ZPC-20

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Instructions

Please read the following instructions carefully before proceeding further:

1. The test is of 2 hours. It will end *sharp* at **8:00 pm**.
2. It is a closed web and closed book test (Except the relevant reading material provided of course).
3. The answers to all the questions are subjective. Please make a pdf (hand-written, word, or LaTeX) of your solutions and email them to evariste@sc.iiitd.ac.in. Please include your *name, roll number, branch, year* in the email.
4. Any submission received after **8:05 pm** will not be evaluated.
5. Relevant reading material for all the questions has been provided in the document itself.
6. If you are stuck on a question that you cannot figure out, move on. Nothing good ever comes out of being hung up on something.
7. Please feel free to reach out to us on the Google Meet in case of any queries.
8. We will provide hints to questions in the last hour, depending on the participation and responses.
9. All the best. GL HF :)

RECURRENCE RELATIONS

1 Introduction to Recurrences

A recurrence relation is a mathematical equation that recursively defines the next term of a mathematical sequence as a function of its previous terms. In general, if a_n is a sequence then $a_n = f(a_{n-1}, a_{n-2}, \dots, a_1, a_0)$ is a recurrence relation for the n^{th} term of this sequence. In the last ZPC we talked about the Fibonacci Sequence and the Binet's formula but there we just stated the result and today we'll see how such recurrences work and how do we get to some fascinating results like the Binet's formula. Whenever we talk about a recurrence relation, a very natural question that might come to your mind is that whether we can define the recursive sequence without using the recurrence relation. For example, the Fibonacci Sequence is defined by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ but it can also be defined without recurrences using the Binet's formula as follows.

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

This is what recurrence relations is all about. Finding the general term without using recurrence relations. Recurrence relations is a pretty interesting topic and is used in almost every field of mathematics. But considering the limited amount time we have, we will only be looking at some common type of recurrences(this part is sad :(but we cannot do anything about it).

2 Classification Of Recurrence Relations

For classification of recurrences, we first need to know some common terminology.

Order: It is the difference in the biggest and the lowest subscript in a recurrence relation. For example, $a_n = a_{n-1} + a_{n-2}$ has the order $n - (n-2) = 2$. Note that for recurrences like $a_n = 2a_{\frac{n}{2}}$, the order is not defined.

Linearity: We say that a recurrence relation is linear if the the recurrence relation does not have the product of various terms of a sequence in it. For example, $a_n a_{n-1} = n$ is non-linear and has order 1 and $a_n = 2a_{n-3}$ is linear and has order 3.

Homogeneity: Consider the recurrence relation $f_0(n)a_n + f_1(n)a_{n-1} + \cdots + f_r(n)a_{n-r} = g(n)$ where $f_i(n)$ and $g(n)$ are some arbitrary known functions of n. Then, the recurrence relation is homogeneous if $g(n) = 0$. In other words a recurrence relation is homogeneous if there is no term having a function of n but not having a term of sequence in multiplication. For example, $a_n = -na_{n-1}$ is a homogeneous recurrence relation and $a_n + 2a_{n-1} = 1$ is non-homogeneous.

For today, we will restrict ourselves to recurrences of order 1 and 2 only.

3 First Order Linear Recurrence Relations

Such equations are of the form $a_n = f(n)a_{n-1} + g(n), n \geq 2$ where $f(n)$ and $g(n)$ are known functions of n and $f(n) \neq 0$.
Divide the whole equation by $p_n = f(1).f(2).\dots.f(n)$ and then rewrite as follows:

$$\frac{a_n}{p_n} - \frac{a_{n-1}}{p_{n-1}} = \frac{g(n)}{p_n}$$

Now, let $v_n = \dfrac{a_n}{p_n} \Rightarrow v_n - v_{n-1} = \dfrac{g(n)}{p_n} \Rightarrow \displaystyle\sum_{r=2}^{n} v_r - v_{r-1} = \sum_{r=2}^{n} \frac{g(r)}{p_r}$
$\Rightarrow v_n - v_1 = \displaystyle\sum_{r=2}^{n} \frac{g(r)}{p_r} \Rightarrow v_n = v_1 + \sum_{r=2}^{n} \frac{g(r)}{p_r} \Rightarrow a_n = p_n \left(\frac{a_1}{f(1)} + \sum_{r=2}^{n} \frac{g(r)}{p_r} \right)$
Now, if the equation is homogeneous then $g(r) = 0, \forall r$
$\Rightarrow a_n = [f(n).f(n-1)\dots f(2)]a_1$
If $f(n) = c_1$ and $g(n) = c_2$ i.e. if they are constants then we can further simplify the formula as follows:

$$a_n = a_1 c_1^{n-1} + c_2 \left(c_1^{n-2} + c_1^{n-3} + \cdots + 1\right)$$

Now, if $c_1 = 1$ then $a_n = a_1 + (n-1)c_2$ which is the formula for n^{th} term of an Arithmetic progression. If $c_1 \neq 1$, then $a_n = a_1 c_1^{n-1} + c_2 \dfrac{(c_1^{n-1} - 1)}{c_1 - 1}$

4 Second Order Linear Recurrence Relations

Consider the recurrence relation $a_n = pa_{n-1} + qa_{n-2}$ where p and q are constants. Assume that the solutions of this equation are of the form $a_n = cx^n$.
$\Rightarrow cx^n = pcx^{n-1} + qcx^{n-2} \Rightarrow x^2 - px - q = 0$. This quadratic equation is called the characteristic equation of this second order linear homogeneous recurrence relation. Let us say that α and β are the roots of this equation, then the solution of this recurrence is given by $a_n = \lambda\alpha^n + \mu\beta^n$ if $\alpha \neq \beta$ and for $\alpha = \beta, a_n = (\lambda + \mu n)\alpha^n$.
For example, let's take the fibonacci sequence. The characteristic equation would then be $x^2 - x - 1 = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{5}}{2} \Rightarrow a_n = \lambda \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \mu \left(\dfrac{1-\sqrt{5}}{2}\right)^n$.

Now, $a_1 = 1 \Rightarrow \lambda \left(\dfrac{1+\sqrt{5}}{2}\right) + \mu \left(\dfrac{1-\sqrt{5}}{2}\right) = 1$ and $a_2 = 1 \Rightarrow \lambda \left(\dfrac{1+\sqrt{5}}{2}\right)^2 + \mu \left(\dfrac{1-\sqrt{5}}{2}\right)^2 = 1 \Rightarrow \lambda \left(\dfrac{3+\sqrt{5}}{2}\right) + \mu \left(\dfrac{3-\sqrt{5}}{2}\right) = 1 \Rightarrow \lambda = -\mu = \dfrac{1}{\sqrt{5}}$.

$$\therefore a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right)$$

5 Integral Reduction recurrences

Integrals are almost found everywhere in mathematics. In some cases of integration, we take recourse to the method of successive reduction of the integrand which mostly depends on the repeated application of integration by parts or algebraic manipulations. We apply integration by parts or algebraic manipulations to get to a recurrence relation through which the evaluation of the integral becomes a lot simpler. Let us take example of integrating $\tan^n(\theta)$ with respect to θ.
$I_n = \displaystyle\int \tan^n(\theta) d\theta = \int \tan^{n-2}(\theta)(\sec^2(\theta) - 1) d\theta$

$$= \int \tan^{n-2}(\theta) d\tan(\theta) - \int \tan^{n-2}(\theta) d\theta = \frac{\tan^{n-1}(\theta)}{n-1} - \int \tan^{n-2}(\theta) d\theta$$
$$\Rightarrow I_n = \frac{\tan^{n-1}(\theta)}{n-1} - I_{n-2}.$$
Using the above formula we can now solve this integral for higher values of n with much ease. Now, we know that integration in itself is a vast topic and integration of this type requires time so we provide you with the reduction formulae for some well known integrals and you may use them anywhere in your solutions. Note that you don't require any other results except the ones stated below

5.1 Some Common Reduction Formulae

1. $\int \tan^n(\theta) d\theta : I_n = \dfrac{\tan^{n-1}(\theta)}{n-1} - I_{n-2}$
2. $\int \cot^n(\theta) d\theta : I_n = -\dfrac{\cot^{n-1}(\theta)}{n-1} - I_{n-2}$
3. $\int \dfrac{dx}{(x^2+k)^n} : 2k(n-1)I_n = \dfrac{x}{(x^2+k)^{n-1}} + (2n-3)I_{n-1}$
4. $\int \sin^n(\theta) d\theta : I_n = -\dfrac{\sin^{n-1}(\theta) cos(\theta)}{n} + \dfrac{n-1}{n} I_{n-2}$
5. $\int_0^{\pi/4} \tan^n(\theta) d\theta : I_n = \dfrac{1}{n-1} - I_{n-2}$
6. $\int_0^{\pi/2} \sin^{2n+1}(\theta) \cos^{2n+1}(\theta) d\theta : I_n = \dfrac{n}{2(2n+1)} I_{n-1}$

6 Trigonometric Substitutions In Recurrences

Some times we may come across recurrences that are not linear or they have a complex characteristic equation. At those times we can use trigonometric substitutions to make our job easier.

6.1 Some Common Trigonometric Substitutions

1. $a^2 + x^2$: substitute $x = a\tan\theta$ or $x = a\cot\theta$
2. $a^2 - x^2$: substitute $x = a\sin\theta$ or $x = a\cos\theta$
3. $x^2 - a^2$: substitute $x = a\sec\theta$ or $x = a\csc\theta$

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PROBLEMS

- The questions here are all subjective, and will be graded on how well you can convince the checker of the mathematical rigour of your solution.
- The corresponding points of each question are mentioned with it. Do note that the points may or may not be proportionate to the relative difficulty of the question
- The number of problems presented is much more than what is humanly possible to solve in the given time frame. Hence, prioritise correctness and rigour over number of problems solved.
- Note that any significant conclusion derived in the right direction to solve the problem may fetch you some points, so try to attempt all questions.

1. Let a, b, c be real numbers such that $a + b + c = 0$. Let $S_n = \dfrac{a^n + b^n + c^n}{n}$. Prove that $S_5 = S_2.S_3$.(1 point)
2. Let $a_n = \dfrac{2}{3}a_{n-1} + n^2 - 15$, $n \geq 2, a_1 = 1$.Find a_n.(2 points)
3. Prove that $\displaystyle\int_0^1 x^n(1-x)^n dx = \frac{(n!)^2}{(2n+1)!}$. (2 points)
4. Let $a_0 = 0, b_0 = 1$, and let $a_n = a_{n-1} + 2b_{n-1}$ and $b_n = -a_{n-1} + 4b_{n-1}$. Find a_n and b_n.(3 points)
5. The first term of a sequence is 2020. The next terms are defined using the recurrence relation $x_{n+1} = \dfrac{x_n(\sqrt{2}+1) - 1}{(\sqrt{2}+1) + x_n}$. Find the 2021^{th} term.(4 points)
6. Let $\{x_n\}$ and $\{y_n\}$ be two real sequences defined as follows:

$$x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1 + x_n^2}, \quad y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}}$$

. Prove that $2 < x_n.y_n < 3$ for all $n > 1$. (5 points)

7. Let $\{a_n\}$ be a sequence such that $a_1 = 0$ and $a_{n+1} = 5a_n + \sqrt{1 + 24a_n^2}$. Find a_n.(6 points)

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