Évariste

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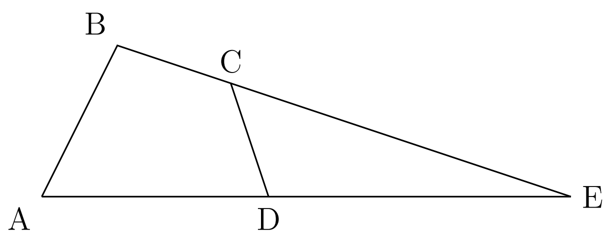
August 27, 2024

Speed Proving Tournament

SPT is a knockout round problem solving competetion where one team tackles a given problem on the blackboard while others do it on piece of paper, the first team to solve one with get half a point (one if done on the board).

1 GEOMETRY

1. The sides AD and BC of a convex polygon are extended to meet at E. Let H and G be the midpoints of BD and AC respectively. Find the ratio of the area of the triangle EGH to that of the quadrilateral ABCD.



2. Given a triangle *ABC*, find the area of the triangle formed by the reflections of D, E, F along A, B, C respectively, where *AD, BE, CF* are medians of the triangle *ABC*.

2 NUMBER THEORY

1. 3 more than the *concatenation of two* (not necessarily distinct) primes q *and* r is the square of a prime p. Find all possible values of p. *(for example, if $q = 7$ and $r = 17$, the concatenation of q and r is 717).*

2. Prove that if ab is a perfect square and $gcd(a, b) = 1$, a and b must both be perfect squares.

3 ALGEBRA

1. $P(x)$ is a 4th degree polynomial with real coefficients such that $P(x) \geq x$. $P(1) = 1$, $P(2) = 4$, $P(3) = 3$. Find the value of $P(4)$.
2. The coefficients of the polynomial *P(x)* are non-negative integers, each less than 100. Given that P(10) = 331633 and $P(-10) = 273373$, compute P(1).
3. Let *a, b, c* be positive real numbers such that
$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$$
Prove that $abc < \frac{1}{8}$.
4. Find all real numbers a such that $3 < a < 4$ and $a(a - 3\{a\})$ is an integer. ($\{x\}$ *denotes the fractional part of* x).
5. Find all fractions that can be written simultaneously in the forms
$$\frac{7k-5}{5k-3} \quad \text{and} \quad \frac{6l-1}{4l-3}$$
for some integers *k, l*.

4 PIGEONHOLE PRINCIPLE

1. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day, but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.
2. Given any set of n integers $a_1, a_2, \cdots, a_n$, prove that there must exist $0 \leq k < l \leq n$ such that $\sum_{j=k+1}^{l} a_j$ is divisible by n.

5 SEQUENCES

1. Given are the positive integers $a_0, \ldots, a_{100}$ such that $a_1 > a_0$, $a_2 = 3a_1 - 2a_0$, $a_3 = 3a_2 - 2a_1, \ldots, a_{100} = 3a_{99} - 2a_{98}$. Prove that $a_{100} > 299$.

2. Evaluate
$$\sum_{k=1}^{\text{inf}} \frac{k^2}{2^k}$$

6 COMBINATORICS

1. How many bit strings of length 10 contain either five consecutive 0's or five consecutive 1's?

2. You have a fair 2019-sided die with faces labeled $1, 2, \ldots, 2019$. After each roll, replace the n-sided die with *(n+1)*-sided die that has the n sides of the previous die here and the additional side being the number she just rolled. What is the probability that your 2019^{th} roll is a 2019?