ZPC-25

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Instructions

Please read the following instructions carefully before proceeding further:

1. The test is of 2 hours. It will end *sharp* at **8:00 pm**.
2. Relevant reading material for all the questions has been provided in the document itself.
3. If you are stuck on a question that you cannot figure out, move on. Nothing good ever comes out of being hung up on something.
4. Please feel free to contact the invigilators in case of any queries.
5. We will provide hints to questions in the last hour, depending on the participation and responses.
6. All the best. GL HF :)

PRINCIPLE OF INCLUSION-EXCLUSION

1 Introduction

Counting things is a very old problem in mathematics but many a times, the counting involved in the problems is not straightforward. For example, let us consider the following problem:

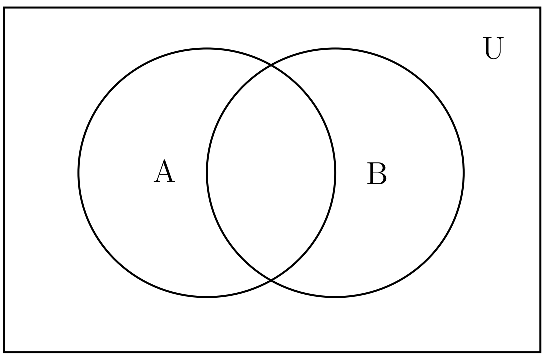
Example: You know that 20 people read Navbharat Times, 40 people read Times of India , the total number of people in the society is 100 and the total number of people who don't read any newspaper out of the two in the society is 50. You need to find the number of people who read newspaper in total. You need to find the number of people in the society who read both these newspapers.

We can first find the number of people who read any newspaper out of the mentioned.The number of people who read any newspaper out of the two mentioned = Total number of people in the society - people who don't read any newspaper out of the mentioned.
So, The number of people who read any newspaper out of the two mentioned = $100 - 50 = 50$

U=100

A=20

B=30



let A = number of people who read Navbharat Times = 20

let B = number of people who read Times of India = 40

Then, The number of people who read any newspaper out of the two mentioned

$= A + B - A \cap B$

So, Number of people who read both these newspapers $= 20 + 40 - 50 = 10$

From the above calculation, we can see that the counting process is not a very straightforward one. We had to add a bunch of things and then subtract a bunch of things to get to the correct number.

2 The Inclusion Exclusion Principle

The common problem in counting arises when we know the number of elements in the union or intersection of various overlapping sets and we have to calculate the number of elements in a set whose cardinality is unknown. In this case we have to be careful whether or not we are over-counting or under-counting something. In situations like these the inclusion exclusion principle makes our life easier.

Theorem (The inclusion exclusion principle): Let $A_1, \ldots, A_n$ be a family of finite sets. Then the number of elements in the union $A_1 \cup \cdots \cup A_n$ is given by

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|.$$

<u>Proof:</u>

We prove this theorem by the use of mathematical induction. First for the base case which in this case is $n = 2$, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

This follows from the fact that $A_1 \cup A_2$ is the union of the disjoint sets A_1 and $A_2 \backslash (A_1 \cap A_2)$, while A_2 is the union of the disjpints sets $A_2 \backslash (A_1 \cap A_2)$ and $A_1 \cap A_2$. From the equalities

$$\begin{aligned}|A_2| &= |A_2 \backslash (A_1 \cap A_2)| + |A_1 \cap A_2|\,,\\ |A_1 \cup A_2| &= |A_1| + |A_2 \backslash (A_1 \cap A_2)|\,,\end{aligned}$$

we obtain the desired equality. Suppose the statement is true for families of $n-1$ sets. We will prove it for families of n sets:

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= \left|\bigcup_{i=1}^{n-1} A_i \cup A_n\right| = \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n\right|\\ &= \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right|\\ &= \sum_{\substack{I \subseteq \{1,\ldots,n-1\}\\ I \neq \emptyset}} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right| + |A_n|\\ &- \sum_{\substack{I \subseteq \{1,\ldots,n-1\}\\ I \neq \emptyset}} (-1)^{|I|+1} \left|\bigcap_{i \in I} (A_i \cap A_n)\right|.\end{aligned}$$

By grouping the sums having the same numbers of factors in their intersections, we obtain the above formula.

So for $n = 2$, we have $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

For $n = 3$, we have $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$

For $n = 4$, we have $|A_1 \cup A_2 \cup A_4 \cup A_3| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| - |A_1 \cap A_4| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_4 \cap A_3| + |A_4 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3 \cap A_4|$

Corollary: Let $A_1, \ldots, A_n$ be a family of finite subsets of the set S, then the probability of the union $\mathbb{P}[A_1 \cup \cdots \cup A_n]$ is given by

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mathbb{P}\left[\bigcap_{i \in I} A_i\right].$$

<u>**Proof:**</u>

Simply divide both sides of the equation by $|S|$. Note that in this case $|S|$ is the size of the sample space. So, the term $\left|\bigcup_{i=1}^{n} A_i\right|$ becomes $\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right]$ and similarly the term $\left|\bigcap_{i \in I} A_i\right|$ becomes $\mathbb{P}\left[\bigcap_{i \in I} A_i\right]$. And therefore the result of the corollary follows.

Theorem: Let $A_1, \ldots, A_n$ be a family of finite subsets of the set S, and let $\overline{A_i} = S - A_i$ be the complement of A_i. Then

$$\left|\bigcap_{i=1}^{n} \overline{A_i}\right| = |S| + \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right|.$$

<u>**Proof:**</u>

Because $\bigcap_{i=1}^{n} \overline{A_i}$ and $\bigcup_{i=1}^{n} A_i$ form a partition of S, the desired result follows from the previous theorem.

Example: Find the number of 5 letter names that do not start with an "M" and have unique letters.

A concise answer to this would be to find the number of possible 5 letter names and to then, exclude those which start with an "M". Since there are 26 letters in the English alphabet and 5 spots to select, the total number of such selections is

$26 \times 25 \times 24 \times 23 \times 22$ or $5! \times \binom{26}{5}$.
Fixing "M" in the first spot, we are free to choose the remaining four letters, making the number of 5 letter names starting with "M" as $1\times, 26 \times 25 \times 24 \times 23$ or $4! \times \binom{26}{4}$.
Thus, the final answer is $5! \times \binom{26}{5} - 4! \times \binom{26}{4} = 26 \times 25 \times 24 \times 23 \times 21 = 7534800$

PROBLEMS

- The questions here are all subjective, and will be graded on how well you can convince the checker of the mathematical rigour of your solution.
- The corresponding points of each question are mentioned with it. Do note that the points may or may not be proportionate to the relative difficulty of the question.
- The number of problems presented is much more than what is humanly possible to solve in the given time frame. Hence, prioritise correctness and rigour over number of problems solved.
- Note that any significant conclusion derived in the right direction to solve the problem may fetch you some points, so try to attempt all questions.

1. a) How many of the positive integers less than 2022 are either a multiple of 3 or 17?

   b) How many perfect powers are less than 2022?

   (1+1 points)

2. What is the probability that upon rearrangement of the integers $\{1, 2, 3, \ldots, 10\}$,

   a) The numbers alternate between odd and even?

   b) None of 1,4,9,6,5 are in their natural positions?

   c) None of the numbers are in their natural positions?

   (2x1+2 points)

3. How many positive integers less than 500 can be written as a sum of two

positive non-zero perfect cubes? (3 points)

4. The Euler ϕ function is defined on the positive integers by $\phi(n) = |\{k : 1 \leq k \leq n, gcd(k, n) = 1\}|$. For example, $\phi(12) = 4$, because there are four positive integers (namely 1, 5, 7 and 11) that are less than or equal to 12 and co-prime to it)

   Verify that $\phi(n) = n \prod_{i=1}^{k}(1 - \frac{1}{p_i})$, where $p_1, \ldots, p_k$ are the distinct primes dividing n (3 points)

5. Find the number of surjective functions from domain A of m elements to range B of n elements ($m > n$) (4 points)

6. Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 50$, where $x_2 \leq 20$ and $x_4 \geq 10$ (4 points)

7. Let n and k be positive integers such that $n > 3$ and $\frac{n}{2} < k < n$. There are n points on a plane, out of which no 3 are collinear. If every point is connected to k other points by line segments, prove that there exists a triangle in the plane (5 points)

8. Let S=$\{1, 2, 3, \ldots, 250\}$. Let the minimal and maximal natural number n such that in any n-element subset of S there are six numbers that are pairwise relatively prime be m and M. Give the value of $\int_0^{M-m} \frac{1}{\sqrt{x}}\, dx$ (5 points)

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