ZPC-19

Himanshi,Sanyam,Aditya

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Instructions

Please read the following instructions carefully before proceeding further:

1. The test is of 2 hours. It will end *sharp* at **8:00 pm**.
2. It is a closed web and closed book test (Except the relevant reading material provided of course).
3. The answers to all the questions are subjective. Please make a pdf (hand-written, word, or LaTeX) of your solutions and email them to evariste@sc.iiitd.ac.in. Please include your *name, roll number, branch, year* in the email.
4. Any submission received after **8:05 pm** will not be evaluated.
5. Relevant reading material for all the questions has been provided in the document itself.
6. If you are stuck on a question that you cannot figure out, move on. Nothing good ever comes out of being hung up on something.
7. Please feel free to reach out to us on the Google Meet in case of any queries.
8. We might (or might not) provide hints to questions in the last hour, depending on the participation and responses.
9. All the best. GL HF :)

MATHEMATICAL INDUCTION IN SEQUENCES

1 The Origins of Mathematical Induction

Mathematical induction is a proving technique(actually it is an axiom) that is used to prove propositions over the set of integers(positive integers in general). The statement of mathematical induction is pretty fundamental yet it is effective in attacking some very tricky and complex propositions that look very difficult to prove otherwise.

Mathematical induction is one of the five basic Peano's axioms upon which the whole arithmetic of natural numbers is based. The form presented by Peano is not exactly the statement that we used to prove propositions but it is the original statement from which all forms of induction can be derived. It says that "If $\mathbb{S}$ is a subset of natural numbers that has the element 1 and if $n \in \mathbb{S} \Rightarrow n+1 \in \mathbb{S}$ then $\mathbb{S} = \mathbb{N}$". From this we derive the first and the second form of mathematical induction that is used for proving statements as follows:
First form: Let $\mathbb{P}(n)$be a proposition about the number n. Then:

1. if $\mathbb{P}(1)$ is true
2. if $\mathbb{P}(n) \Rightarrow \mathbb{P}(n+1) \forall n \in \mathbb{N}$

then $\mathbb{P}(n)$ is true for all $n \in \mathbb{N}$
Second form: Let $\mathbb{P}(n)$be a proposition about the number n. Then:

1. if $\mathbb{P}(1)$ is true
2. if $\mathbb{P}(k)$ is true $\forall k < n \Rightarrow \mathbb{P}(n)$ is true.

then $\mathbb{P}(n)$ is true for all $n \in \mathbb{N}$

2 General Terms and Summation of Sequences

A row of numbers arranged in a certain order is called sequence. Each number in the sequence is called term. The numbers are called, in order, the first term, the second term,... and so on. If the n^{th} term can be represented by an algebraic formula then it is called the general formula. let S_n be the sum of the first n terms of the sequence $\{a_n\}$, then the following relation holds:

$$a_1 = S_1 \ , a_n = S_n - S_{n-1} \ \forall \ n = 2, 3, \ldots$$

Note: A sequence whose general term stays between two specific finite numbers then the sequence is called a bounded sequence and the numbers bounding the sequence are called the bounds of the sequence.

3 The Arithmetic and Geometric Sequences

3.1 Arithmetic Sequence

The general term of the arithmetic sequence $\{a_n\}$ is given by $a_n = a_1 + (n-1)d$ where d is any non-zero number(known as the common difference) and a_1, a_n are the first and n^{th} term of the sequence respectively. It can be shown that the sum of first n terms (S_n) can be given by:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d) = \frac{n}{2}(a_1 + a_n)$$

Proof: let $\mathbb{P}(n) : S_n = \frac{n}{2}(2a_1 + (n-1)d)$. Now $\mathbb{P}(1)$ is clearly true as $S_1 = a_1 = \frac{1}{2}(2a_1 + (1-1)d)$. Now suppose that $\mathbb{P}(n)$ is true, then $S_n = \frac{n}{2}(2a_1 + (n-1)d)$.

$$\Rightarrow S_{n+1} = a_{n+1} + S_n = a_1 + nd + \frac{n}{2}(2a_1 + (n-1)d) = (n+1)a_1 + \frac{n(n+1)}{2}d$$

$$\Rightarrow S_{n+1} = \frac{n+1}{2}(2a_1 + nd) = \frac{n+1}{2}(2a_1 + (n+1-1)d) \Rightarrow \mathbb{P}(n+1) \text{ is true.}$$

Now, by the first principle of mathematical induction one can directly see the truth of the proposition about sum of an arithmetic sequence.

3.2 Geometric Sequence

The general term of the geometric sequence $\{a_n\}$ is given by $a_n = a_1 r^{n-1}$ where r is any non-zero number(called the common ratio) and a_1, a_n are the first and n^{th} term of the sequence respectively. It can be shown that the sum of first n terms (S_n) can be given by:

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

Proof: let $\mathbb{P}(n) : S_n = \frac{a_1(r^n-1)}{r-1}$. Now $\mathbb{P}(1)$ is clearly true as $S_1 = a_1 = \frac{a_1(r^1-1)}{r-1}$. Now suppose that $\mathbb{P}(n)$ is true, then $S_n = \frac{a_1(r^n-1)}{r-1}$.

$$\Rightarrow S_{n+1} = a_{n+1} + S_n = a_1 r^n + \frac{a_1(r^n-1)}{r-1} = \frac{a_1(r^n-1) + a_1 r^n (r-1)}{r-1}$$

$$\Rightarrow S_{n+1} = \frac{a_1 r^n - a_1 + a_1 r^{n+1} - a_1 r^n}{r-1} = \frac{a_1(r^{n+1}-1)}{r-1} \Rightarrow \mathbb{P}(n+1) \text{ is true.}$$

Now, by the first principle of mathematical induction one can directly see the truth of the proposition about sum of a geometric sequence. Also from the above result one can see that if $|r| < 1$ then $S_\infty = \lim_{n\to\infty} \frac{a_1(r^n-1)}{r-1} = \frac{a_1}{1-r}$ (as $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$).

4 The Fibonacci Sequence

The Fibonacci sequence is one of the most intriguing and interesting sequences in mathematics. There can be endless discussions about the beauty of this sequence and one can draw numerous interesting conclusions by working on this sequence. The Fibonacci sequence is defined as follows:

$$F_1 = F_2 = 1\ ,\ F_{n+2} = F_{n+1} + F_n,\ \ \forall\ n = 1, 2, 3, 4, \ldots$$

Having a little knowledge about solving recurrence relations will enable you to define the general term of this sequence without using the previous terms(known as the binet's formula) as follows:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Note: you are not required to know how to solve recurrence relations or any special properties of the fibonacci sequence (afterall it is a **zero-prerequisite contest**) and you may use the above formula anywhere you like in the problems ;).

5 Intuition and Proofs

Intuition is the first step to proving a mathematical statement. Many a times people encounter things in mathematics that have never been seen before and in order to discover their mystery, mathematicians use their intuition to get a direction to proceed in order to unravel the mystery of such new examples. Although intuitions are great from a mathematical point of view but one must note the fact that intuitions can only provide you with a direction to proceed and they are not always true. For example- The great mathematician Fermat had an intuition that numbers of the form $2^{2^n} + 1$ are always prime. He had this intuition after viewing the first few terms of the sequence that are 3, 5, 17, 257, 65537, and so on. But then mathematicians were soon able to prove that the 6^{th} terms of this sequence is not a prime and nothing could be said about the primality of further terms in general(again this was just an anecdote and you need not know anything about it). So, intuitions are great as this intuition was not true but we still got to know that terms after the fifth term in this sequence need not be prime. So, this intuition was indeed awesome.

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PROBLEMS

- The questions here are all subjective, and will be graded on how well you can convince the checker of the mathematical rigour of your solution.
- The corresponding points of each question are mentioned with it. Do note that the points may or may not be proportionate to the relative difficulty of the question
- The number of problems presented is much more than what is humanly possible to solve in the given time frame. Hence, prioritise correctness and rigour over number of problems solved.
- Note that any significant conclusion derived in the right direction to solve the problem may fetch you some points, so try to attempt all questions.

1. Prove the binet's formula stated in the above text.(1 point)
2. Show that if $x + \dfrac{1}{x} \in \mathbb{Z}^+$ then $x^n + \dfrac{1}{x^n} \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}^+$. (1 points)
3. Evaluate the limit: $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n}$.(1+1=2 points)
4. Consider $a_n = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$, where a_n has $\sqrt{2}$ exactly n times in form of exponent tower. Prove that the sequence $\{a_n\}$ is increasing and has the upper bound as 2. A few starting terms of a_n rounded off to two decimal places are 1.41, 1.63, 1.76,... and so on. (1+2=3 points)
5. Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying the property $a_{i+j} \leq a_i + a_j$ for all $i, j \in \mathbb{N}$. Then for any n, prove the following result:

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

   (2+3=5 points)
6. We are provided with a sequence $\{a_n\}$ which has the first few terms $a_0 = 0,\ a_1 = 1$ and other terms are defined as follows:

$$\frac{a_{n+1} - 3a_n + a_{n-1}}{2} = (-1)^n\ ,$$

   for all integers $n > 0$.Prove that a_n is a perfect square for all $n \geq 0$.
   (3+5=8 points)
7. We are provided with a sequence $\{a_n\}$ which has the first few terms $a_0 = 0,\ a_1 = 1,\ a_2 = 2,\ a_3 = 6$ and other terms are defined as follows:

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n\ \forall\ n \in \mathbb{Z}^+$$

   Prove that n divides a_n for all $n > 0$. (5+8=13 points)

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