# ZPC-21 solutions

Ramanjeet,Raunak,Ishaan,Sanyam,Himanshu,Mudit

March 12, 2022

## 1 PROBLEM 1

Since $y$ is the mean of the numbers $x_1, x_2, \ldots, x_n$, so we have $y = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.
$\Rightarrow ny = x_1 + x_2 + \cdots + x_n$. Now, if all $x_i < y$, then $x_1 + x_2 + \cdots + x_n < ny$ which is not possible, so there must be an $x_i \geq y$. A similar argument can be used for the case when all $x_i > y$, where $x_1 + x_2 + \cdots + x_n > ny$ would lead to a contradiction.

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## 2 PROBLEM 2

Note that m-n+1 represents the number of natural numbers starting from n and ending at m

Suppose m-n+1 $\in \mathbb{E}$, the set of even numbers, form pairs like this:

$(n, m)$

$(n+1, m-1)$

$(n+2, m-2)$

.

.

$\frac{m-n+1}{2}$ pairs. Now, each pair is a hole, and what we select is a pigeon, Since m-n+1 $\in \mathbb{E}$, $\frac{m-n+1}{2} \in \mathbb{Z}$

$$\implies \left\lceil \frac{m-n+1}{2} \right\rceil + 1 = \frac{m-n+1}{2} + 1$$

It is clear that there are more pigeons than holes and so, at-least one hole has more than one pigeon, and so there must be at-least one pair that is completely selected. Now, note that each pair adds up to m+n, so we are done

A Similar approach can be followed for odd m-n+1

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# 3 PROBLEM 3

The expression $\dfrac{y_i - y_j}{1 + y_i . y_j}$ is similar to the formula for $tan(a-b)$. So we proceed as follows. Divide the interval $(-\pi/2, \pi/2]$ into 6 intervals $(-\pi/2, -2\pi/6]$, $(-2\pi/6, -\pi/6]$, $(-\pi/6, 0]$, $(0, \pi/6]$, $(\pi/6, 2\pi/6]$ and $(2\pi/6, \pi/2]$.

Now, let $y_i = tan(x_i)$ for $i = 1, 2, \ldots, 7$. Since we have 7 numbers and 6 subintervals, then by PHP we can say that two of the $x_i$'s must lie in one of the 6 subintervals. Then we have $0 \leq x_i - x_j \leq \pi/6$

$\Rightarrow 0 \leq tan(x_i - x_j) \leq tan(\pi/6)$

$\Rightarrow 0 \leq \dfrac{y_i - y_j}{1 + y_i y_j} \leq \dfrac{1}{\sqrt{3}}$.

Note: Here we need not take any other interval as tan() function goes from $-\infty$ to $+\infty$, so all possible values will lie within this interval.

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# 4 PROBLEM 4

There are n sets of k (adjacent) numbers. Lets denote the numbers as $P_1, P_2, P_3....P_n$

$D_1 = P_1P_2P_3..........P_{k-1}P_k$

$D_2 = P_2P_3..........P_kP_{k+1}$

.

.

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$D_{n-1} = P_{n-1}P_nP_1P_2..........P_{k-3}P_{k-2}$

$D_n = P_nP_1P_2P_3..........P_{k-2}P_{k-1}$

Now, consider the product $s =$

$D_1D_2.......D_n = (P_1P_2P_3...........P_n)^k$

This equality holds because $P_i$ occurs exactly k times in the product s.

$\because P_i = i, \implies (P_1P_2P_3...........P_n)^k = (n!)^k$

**Claim:** All $D_i < (n!)^{\frac{k}{n}}$

$\implies \prod_{i=1}^{n} D_i < (n!)^{\frac{k}{n}\times n} = (n!)^k$

This is a contradiction as $\prod_{i=1}^{n} D_i = (n!)^k$

This means our claim is false,

$\implies \exists D_i \geq (n!)^{\frac{k}{n}}$ and we are done

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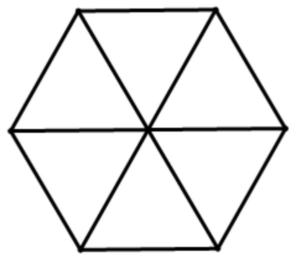
# 5 PROBLEM 5

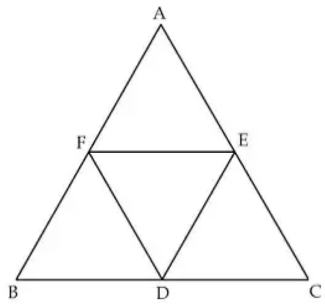
Consider dividing the regular hexagon into 6 equilateral triangles:



We have $6n^2+1$ points, and holes(equilateral triangle) to put them in.

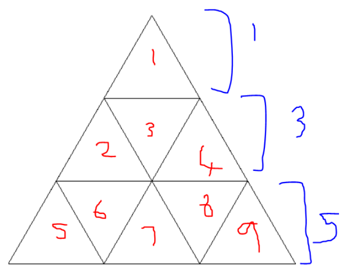
$\implies$ one triangle must have $\left\lceil \dfrac{6n^2+1}{6} \right\rceil = \left\lceil \dfrac{6n^2}{6} + \dfrac{1}{6} \right\rceil = n^2+1$ points.

Consider this equilateral triangle,




It has 4 sections, each section itself is an equilateral triangle, the side length of each triangle is $\frac{a}{2}$

This can be further extended to form $n^2$ sections, here's an example for n=3,




So, divide the main triangle into $n^2$ equilateral triangles like this, by forming n rows of equilateral triangles.

Note, that this works because $\sum_{i=1}^{n}(2i-1) = n^2$

This gives $n^2$ holes(equilateral triangles) and $n^2+1$ pigeons (points)

So, by the Pigeonhole Principle,

One triangle must contain $\left\lceil \frac{n^2+1}{n^2} \right\rceil = 2$ points.

The longest distance between any two points on the equilateral triangle is if the points are placed on two of the vertices (aka the side length)

Since we divided a into n different rows,

Sidelength of the smaller equilateral triangles $= \frac{a}{n}$,

So these two points which lie in this small equilateral triangle has distance $\leq \frac{a}{n}$ and we are done

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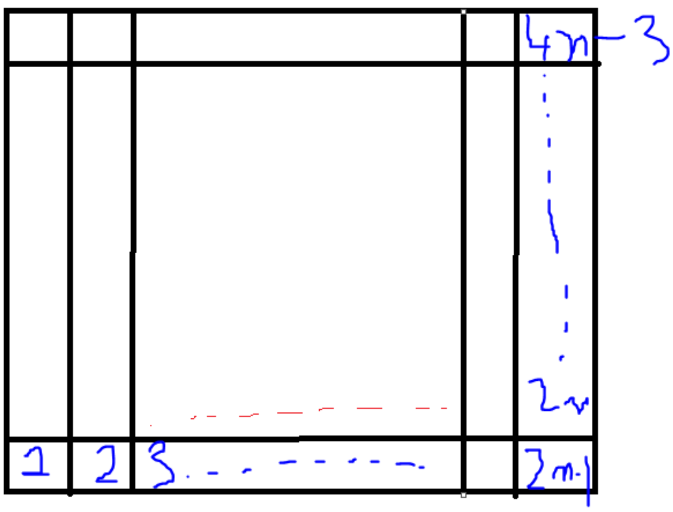
# 6 PROBLEM 6

**Solution:** The maximum different types of numbers on the board is 4n-3

Suppose the bottom left corner of the square is 1, The maximum difference can be achieved as follows:



i.e. by strictly increasing or decreasing while moving from one corner to the opposite, but the max unique elements will always be 4n-3

There are $(2n-1)^2 = 4n^2 - 4n + 1$ squares in the grid,

This makes $4n^2 - 4n + 1$ pigeons and 4n-3 holes,

so at least one hole must have $\left\lceil \dfrac{4n^2 - 4n + 1}{4n - 3} \right\rceil = \left\lceil n - \dfrac{1}{4} + \dfrac{1}{4\,(4n-3)} \right\rceil = n$ pigeons.

In other words, at least one number occurs at least n times, and so we are done

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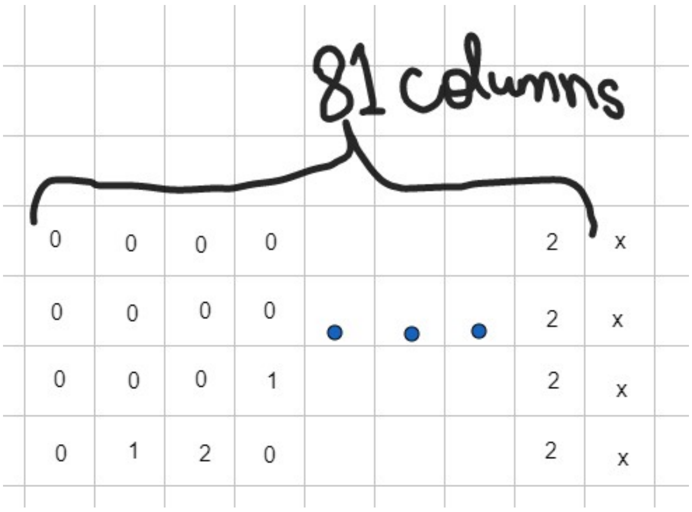
# 7 PROBLEM 7

Consider a $4 \times 82$ rectangle in this infinite grid. Now, each column in this subgrid has 4 rows. So, each column in this subgrid would be containing a string with possible characters 0,1,2. Note that we have only 3 characters and 4 rows in each column, so atleast one character is repeated in every column. Now, there are only $3 \times 3 \times 3 \times 3 = 81$ possible strings and we have to fill 82 columns, so atleast one string will be repeated and that would again have 2 digits same. Also, these 2 same digits were also same in the previous string, so we get a total of 4 squares in the subgrid that form a rectangle and have same numbers at the corners. For example, in the below scenario, if xxxx=0010 then there are 4 squares with same number.

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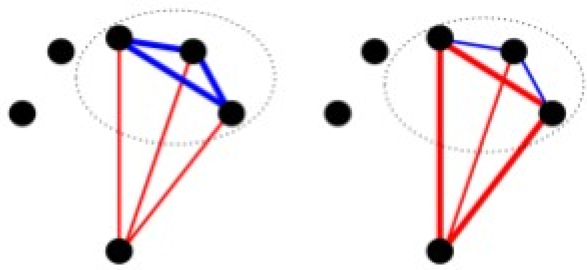
# 8 PROBLEM 8

Let us say that no one wins and the game ends in a draw. Then, there would be a total of 15 line segments joining the points (All points have a line segment between them). Now, consider a point $P_1$ which is connected to $P_2, P_3, \ldots, P_6$. Now we have 5 line segments that can have red or blue colour. Thus by pigeonhole principle, one colour appears in more than two line segments. So, without loss of generality assume that we have more red line segments than blue line segments from $P_1$ and let us say that points $P_2, P_3, P_4$ are the ones connected by the red line segments to $P_1$. Now, $P_2, P_3, P_4$ will not have any red line segment between them because in that case a triangle is formed with those two vertices and $P_1$ with all line segments red, which leads to a contradiction. So, the three line segments between $P_2, P_3, P_4$ will be blue in colour and the triangle with these 3 points will have all edges blue. This contradicts our assumption. So, someone must definitely win this game.



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# 9 PROBLEM 9

Note that the fractional part is less than $1/10^x$ when the decimal point is followed by $x$ zeroes. Now, if we find two numbers in the sequence such that the first x digits after the decimal are same in both of them, then the difference between those two multiples of $\pi$ would be a multiple of $\pi$ whose first x digits are zero. Now, to fill the first x digits after the decimal, we have $10 \times 10 \times \cdots \times 10 = 10^x$ possibilities. So, by pigeonhole principle, any $10^x + 1$ multiples of $\pi$ would contain at least two multiples with same first x digits after the decimal.

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