ZPC-25: Principle of Inclusion-Exclusion

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1 PROBLEM 1

a) Include multiples of 3 and of 17 but exclude multiple of 51 (as common)

$\lfloor\frac{2022}{3}\rfloor + \lfloor\frac{2022}{7}\rfloor - \lfloor\frac{2022}{51}\rfloor = 674 + 118 - 39 = 753$

However, we want integers less than 2022 and 2022 is perfectly divisible by 3 and not 17, so the final answer is <u>752</u>.

b) The number of perfect powers of exponent n greater than 1 and less than 2022 is $\left\lfloor 2022^{\frac{1}{n}} \right\rfloor - 1$. The trick is to include 1 and perfect powers of prime exponents and exclude square-free powers (product of distinct primes) as they are shared perfect powers

Including prime, $\left\lfloor 2022^{\frac{1}{2}} \right\rfloor - 1 + \left\lfloor 2022^{\frac{1}{3}} \right\rfloor - 1 + \left\lfloor 2022^{\frac{1}{5}} \right\rfloor - 1 + \left\lfloor 2022^{\frac{1}{7}} \right\rfloor - 1 = 43 + 11 + 3 + 1 = 58$

Excluding square-free, $\left\lfloor 2022^{\frac{1}{6}} \right\rfloor - 1 + \left\lfloor 2022^{\frac{1}{10}} \right\rfloor - 1 = 2 + 1 = 3$

Total perfect powers less than 2022 is 58-3+1, so final answer is <u>56</u>.

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2 PROBLEM 2

a) Include odd numbers in odd number positions and even numbers in even number positions (they are free to permute among themselves), to make the number of possible permutations $(5!)^2$

Since the question just states alternation and not whether the sequences starts with an odd number or even, the final answer would be, $\frac{2\times(5!)^2}{10!}$

b) Exclude the number of permutations in which at least one of 1,4,9,6 and 5 are fixed in their natural position (which is found by more inclusion and exclusion)

For at least one of 1,4,9,6 or 5 to be fixed, $\binom{5}{1}\times 9! - \binom{5}{2}\times 8! + \binom{5}{3}\times 7! - \binom{5}{4}\times 6! + \binom{5}{5}\times 5!$

Thus, the final answer will be $\frac{10! - \binom{5}{1}\times 9! + \binom{5}{2}\times 8! - \binom{5}{3}\times 7! + \binom{5}{4}\times 6! - \binom{5}{5}\times 5!}{10!}$

c) Similarly, exclude the number of permutations in which at least one of the numbers are fixed in their natural position to get the final answer as a summation $\frac{1}{10!}\times\sum_{i=0}^{10}(-1)^i\binom{10}{i}(10-i)!$ or $\sum_{i=0}^{10}\frac{(-1)^i}{i!}$

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3 PROBLEM 3

We can notice that $8^3 = 512 > 500$, so we must include numbers to be cubed from 1 to 7 only. Out of all combinations, we can exclude $6^3 + 7^3$ and $7^3 + 7^3$ as their sum exceeds 500 and every other combination has a sum lower than these three, making the number of combinations as $7^2 - 3$ or 46 and the final answer as $46 - 20 = 26$ numbers

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4 PROBLEM 4

If a number is co-prime to n, it has to be co-prime to all of the prime powers present in the prime factorisation of $n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$, giving us the formula $\phi(n) = \phi(p_1^{a_1}) + \cdots + \phi(p_r^{a_r})$ (here we multiply instead of add for inclusion, as each combination of one element chosen from across all the sets of $\phi(p_i^{a_i})$ gives a number co-prime to n, Chinese Remainder Theorem)

And we can see that $\phi(p_i^{a_i}) = p_i^{a_i} - p_i^{a_i - 1} = p_i^{a_i}(1 - \frac{1}{p_i})$ (There are $p_i^{a_i}$ elements, from which we exclude smaller powers of p_i, because they are the only ones which will give a gcd greater than 1 when compared with $p_i^{a_i}$)

This gives us our identity $\phi(n) = \prod_{i=1}^{r} p_i^{a_i}(1 - \frac{1}{p_i}) = n \prod_{i=1}^{r}(1 - \frac{1}{p_i})$

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5 PROBLEM 5

There are multiple ways to give a mapping from subsets of W, the set of functions from A=$\{a_1, \ldots, a_m\}$ to B=$\{b_1, \ldots, b_n\}$ to natural numbers, from which we can include and exclude cases to get the number of onto functions (like some of the examples given below):

- $\psi : \epsilon_i \longrightarrow i$, where $\epsilon_i \in W$ such that b_i is not in the range of ϵ_i and $1 \leq i \leq n$
- $\xi : \epsilon_i \longrightarrow i$, where $\epsilon_i \in W$ such that b_i is in the range of ϵ_i and $1 \leq i \leq n$
- $\nu : \epsilon_i \longrightarrow i$, where $\epsilon_i \in W$ such that the size of the range of ϵ_i is less than or equal to i and $1 \leq i \leq n$

We will be working with only ν for convenience. The total number of functions from A to B will be n^m or $|W| = n^m$, from which we will exclude all functions of range with size less than n.

The number of pre-images of i in ν will be $\binom{n}{i} \times i^m$

Thus, the final answer will be $\sum_{i=0}^{n-1}(-1)^i\binom{m}{i}(n-i)^m$

Note: Since there are multiple mappings, you can try to find the given result by yourself, using a different mapping

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6 PROBLEM 6

We have the constraints $x_1 \geq 0, 0 \leq x_2 \leq 20, x_3 \geq 0, x_4 \geq 10$

Take $x_4' = x_4 - 10$, to get $x_1 + x_2 + x_3 + x_4' = 40$

With the constraints $x_1 \geq 0, 0 \leq x_2 \leq 20, x_3 \geq 0, x_4 \geq 0$

In $x_2 \geq 0$, the only two cases are $0 \leq x_2 \leq 20$ and $x_2 \geq 21$, which tells us what to exclude and from what

For $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$, the number of solutions is $\binom{40+4-1}{4-1}$

For $x_1 \geq 0, x_2 \geq 21, x_3 \geq 0, x_4 \geq 0$, the number of solutions is $\binom{40+4-21-1}{4-1}$

Thus, the final answer is $\binom{43}{3} - \binom{22}{3}$

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7 PROBLEM 7

Randomly select two connected points on the plane p_1 and p_2 and the set of all points connected to p_1 be A and that to p_2 be B.

From question, $|A| \geq k-1$ and $|B| \geq k-1$, so $n-2 \geq |A \cup B| = |A|+|B|-|A \cap B|$

$\implies n-2 \geq |A|+|B|-|A \cap B| \geq 2(k-1)-|A \cap B|$

$\implies |A \cap B| \geq 2k-n > 0$

Thus there exists at least one point that is connected to both A and B, forming a triangle.

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8 PROBLEM 8

Let's look at the problem in an another way. We aim to find the least number of integers we can remove from S to leave a set which does not contain 6 pairwise coprime integers. Let $S_p = \{p^k | p^k \leq 280\}$ and more generally $S_{p_1, p_2, \ldots, p_l} = \{n = p_1^{q_1} p_2^{q_2} \ldots . p_l^{q_l} | n \leq 280\}$ and finally $S_1 = \{1\}$. It is clear that these sets taken over all collections of primes and 1 forms a partition of S.

If there are representatives from five sets S_p, there are six mutually coprime integers, so all but five of the sets S_p must be completely removed. It is clear that (with the exception of 1, which can clearly be removed first) if $p > q$, $|S_p| \leq |S_q|$ so it is worth removing S_p before removing S_q (if we are striving for a minimum).

Furthermore, if we remove two sets S_p, S_q, we must also (to stop there from being 6 mutually coprime integers) remove $S_{p,q}$ and so on, and these have similar ordering relations. So once we have removed everything we need to keeping sizes of sets removed to a minimum, we are left with $S_2, S_3, S_5, S_7, S_{11}$ and the multi-index sets not coprime to all of these. In other words, we have the set of multiples of 2,3,5,7 and 11, which can be calculated (by lots and lots of inclusion-exclusion) to have cardinality 198. Therefore, a subset of S of size 199 must contain 6 coprime integers, i.e. $m = 199$.

Trivially, $M = 250$ as there is only one subset of S that has 250 elements and there are definitely at least 6 primes in S, which makes the final answer as $2\sqrt{M - m} = 2\sqrt{250 - 199} = 2\sqrt{51}$

Note: This question is a variation of IMO 1991/3. Do check out that question as well)

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