Sequence and Series

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Telescoping

Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence whose general term, a_k can be written as $b_{k+1} - b_k$, where $\{b_n\}_{n=0}^{\infty}$ is another infinite sequence. Then,

$$\sum_{k=0}^{\infty} a_k = B - b_0$$

where

$$B = \lim_{n\to\infty} b_n,$$

also known as limit of the series. You could easily show this by writing a few terms and seeing every other term cancel out.

Partial fractions

The method of breaking down a fraction into a sum of lesser degree fractions. Very helpful in converting a series to a telescoping sum if we can find a breakdown which admits the property of a telescoping series. Consider

$$\underbrace{\frac{6^n}{(3^{n+1} - 2^{n+1})(3^n - 2^n)}}_{a_n} = \underbrace{\frac{2^n}{3^n - 2^n}}_{b_n} - \underbrace{\frac{2^{n+1}}{3^{n+1} - 2^{n+1}}}_{b_{n+1}}$$

This gives us

$$\sum_{k=\alpha}^{\beta} a_k = b_\alpha - b_\beta.$$

Note that

$$B = \lim_{n\to\infty} \frac{2^n}{3^n - 2^n} = 0,$$

which also tells us $\sum_{k=0}^{\infty} b_k$ converges. (Since if $B \neq 0$, we would keep on adding something forever, and the sum would keep increasing/decreasing.) Also note that the to try and get this decomposition of a_n, we try to be desperate and let

$$\frac{6^n}{(3^{n+1} - 2^{n+1})(3^n - 2^n)} = \frac{A}{3^{n+1} - 2^{n+1}} - \frac{B}{3^n - 2^n}$$

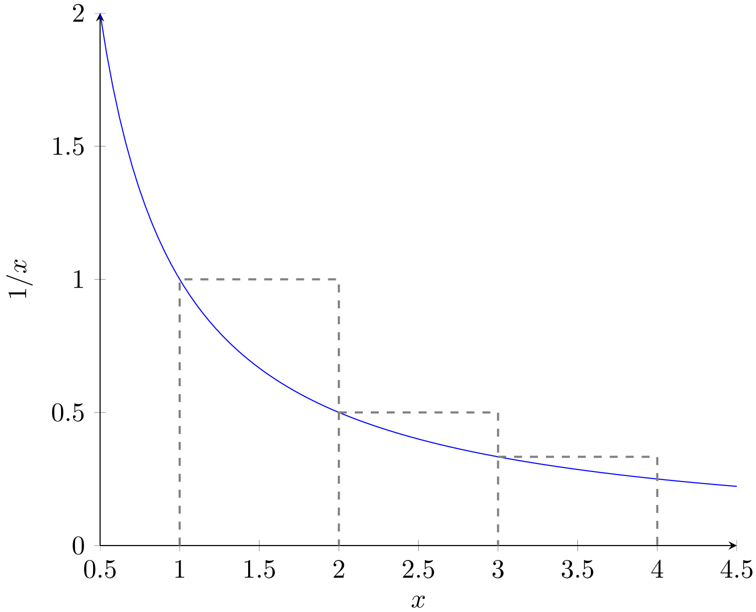
because we're trying to get something exactly like that (for the telescoping to work). Solving for A and B now is simple, $A = -2^{n+1}$ and $B = -2^n$ will work. $A = 3^{n+1}$ and $B = 3^n$ will also work (they are equivalent, you could get one from the other).

P-series

Or the Riemann-Zeta function for complex valued inputs,

$$\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for $p > 1$, and diverges otherwise. For $p = 1$, it is the harmonic series.



The sum

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

can be thought of as the area of a rectangle with unit width and $1/k$ height, or the upper Darboux sum approximation of the curve $y = 1/x$. In other words,

$$\int_1^{\infty} \frac{1}{x}\,\mathrm{d}x < \sum_{k=1}^{\infty} \frac{1}{k}$$

where the integral on the left is $\ln x\Big|_1^{\infty}$, which is infinite. And since the sum on the right is larger, it too, must be infinite.

Ratio test

Let $\{a_n\}$ be a sequence,

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

For $r > 1$, the sequence diverges, for $r < 1$ the sequence converges. For $r = 1$ the test is inconclusive.

Root test

Let $A = \sum_{k=1}^{\infty} a_n$, consider

$$r = \lim_{n\to\infty} a_n^{\frac{1}{n}}.$$

If $r < 1$, A converges and if $r > 1$, it diverges. For $r = 1$ the test is inconclusive.

Absolute convergence

Since

$$a_n \leq |a_n|,$$

if $\sum |a_n|$ converges then so does $\sum a_n$. Such a sequence $\{a_n\}$ is called absolutely convergent. A sequence is called conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ does not, for example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Cauchy's condensation test

For an increasing, positive sequence

$$0 \leq a_1 \leq a_2 \leq a_3 \leq \ldots$$

$\sum a_n$ converges if

$$\sum_{k=0}^{\infty} 2^k a_{(2^k)}$$

converges. (Take a moment to verify this is indeed trivial.)

Questions

The questions discussed were

- SMMC 2023 A1
- SMMC 2019 A1
- SMMC 2019 A2
- SMMC sample question # 9
- SMMC sample question # 13
- BMO 2023 2