ZPC-19 solutions

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1 PROBLEM 1

Assume $\mathbb{P}(n): \ F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$. Now $\mathbb{P}(1)$ is clearly true after putting $n = 1$ in the formula. Now, assume $\mathbb{P}(n),\ \mathbb{P}(n-1),\ \mathbb{P}(n-2),\ \ldots,\mathbb{P}(1)$ are all true.

$\Rightarrow F_{n+1} = F_n + F_{n-1}$

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) + \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)$$

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right)\right)$$

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right)$$

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right)$$

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \Rightarrow \mathbb{P}(n+1) \text{ is true.}$$

Therefore by the second form of induction, the binet's formula is true for all positive integers.

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2 PROBLEM 2

$\mathbb{P}(n) : x^n + \frac{1}{x^n} \in \ \mathbb{Z}^+$

Now, $\mathbb{P}(1)$ is a given fact, the truth of $\mathbb{P}(2)$ can be easily obtained by the following algebraic manipulation:

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

Now, assume that $\mathbb{P}(n),\ \mathbb{P}(n-1),\ \mathbb{P}(n-2),\ \ldots,\mathbb{P}(1)$ are all true.
Now $x^{n+1}+\dfrac{1}{x^{n+1}}=\left(x+\dfrac{1}{x}\right)\left(x^n+\dfrac{1}{x^n}\right)-\left(x^{n-1}+\dfrac{1}{x^{n-1}}\right)$
Now, all the terms on the RHS are positive integers, therefore LHS must be a positive integer $\Rightarrow \mathbb{P}(n+1)$ is true.
Therefore by the second form of induction, the given statement is true for all positive integers.

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3 PROBLEM 3

let $\lim_{n\to\infty}\dfrac{F_{n+1}}{F_n}=\alpha \Rightarrow \lim_{n\to\infty}\dfrac{F_{n+1}}{F_n}=\lim_{n\to\infty}\dfrac{F_n+F_{n-1}}{F_n}=1+\lim_{n\to\infty}\dfrac{F_{n-1}}{F_n}=1+\dfrac{1}{\alpha}$

$\alpha = 1+\dfrac{1}{\alpha} \Rightarrow \alpha^2-\alpha-1=0 \Rightarrow \alpha=\left(\dfrac{1\pm\sqrt{5}}{2}\right)$. Now, since α is a ratio of positive numbers so it must be positive. Thus, $\alpha=\left(\dfrac{1+\sqrt{5}}{2}\right)$. (This is also called the golden ratio)

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4 PROBLEM 4

Now, note that $1.414>1 \Rightarrow \sqrt{2}^{\sqrt{2}}>\sqrt{2} \Rightarrow a_2>a_1$
Now, let $\mathbb{P}(n) : a_n<a_{n+1}\ \ and\ \ a_n<2$. Thus, the truth of $\mathbb{P}(1)$ is a trivial thing to see. Now, assume that $\mathbb{P}(n),\ \mathbb{P}(n-1),\ \mathbb{P}(n-2),\ \ldots,\mathbb{P}(1)$ are all true.
$\Rightarrow 2>a_n>a_{n-1}>a_{n-2}>\cdots>a_1\ \ and\ \ a_{n+1}>a_n \Rightarrow a_{n+1}=\sqrt{2}^{a_n}<\sqrt{2}^2 \Rightarrow a_{n+1}<2$.
Also, note that $a_{n+2}=\sqrt{2}^{a_{n+1}}>\sqrt{2}^{a_n}=a_{n+1} \Rightarrow \mathbb{P}(n+1)$ is true.
Therefore by the second form of induction, the given statement is true for all positive integers.

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5 PROBLEM 5

let $\mathbb{P}(n) : a_1+\dfrac{a_2}{2}+\dfrac{a_3}{3}+\cdots+\dfrac{a_n}{n}\geq a_n$. Now, $\mathbb{P}(1)$ is trivially true as $a_1\geq a_1$.
let $b_n=a_1+\dfrac{a_2}{2}+\dfrac{a_3}{3}+\cdots+\dfrac{a_n}{n}$ and assume that $\mathbb{P}(n),\ \mathbb{P}(n-1),\ \mathbb{P}(n-2),\ \ldots,\mathbb{P}(1)$ are all true.
Then $b_k\geq a_k\ \forall\ k\leq n \Rightarrow \sum_{k=1}^{n} b_k \geq \sum_{k=1}^{n} a_k$. Now the term $\dfrac{a_i}{i}$ appears $n-i+1$ times in the LHS of this inequality.
$\Rightarrow \sum_{k=1}^{n}\dfrac{n-k+1}{k}a_k \geq \sum_{k=1}^{n}a_k \Rightarrow (n+1)\sum_{k=1}^{n}\dfrac{a_k}{k}-\sum_{k=1}^{n}a_k\geq\sum_{k=1}^{n}a_k$

$$\Rightarrow (n+1)\sum_{k=1}^{n}\frac{a_k}{k} \geq 2\sum_{k=1}^{n} a_k \Rightarrow (n+1)\sum_{k=1}^{n+1}\frac{a_k}{k} \geq a_{n+1} + 2\sum_{k=1}^{n} a_k$$

Now, $a_i + a_{n+1-i} \geq a_{n+1-i+i} = a_{n+1} \Rightarrow \sum_{i=1}^{n} a_i + a_{n+1-i} \geq na_{n+1}$

$\Rightarrow 2\sum_{k=1}^{n} a_k \geq na_{n+1}$. Using this fact in the previous equation we get the following:

$\Rightarrow (n+1)\sum_{k=1}^{n+1}\frac{a_k}{k} \geq a_{n+1} + 2\sum_{k=1}^{n} a_k \geq (n+1)a_{n+1} \Rightarrow \sum_{k=1}^{n+1}\frac{a_k}{k} \geq a_{n+1} \Rightarrow \mathbb{P}(n+1)$ is true. Therefore by the second form of induction, the given statement is true for all positive integers.

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6 PROBLEM 6

This problem requires a strong intuition and in order to get that intuition one must consider observing the first few terms of this sequence. Here, $a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 4,\ a_4 = 9,\ a_5 = 25,\ a_6 = 64,\ a_7 = 169$ and so on. Now, from only these terms one might get a strong intuition that $a_0 = 0$ and for rest of the terms $a_n = F_n^2$. Now, we already see that it is true for the first 8 terms of this sequence. let $\mathbb{P}(n) : a_n = F_n^2$, So, $\mathbb{P}(1),\ \mathbb{P}(2),\ \mathbb{P}(3)$ are true. Assume that $\mathbb{P}(n),\ \mathbb{P}(n-1),\ \mathbb{P}(n-2),\ \ldots,\mathbb{P}(1)$ are all true. So, $\dfrac{a_{n+1} - 3a_n + a_{n-1}}{2} + \dfrac{a_n - 3a_{n-1} + a_{n-2}}{2} = (-1)^n + (-1)^{n-1} = 0$.
$\Rightarrow a_{n+1} = 2a_n + 2a_{n-1} - a_{n-2} = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 = 2F_n^2 + 2F_{n-1}^2 - (F_n - F_{n-1})^2$

$$\Rightarrow a_{n+1} = 2F_n^2 + 2F_{n-1}^2 - F_n^2 - F_{n-1}^2 + 2F_nF_{n-1} = (F_n + F_{n-1})^2 = F_{n+1}^2$$

$\Rightarrow \mathbb{P}(n+1)$ is true. Therefore by the second form of induction, the given statement is true for all positive integers. So, every term in this sequence will be a perfect square.

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7 PROBLEM 7

Again like the previous problem this problem requires a strong intuition, so we start by writing the first few terms of this sequence. We have $a_0 = 0,\ a_1 = 1;\ a_2 = 2,\ a_3 = 6,\ a_4 = 12,\ a_5 = 25,\ a_6 = 48,\ a_7 = 91$ and so on. We can see that if $n > 0$ then a_n is divisible by n. So let us consider the sequence of terms $b_n = \dfrac{a_n}{n}$. Now, $b_1 = 1,\ b_2 = 1,\ b_3 = 2,\ b_4 = 3,\ b_5 = 5,\ b_6 = 8,\ b_7 = 13$ and so on.

Again this gives the person a strong intuition that $b_n = F_n \Rightarrow a_n = nF_n\ \forall\ n > 0$. let $\mathbb{P}(n) : a_n = nF_n$. So, the cases of $\mathbb{P}(1),\ \mathbb{P}(2)$ *and* $\mathbb{P}(3)$ are trivial. Assume that $\mathbb{P}(n+3),\ \mathbb{P}(n+2),\ \mathbb{P}(n+1),\ \ldots,\mathbb{P}(1)$ are all true.
Then, $a_{n+4} = 2(n+3)F_(n+3) + (n+2)F_{n+2} - 2(n+1)F_{n+1} - nF_n$

$\Rightarrow a_{n+4} = 2(n+3)F_{n+3} + (n+2)F_{n+2} - 2(n+1)F_{n+1} - n(F_{n+2} - F_{n+1})$
$\Rightarrow a_{n+4} = 2(n+3)F_{n+3} + 2F_{n+2} - (n+2)F_{n+1}$
$\Rightarrow a_{n+4} = 2(n+3)F_{n+3} + 2F_{n+2} - (n+2)(F_{n+3} - F_{n+2})$
$\Rightarrow a_{n+4} = (n+4)F_{n+3} + (n+4)F_{n+2} = (n+4)(F_{n+3} + F_{n+2})$
$\Rightarrow a_{n+4} = (n+4)F_{n+4} \Rightarrow \mathbb{P}(n+4)$ is true.
Therefore by the second form of induction, the given statement is true for all positive integers. So, every term a_n in this sequence will be divisible by n.

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