ZPC-22 solutions

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July 21, 2022

1 PROBLEM 1

We know that $a_i = f(i)$, now let $q(x) = f(x) - x^2$. Now, clearly $q(x)$ has roots $1,\ 2,\ 3$ and because $f(x)$ is a cubic polynomial, so $q(x)$ is also cubic.

$\Rightarrow q(x) = a(x-1)(x-2)(x-3)$ for some $a \neq 0$.

$\Rightarrow f(x) = a(x-1)(x-2)(x-3) + x^2$

Substitute $x = 4$, to get $f(4) = \lambda = 16 + 6a \Rightarrow a = \dfrac{\lambda - 16}{6}$

$\Rightarrow f(x) = \dfrac{\lambda - 16}{6}(x-1)(x-2)(x-3) + x^2$

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2 PROBLEM 2

Suppose $\exists x_0$ such that $f(x_0) = 0$

$\implies \frac{1}{(f(x_0))^2}$ is not defined.

But this is a contradiction as a polynomial is defined for all real values of the input.

$\implies \forall x f(x) \neq 0 \in \mathbf{R}$

$\implies$ f(x) can not be the zero polynomial

Suppose deg(f) = n $\geq$ 1 (1)

Now, consider $\frac{1}{f(x)^2}$,

Clearly, $\frac{1}{f(x)^2} \neq 0$, So $\frac{1}{f^2}$ is not the zero polynomial

$\implies$ deg$\left(\frac{1}{f^2}\right)$ = m ≥ 0

$\because$ deg$(f_1 f_2)$ = deg(f_1) + deg(f_2),

Using (1),

$\implies$ deg(f^2) = 2n

$\implies$ deg$\left(f^2 \times \frac{1}{f^2}\right)$ = deg(f^2) + deg$\left(\frac{1}{f^2}\right)$ = 2n + m $\geq 2 + 0 = 2$,

but since f is never 0, $f^2 \times \frac{1}{f^2} = 1$, and deg(1) = 0

$\implies$ (1) is contradicted!

$\implies$ deg(f)<1, and since f is not the zero polynomial $\implies$ deg(f) = 0

Proved

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3 PROBLEM 3

By the quadratic formula, the roots of a quadratic equation $ax^2 + bx + c = 0$ are given as: $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

If one root is rational, say equal to rational number k, then:

(suppose the plus case is rational)

$\frac{-b+\sqrt{b^2-4ac}}{2a} = k \implies \sqrt{b^2 - 4ac} = 2a \cdot k + b$

$\because$ 2, a, b and k are rationals, 2ak+b is also a rational

$\implies \sqrt{b^2 - 4ac}$ is rational.

$\implies -b - \sqrt{b^2 - 4ac}$ is rational $\implies \frac{-b-\sqrt{b^2-4ac}}{2a}$ is rational.

The minus case can be proved similarly

This means, if one root of a quadratic equation is rational, then both of them must be rational.

Now, suppose for a generic polynomial $ax^2 + bx + c = 0$, x=1 is a root,

$\because$ 1 is rational, the other root must also be rational. As 1 is a root, $a + b + c = 0$, so any set of coefficients a,b,c such that

$a+b+c = 0$ and $a, b, c \neq 0$ (to preserve quadratic nature in case of rearrangement), are valid answers.

Five such sets are:

$\{1, 1, -2\}$, $\{1, 2, -3\}$, $\{1, 5, -6\}$, $\{3, -1, -2\}$, $\{11, 11, -22\}$

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4 PROBLEM 4

Let p be an integer and let $S = \{x : gcd(x, p) = 1\}$ (set of all positive integers coprime to p).

Claim 1: S contains infinitely many elements

Proof: All integers $z = np + 1$ where $n \in \mathbb{N}$ are in S, so S has infinitely many elements. Now, let $g(x) = f(x) - p$

Claim 2: $g(x)$ has infintely many roots

Proof: Note that for any integer $q \in S$, $g\left(\dfrac{p^2}{q^2}\right) = f\left(\dfrac{p^2}{q^2}\right) - p = p - p = 0$. Since, there are infinitely many integers in S, so $g(x)$ has infinitely many roots.

Since $g(x)$ has infinitely many roots, so it must be a constant function otherwise it will have infinite degree. So, $f(x) = p\ \forall x$, but $f(q^2/p^2) = q$. This contradicts the given facts. So, there is no such polynomial $f(x)$. Hence, disproved.

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5 PROBLEM 5

Let $f(x) = P(x) - 5 \Rightarrow a,\ b,\ c,\ d$ are the roots of $f(x)$

$\Rightarrow P(x) = g(x)(x-a)(x-b)(x-c)(x-d) + 5$ for some polynomial $g(x)$.

Substituting $x = k$ we get, $P(k) = g(k)(k-a)(k-b)(k-c)(k-d) + 5$

$\Rightarrow 8 = g(k)(k-a)(k-b)(k-c)(k-d) + 5 \Rightarrow 3 = g(k)(k-a)(k-b)(k-c)(k-d)$

Now,$P(x)$ has only integer coefficients and all of $g(k),\ (k-a),\ (k-b),\ (k-c),\ (k-d)$ are integers. So, one of them will be equal to ± 3 and rest all will be ± 1. Now, this is noly possible if atleast two factors of the form $k - x$ are equal to either 1 or -1. Without loss of generality let us assume $k - a = k - b \Rightarrow a = b$. But this would be a contradiction as $a,\ b,\ c,\ d$ are all distinct. So, there is no such integer k.

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6 PROBLEM 6

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \Rightarrow f(1) = a_n + a_{n-1} + \cdots + a_1 + a_0$

$f(f(n)+1) = a_n(f(n)+1)^n + a_{n-1}(f(n)+1)^{n-1} + \cdots + a_1(f(n)+1) + a_0$

$\Rightarrow f(f(n)+1) = kf(n) + a_n + a_{n-1} + \cdots + a_1 + a_0 = kf(n) + f(1)$ for some $k \in \mathbb{N}$

$\Rightarrow f(f(n)+1) = f(1) \mod f(n)$. So, for $f(n)$ to divide $f(f(n)+1)$, $f(n)$ must divide $f(1)$.

For $n = 1$, $f(n) = f(1)$. So, $f(n)$ divides $f(f(n)+1)$ in this case.

Now, if $f(n)$ divides $f(f(n)+1) \Rightarrow f(n)$ divides $f(1)$. Also, note that because $f(x)$ has all positive coefficients, so $f(n) > f(1)\ \forall\ n > 1$. Thus for any $n > 1$, $f(n) > f(1) \Rightarrow f(n)$ cannot divide $f(1)$.

Thus, $f(n)$ divides $f(f(n)+1) \Rightarrow n = 1$.

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7 PROBLEM 7

Note that all such polynomials $f(x)$ will have a positive leading coefficient because if $f(x)$ has a negative leading coefficient then $\lim_{x\to\infty} f(x) + f'(x) < 0$.
Also note that all such $f(x)$ will have even degree because if $f(x)$ has odd degree then $\lim_{x\to-\infty} f(x) + f'(x) < 0$.
Since, $f(x)$ has even degree , so it must have even number of real roots. Let $\alpha,\ \beta,\ \ldots$ be the real roots of $f(x) \Rightarrow f'(\alpha) > 0$ and $f'(\beta) > 0$ and so on (because $f(\alpha) = f(\beta) = \cdots = 0$). But this implies that for every real zero of $f(x)$, $f(x)$ goes from negative to positive value, but to go from negative to positive value , we first need to go to some negative value. Now, this is a contradiction as this implies that at some zero, we will be going from positive to negative value.
Thus, $f(x)$ must have no real roots.
Now, $\lim_{x\to\infty} f(x) > 0$ and $f(x)$ has no real roots, so, $f(x)$ is always positive(always above the x-axis).

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