ZPC-20 solutions

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1 PROBLEM 1

Note that when $a+b+c=0$ then $a^3+b^3+c^3=3abc$ or $S_3=abc$ and

$a^2+b^2+c^2=-2(ab+bc+ca)$ or $S_2=-(ab+bc+ca)$

Now, $a^5+b^5+c^5=(a^2+b^2+c^2)(a^3+b^3+c^3)-a^3b^2-a^3c^2-b^3a^2-b^3c^2-c^3a^2-c^3b^2$

$\Rightarrow a^5+b^5+c^5=(a^2+b^2+c^2)(a^3+b^3+c^3)-a^2b^2(a+b)-b^2c^2(b+c)-c^2a^2(c+a)$

$\Rightarrow a^5+b^5+c^5=(a^2+b^2+c^2)(a^3+b^3+c^3)-a^2b^2(-c)-b^2c^2(-a)-c^2a^2(-b)$

$\Rightarrow a^5+b^5+c^5=(a^2+b^2+c^2)(a^3+b^3+c^3)+abc(ab+bc+ca)$

$\Rightarrow a^5+b^5+c^5=(a^2+b^2+c^2)(a^3+b^3+c^3)-\dfrac{(a^3+b^3+c^3)}{3}.\dfrac{(a^2+b^2+c^2)}{2}$

$\Rightarrow a^5+b^5+c^5=\dfrac{5(a^2+b^2+c^2)(a^3+b^3+c^3)}{6}$

$\Rightarrow \dfrac{a^5+b^5+c^5}{5}=\dfrac{(a^3+b^3+c^3)}{3}.\dfrac{(a^2+b^2+c^2)}{2} \Rightarrow S_5=S_2.S_3$

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2 PROBLEM 2

Compare the given recurrence relation with $a_n=f(n)a_{n-1}+g(n)$

So, $f(n)=\dfrac{2}{3}$ and $p_r=\left(\dfrac{2}{3}\right)^r \Rightarrow a_n=\left(\dfrac{2}{3}\right)^n\left(\dfrac{3}{2}+\displaystyle\sum_{r=2}^{n}\left(\dfrac{3}{2}\right)^r(r^2-15)\right)$

Now, let $S=\displaystyle\sum_{r=2}^{n}\left(\dfrac{3}{2}\right)^r(r^2-15)$

$\Rightarrow S - \frac{3}{2}S = (2^2 - 15)\frac{9}{4} + \sum_{r=3}^{n} \left(\frac{3}{2}\right)^r (r^2 - 15) - \sum_{r=3}^{n} \left(\frac{3}{2}\right)^r ((r-1)^2 - 15) - (n^2 - 15)\left(\frac{3}{2}\right)^{n+1}$

$\Rightarrow -\frac{1}{2}S = -11\left(\frac{9}{4}\right) + 5\left(\frac{3}{2}\right)^3 + 7\left(\frac{3}{2}\right)^4 + \cdots + (2n-1)\left(\frac{3}{2}\right)^n - (n^2 - 15)\left(\frac{3}{2}\right)^{n+1}$

$\Rightarrow -\frac{1}{2}S + \frac{3}{2}.\frac{1}{2}S = -11\left(\frac{3}{2}\right)^2 + 16\left(\frac{3}{2}\right)^3 + 2\left(\frac{3}{2}\right)^4 + 2\left(\frac{3}{2}\right)^5 + \cdots + 2\left(\frac{3}{2}\right)^n - (n^2 + 2n - 16)\left(\frac{3}{2}\right)^{n+1} + (n^2 - 15)\left(\frac{3}{2}\right)^{n+2}$

Solving the above expression by summing the geometric progression, we get

$\Rightarrow S = 36 + \left(\frac{3}{2}\right)^n (3n^2 - 12n - 15) \Rightarrow a_n = 25\left(\frac{2}{3}\right)^{n-1} + 3n^2 - 12n - 15$

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3 PROBLEM 3

We use the trigonometric substitution $x = \sin^2\theta$ to simplify the following integral. Note that in this case $dx = 2\sin\theta\cos\theta$ and at $x = 0$, $\theta = 0$ and at $x = 1$, $\theta = \frac{\pi}{2}$.

$\int_0^1 x^n(1-x)^n dx = 2\int_0^{\pi/2} \sin^{2n}\theta\cos^{2n}\theta\sin\theta\cos\theta d\theta = 2\int_0^{\pi/2} \sin^{2n+1}\theta\cos^{2n+1}\theta d\theta$

Now, let $\int_0^{\pi/2} \sin^{2n+1}\theta\cos^{2n+1}\theta d\theta = I_n$. Then, we have $I_n = \frac{n}{2(2n+1)}I_{n-1}$ and $I_0 = \frac{1}{2}$

Now, $I_n = \frac{n}{2(2n+1)}I_{n-1} = \frac{n}{2(2n+1)}.\frac{n-1}{2(2n-1)}I_{n-2} = \ldots$

$= \frac{1}{2^n}\frac{n!}{(2n+1).(2n-1)\ldots 3.1}I_0 = \frac{1}{2^n}\frac{n!.n!}{(2n+1).(2n-1)\ldots 3.1.n!}I_0 = \frac{(n!)^2}{(2n+1)!}I_0$

$\int_0^1 x^n(1-x)^n dx = 2I_n = 2\frac{(n!)^2}{(2n+1)!}I_0 = 2\frac{(n!)^2}{(2n+1)!}\frac{1}{2} = \frac{(n!)^2}{(2n+1)!}$

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4 PROBLEM 4

We have $a_n = a_{n-1} + 2b_{n-1} \Rightarrow b_{n-1} = \dfrac{a_n - a_{n-1}}{2}$ and $b_n = \dfrac{a_{n+1} - a_n}{2}$
Substituting the above values of b_n and b_{n-1} in $b_n = -a_{n-1} + 4b_{n-1}$, we get $a_{n+1} = 5a_n - 6a_{n-1}$. Now this is a normal recurrence relation with characteristic equation $x^2 - 5x + 6 = 0$. Therefore on solving the equation we get $x = 2, 3$. So, $a_n = c_1 2^n + c_2 3^n$. Now, $a_0 = 0 \Rightarrow c_1(1) + c_2(1) = 0 \Rightarrow c_1 = -c_2$. Also, $a_1 = 2 \Rightarrow 2 = c_1(2) - c_1(3) \Rightarrow c_1 = -2$ and $c_2 = 2$. So, $a_n = 2(3^n - 2^n)$. Now, $b_n = -a_{n-1} + 4b_{n-1} \Rightarrow 4b_{n-1} - b_n = a_{n-1}$ and $4b_n - b_{n+1} = a_n$. Substituting these two values in $a_n = a_{n-1} + 2b_{n-1}$ we get $b_{n+1} = 5b_n - 6b_{n-1}$. This recurrence has the characteristic equation $x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3 \Rightarrow b_n = d_1(2)^n + d_2(3)^n$. Now , $b_0 = 1$ and $b_1 = 4 \Rightarrow d_2 = 2, d_1 = -1$. So, $b_n = 2(3)^n - 2^n$.

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5 PROBLEM 5

Rearrange the given expression as $x_{n+1} = \dfrac{x_n - (\sqrt{2} - 1)}{1 + (\sqrt{2} - 1)x_n}$ and write $(\sqrt{2} - 1)$ as $\tan \dfrac{\pi}{8}$. Now, use the trigonometric substitution $x_n = \tan \theta_n$. After doing all this the expression reduces to the following expression:
$x_{n+1} = \dfrac{\tan \theta_n - \tan \frac{\pi}{8}}{1 + \tan \frac{\pi}{8} \tan \theta_n} = \tan\left(\theta_n - \dfrac{\pi}{8}\right) = \tan \theta_{n+1}$ and similarly, $\tan \theta_n = \tan\left(\theta_{n-1} - \frac{\pi}{8}\right)$

$$\Rightarrow \frac{\tan \theta_n - \tan \frac{\pi}{8}}{1 + \tan \frac{\pi}{8} \tan \theta_n} = \frac{\tan\left(\theta_{n-1} - \frac{\pi}{8}\right) - \tan \frac{\pi}{8}}{1 + \tan \frac{\pi}{8} \tan\left(\theta_{n-1} - \frac{\pi}{8}\right)} \Rightarrow \tan \theta_{n+1} = \tan\left(\theta_{n-1} - 2.\frac{\pi}{8}\right).$$

Continuing this procedure over and over, we get

$$\tan \theta_{n+1} = \tan\left(\theta_n - \frac{\pi}{8}\right) = \tan\left(\theta_{n-1} - 2.\frac{\pi}{8}\right) = \tan\left(\theta_{n-2} - 3.\frac{\pi}{8}\right) \ldots$$

$$\Rightarrow \tan \theta_{n+1} = \tan\left(\theta_1 - n.\frac{\pi}{8}\right) \Rightarrow x_{2021} = \tan \theta_{2021} = \tan\left(\theta_1 - 2020.\frac{\pi}{8}\right)$$

$$\Rightarrow x_{2021} = \tan\left(\theta_1 - \frac{\pi}{2}\right) = -\cot \theta_1 = -\frac{1}{\tan \theta_1} = -\frac{1}{x_1} = -\frac{1}{2020}$$

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6 PROBLEM 6

First point to note is that both x_n and y_n are always positive. Here we use the trigonometric substitutions $x_n = \cot\alpha_n$ and $y_n = \tan\theta_n$ to simplify the problem.

$\Rightarrow x_{n+1} = \cot\alpha_{n+1} = \cot\alpha_n + \sqrt{1+\cot^2\alpha_n} = \cot\alpha_n + \csc\alpha_n = \dfrac{1+\cos\alpha_n}{\sin\alpha_n}$

$= \dfrac{2\cos^2(\alpha_n/2)}{2\sin(\alpha_n/2)\cos(\alpha_n/2)} = \cot\dfrac{\alpha_n}{2} \Rightarrow \cot\alpha_{n+1} = \cot\dfrac{\alpha_n}{2} = \cdots = \cot\dfrac{\alpha_1}{2^n}$

Now , since all the terms of the sequence are positive and $\cot\alpha_1 = \sqrt{3}$

$\Rightarrow \alpha_1 = \dfrac{\pi}{6} \Rightarrow x_n = \cot\left(\dfrac{\pi}{3.2^n}\right)$. Now, we do the similar operations on y_n.

$y_{n+1} = \tan\theta_{n+1} = \dfrac{\tan\theta_n}{1+\sqrt{1+\tan^2\theta_n}} = \dfrac{\tan\theta_n}{1+\sec\theta_n} = \dfrac{\sin\theta_n}{1+\cos\theta_n} = \tan\dfrac{\theta_n}{2}$

$\Rightarrow \tan\theta_{n+1} = \tan\dfrac{\theta_n}{2} = \tan\dfrac{\theta_{n-1}}{2^2} = \cdots = \tan\dfrac{\theta_1}{2^n}$. Now, since all terms are positive and $\tan\alpha_1 = \sqrt{3} \Rightarrow \theta_1 = \dfrac{\pi}{3} \Rightarrow y_n = \tan\left(\dfrac{\pi}{3.2^{n-1}}\right)$

Now, $x_n y_n = \cot\left(\dfrac{\pi}{3.2^n}\right)\tan\left(\dfrac{\pi}{3.2^{n-1}}\right)$. Assume $\dfrac{\pi}{3.2^n} = k$

$x_n y_n = \cot(k)\tan(2k) = \dfrac{2}{1-\tan^2 k}$. Now, note that for $n > 1, 0 < k < \dfrac{\pi}{6}$

$\Rightarrow \tan k < \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}} \Rightarrow 1-\tan^2 k > \dfrac{2}{3} \Rightarrow \dfrac{2}{1-\tan^2 k} < 3 \Rightarrow x_n y_n < 3 \;\forall\; n > 1$

. Also, $0 < \tan k \Rightarrow 1-\tan^2 k < 1 \Rightarrow \dfrac{1}{1-\tan^2 k} > 1 \Rightarrow \dfrac{2}{1-\tan^2 k} > 2$

$\Rightarrow x_n y_n > 2 \Rightarrow 2 < x_n y_n < 3 \;\forall\; n > 1$

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7 PROBLEM 7

We have $a_{n+1} = 5a_n + \sqrt{1+24a_n^2} \Rightarrow (a_{n+1}-5a_n)^2 = 1+24a_n^2$

$\Rightarrow a_{n+1}^2 + a_n^2 - 10a_na_{n+1} = 1 \Rightarrow a_{n-1}^2 + a_n^2 - 10a_na_{n-1} = 1 = a_{n+1}^2 + a_n^2 - 10a_na_{n+1}$

$a_{n-1}^2 + a_n^2 - 10a_na_{n-1} = a_{n+1}^2 + a_n^2 - 10a_na_{n+1} \Rightarrow a_{n-1}^2 - 10a_na_{n-1} = a_{n+1}^2 - 10a_na_{n+1}$

$\Rightarrow (a_{n+1}-a_{n-1})(a_{n+1}+a_{n-1}) = 10a_n(a_{n+1}-a_{n-1}) \Rightarrow a_{n+1} = 10a_n - a_{n-1}$

Now the recurrence thus obtained has the characteristic equation $x^2 - 10x + 1 = 0$.

So, we get $x = 5 \pm 2\sqrt{6} \Rightarrow a_n = c_1(5+2\sqrt{6})^n + c_2(5-2\sqrt{6})^n$

Now, $a_1 = 0$ and $a_2 = 1 \Rightarrow 5(c_1+c_2)+2\sqrt{6}(c_1-c_2) = 0$ and $49(c_1+c_2)+20\sqrt{6}(c_1-c_2) = 1$. Solving these two equations we get, $c_1 = \frac{1}{4\sqrt{6}}\left(5-2\sqrt{6}\right)$ and the value of $c_2 = -\frac{1}{4\sqrt{6}}\left(5+2\sqrt{6}\right)$.

Using these values we get, $a_n = \frac{1}{4\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^n - \left(5-2\sqrt{6}\right)^n\right)$

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