f(5) = 101$.

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PROBLEMS

- The questions here are all subjective, and will be graded on how well you can convince the checker of the mathematical rigour of your solution.
- The corresponding points of each question are mentioned with it. Do note that the points may or may not be proportionate to the relative difficulty of the question
- The number of problems presented is much more than what is humanly possible to solve in the given time frame. Hence, prioritise correctness and rigour over number of problems solved.
- Note that any significant conclusion derived in the right direction to solve the problem may fetch you some points, so try to attempt all questions.

1. Find a formula for a_n, given that $a_1 = 1^2, a_2 = 2^2, a_3 = 3^2$ and $a_4 = \lambda$, where λ can be any given real number. (Assume that every term of the sequence a_n can be generated by a unique cubic polynomial, say $f(x)$) (1 point)
2. If $f(x)$ and $\dfrac{1}{(f(x))^2}$ are both polynomials in one variable, show that f must be a non-zero constant polynomial (3 points)
3. Prove: There exist three non-zero rational numbers acting as coefficients of a quadratic equation with exactly 2 distinct rational roots, no matter which order you arrange the coefficients.
   Find five such sets of real numbers. (3 points)
4. Prove or disprove: There exists a polynomial $f(x)$ such that $f(x) = p$ whenever $x = \left(\dfrac{p^2}{q^2}\right)$. Where p and q are any co-prime positive integers. (4 points)
5. Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be a polynomial with integral coefficients. Suppose that there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Find an integer k with $P(k) = 8$. (4 points)
6. Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer,
   then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$. (5 points)
7. Let $f(x)$ be a polynomial with real coefficients, and suppose that
   $f(x) + f'(x) > 0$ for all x.
   Prove that $f(x) > 0$ for all x and give three examples of such polynomials $f(x)$. (5 points)

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