ZPC-21

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Instructions

Please read the following instructions carefully before proceeding further:

1. The test is of 2 hours. It will end *sharp* at **8:00 pm**.
2. It is a closed web and closed book test (Except the relevant reading material provided of course).
3. The answers to all the questions are subjective. Please make a pdf (handwritten, word, or LaTeX) of your solutions and email them to evariste@sc.iiitd.ac.in. Please include your *name, roll number, branch, year* in the email.
4. Any submission received after **8:10 pm** will not be evaluated.
5. Relevant reading material for all the questions has been provided in the document itself.
6. If you are stuck on a question that you cannot figure out, move on. Nothing good ever comes out of being hung up on something.
7. Please feel free to reach out to us on the Google Meet in case of any queries.
8. We will provide hints to questions in the last hour, depending on the participation and responses.
9. All the best. GL HF :)

THE PIGEONHOLE PRINCIPLE

1 Introduction to Pigeonhole Principle

A very fascinating thing about mathematics is how very simple looking mathematical statements often are very useful in solving a lot of real life problems. The pigeonhole principle is one such useful tool that can be used to solve a ton of difficult problems just by using a simple mathematical statement. For example, it can be used to prove that at least two WordPress.com blogs have the same number of yearly comments. Suppose, a person decides to participate in a race and for that he needs to practice completing sprints of 100 metres. He has thirty days left and he pledges to sprint at least once every day. At the end he knows that in total he sprinted 45 times. Then, there might be a series of consecutive days in which he sprinted **"exactly"** 14 times. This is not at all something obvious and you may try solving this puzzle. The solution to this opening puzzle will be available at the end of the document.

2 Simple Form Of Pigeonhole Principle

Pigeonhole principle(PHP) basically is nothing but a very naive statement that if you want to put more than n pigeons into n holes then at least one hole must contain more than one pigeon. People might think that how can something so trivial and obvious said to be very useful but as you go through this document you'll be amazed to see that it is not at all simple in the first place. We start by mathematically defining the simple form of the pigeonhole principle as follows:

Theorem: If n pigeonholes contain more than n pigeons then at least one of the hole has more than one pigeons.

Proof: To the contrary assume that it is not true then all holes contain less than or equal to 1 pigeons.

Now, let a_i denote the number of pigeons in i^th pigeonhole. Then $\sum_{i=0}^{n} a_i \leq \sum_{i=0}^{n} 1 = n$. But $\sum_{i=0}^{n} a_i$ can't be less than n as it is the number of pigeons which is greater than n. So, by contradiction, the theorem holds.

We can solve a variety of problems by using PHP. The main motive while solving a problem using PHP is to generate the pigeoholes or select pigeons or pigeonholes in a way such that we can apply PHP. After we have the pigeons and the pigeonholes then we just have to apply the pigeonhole principle to get to our final conclusion. To demonstrate this let us take an example.

Example: 51 distinct numbers are chosen from the set $\{1, 2, 3, \ldots, 100\}$. Prove that we can find two numbers out of the selected numbers that are coprime(Two numbers are coprime if their GCD or HCF is 1).

Here we use a fact that the GCD of two consecutive integers is always 1 because of the euclidean algorithm to find GCD.Note that $n+1 = n \times 1 + 1 \Rightarrow$ GCD(n+1,n)=1. Now partition the original set into 50 sets as : $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$. Now, we have numbers as pigeons and these sets are the pigeonholes. So, if we want to select 51 distinct integers from these 50 sets then we must select at least two elements that are from the same set. And those two elements come out to be coprime proving the example.
In the above set, we can also show that if 51 numbers are chosen then one can find two numbers that have an odd sum. We can again see that two of then cho-

sen numbers are from the same set and any two consecutive integers have an odd sum.(Think why? :)).

3 Strong Form Of Pigeonhole Principle

Theorem: If k pigeonholes contain more than or equal to n pigeons then at least one of the hole has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

Proof: Let us assume that this statement is false, then any hole contains at most $\left(\left\lceil \frac{n}{k} \right\rceil - 1\right)$ pigeons $= \left\lfloor \frac{n}{k} \right\rfloor$ pigeons. Now, Total number of pigeons will be less than or equal to $k \times \left\lfloor \frac{n}{k} \right\rfloor \leq k \times \frac{n}{k} = n$. But, total number of pigeons is more than or equal to n, therefore this must only hold when $\left\lfloor \frac{n}{k} \right\rfloor = n$ or when k divides n. But in that case also each hole has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons. So for this case the theorem holds true. For all the other cases, this is a contradiction as the number of pigeons cannot be less than n. So the theorem is true for all n and k.

The basic idea behind all the problems of pigeonhole principle is proof by contradiction by assuming the incorrect number of pigeons.

Example: Sarah writes 3001 4 digit numbers on a sheet of paper, Prove that we can find more than one pair of numbers out of the written numbers such that the numbers in pair have difference divisible by 1000.

Now, Note that there are only $10 \times 10 \times 10 = 1000$ possibilities for the last three digits of any four digit number that are $000, 001, 002, \ldots, 999$. Now, if we select 3001 four digit numbers, then at least $\left\lceil \frac{3001}{1000} \right\rceil = 4$ must have the same last three digits. Now, any numbers with the last three digits same have difference divisible by 1000(if x is the number formed by last three digits then $(1000a + x) - (1000b + x) = 1000(a - b)$). Now, since 4 numbers have same last three digits, we can find $\binom{4}{2} = 6$ pairs of numbers satisfying the given property.

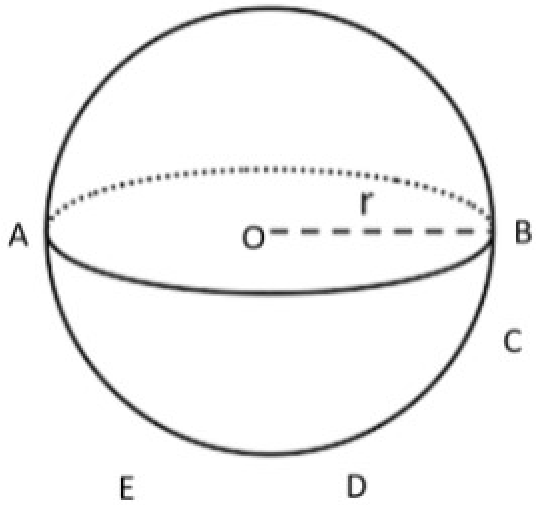
4 Thinking With Functions

We can simply think of pigeonhole principle in terms of injectivity and surjectivity of functions. Consider a mapping f from set A to set B. Now if set A has more elements than set B, then the mapping $f : A \to B$ cannot be injective (one-one) because there will be two elements with the same image in set B. Also, if set B has more elements than set A, then this function cannot be surjective(onto) because not all elements of B will have a preimage in A. This proves that for a function to be a bijection(both one-one and onto at the same time), the two sets involved in mapping must have the same cardinality(number of elements). This gives rise to a proof technique called the bijection proof. To prove that two sets have same cardinality, we define a function between the two sets and then prove that it is a bijection due to which the two sets need to have the same cardinality. Many great theorems use the bijection proof method like multiplicativity of euler's totient function, pentagonal number theorem, etc. You may or may not use bijection proof in your solutions to problems of this ZPC but all of the problems this time can be solved without using the bijection proof because we know that coming up with bijections takes up a lot of time.

5 Pigeonhole Principle In Geometry

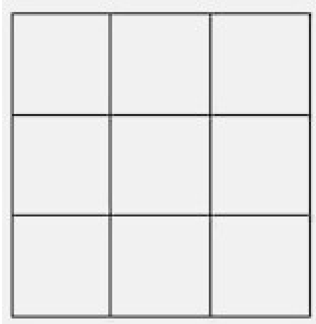
These problems generally involve dividing the geometric region into parts such that the pigeonhole principle can be applied on those parts. We would take a few examples that have geometry involved in them.
Example: Five points are placed on the surface a a sphere. Prove that 4 of them must lie in the same hemisphere.

Proof: Let the five points be A, B, C, D and E. Now, draw a circle passing from A and B(It can be proven that this circle is unique). Now, we have 3 points remaining and two hemispheres. So,one hemishpere must contain at least 2 out of the three given points and A and B are lie in both of the hemispheres. Therefore,we can get at least 2+2=4 points in the same hemisphere.



Example: 10 points are placed in a rectangle of dimensions 3×3 square units. Prove that we can find two points in the rectangle which have distance between them less than 1.415 units.

Proof: Divide the grid into 9 equal parts as shown. Now Atleast one part must contain 2 or more points. That means we can find two points that lie inside the same part. Now, the longest distance between two points in any part can be $\sqrt{2}$(length of diagonal). So, there must be two points that are at a distance less than or equal to $\sqrt{2}$ and $\sqrt{2} < 1.415$. So, there must be two points that are at a distance less than 1.415.



6 Examples From Number Theory And Algebra

The number theoretic examples generally deal with the facts related to divisibility and remainders. We try to find the properties that can be treated as pigeons and the pigeonholes. The following are a few such examples:

Example: Show that among any 101 positive integers not exceeding 200 there must be an integer that divides one of the other integers.

Proof: Partition the whole set as follows:

1. Each set contains an odd number say x and other elements are 2x,4x,8x,etc.
2. So the sets are like $\{1, 2, 4, \dots\}$, $\{3, 6, 12, \dots\}, \dots, \{197\}, \{199\}$.

Now since there are 100 odd numbers, so we have 100 sets. Since we are choosing 101 elements from 100 sets, two elements must be from the same set. Now note that if two elements are chosen from the same set then the smaller one will divide the bigger one. This proves the given statement.

Example: Prove that in a sequence of $n \in \mathbb{N}$ integers $a_1, a_2, \dots, a_n$, we can find several consecutive terms $a_i, a_{i+1}, \dots, a_j$ such that their sum is divisible by n.

Proof: Define a sequence $\{b_{ij}\}_{i,j=1,1}^{i,j=n,n}$ where $b_{ij} = a_i + a_{i+1} + \cdots + a_j$.Now note that if for any i, if b_{ii} is divisible by n then this means that a_i is divisible by n, so our work would be done.
Now, consider $b_{11}, b_{12}, b_{13}, \dots, b_{1n}$. If any b_{1i} is divisible by n then our work is done. Now, if no b_{1i} is divisible by n then the only possible remainders for these n elements are $1, 2, \dots, n-1$. So, by pigeonhole principle, since we have $n-1$ remainders and n numbers, two numbers say b_{1k} and b_{1m} must have the same remainders. Let that remainder be r. So, $b_{1k} = n.q_1 + r, b_{1m} = n.q_2 + r \Rightarrow b_{1k} - b_{1m} = n(q_1 - q_2) \Rightarrow b_{mk} = n(q_1 - q_2)$. So, we get a series of consecutive numbers that is divisible by n.

Example: Prove that for any $n \in \mathbb{N}$, we can find a multiple of n that has only $0s$ and $1s$ in its decimal representation.

Proof: Consider the sequence $1, 11, 111, \dots, 1 \dots (n \; digits) \dots 1$. Now if some

element in this sequence is divisible by n then we are done.
Now, if no element in this sequence is divisible by n then the n elements must leave remainders from $1, 2, \ldots, n-1$ when divided by n. So, two numbers say $1 \ldots 1(k\ 1s)$ and $1 \ldots 1(m\ 1s)$ must have the same remainder. So, their difference would be divisible by n. and note that their difference would only contain 1s and 0s.

Example: The product of five given polynomials is a polynomial of degree 21. Prove that we can choose two of those polynomials so that the degree of their product is at least nine.

Proof: For 5 polynomials, there will be $\binom{5}{2} = 10$ pairs of polynomials. Let those pairs have products $g_1, g_2, \ldots, g_{10}$ and let their degrees be $d_1, d_2, \ldots, d_{10}$. Let the polynomials be represented by p_1, p_2, p_3, p_4, p_5. Now, $g_1.g_2.g_3 \ldots g_{10} = (p_1.p_2.p_3.p_4.p_5)^4$, so the product of all g_i results in a polynomial of degree $4 \times 21 = 84 \Rightarrow \sum_{i=1}^{10} d_i = 84$. Now, we have to divide degree 84 in 10 $d_i s$. So, at least one d_i should have degree more than or equal to $\left\lceil \frac{84}{10} \right\rceil = 9$.

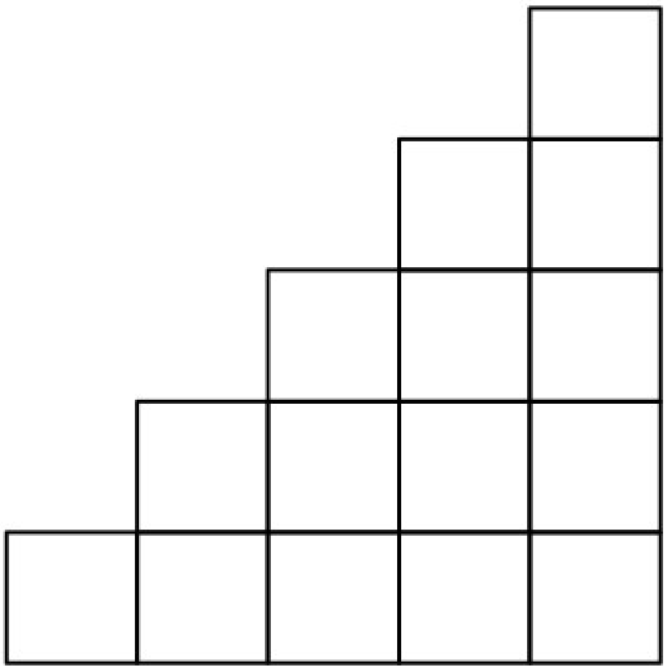
Note: For many algebraic and number theory problems, we often take the sum of product sequence of the given sequence and then apply pigeonhole principle as you saw in many examples above.

7 Puzzles and Games Using PHP

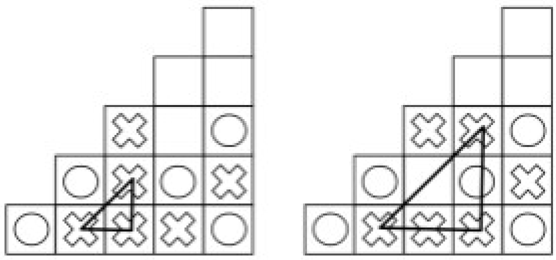
Example: Suppose there are n people in a room and they shake hands with each other(not necessarily everyone else). So. prove that there are exactly two people in the room who have exactly same number of handshakes.

Proof: The maximum possible number of handshakes a person can do is $n-1$. And the minimum possible handshakes are 0. So, the number of handshakes are from the numbers $0, 1, 2, \ldots, n-1$ which are n numbers in total with n people. But if someone shook hands zero times then there cannot be anyone with $n-1$ handshakes. So, we would only have $n-1$ possibilities for handshakes and thus, some two people must have the same number of handshakes.

Example: Consider a modified tic-tac-toe game with the following board. Players take turns to add circles and crosses in the board. A player wins when three of their marks form the corners of a right-angled isosceles triangle of any size in the same orientation as the board (the corners may be touching or spread out). Prove that this game can never end in a tie in the end.

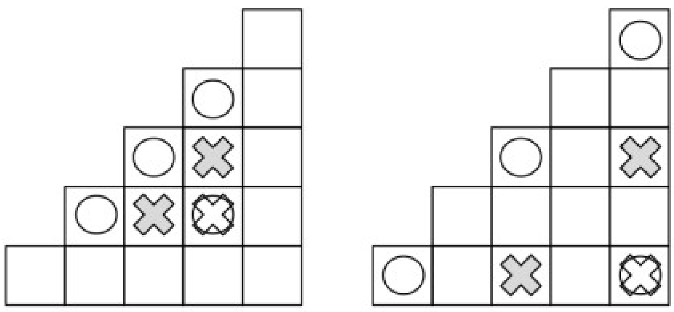


Starting grid



Some of the possible ways for Crosses to win.

Proof: Assume that the game end in a tie. Note that at least three squares out of the five on the longest diagonal are the same symbol. Without loss of generality, let us assume that they are circles. So, now we have 3 or more circles on the longest diagonal. Now, if three circles are there in the longest diagonal then we must have a pair of crosses to prevent circles from winning. But then we must have circle for the pair of crosses and a cross for preventing circles from winning but since each sqaure has only one symbol, so this would make someone victorious.



7.1 Solution To Opening Problem

Let x_i denote the number of sprints done in the first i days for $1 \leq i \leq 30$. Then $x_{30} = 45$ and the 60 numbers $x_i, x_i + 14 (1 \leq i \leq 30)$ all lie between 1 to 59(45+14). So, two numbers should be same. Note that because one sprint is compulsory everyday so $x_i s$ form an increasing sequence. Thus $x_i \neq x_j$ if $i \neq j$. So, we must have $x_i + 14 = x_j$ for some $i < j$. So, exactly 14 sprints were done from day $i + 1$ to j.

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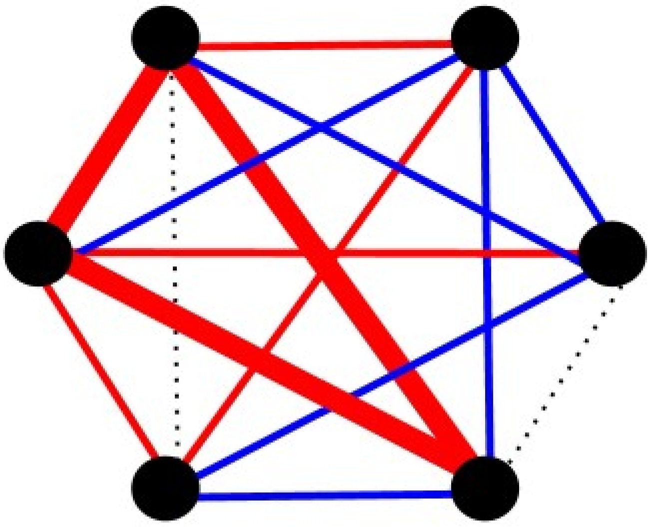
PROBLEMS

- The questions here are all subjective, and will be graded on how well you can convince the checker of the mathematical rigour of your solution.
- The corresponding points of each question are mentioned with it. Do note that the points may or may not be proportionate to the relative difficulty of the question
- The number of problems presented is much more than what is humanly possible to solve in the given time frame. Hence, prioritise correctness and rigour over number of problems solved.
- Note that any significant conclusion derived in the right direction to solve the problem may fetch you some points, so try to attempt all questions.

1. Suppose that $x_1, x_2, \ldots, x_n$ is a sequence of real numbers with mean y then there must exist an $x_i \leq y$ and an $x_i \geq y$. (1 point)
2. We are given natural numbers from n to m. Prove that if we select out $\left\lceil \frac{m-n+1}{2} \right\rceil + 1$ numbers from the given numbers then there must be at least two numbers of them whose sum is exactly $n+m$. (2 points)
3. You are given real numbers $y_1, y_2, \ldots, y_7$, Prove that there would be two of them that satisfy $0 \leq \frac{y_i - y_j}{1 + y_i . y_j} \leq \frac{1}{\sqrt{3}}$. (2 points)
4. Suppose the integers from 1 to n are arranged in some order around a circle, and let k be an integer with $1 \leq k \leq n$. Show that there must exist a sequence of k adjacent numbers in the arrangement whose product is at least $\lceil (n!)^{k/n} \rceil$. (3 points)
5. $6n^2 + 1$ points are placed inside a regular hexagon of side length a. Prove that there are two points in the hexagon which have distance between them less than $\frac{a}{n}$. (3 points)
6. We have a $(2n-1) \times (2n-1)$ square grid. Each small square in the grid is filled with an integer such that the difference of any two adjacent squares is less than or equal to 1. Prove that there exists an integer that shows up at least n times on the board. (4 points)
7. Consider an infinite grid in which each box in the grid is filled with one integer out of 0, 1 and 2. Prove that there exists a rectangular portion of the grid with same numbers at each of the 4 corner boxes. (4 points)
8. We have a set of six points A, B, C, D, E, F which form the vertices of a regular hexagon. Mudit and Himanshu are very much interested in playing games so they decide to play a game with these six points. If one person wins then the other becomes sad. The only case when both are happy is when there is a tie. The game involves joining the vertices of the hexagon. Mudit

draws red lines between vertices and Himanshu draws blue lines. Both take turns drawing lines and in each move a player can draw only one line. A player loses if there is a triangle formed from his colour in the board(For example in the example below Himanshu is the winner). Prove that this game will definitely make someone sad. :((4 points)



9. Let $S = \{\pi, 2\pi, 3\pi, \ldots\}$ be a sequence of real numbers where π is the regular circle constant whose value is approximately 3.14159265 (π is an irrational number and these are just a few). Let $x \in \mathbb{N}$ be a natural number, then prove that there is some term in the sequence which has a fractional part less than $\frac{1}{10^x}$. (5 points)

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