ZPC-26

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Instructions

Please read the following instructions carefully before proceeding further:

1. The test is of 2 hours. It will end *sharp* at **8:00 pm**.

2. Relevant reading material for all the questions has been provided in the document itself.

3. If you are stuck on a question that you cannot figure out, move on. Nothing good ever comes out of being hung up on something.

4. Please feel free to contact the invigilators in case of any queries.

5. We will provide hints to questions in the last hour, depending on the participation and responses.

6. All the best. GL HF :)

HOMOTHETY

1 Introduction

Homothety is one of the most interesting things in Euclidean geometry. Before formally defining homothety, we give some examples to give the reader a feeling of what homothety is.

Speaking in simpler terms, Homothety is nothing but a dilation of a geometric figure about a center. You can think of Homothety as scaling a mathematical figure about a point. For example, see the following figure:
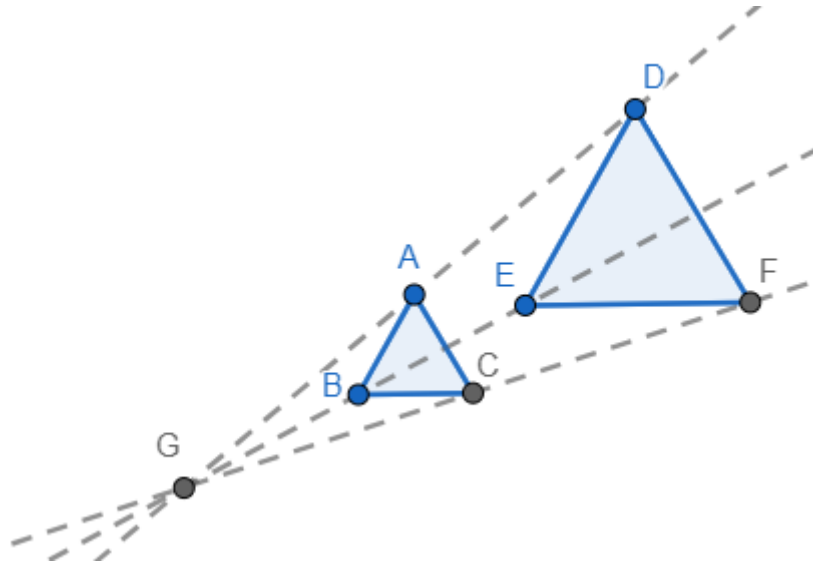
The above figure can be called a homothety from ΔABC to ΔDEF about point G. It can also be seen as a homothety from ΔDEF to ΔABC about point G.

That gives you the basic intuition of Homothety. Now, we move to the formal definition.

2 Formal Definition of Homothety

From the previous section, we understand that a homothety or dilation or central similarity is a special type of similarity, in which a figure is dilated from a center.

It is just equivalent to scaling a figure about a center. The formal definition of Homothety is as follows:

Definition 1 (Homothety). As homothety h is a geometric transformation defined by a center O and a real number k. It sends a point P to another point $h(P)$, multiplying the distance from O by k. The number k is called the scale factor of the homothety.

For example, the below figure shows a homothety h that maps the points of a triangle $\triangle ABC$ to $\triangle h(A)h(B)h(C)$ about the point O and scales it by a factor of $k = 3$.
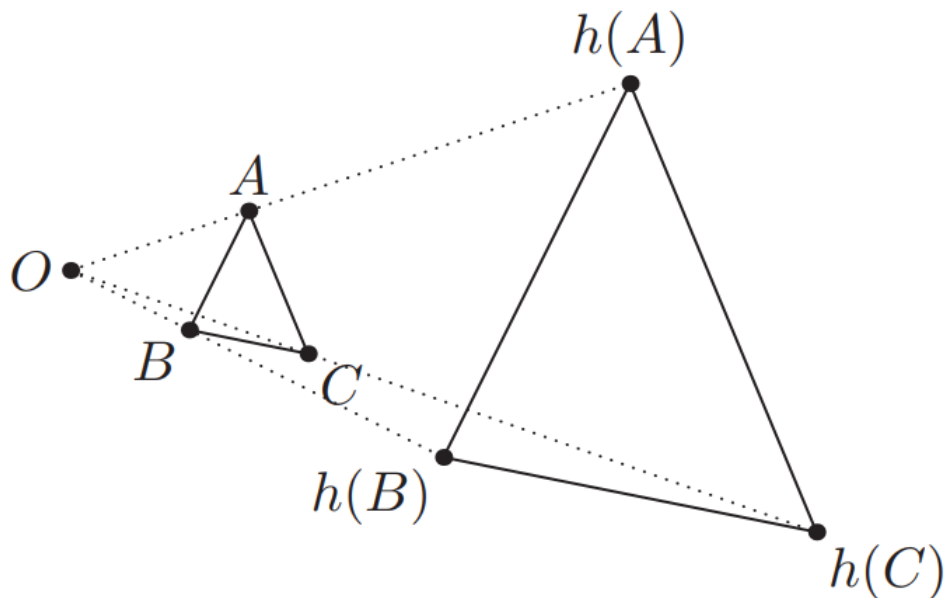
Note that on changing the center O the homothety can completely change. Also, by changing the factor k, the homothety changes. It is important to note that k can be negative, in which case we have a negative homothety and the two figures lie on the opposite sides of the center O. The following figure shows a negative Homothety.
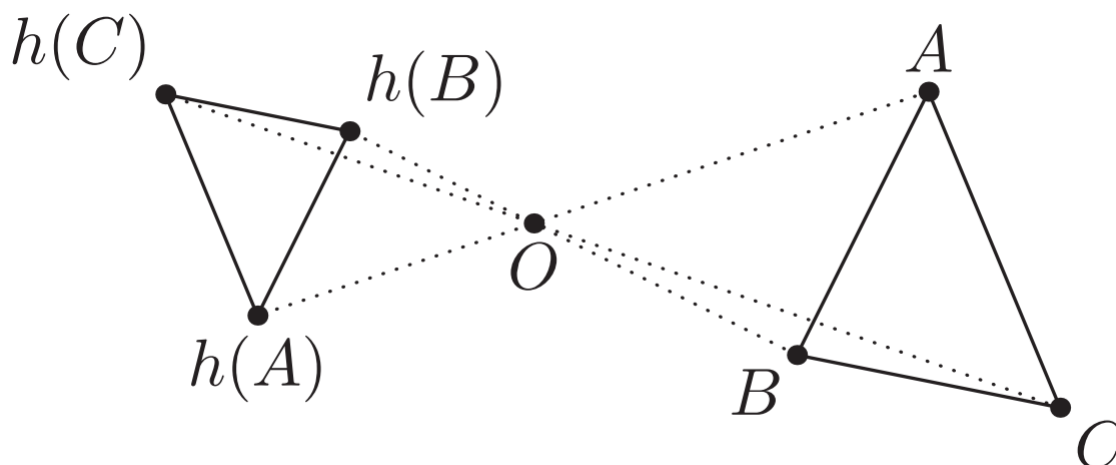
Homothety preserves many things, including but not limited to tangency, angles, circles, and so on. They do not preserve length, but they work well enough: the lengths are simply all multiplied by k.

3 More about Homothety

Given non-congruent parallel segments $\overline{AB}$ and $\overline{XY}$ (what happens if $\overline{AB} = \overline{XY}$?), we can consider the intersection point O of lines $\overline{AX}$ and $\overline{BY}$. This is the center of a homothety sending one segment to the other. (As is the intersection of lines $\overline{AY}$ and $\overline{BX}$—one of these is negative.) As a result, parallel lines are often indicators of homotheties. It is useful for proving collinearity, concurrency, determining ratios, and constructing points.

Any homothety satisfies the following properties:

1. Under a homothety, parallel lines map onto parallel lines.

2. Homothety preserves angles.

3. Homothety preserves orientation.

4. Every homothety has an inverse.(Homothety with factor k will be cacelled by a homothety of factor $\frac{1}{k}$)

5. Successive application of two homotheties with factors k_1 and k_2 can be thought as a homothety with factor $k_1 k_2$. This is called the multiplication of homotheties. Note that this multiplication of homotheties is not commutative because even though the resulting homotheties will have the same factors but they my have different positions in space.

6. Product of homotheties is not associative.

PROBLEMS

- The questions here are all subjective, and will be graded on how well you can convince the checker of the mathematical rigour of your solution.

- The corresponding points of each question are mentioned with it. Do note that the points may or may not be proportionate to the relative difficulty of the question.

- The number of problems presented is much more than what is humanly possible to solve in the given time frame. Hence, prioritise correctness and rigour over number of problems solved.

- Note that any significant conclusion derived in the right direction to solve the problem may fetch you some points, so try to attempt all questions.

1. For every two non-intersecting circles, prove that there exists at least 2 homotheties mapping one to each other (1 point)

2. Let $ABCD$ be a parallelogram and let E be a point on the side AB for which $AE : EB = 2 : 1$. Suppose that BD and CE intersect at F. Determine $BF : FD$ (2 points)

3. Prove Pythagoras' theorem using homotheties (2 points)

4. Let the incircle of triangle PQR touch side QR at D, and let DT be a diameter of the circle. If line pT meets qr at X, then $QD = RX$ (3 points)

5. Let H be the orthocentre of triangle ABC, then prove that the midpoints of the sides, the pedal points (feet of the altitudes) and the midpoints of AH,

BH, CH all lie on a common circle (3 points)

6. ABC and $A'B'C$ are similarly oriented equilateral (differ only by a rotation) triangles intersecting only at C; P,Q,R are the respective midpoints of AB', BC, $A'C$. Prove that triangle PQR is equilateral (4 points)

7. Triangle ABC has orthocenter H, incenter I and circumcenter O. Let K be the point where the incircle touches BC. If $IO \parallel BC$, then prove that $AO \parallel HK$ (5 points)

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