SYNCHRONIZATION FOR KPZ

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Abstract. We study the large-times behavior of the KPZ equation:

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \eta(t,x), \qquad h(0,x) = h_0(x), \qquad (t,x) \in [0,\infty) \times \mathbb{T}^d,$$

on the $d-$dimensional torus $\mathbb{T}^d$ driven by some ergodic noise η (e.g. space-time white or fractional in time). We use infinite-dimensional extensions of well known results for positive random matrices to show that solutions to such equation with different initial conditions synchronize at an exponential rate. Furthermore, we establish a one force, one solution principle for $t \in \mathbb{R}$. In particular, these results imply ergodicity of the solution to the equation.

Introduction

In this short work we present an elementary approach to study the large-times behavior of solutions $h\colon \mathbb{R}_{\geqslant} \times \mathbb{T}^d \to \mathbb{R}$ to KPZ-like equations:

$$(\partial_t - \Delta_x) h(t,x) = |\nabla_x h|^2(t,x) + \eta(t,x), \qquad h(0,x) = h_0(x), \qquad (t,x) \in \mathbb{R}_{\geqslant} \times \mathbb{T}^d, \tag{1}$$

where η is a random forcing, $\mathbb{T}^d$ is the d-dimensional torus and $\mathbb{R}_{\geqslant} = [0,\infty)$. This equation is linked to the linear heat equation with multiplicative noise:

$$(\partial_t - \Delta_x) u(t,x) = \eta(t,x) \cdot u(t,x), \qquad u(0,x) = u_0(x), \qquad (t,x) \in \mathbb{R}_{\geqslant} \times \mathbb{T}^d \tag{2}$$

via the Cole-Hopf transform $u = \exp(h)$. In particular, since the latter equation is linear, the KPZ equation is shift invariant, so most results hold "modulo constants", that is for the solution $v = \nabla_x h$ to Burgers' equation:

$$(\partial_t - \Delta_x) v(t,x) = \nabla_x |v|^2(t,x) + \nabla_x \eta(t,x), \qquad v(0,x) = v_0(x), \qquad (t,x) \in \mathbb{R}_{\geqslant} \times \mathbb{T}^d. \tag{3}$$

These equations have obtained much interest in past. For example, Burgers' equation is a toy model for fluid dynamics and the KPZ equation (with η being space-time white noise) arises as a weak scaling limit in many asymmetric growing interface models. Particular interest lies in the large times behavior of the equation. Here one should distinguish two lines of literature. In the first one η is chosen "rough", namely space-time white noise, which makes the solution theory for the equation considerably more intricate (see [17, 18, 14] for celebrated results). The ergodic behavior of this equation was analyzed by Hairer and Mattingly, who proved convergence to the invariant measure "modulo constants" by means of a strong Feller property [19], which has a much broader applicability than just the equations considered here. Funaki and Quastel [13] moreover identify the invariant measure as the Brownian bridge, up to a constant shift. In addition, Gubinelli and Perkowski prove the existence of a spectral gap for Burgers' equation [16], implying exponential convergence to the invariant measure.

On the other side, a vast literature studied the case of smooth noise, e.g. $\eta(t,x) = V(x)\,\mathrm{d}\beta_t$ for $V \in C^\infty(\mathbb{T}^d)$ and a Brownian motion β, establishing further properties deeply intertwined with the structure of the equation. In this direction, a seminal work by Sinai [25]

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proved synchronization for (3). Namely, there exists a function $\overline{v}(t,x)$, defined for all times $t \in \mathbb{R}$, such that for a large class of initial conditions v_0:

$$\lim_{t\to\infty} v(t,x) - \overline{v}(t,x) = 0.$$

If one starts Burgers' equation at time $-n$ with $v^{-n}(-n,x) = v_0(x)$ a similar law holds (called a one force, one solution principle), namely:

$$\lim_{n\to\infty} v^{-n}(t,x) = \overline{v}(t,x), \qquad \forall (t,x) \in (-\infty,\infty) \times \mathbb{T}^d.$$

This implies that $\overline{v}$ is the unique solution to (3) on $\mathbb{R}$. Results of this kind have subsequently been generalized in many directions, most notably to the inviscid case by Weinan, Khanin, Mazel and Sinai [26] or to the non-compact setting, for example by Bakhtin, Cator and Khanin [5].

At this point we should observe that the latter extensions lie beyond the capacities of the approach we use here, since both the compactness of the space and the smoothing property of the Laplacian play a crucial role, as they in turn guarantee compactness in certain function spaces. We instead present here a general approach to study the two properties above in the compact setting, allowing any dimension and any reasonable choice of noise (including the "rough" case, or non-Markovian noises).

Our methods are directly inspired by the work of Sinai [25], where the solution u to (2) is represented by $u(1,x) = Au_0(x)$ for a compact strictly positive operator A, although we do not make use of the representation of A via the Feynman-Kac formula. If η were a time-independent (static) noise, the synchronization of the solution v to (3) would amount to the convergence, upon rescaling, of u to the (random) eigenfunction of A associated to its largest eigenvalue: An instance of the Krein-Rutman Theorem. The main observation in this work is that such convergence can be lifted directly to the non-static case as an application of the theory of random dynamical systems. This follows by a well-known contraction principle for positive operators in projective spaces under Hilbert's projective metric (see [6] for an overview). Indeed such method was already developed by Arnold, Demetrius and Gundlach [3] and later refined by Hennion [20] for random matrices. Their proofs naturally extend to the infinite-dimensional case, giving rise to an ergodic version of the Krein-Rutman theorem (see Theorem 3.2). We can embed this theorem (which provides convergence only in weak topologies) in our PDE setting, to obtain convergences in appropriate Hölder spaces, depending on the regularity of the driving noise (see Theorem 4.3).

It is notable that we can immediately deduce almost sure exponential rates of convergence, cf. [16]. Moreover, we study the convergences at the level of the KPZ Equation (1) instead of only for Burgers' Equation (3) and provide some results concerning the normalization constants, at least for smooth noise: see Corollary 5.7.

Albeit within the restrictions we already addressed, the power of the random dynamical systems approach lies in the capacity of treating almost every reasonable choice of driving noise, namely such that:

(1) The noise η is ergodic (see Proposition 5.8 for a classical condition if η is Gaussian).
(2) Equation (2) is almost surely well-posed (there exists a unique, global in time solution for every $u_0 \in C(\mathbb{T}^d)$), the solution map being a linear, compact, strictly positive operator on $C(\mathbb{T}^d)$.

As an example we treat the case of η being space-time white noise and $\eta(t,x) = V(x)\,\mathrm{d}\beta^H_t$ for β^H a fractional Brownian motion of Hurst parameter H and $V \in C^\infty(\mathbb{T})$. In the latter case the solution is not Markovian, and ergodic results are rare, see for example a work by Maslowski and Pospíšil [21] for ergodicity of linear SPDEs with additive fractional noise.

Finally, let us remark that there are several instances of applications of the theory of random dynamical systems to stochastic PDEs. Particularly related to our work is the study of order-preserving systems which admit some random attractor, first addressed by

Arnold and Chueshov [2], then by Flandoli, Gess and Scheutzow [11] and very recently by Butkovsky and Scheutzow [7]. The spirit of these results is much similar to ours, but although the linearity of (2) on one hand guarantees order preservation, on the other hand it does not allow the existence of a random attractor. In this sense our, essentially linear, case appears to be a degenerate example of the synchronization addressed in the just quoted works.

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1. Notations

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_{\geqslant} = [0, +\infty)$ and $\iota = \sqrt{-1}$. Furthermore, for $d \in \mathbb{N}$ let $\mathbb{T}^d$ be the torus $\mathbb{T}^d = [-1/2, 1/2]^d / \sim$, where $\sim$ is the equivalence relation which glues two opposite edges. The case $d = 1$ is of particular interest, so we write $\mathbb{T} = \mathbb{T}^1$.

For a general set $\mathcal{X}$ and functions $f, g \colon \mathcal{X} \to \mathbb{R}$ we write $f \lesssim g$ is $f(x) \leqslant Cg(x)$ for all $x \in \mathcal{X}$ and a constant C independent of x. To clarify on which parameters C is allowed to depend we might add them as subscripts to the "$\lesssim$" sign.

For $\alpha > 0$ let $\lfloor \alpha \rfloor$ be the smallest integer beneath α and for a multiindex $k \in \mathbb{N}^d$ write $|k| = \sum_{i=1}^d k_i$. Denote with $C(\mathbb{T}^d)$ the space of continuous real-valued functions on $\mathbb{T}^d$, and, for $\alpha > 0$, with $C^\alpha(\mathbb{T})$ the space of $\lfloor \alpha \rfloor-$ differentiable functions f such that $\partial^k f$ is $(\alpha - \lfloor \alpha \rfloor)-$Hölder continuous for every multiindex $k \in \mathbb{N}^d$ such that $|k| = \lfloor \alpha \rfloor$, if $\alpha - \lfloor \alpha \rfloor > 0$, or simply continuous if $\alpha \in \mathbb{N}_0$. For $\alpha \in \mathbb{R}_{\geqslant} \backslash \mathbb{N}_0$ we obtain the seminorms on $C^\alpha(\mathbb{T}^d)$:

$$[f]_\alpha = \max_{|k| = \lfloor \alpha \rfloor} \|\partial^k f\|_\infty 1_{\{|k|>0\}} + \sup_{x,y \in \mathbb{T}^d} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^{\alpha - \lfloor \alpha \rfloor}}.$$

Now, let X be a Banach space. We denote with $\mathcal{B}(X)$ the Borel $\sigma-$algebra on X. Furthermore, let $[a, b] \subseteq \mathbb{R}$ be an interval, then we define $C([a, b]; X)$ the space of continuous functions $f \colon [a, b] \to X$. We write $C_{\mathrm{loc}}((-\infty, b], X)$ for the space of continuous functions with the topology of uniform convergence on compact sets (similarly if $b = \infty$).

Finally, let us introduce a more sophisticated family of space (resp. space-time) distributions: Besov spaces. Following [4, Section 2.2] choose a smooth dyadic partition of the unity on $\mathbb{R}^d$ (resp. $\mathbb{R}^{d+1}$) $(\chi, \{\varrho_j\}_{j \geqslant 0})$ and define $\varrho_{-1} = \chi$. We define the Fourier transforms for $f \colon \mathbb{T}^d \to \mathbb{R}$ and $g \colon \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}$ - these definitions extend naturally to spatial (resp. space-time) tempered distributions $\mathcal{S}'(\mathbb{T}^d)$ (resp. $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$):

$$\mathcal{F}_{\mathbb{T}^d} f(k) = \int_{\mathbb{T}^d} e^{-2\pi\iota\langle k, x\rangle} f(x) \, \mathrm{d}x, \qquad k \in \mathbb{Z}^d,$$

$$\mathcal{F}_{\mathbb{R} \times \mathbb{T}^d} g(\tau, k) = \int_{\mathbb{R} \times \mathbb{T}^d} e^{-2\pi\iota(\tau t + \langle k, x\rangle)} g(t, x) \, \mathrm{d}t \, \mathrm{d}x, \qquad (\tau, k) \in \mathbb{R} \times \mathbb{Z}^d.$$

And similarly its well-known inverses. We then define the space (resp. space-time) Paley blocks:

$$\Delta_j f(x) = \mathcal{F}^{-1}_{\mathbb{T}^d}[\varrho_j \cdot \mathcal{F}_{\mathbb{T}^d} f](x), \qquad \Delta_j g(t, x) = \mathcal{F}^{-1}_{\mathbb{R} \times \mathbb{T}^d}[\varrho_j \cdot \mathcal{F}_{\mathbb{R} \times \mathbb{T}^d} g](t, x).$$

Then we can define the spaces $B^\alpha_{p,q}(\mathbb{T}^d)$ and $B^{\alpha,a}_{p,q}(\mathbb{R} \times \mathbb{T}^d)$ of tempered distributions with, respectively, the following norms:

$$\|f\|_{B^\alpha_{p,q}(\mathbb{T}^d)} = \|(2^{j\alpha} \|\Delta_j f\|_{L^p(\mathbb{T})})_{j \geqslant -1}\|_{\ell^q}, \qquad \|g\|_{B^{\alpha,a}_{p,q}(\mathbb{R} \times \mathbb{T}^d)} = \|(2^{j\alpha} \|\Delta_j f(\cdot)/\langle \cdot \rangle^a\|_{L^p(\mathbb{R} \times \mathbb{T}^d)})_{j \geqslant -1}\|_{\ell^q},$$

where we define the weight $\langle (t, x) \rangle = 1 + |t|$. We can thus define the following Hilbert spaces: $H^\alpha(\mathbb{T}^d) = B^\alpha_{2,2}(\mathbb{T}^d)$ and $H^\alpha_a(\mathbb{R} \times \mathbb{T}^d) = B^{\alpha,a}_{2,2}(\mathbb{R} \times \mathbb{T}^d)$.

2. Setting

We follow closely the work of Bushell [6]. Let X be a Banach space and $K \subseteq X$ a closed cone such that $K \cap (-K) = \{0\}$. Denote with $\mathring{K}$ the interior of K and write $K^+ = K \backslash \{0\}$. Such cone induces a partial order in X by defining for $x, y \in X$:

$$x \leqslant y \iff y - x \in K \quad \text{and} \quad x < y \iff y - x \in \mathring{K}.$$

We furthermore define for $x, y \in K^+$:

$$M(x,y) = \inf\{\lambda \geqslant 0 \colon x \leqslant \lambda y\}, \qquad m(x,y) = \sup\{\mu \geqslant 0 \colon \mu y \leqslant x\},$$

with the convention $\inf \varnothing = \infty$. Then $M(x,y) \in (0,\infty]$ and $m(x,y) \in [0,\infty)$ so that we can define Hilbert's projective distance:

$$d_H(x,y) = \log\left(M(x,y)\right) - \log\left(m(x,y)\right) \in [0,\infty], \qquad \forall\, x,y \in K^+.$$

This metric is only semidefinite positive on K^+, and may be infinite. A remedy for the first issue is to consider a linear space $U \subseteq X$ which intersects transversely K^+, that is:

$$\forall x \in K^+, \quad \exists ! \lambda > 0 \quad \text{s.t.} \quad \lambda x \in U.$$

We write $\lambda(x)$ for the normalization constant above. As for the second issue, one can observe that the distance is finite on the interior of K, cf. [6, Theorem 2.1] and thus, defining $E = \mathring{K} \cap U$ one has that (E, d_H) is a metric space. Consider $\mathcal{L}(X)$ the set of linear bounded operators on X. We use the following definitions of positive operators:

$$\begin{aligned} A(K) \subseteq K \quad &\Rightarrow \quad A \text{ nonnegative.} \\ A(\mathring{K}) \subseteq \mathring{K} \quad &\Rightarrow \quad A \text{ positive.} \\ A(K^+) \subseteq \mathring{K} \quad &\Rightarrow \quad A \text{ strictly positive.} \end{aligned}$$

The projective action of a positive operator A on X is then defined by: $A \cdot x = Ax / \lambda(Ax)$. We denote with $\tau(A)$ the projective norm:

$$\tau(A) = \sup_{\substack{x,y \in E \\ x \neq y}} \frac{d_H(A \cdot x, A \cdot y)}{d_H(x,y)}. \tag{4}$$

The backbone of our approach is Birkhoff's theorem for positive operators [6, Theorem 3.2]:

$$\tau(A) = \tanh\Big(\frac{1}{4}\Delta(A \cdot E)\Big) \leqslant 1, \quad \text{where} \quad \Delta(F) = \sup_{x,y \in F}\{d_H(x,y)\}.$$

We denote with $\mathcal{L}_{\mathrm{cp}}(X)$ the space of positive operators A which are contractive in (E, d_H):

$$A \in \mathcal{L}_{\mathrm{cp}}(X) \iff \tau(A) < 1.$$

Example 2.1. The only example considered in this work is $X = C(\mathbb{T}^d)$ the space of real-valued continuous functions on the torus. Here K is the cone of nonnegative functions, and:

$$U = \Big\{ f \in X \colon \int_{\mathbb{T}^d} f(x)\,\mathrm{d}x = 1 \Big\}.$$

It is easy to see that in this setting the projective distance is bounded by:

$$\|\log(f) - \log(g)\|_\infty \leqslant d_H(f,g) \leqslant 2\|\log(f) - \log(g)\|_\infty, \qquad \forall f,g \in E. \tag{5}$$

In particular, this inequality implies that (E, d_H) is a complete metric space. Moreover strictly positive kernels are contractions on E (cf. [6, Section 6]), i.e. if for some $0 < \alpha \leqslant \beta < \infty$:

$$A \in \mathcal{L}(X),\ A(f)(x) = \int_{\mathbb{T}^d} K(y,x) f(y)\,\mathrm{d}y,\ 0 < \alpha \leqslant K(y,x) \leqslant \beta,\ \Rightarrow\ A \in \mathcal{L}_{\mathrm{cp}}(X). \tag{6}$$

Remark 2.2. *For the sake of simplicity and concreteness we did not address the general question of completeness of the space* (E, d_H)*, since in the case of interest to us it follows from Equation* (5)*. There are known criteria for completeness, cf.* [6, Section 4] *and the references therein.*

Remark 2.3. *In view of* (4)*, an application of Banach's fixed point theorem in* (E, d_H) *to operators satisfying* (6) *delivers the existence of a unique positive eigenfunction. This is a variant of the Krein-Rutman theorem. The formulation we propose here is convenient because of its natural extension to random dynamical systems.*

3. A Random Krein-Rutman Theorem

In this section we reformulate the results of [3, 20] for positive operators on Banach spaces.

An *invertible metric discrete dynamical system* (IDS) $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a measurable map $\vartheta \colon \mathbb{Z} \times \Omega \to \Omega$ such that $\vartheta(z+z', \cdot) = \vartheta(z, \vartheta(z', \cdot))$ and $\vartheta(0, \omega) = \omega$ for all $\omega \in \Omega$, and such that $\mathbb{P}$ is invariant under $\vartheta(z, \cdot)$ for $z \in \mathbb{Z}$. For brevity we write $\vartheta^z(\cdot)$ for the map $\vartheta(z, \cdot)$. A set $\widetilde{\Omega} \subseteq \Omega$ is said to be *invariant* for ϑ if $\vartheta^z \widetilde{\Omega} = \widetilde{\Omega}$, for all $z \in \mathbb{Z}$ and an IDS is said to be *ergodic* if any invariant set $\widetilde{\Omega}$ satisfies $\mathbb{P}(\widetilde{\Omega}) \in \{0, 1\}$ (cf. [1, Appendix A]).

Consider X, E as in the previous section and, for a given IDS, a random variable $A \colon \Omega \to \mathcal{L}(X)$. This generates a measurable, linear, discrete random dynamical system (RDS) (see [1, Definition 1.1.1]) φ on X by defining $A_0(\omega) = \omega$ and $A_n(\omega) = A(\vartheta^n \omega), n \in \mathbb{N}$ and letting:

$$\varphi_n(\omega)x = A_n(\omega) \cdots A_0(\omega)x, \qquad n \in \mathbb{N}_0.$$

If $A(\omega)$ is in addition positive for every $\omega \in \Omega$ (we then simply say that A is positive), we can interpret φ as an RDS on E via the projective action:

$$\varphi_n(\omega) \cdot x = A_n(\omega) \cdot \cdots \cdot A_0(\omega) \cdot x, \qquad n \in \mathbb{N}_0.$$

Assumption 3.1. *Assume we are given* X, K, U, E *as in the previous section and that* (E, d_H) *is a* complete *metric space. Assume in addition that there exists an* ergodic *IDS* ϑ*. Let* φ_n *be a RDS defined via a random* positive *operator* A *as above, such that:*

$$\mathbb{P}\Big(A \in \mathcal{L}_{cp}(X)\Big) > 0.$$

In this setting the following is a random version of the Krein-Rutman theorem.

Theorem 3.2. *Under Assumption 3.1 there exists a* $\vartheta-$*invariant set* $\widetilde{\Omega} \subseteq \Omega$ *of full* $\mathbb{P}-$*measure and a random variable* $u \colon \Omega \to \mathring{K}$ *such that:*

(1) *For all* $\omega \in \widetilde{\Omega}$ *and* $f, g \in E$*:*

$$\limsup_{n \to \infty} \frac{1}{n} \sup_{f,g \in E} \log d_H(\varphi_n(\omega) \cdot f, \varphi_n(\omega) \cdot g) \leqslant \mathbb{E} \log \big(\tau(A)\big) < 0.$$

(2) u *is measurable w.r.t. to the* $\sigma-$*field* $\mathcal{F}^- = \sigma((A(\vartheta^{-n} \cdot))_{n \in \mathbb{N}})$ *and:*

$$\varphi_n(\omega) \cdot u(\omega) = u(\vartheta^n \omega).$$

(3) *For all* $\omega \in \widetilde{\Omega}$*:*

$$\limsup_{n \to \infty} \frac{1}{n} \sup_{f \in E} \log d_H(\varphi_n(\vartheta^{-n}\omega) \cdot f, u(\omega)) \leqslant \mathbb{E} \log \big(\tau(A)\big) < 0$$

as well as:

$$\limsup_{n \to \infty} \frac{1}{n} \sup_{f \in E} \log d_H(\varphi_n(\omega) \cdot f, u(\vartheta^n \omega)) \leqslant \mathbb{E} \log \big(\tau(A)\big) < 0.$$

(4) *The measure* $\delta_{u(\omega)}$ *on* E *is the unique invariant measure for the RDS* φ *on* E*.*

Notation 3.3. *We refer to the first property as* asymptotic synchronization *and to the third property as* one force, one solution *principle.*

Proof. As for the first property, we can compute:

$$\begin{aligned} d_H(\varphi_n(\omega)\cdot f, \varphi_n(\omega)\cdot g) \leqslant& \tau(A_n(\omega)) d_H(\varphi_{n-1}(\omega)\cdot f, \varphi_{n-1}(\omega)\cdot g) \leqslant \cdots \\ \leqslant& \prod_{i=0}^{n} \tau(A(\vartheta^i\omega)) d_H(f,g). \end{aligned}$$

Then, applying the logarithm and Birkhoff's ergodic theorem we find:

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\tau(\varphi_n(\omega))\right) \leqslant \mathbb{E}\log\left(\tau(A)\right) < 0.$$

Note that if $\mathbb{E}\log\big(\tau(A)\big) = -\infty$ we can instead use the previous computation with $\tau(A(\vartheta^i\omega))$ replaced by $\tau(A(\vartheta^i\omega)) \wedge e^{-M}$ and eventually pass to the limit $M \to \infty$. To obtain the result uniformly over f, g we simply apply a Taylor expansion to:

$$\Delta(\varphi_n \cdot E) = 4 \text{ arctanh } (\tau(\varphi_n(\omega))).$$

The second point as well as the first property of (3) follow from Lemma 3.4 below. The invariant sets in all points can be chosen to be equal to the same $\widetilde{\Omega}$ up to taking intersections of invariant sets, which are still invariant. Point (4) and the last point of (3) follow the from the previous properties.

□

Lemma 3.4. *There exists a $\vartheta-$invariant set $\widetilde{\Omega} \subseteq \Omega$ of full $\mathbb{P}-$measure and an $\mathcal{F}^{-}-$adapted random variable $u : \Omega \to \mathring{K}$ such that:*

$$\varphi_n(\omega)u(\omega) = u(\vartheta^n\omega), \qquad \forall \omega \in \widetilde{\Omega}, n \in \mathbb{N}.$$

Moreover for all $\omega \in \widetilde{\Omega}$:

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{f\in E} \log d_H(\varphi_n(\vartheta^{-n}\omega)\cdot f, u(\omega)) \leqslant \mathbb{E}\log\big(\tau(A)\big) \in [-\infty, 0).$$

Proof. We start by observing (as in [20, Proof of Lemma 3.3]) that the sequence of sets $F_n(\omega) = \varphi_n(\vartheta^{-n}\omega)\cdot E$ is decreasing, i.e. $F_{n+1} \subseteq F_n$. Let us write $F(\omega) = \bigcap_{n\geqslant 1} F_n(\omega)$. It is possible to estimate:

$$\Delta(F) \leqslant \lim_{n\to\infty} \Delta(F_n) = \lim_{n\to\infty} 4 \text{ arctanh } (\tau(\varphi_n(\vartheta^{-n}\omega))).$$

Now, there exists a $\vartheta-$invariant set $\widetilde{\Omega}$ of full $\mathbb{P}-$measure such that for all $\omega \in \widetilde{\Omega}$:

$$\lim_{n\to\infty} \frac{1}{n} \log\Big(\tau(\varphi_n(\vartheta^{-n}\omega))\Big) \leqslant \lim_{n} \frac{1}{n} \sum_{i=0}^{n} \log\tau(A(\vartheta^{-i}\omega)) = \mathbb{E}\log\big(\tau(A)\big) < 0. \tag{7}$$

In particular $\Delta(F) = 0$. By completeness of E it follows that F is a singleton. Let us write $F(\omega) = \{u(\omega)\}$ and extend u trivially outside of $\widetilde{\Omega}$: it is clear that u is adapted to $\mathcal{F}^{-}$. Since for $k \in \mathbb{N}$ and $n \geqslant k$

$$\varphi_n(\vartheta^{-n}\vartheta^k\omega) = \varphi_k(\omega)\cdot\varphi_{n-k}(\vartheta^{-(n-k)}\omega),$$

passing to the limit we have: $u(\vartheta^k\omega) = \varphi_k(\omega)u(\omega)$.

Finally, a Taylor expansion guarantees that:

$$\Delta(\varphi_n(\vartheta^{-n}\omega)\cdot E) = 4 \text{ arctanh } (\tau(\varphi_n(\vartheta^{-n}\omega))) \leqslant \tau(\varphi_n(\vartheta^{-n}\omega))(4+\mathcal{O}(1)).$$

This estimate, combined with the fact that

$$\sup_{f\in E} d_H(\varphi_n(\vartheta^{-n}\omega)\cdot f, u(\omega)) = \sup_{f\in E} d_H(\varphi_n(\vartheta^{-n})\cdot f, \varphi_n(\vartheta^{-n}\omega)\cdot u(\vartheta^{-n}\omega)) \leqslant \Delta(\varphi_n(\vartheta^{-n}\omega)\cdot E)$$

and (7) provides the required convergence result.

□

4. Application to SPDEs

In this section we discuss how to apply the previous results to stochastic PDEs. Concrete examples will be covered in the next section. For clarity, nonetheless, the reader should keep in mind that we want to study ergodic properties of solutions to Equation (1). Since the associated heat equation with multiplicate noise (2) is linear and the solution map is expected to be strictly positive, we may assume that such solution map generates a continuous random dynamical system φ.

In general it is not trivial to prove that a stochastic system generates a continuous-time random dynamical system over a *continuous* dynamical system ϑ (the problem is known as perfection, cf. [1, Section 1.3]). Since we want to use the solution theory for (2) with η space-time white noise as a black box, we want to assume only that we are guaranteed the existence of a solution outside of a null-set N_0 depending on the initial time $t = 0$. In this case the system satisfies only what is known as a *crude* cocycle property. To avoid the issue of continuous perfection we then restrict our attention to *continuous* RDS over *discrete* IDS ϑ, as described below.

Definition 4.1. *A* continuous *RDS over a* discrete *IDS* $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ *and on a measure space* $(X, \mathcal{B})$ *is a map*

$$\varphi \colon \mathbb{R}_{\geqslant} \times \Omega \times X \to X$$

such that the following two properties hold:

(1) Measurability: φ *is* $\mathcal{B}(\mathbb{R}_{\geqslant}) \otimes \mathcal{F} \otimes \mathcal{B}-$ *measurable.*
(2) Cocycle property: $\varphi(0, \omega) = \mathrm{Id}_X$, *for all* $\omega \in \Omega$ *and:*

$$\varphi(t+n, \omega) = \varphi(t, \vartheta^n \omega) \circ \varphi(n, \omega), \qquad \forall t \in \mathbb{R}_{\geqslant}, n \in \mathbb{N}_0, \omega \in \Omega.$$

We then formulate the following assumptions, under which our main result will hold.

Assumption 4.2. *Let* $d \in \mathbb{N}$ *and* $\beta > 0$. *Let* $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P}, \vartheta)$ *be a discrete ergodic IDS, over which is defined a continuous RDS* φ*:*

$$\varphi : \mathbb{R}_{\geqslant} \times \Omega_{\mathrm{kpz}} \to \mathcal{L}(C(\mathbb{T}^d)).$$

There exists a $\vartheta-$*invariant set* $\widetilde{\Omega} \subseteq \Omega_{\mathrm{kpz}}$ *of full* $\mathbb{P}-$*measure such that following properties are satisfied for all* $\omega \in \widetilde{\Omega}$ *and any* $T > S > 0$*:*

(1) *There exists a kernel* $K : \Omega_{\mathrm{kpz}} \to C([S,T]; C(\mathbb{T}^d \times \mathbb{T}^d))$ *such that for all* $S \leqslant t \leqslant T$*:*

$$\varphi_t(\omega) f(x) = \int_{\mathbb{T}^d} K(\omega, t, x, y) f(y)\, \mathrm{d}y, \qquad \forall f \in C(\mathbb{T}^d), x \in \mathbb{T}^d.$$

(2) *There exist* $0 < \gamma(\omega, S, T) \leqslant \delta(\omega, S, T)$ *such that:*

$$\gamma(\omega, S, T) \leqslant K(\omega, t, x, y) \leqslant \delta(\omega, S, T), \qquad \forall x, y \in \mathbb{T}^d,\ S \leqslant t \leqslant T,$$

which implies that $\mathbb{P}\big(\varphi_t \in \mathcal{L}_{\mathrm{cp}}(C(\mathbb{T}^d)), \forall t \in (0, \infty)\big) = 1$.

(3) *There exists a constant* $C(\beta, \omega, S, T)$ *such that:*

$$\|\varphi_t f\|_\beta \leqslant C(\beta, \omega, S, T) \|f\|_\infty, \qquad \forall f \in C(\mathbb{T}^d),\ S \leqslant t \leqslant T.$$

(4) *The following moment estimates are satisfied for any* $f \in C(\mathbb{T}^d)$*:*

$$\mathbb{E} \log \big(C(\beta, S, T)\big) + \mathbb{E} \sup_{S \leqslant t \leqslant T} d_H(\varphi_t \cdot f, f) < +\infty.$$

The first two assumptions essentially imply that we can use the results from the previous section. The last two will allow us to lift the convergence From $C(\mathbb{T}^d)$ to $C^\beta(\mathbb{T}^d)$. We now state the main result of this section. In view of the motivating example and in the setting of the previous assumption, we say that for $z \in \mathbb{Z}$ the map

$$[z, +\infty) \times \mathbb{T}^d \ni (t, x) \mapsto h^z(\omega, t, x), \qquad h^z(\omega, z, x) = h_0(x)$$

solves Equation (1) if $h^z(\omega, t) = \log\big(\varphi_t(\vartheta^z \omega) \exp(h_0)\big)$ for φ_t as in the previous assumption.

Theorem 4.3. *Under Assumption 4.2, for $h_0^1, h_0^2 \in C(\mathbb{T}^d)$, $n \in \mathbb{N}, i \in \{1,2\}$, let $h_i(t) \in C(\mathbb{T}^d)$ be the random solution to Equation (1) started at time 0 in h_0^i and evaluated at time $t \geqslant 0$. Similarly, let $h_i^{-n}(t) \in C(\mathbb{T}^d)$ be the solution started in $-n$ in h_0^i and evaluated at time $t \geqslant -n$. There exists an invariant set $\widetilde{\Omega} \subseteq \Omega_{\mathrm{kpz}}$ of full $\mathbb{P}-$measure such that for any $\alpha < \beta,\ \alpha \notin \mathbb{N}$:*

(1) *For any $h_0^i \in C(\mathbb{T}^d), i \in \{1,2\}$ there exists a map $c(h_0^1, h_0^2) \colon \Omega_{\mathrm{kpz}} \times \mathbb{R}_{\geqslant} \to \mathbb{R}$ such that for any $T > 0, \omega \in \widetilde{\Omega}$:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{t\in[n,n+T]} \|h_1(\omega,t) - h_2(\omega,t) - c(\omega,t,h_0^1,h_0^2)\|_{C^\alpha(\mathbb{T}^d)} < 0,$$

as well as:

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{t\in[n,n+T]} [h_i(\omega,t)]_\alpha \leqslant 0.$$

(2) *There exists a random function $h_\infty \colon \Omega_{\mathrm{kpz}} \to C_{\mathrm{loc}}((-\infty,\infty); \mathcal{C}^\alpha(\mathbb{T}^d))$ such that for any $T > 0, \omega \in \widetilde{\Omega}$ as well as $h_0^1 \in C(\mathbb{T}^d)$, there exists a sequence of maps $c^{-n}(h_0^1) \colon \Omega_{\mathrm{kpz}} \times \mathbb{R}_{\geqslant} \to \mathbb{R}$ for which:*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{h_0^1 \in C(\mathbb{T}^d)} \log \sup_{t\in[(-T)\vee(-n),T]} \|h_1^{-n}(\omega,t) - h_\infty(\omega,t) - c^{-n}(\omega,t,h_0^1)\|_{C^\alpha(\mathbb{T}^d)} < 0.$$

Passing to the gradient we can omit the constants and find the following principles for Burgers' Equation (3).

Corollary 4.4. *In the same setting as before, it immediately follows that also:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{t\in[n,n+T]} \|\nabla_x h_1(\omega,t) - \nabla_x h_2(\omega,t)\|_{C^{\alpha-1}(\mathbb{T}^d)} < 0,$$
$$\limsup_{n\to\infty} \frac{1}{n} \sup_{h_0^1 \in C(\mathbb{T}^d)} \log \sup_{t\in[(-T)\vee(-n),T]} \|\nabla_x h_1^{-n}(\omega,t) - \nabla_x h_\infty(\omega,t)\|_{C^{\alpha-1}(\mathbb{T}^d)} < 0,$$

where the space $C^{\alpha-1}(\mathbb{T}^d)$ has to be interpreted as the Besov space $B^{\alpha-1}_{\infty,\infty}(\mathbb{T}^d)$ for $\alpha \in (0,1)$.

Proof of Theorem 4.3. Step 1. Define:

$$u_0^i = \exp(h_0^i)/\|\exp(h_0^i)\|_{L^1} \in E,$$

so that $h_i(\omega,t) = \log\left(\varphi_t(\omega)\cdot u_0^i\right) + c_i(\omega,t)$, where $c_i(\omega,t) \in \mathbb{R}$ is the normalization constant:

$$c_i(\omega,t) = \log\left(\int_{\mathbb{T}^d} \left(\varphi_t(\omega) u_0^i\right)(x)\,\mathrm{d}x\right) + \log\left(\int_{\mathbb{T}^d} \exp(h_0^i)(x)\,\mathrm{d}x\right).$$

Let us write $c(\omega,t,h_0^1,h_0^2) = c_1(\omega,t) - c_2(\omega,t)$. Similarly, for $-n \leqslant t \leqslant 0$ we have

$$h_i^{-n}(\omega,t) = \log\left(\varphi_{n+t}(\vartheta^{-n}\omega)\cdot u_0^i\right) + c_i^{-n}(\omega,t) = h_i(\vartheta^{-n}\omega, n+t),$$

where $c_i^{-n}(\omega,t) = c_i(\vartheta^{-n}\omega, n+t)$. We also write $c^{-n}(\omega,t,h_0^1,h_0^2) = c_1^{-n}(\omega,t) - c_2^{-n}(\omega,t)$. Now we prove the following simpler version of the required result:

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \sup_{t\in[n,n+T]} \|h_1(\omega,t) - h_2(\omega,t) - c(\omega,t,h_0^1,h_0^2)\|_\infty < 0\\
&\limsup_{n\to\infty} \frac{1}{n} \sup_{h_0^1,h_0^2 \in C(\mathbb{T}^d)} \log \sup_{t\in[(-T)\vee(-n),T]} \|h_1^{-n}(\omega,t) - h_2^{-n}(\omega,t) - c^{-n}(\omega,t,h_0^1,h_0^2)\|_\infty < 0.
\end{aligned} \tag{8}$$

First we eliminate the time supremum. Indeed we have, in view of Inequality (5):

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log\sup_{t\in[n,n+T]}\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\\&\quad\leqslant\limsup_{n\to\infty}\frac{1}{n}\log\sup_{t\in[n,n+T]}d_H(\varphi_t(\omega)\cdot u_0^1,\varphi_t(\omega)\cdot u_0^2)\leqslant\limsup_{n\to\infty}\frac{1}{n}\log d_H(\varphi_n(\omega)\cdot u_0^1,\varphi_n(\omega)\cdot u_0^2)\end{aligned}$$

since one can estimate

$$\sup_{t\in[n,n+T]}d_H(\varphi_t(\omega)\cdot u_0^1,\varphi_t(\omega)\cdot u_0^2)\leqslant\sup_{t\in[n,n+T]}\tau(\varphi_{t-n}(\vartheta^n\omega))\cdot d_H(\varphi_n(\omega)\cdot u_0^1,\varphi_n(\omega)\cdot u_0^2)$$

and $\tau(\varphi_{t-n}(\vartheta^n\omega))\leqslant 1$. Similarly, also for the backwards case. At this point, in view of Assumption 4.2, we can apply Theorem 3.2 in the setting of Example 2.1 with $A(\omega)=\varphi_1(\omega)$ to see that there exists a $u_\infty=\exp(h_\infty)\colon\Omega_{\text{kpz}}\to C(\mathbb{T}^d)$ such that:

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log d_H(\varphi_n(\omega)\cdot u_0^1,\varphi_n(\omega)\cdot u_0^2)<0,\\&\limsup_{n\to\infty}\frac{1}{n}\sup_{u_0^1\in E}\log d_H(\varphi_n(\vartheta^{-n}\omega)\cdot u_0^1,u_\infty(\omega))<0,\end{aligned}$$

which via the previous calculation implies (8).

Step 2. We now prove convergence in $C^\alpha(\mathbb{T}^d)$ for $\alpha<\beta,\alpha\notin\mathbb{N}$. Since the same arguments extend to the more general case, we may assume that $\beta\in(0,1)$. Thus fix α and define $\theta\in(0,1)$ by $\alpha=\beta\theta$. Since we already proved convergence in $\|\cdot\|_\infty$, to prove convergence in $C^\alpha(\mathbb{T}^d)$ we only have to control the $\alpha-$seminorm $[\cdot]_\alpha$. We treat the forwards and backwards in time cases differently. Let us start with the first case. We bound the Hölder seminorm via:

$$\begin{aligned}&[h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)]_\alpha\\&\quad\leqslant\Big(2\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\Big)^{1-\theta}\Big([\log(\varphi_t(\omega)\cdot u_0^1)]_\beta+[\log(\varphi_t(\omega)\cdot u_0^2)]_\beta\Big)^\theta\end{aligned}\tag{9}$$

Then fix n,T and $t\in[n,n+T]$, denote $t=n-1+\tau$ and rewrite the last terms as:

$$\begin{aligned}[\log(\varphi_t(\omega)\cdot u_0^1)]_\beta&\leqslant\frac{1}{m(\varphi_t(\omega)\cdot u_0^1)}[\varphi_t(\omega)\cdot u_0^1]_\beta=\frac{1}{m(\varphi_t(\omega)\cdot u_0^1)}[\varphi_\tau(\vartheta^{n-1}\omega)\varphi_{n-1}(\omega)\cdot u_0^1]_\beta\\&\leqslant\frac{C(\beta,\vartheta^{n-1}\omega,1,T+1)}{m(\varphi_t(\omega)\cdot u_0^1)}\|\varphi_{n-1}(\omega)\cdot u_0^1\|_\infty\end{aligned}$$

where $m(\cdot)$ indicates the minimum of a function. Then estimate:

$$\begin{aligned}\log\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\alpha\leqslant&(1-\theta)\log 2\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\\&+\theta\log\Big(\sum_{i=1,2}\frac{C(\beta,\vartheta^{n-1}\omega,1,T+1)}{m(\varphi_t(\omega)\cdot u_0^i)}\|\varphi_{n-1}(\omega)\cdot u_0^i\|_\infty\Big).\end{aligned}$$

To conclude, in view of the result from the previous step, we have to prove that:

$$\limsup_{n\to\infty}\sup_{n\leqslant t\leqslant n+T}\frac{1}{n}\log\Big(\sum_{i=1,2}\frac{C(\beta,\vartheta^{n-1}\omega,1,T+1)}{m(\varphi_t(\omega)\cdot u_0^i)}\|\varphi_{n-1}(\omega)\cdot u_0^i\|_\infty\Big)\leqslant 0.$$

By the means of considerations on the line of $\log\max_i x_i=\max_i\log x_i$ it is sufficient to prove that for any $f\in C(\mathbb{T}^d)$:

$$\limsup_{n\to\infty}\sup_{n\leqslant t\leqslant n+T}\frac{1}{n}\Big[\log\Big(C(\beta,\vartheta^{n-1}\omega,1,T+1)\Big)+\log\Big(\|\varphi_{n-1}(\omega)\cdot f\|_\infty\Big)-\log\Big(m\big(\varphi_t(\omega)\cdot f\big)\Big)\Big]\leqslant 0,$$

which is once more equivalent to the following:

$$\limsup_{n\to\infty}\frac{1}{n}\left[\log\Big(C(\beta,\vartheta^{n-1}\omega,1,T{+}1)\Big)+\sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi_{n-1+\tau}(\omega)f,f)\right]\leqslant 0. \tag{10}$$

Let us start with the last term. We can bound:

$$\begin{aligned} d_H(\varphi_{n-1+\tau}(\omega)\cdot f,f) &\leqslant \tau(\varphi_\tau(\vartheta^{n-1}\omega))d_H(\varphi_{n-1}(\omega)\cdot f,f)+d_H(\varphi_\tau(\vartheta^{n-1}\omega)\cdot f,f)\leqslant\dots\\ &\leqslant \sum_{i=0}^{n-1}\prod_{j=i+1}^{n-1}\tau(\varphi_1(\vartheta^j\omega))d_H(\varphi_1(\vartheta^i\omega)\cdot f,f)+\sup_{1\leqslant\tau\leqslant T+1}d_H(\varphi_\tau(\vartheta^{n-1}\omega)\cdot f,f). \end{aligned}$$

By Assumption 4.2 $\sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi_\tau(\omega)\cdot f,f)\in L^1$, hence:

$$\lim_{n\to\infty}\frac{1}{n}\sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi_\tau(\vartheta^{n-1}\omega)\cdot f,f)=0$$

by the ergodic theorem. Now note that by Lebesgue dominated convergence, since $d_H(\varphi_1 f,f)\in L^1$, it holds that:

$$\lim_{c\to\infty}\mathbb{E}\left[\prod_{j=1}^{c}\tau(\varphi_1(\vartheta^j\omega))d_H(\varphi_1 f,f)\right]=0.$$

Hence fix $\varepsilon>0$ and choose $c\in\mathbb{N}$ so that the average above is bounded by ε. We can then estimate once more via the ergodic theorem:

$$\begin{aligned} &\limsup_{n\to\infty}\frac{1}{n}\sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi_{n-1+\tau}(\omega)\cdot f,f)\\ &\qquad\leqslant \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1-c}\prod_{j=i+1}^{i+c}\tau(\varphi_1(\vartheta^j\omega))d_H(\varphi_1(\vartheta^i\omega)\cdot f,f)\leqslant\varepsilon. \end{aligned}$$

This delivers the required result. To deduce (10) we are left with the term containing $C(\beta,\vartheta^n\omega)$. Once more the ergodic theorem and Assumption 4.2 we immediately have that :

$$\lim_{n\to\infty}\frac{1}{n}\log C(\beta,\vartheta^n\omega,1,T{+}1)=0.$$

This concludes the proof of (10).

We pass to the bound backwards in time. Up to replacing T with $\lceil T\rceil$ assume $T\in\mathbb{N}$. We find for $T<n{-}1$ and $-T\leqslant t\leqslant T$ so that $t=-T{-}1{+}\tau$ with $1\leqslant\tau\leqslant 2T{+}1$:

$$\begin{aligned} &[h_1^{-n}(\omega,t)-h_\infty(\omega,t)-c^{-n}(\omega,t,h_0^1,h_0^2)]_\alpha=\\ &\quad=\Big[\log\Big(\varphi_\tau(\vartheta^{-T-1}\omega)\big(\varphi_{n-T-1}(\vartheta^{-n}\omega)\cdot u_0^1\big)\Big)-\log\Big(\varphi_\tau(\vartheta^{-T-1}\omega)\big(u_\infty(\omega,-T{-}1)\big)\Big)\Big]_\alpha. \end{aligned}$$

Now since $\varphi_{n-T-1}(\vartheta^{-n}\omega)\cdot u_0^1\to u_\infty(\omega,-T{-}1)$ in $C(\mathbb{T}^d)$ uniformly over u_0^1 there exists a $c>0$ such that $\big(\varphi_n(\omega)\cdot u_0^1\big)(x)\geqslant c,\ \forall x\in\mathbb{T}^d, n\in\mathbb{N}, u_0^1\in C(\mathbb{T}^d)$. By Assumption 4.2 this implies that:

$$\inf_{u_0^1\in C(\mathbb{T}^d)}\inf_{n>T+1}\inf_{\substack{1\leqslant\tau\leqslant T+1\\ x\in\mathbb{T}^d}}\varphi_\tau(\vartheta^{-T}\omega)\big(\varphi_{n-1}(\vartheta^{-n}\omega)\cdot u_0^1\big)(x)\geqslant c\gamma(\vartheta^{-T}\omega,1,2T{+}1).$$

With the above estimate we can follow the interpolation bound (9) and a simpler version of the steps that follow (indeed, we do not need to apply the ergodic theorem in this case) to conclude the proof.

□

5. Examples

We treat two prototypical examples, which show the range of applicability of the previous results. First, we treat the KPZ equation driven by space-time white noise. In the second example the noise is taken to be smooth in space (only for simplicity) but fractional in time.

5.1. **KPZ driven by space-time white noise.** Let the random force η in (1) be space-time white noise ξ in one spatial dimension. That is, a Gaussian processes indexed by functions in $L^2(\mathbb{R}\times\mathbb{T})$ such that:

$$\mathbb{E}\Big[\xi(f)\xi(g)\Big] = \int_{\mathbb{R}\times\mathbb{T}} f(t,x)g(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

Then we consider h, u the respective solutions to the equations:

$$(\partial_t - \partial_x^2)h = (\partial_x h)^2 + \xi - \infty, \qquad h(0,x) = h_0(x), \qquad (t,x) \in \mathbb{R}_{\geqslant} \times \mathbb{T} \tag{11}$$

$$(\partial_t - \partial_x^2)u = u\cdot(\xi - \infty), \qquad u(0,x) = u_0(x), \qquad (t,x) \in \mathbb{R}_{\geqslant} \times \mathbb{T} \tag{12}$$

where the presence of the infinity "∞" indicates the necessity of renormalization. The solution theory for this equation has been a celebrated result [17, 18, 14] and requires tools as regularity structures or paracontrolled distributions. We shall use this theory as a black box, via Lemma 5.1 below. First, let us rigorously define the IDS associated to the equation.

Lemma 5.1. *Let $\Omega_{\mathrm{kpz}} = H^\alpha_a(\mathbb{R}\times\mathbb{T})$ for $\alpha < -1, a > \frac{1}{2}$, $\mathcal{F} = \mathcal{B}(H^\alpha_a(\mathbb{R}\times\mathbb{T}))$ and let $\mathbb{P}$ be the law of space-time white noise ξ on Ω_{kpz}. The space $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P})$ is extended to an* ergodic *IDS via the translation group (in the sense of distributions) $\{\vartheta^z\}_{z\in\mathbb{Z}}$, which acts by:*

$$\vartheta^z\omega(t,x) = \omega(t+z,x), \qquad \forall \omega \in \Omega_{\mathrm{kpz}}, \ \ t \in \mathbb{R}, \ \ x \in \mathbb{T}.$$

There exists a null-set $N_0 \subseteq \Omega_{\mathrm{kpz}}$ such that Equation (12) *admits a solution*

$$\mathbb{R}_{\geqslant} \ni t \mapsto \psi_t(\omega)u_0, \qquad \forall\, \omega \in N_0^c, \ \ u_0 \in C(\mathbb{T})$$

in the sense of [15, Theorem 6.15]. *There furthermore exists a continuous RDS φ such that:*

$$\forall \omega \in \Omega_{\mathrm{kpz}}, \qquad \varphi_t(\omega) \in \mathcal{L}(C(\mathbb{T})) \qquad \textit{and} \qquad \mathbb{P}\Big(\varphi_t u_0 = \psi_t u_0, \ \ \forall t \geqslant 0, u_0 \in C(\mathbb{T})\Big) = 1.$$

Proof. The fact that space-time white noise ξ lives in Ω_{kpz} as defined above, as well as the fact that translations induce an ergodic dynamical system is explained in Corollary 5.9. The path-wise well-posedness of the equation outside of a null-set N_0 is proved (among others) in [15, Theorem 6.15]. Up to enhancing N_0 to $\overline{N}_0 = \bigcup_{z\in\mathbb{Z}} \vartheta^z N_0$ we may assume that N_0 is $\vartheta-$invariant. Then we can define $\varphi(\omega) = \psi(\omega)$ on N_0^c and extend it trivially on N_0. □

The RDS φ introduced in the previous lemma falls into the framework of the preceding sections.

Corollary 5.2. *Let φ be defined as in Lemma 5.1. Then φ satisfies Assumption 4.2 in dimension $d = 1$ for any $\beta < \frac{1}{2}$. In particular, the results of Theorem 4.3 apply.*

Since it is slightly technical, we postpone a rigorous proof to the appendix. Here we just address the main ideas at work.

Remark 5.3. *The proof of the previous lemma works as follows. The first property of Assumption 4.2 is a consequence of the linearity of the equation, while the second one is a consequence of a strong maximum principle satisfied by the SPDE* [22, 8]. *The third property is a consequence of the smoothing effect of the Laplacian (the condition $\beta < \frac{1}{2}$ is due to the irregularity of space-time white noise). It is straightforward to make this smoothing effect quantitative and obtain the first average bound appearing in the fourth property. The last bound requires a quantitative lower bound to* (12)*, which was developed in* [23].

Remark 5.4. *In the previous lemma we have proven that we can apply Theorem 4.3. The latter guarantees synchronization up to subtracting constants $c(\omega,t)$. In fact it is possible to choose $c(\omega,t) \equiv \overline{c}(\omega)$ for a time-independent $\overline{c}(\omega)$. For fractional noise this is proved in Corollary 5.7 below, where we make use of the spatial smoothness of the noise to write an ODE for the constant $c(\omega,t)$: Equation* (15)*. Backwards in time the same approach guarantees that*

h_∞ is the unique solution to the KPZ Equation for $t \in \mathbb{R}$, up to constant (in time and space) shifts.

The approach of Corollary 5.7 can be lifted to the space-time white noise setting by defining the product which appears in the ODE for example in a paracontrolled way. To complete the argument one then needs to control the paracontrolled, and not only the Hölder norms in Theorem 4.3. This program appears feasible, but behind the technical scopes of this paper.

5.2. **KPZ driven by fractional noise.** We now consider fractional noise in time. Let $H \in (0,1)$. Formally the noise is given by $\eta(t,x) = \xi^H(t) \cdot V(x)$ for some $V \in C^\infty(\mathbb{T})$ and $\xi^H(t) = \partial_t \beta^H(t)$ for a fractional Brownian motion of parameter H. For convenience, we instead define the noise via its spectral covariance function, see [24, Section 3], namely ξ^H is a Gaussian process indexed by functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |\tau|^{1-2H} |\hat{f}(\tau)|^2 \,\mathrm{d}\tau$ (with $\hat{f}$ being the temporal Fourier transform) with covariance:

$$\mathbb{E}\Big[\xi^H(f)\xi^H(g)\Big] = c_H \int_{\mathbb{R}} |\tau|^{1-2H} \hat{f}(\tau)\overline{\hat{g}(\tau)} \,\mathrm{d}\tau, \qquad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}.$$

We can recover the fractional Brownian motion by:

$$\beta^H_t := \int_0^t \xi^H(\,\mathrm{d}s) = \xi^H(1_{[0,t]}) \quad \text{in} \quad L^2(\Omega_{\mathrm{kpz}}; \mathbb{R}). \tag{13}$$

Analogously to the previous result we can construct an ergodic IDS associated to ξ^H, as well as solve Equation (2) driven by such noise:

$$(\partial_t - \partial_x^2) u(t,x) = \xi^H(t) V(x) \cdot u(t,x), \qquad u(0,x) = u_0(x), \qquad (t,x) \in \mathbb{R}_\geqslant \times \mathbb{T}. \tag{14}$$

We consider mild solutions to this equation, namely u such that:

$$u(\omega,t,x) = \int_0^t P_{t-s}[u(\omega,s,\cdot)V(\cdot)](x) \xi^H(\omega, \,\mathrm{d}s),$$

where P_t is the heat semigroup: $P_t f(x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{T}^d} f(y) e^{-\frac{|x-y|^2}{4t}} \,\mathrm{d}y$.

Remark 5.5. *The motivation behind this example is to prove the concept that the presented techniques apply also in a non-Markovian setting. In particular, one can easily generalize this setting to a much wider class of fractional noises, possibly giving rise to a singular SPDE, as long as there exists a solution theory.*

We can now prove that Equation (14) falls in the framework of the theory in the previous sections.

Lemma 5.6. *Let $\Omega_{\mathrm{kpz}} = H^\alpha_a(\mathbb{R})$ for $\alpha < H-1, a > \frac{1}{2}$, $\mathcal{F} = \mathcal{B}(H^\alpha_a(\mathbb{R}))$ and let $\mathbb{P}$ be the law of the fractional noise ξ^H on Ω_{kpz}. The space $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P})$ is extended to an* ergodic *IDS via the integer translation group (in the sense of distributions) $\{\vartheta^z\}_{z\in\mathbb{Z}}$, which acts by:*

$$\vartheta^z \omega(t) = \omega(t+z), \qquad \forall \omega \in \Omega_{\mathrm{kpz}}, \ \ t \in \mathbb{R}.$$

Moreover, there exists a null-set N_0 such that for every $\omega \in N_0^c$ Equation (12) *admits a solution:*

$$\mathbb{R}_\geqslant \ni t \mapsto \psi_t(\omega) u_0, \qquad \forall\ u_0 \in C(\mathbb{T}).$$

There furthermore exists a continuous RDS φ such that:

$$\forall \omega \in \Omega_{\mathrm{kpz}}, \qquad \varphi_t(\omega) \in \mathcal{L}(C(\mathbb{T})) \qquad \textit{and} \qquad \mathbb{P}\Big(\varphi_t u_0 = \psi_t u_0, \ \ \forall t \geqslant 0, u_0 \in C(\mathbb{T})\Big) = 1.$$

Finally, φ satisfies Assumption 4.2 in $d = 1$ for any $\beta > 0$.

Proof. The fact that ξ^H takes values in Ω_{kpz} and that $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P}, \vartheta)$ generates an ergodic dynamical system follows from Corollary 5.9. Let us pass to the well-posedness of the system.

To solve the equation above, consider the field $X(\omega,t,x)$ solving

$$(\partial_t - \partial_x^2) X(\omega,t,x) = \xi^H(\omega,t) V(x), \qquad X(\omega,0,x) = 0, \qquad \forall (t,x) \in \mathbb{R}_\geqslant \times \mathbb{T}.$$

and let us prove that there exists a $\vartheta-$invariant null-set N_0 such that for all $\omega \in N_0^c$ the process $(t,x) \mapsto X(\omega,t,x)$ lies in $C_{\mathrm{loc}}([0,\infty); C^\gamma(\mathbb{T}))$ for any $\gamma > 0$. Indeed, by the construction in Equation (13) and the Kolmogorov continuity criterion we can find a $\vartheta-$invariant null-set N_0 such that for $\omega \in N_0^c$ there exists a process $t \mapsto \beta^H(\omega,t) \in C_{\mathrm{loc}}((-\infty,\infty);\mathbb{R})$ such that $\xi^H(\omega) = \partial_t \beta^H(\omega)$ in the sense of distributions. Then we write, by integration by parts:

$$X(t,x) = \int_0^t P_{t-s}V(x)\xi^H(\,\mathrm{d}s) = \int_0^t \beta^H(s)(P_{t-s}\partial_x^2 V)(x)\,\mathrm{d}s + V(x)\beta^H(t) - (P_tV)(x)\beta^H(0).$$

Hence the result concerning X is proved. Now, the solution u to Equation (14) can be written as $u = e^X w$, with w solving:

$$(\partial_t - \partial_x^2)w(\omega,t,x) = 2\partial_x X(\omega,t,x)\partial_x w(\omega,t,x) + (\partial_x X)^2(\omega,t,x)w(\omega,t,x), \qquad w(\omega,0,x) = u_0(x).$$

In view of the smoothness of $X(\omega)$ for $\omega \in N_0^c$ the above random PDE can be solved on N_0^c with classical results from PDE theory. Moreover the solution w lives in $C_{\mathrm{loc}}([0,\infty); C^\gamma(\mathbb{T}))$ for any $\gamma > 0$. Since the null-set N_0 is $\vartheta-$invariant and does not depend on the initial time $t = 0$ we can construct φ on Ω_{kpz} by setting it to zero on N_0.

The fact that φ satisfies Assumption 4.2 is simpler than and follows similarly to the proof of Corollary 5.2, so we refer to the latter. □

Corollary 5.7. *The results of Theorem 4.3 hold for φ defined as above. Moreover, for any $h_0^1, h_0^2 \in C(\mathbb{T})$ the constant $c(\omega,t,h_0^1,h_0^2)$ in the theorem can be chosen independent of time. Similarly, $h_\infty(\omega)$ is the unique solution to Equation* (14) *for times $t \in \mathbb{R}$, up to a constant (random) shift.*

Proof. By the previous lemma it is clear that we can apply Theorem 4.3. Let us discuss the constants $c(\omega,t,h_0^1,h_0^2)$. For the first point, it is sufficient to prove that there exists a constant $\bar{c}(\omega,h_0^1,h_0^2)$ such that for every $\omega \in \widetilde{\Omega}$ (for an invariant set $\widetilde{\Omega}$ of full $\mathbb{P}-$measure) and any $T > 0$:

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{t\in[n,n+T]} |c(\omega,t,h_0^1,h_0^2) - \bar{c}(\omega,h_0^1,h_0^2)| < 0.$$

Note that the theorem implies the convergence:

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{t\in[n,n+T]} \log \|\Pi_\times(h_1(\omega,t) - h_2(\omega,t))\|_\alpha < 0,$$

for any $\alpha > 0$, where $\Pi_\times$ is defined for $f \in C(\mathbb{T})$ as $\Pi_\times f = f - \int_{\mathbb{T}} f(x)\,\mathrm{d}x$, so that we can actually choose the constants to be:

$$c(\omega,t,h_0^1,h_0^2) = \int_{\mathbb{T}} h_1(\omega,t,x) - h_2(\omega,t,x)\,\mathrm{d}x.$$

Since h_i is a solution to the KPZ Equation one has:

$$\partial_t c(\omega,t,h_0^1,h_0^2) = \int_{\mathbb{T}} \partial_x(h_1 - h_2)\partial_x(h_1 + h_2)(\omega,t,x)\,\mathrm{d}x. \tag{15}$$

Now, in view of the result (1) of Theorem 4.3 we find that:

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{t\in[n,n+1]} \log |\partial_t c(\omega,t,h_0^1,h_0^2)| < 0.$$

In particular this implies that there exists a constant $\bar{c}(\omega,h_0^1,h_0^2) := \lim_{t\to\infty} c(\omega,t,h_0^1,h_0^2)$ and in addition

$$|\bar{c}(\omega,h_0^1,h_0^2) - c(\omega,t,h_0^1,h_0^2)| \leqslant \int_t^\infty |\partial_s c(\omega,s,h_0^1,h_0^2)|\,\mathrm{d}s \lesssim e^{-d(\omega)t},$$

for some $d(\omega) > 0$, which proves the required result.

We pass to the last statement. Let h_∞ be the solution to the KPZ equation for $t \in \mathbb{R}$ obtained from Theorem 4.3. Suppose $\overline{h}_\infty$ is a second solution. The uniform dependence on h_0^1 in Point (2) of Theorem 4.3 guarantees that there exists $c_\infty(\omega, t)$ such that $h_\infty(\omega, t, x) - \overline{h}_\infty(\omega, t, x) = c_\infty(\omega, t)$. Since $t \mapsto c_\infty(\omega, t)$ solves the same ODE as above with h_1, h_2 replaced by $h_\infty, \overline{h}_\infty$ we have that $c(\omega, t) \equiv c(\omega)$ is time-independent.

□

5.3. **Mixing of Gaussian fields.** Let us state a general criterion which ensures that a possibly infinite-dimensional Gaussian field is mixing (and hence ergodic). This is a simple generalization of a classical result for one-dimensional processes, cf. [9, Chapter 14]. We indicate with $\mathbf{B}^*$ the dual of a Banach space $\mathbf{B}$ and write $\langle \cdot, \cdot \rangle$ for the dual pairing.

Proposition 5.8. *Let* $\mathbf{B}$ *be a separable Banach space. Let* μ *be a Gaussian measure on* $(\mathbf{B}, \mathcal{B}(\mathbf{B}))$ *and* $\vartheta \colon \mathbb{N}_0 \times \mathbf{B} \to \mathbf{B}$ *a dynamical system which leaves* μ *invariant. Denote with* ξ *the canonical process on* $\mathbf{B}$ *under* μ*. The following condition*

$$\lim_{n \to \infty} \mathrm{Cov}(\langle \xi, \varphi \rangle, \langle \vartheta^n \xi, \varphi' \rangle) = 0, \qquad \forall \varphi, \varphi' \in \mathbf{B}^*$$

implies that the system is mixing, that is for all $A, B \in \mathcal{B}(\mathbf{B})$*:*

$$\lim_{n \to \infty} \mu(A \cap \vartheta^{-n} B) = \mu(A)\mu(B).$$

Proof. First, we reduce ourselves to the finite-dimensional case. Indeed, note that the sequence $(\xi, \vartheta^n \xi)$ is tight in $\mathbf{B} \times \mathbf{B}$, because ϑ leaves μ invariant. Furthermore, tightness implies that the sequence is flatly concentrated (cf. [10, Definition 2.1]), that is for every $\varepsilon > 0$ there exists a finite-dimensional linear space $S^\varepsilon \subseteq \mathbf{B} \times \mathbf{B}$ such that:

$$\mathbb{P}\big((\xi, \vartheta^n \xi) \in S^\varepsilon\big) \geqslant 1 - \varepsilon.$$

Hence, it is sufficient to check the mixing property for $A, B \in \mathcal{B}(S^\varepsilon)$.

This means that there exists an $n \in \mathbb{N}$ and $\varphi_i \in \mathbf{B}^*$ for $i = 1, \dots, n$ such that we have to check the mixing property for the vector:

$$((\langle \xi, \varphi_i \rangle)_{i=1,\dots,n}, \quad (\langle \vartheta^n \xi, \varphi_i \rangle)_{i=1,\dots,n}).$$

In this setting and in view of our assumptions the result follows from [12, Theorem 2.3].

□

Corollary 5.9. *Let* ξ *be space-time white noise on* $\mathbb{R} \times \mathbb{T}$*. Then* ξ *is supported in* $H^\alpha_a(\mathbb{R} \times \mathbb{T})$ *for any* $\alpha < -1$ *and* $a > 1/2$ *and its law* $\mathbb{P}$ *induces a mixing IDS* $(H^\alpha_a(\mathbb{R} \times \mathbb{T}), \mathcal{B}(H^\alpha_a), \mathbb{P}, \vartheta)$*, with*

$$\vartheta^z \xi(t, x) = \xi(t+z, x),$$

in the sense of distributions.

Similarly, let ξ^H *be fractional noise on* $\mathbb{R}$*. Then* ξ^H *is supported in* $H^\alpha_a(\mathbb{R})$ *for any* $\alpha < H-1$ *and* $a > 1/2$ *and its law* $\mathbb{P}^H$ *induces a mixing IDS* $(H^\alpha_a(\mathbb{T}), \mathcal{B}(H^\alpha_a), \mathbb{P}^H, \vartheta)$*, with*

$$\vartheta^z \xi^H(t) = \xi^H(t+z)$$

in the sense of distributions.

Proof. Let us start with space-time white noise. The regularity result is actually suboptimal and well understood, so we leave it as an exercise to the reader. We have to check the condition on the covariances. Here it is sufficient to observe that for $\varphi, \varphi' \in C^\infty_c(\mathbb{R} \times \mathbb{T})$: $\mathrm{Cov}(\langle \xi, \varphi \rangle, \langle \xi, \varphi' \rangle) = 0$ by independence in time and compact support, as soon as n is large enough. Let us pass to the fractional noise. For clarity we prove also the regularity result. Here one can estimate:

$$\begin{aligned}\mathbb{E}\Big[\|\Delta_j \xi^H(\cdot)/\langle\cdot\rangle^a\|_{L^2}^2\Big] &= \int_{\mathbb{R}} \frac{1}{(1+|t|)^{2a}} \mathbb{E}\big[|\Delta_j \xi^H(t)|^2\big]\,\mathrm{d}t \lesssim_a \sup_{t\in\mathbb{R}} \mathbb{E}\big[|\Delta_j \xi^H(t)|^2\big] \\ &= c_H \int_{\mathbb{R}} |\tau|^{1-2H} \varrho_j^2(\tau)\,\mathrm{d}\tau \lesssim 2^{j(2-2H)},\end{aligned}$$

where we used that $2a > 1$ and that for $j \geqslant 0$ $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ for a function ϱ with support in an annulus. As for the covariance condition, we have for $\varphi, \varphi' \in C_c^\infty(\mathbb{T})$:

$$\mathrm{Cov}(\langle\xi^H, \varphi\rangle, \langle\vartheta^n \xi^H, \varphi'\rangle) \simeq \int_{\mathbb{R}} |\tau|^{1-2H} e^{in\tau} \hat{\varphi}(\tau)\hat{\varphi}'(\tau)\,\mathrm{d}\tau = (\psi^H * \varphi * \varphi')(n) \to 0 \quad \text{for } n \to \infty.$$

with $\psi^H(t) = (\mathcal{F}^{-1}[|\cdot|^{1-2H}])(t)$. This concludes the proof.

□

Appendix A. Proof of Corollary 5.2

In the following we will prove that the required assumptions are satisfied for all $\omega \in N_0^c$ for a null-set N_0. We can make this set $\vartheta-$invariant by defining $N = \bigcup_{z\in\mathbb{Z}} \vartheta^z N_0$.

Property 1. We can formally define the kernel by $K(\omega, t, x, y) = \varphi_t(\omega)(\delta_y)(x)$, where δ_y indicates a Dirac δ centered at y. This can be given rigorous meaning to, for example in [15, Section 6], where the authors prove that there exists an $\varepsilon > 0$ such that for almost all ω, for any choice of $0 < S < T$ and all $t \in [S, T]$ the function φ_t can be extended to a map:

$$\begin{aligned}&\varphi_t(\omega)\colon B^\zeta_{p,\infty}(\mathbb{T}) \to C^\beta(\mathbb{T}) \quad \text{such that} \\ &\sup_{S\leqslant t\leqslant T} \|\varphi_t(\omega)u_0\|_{C^\beta(\mathbb{T})} \lesssim C(\omega, \beta, \zeta, p, S, T)\|u_0\|_{B^\zeta_{p,\infty}},\end{aligned} \tag{16}$$

for any $\beta < \frac{1}{2}, \zeta > -\varepsilon$ and any $p \geqslant 1$ (note that the target space of φ_t is at first just $B^\beta_{p,\infty}(\mathbb{T})$: by repeatedly applying this result together with Besov embeddings, since we look at $t > S$ and do not require uniform bounds in t near to zero the here claimed result follows). We can thus conclude the continuity of the kernel K if we can prove that $\{\delta_y\}_{y\in\mathbb{T}} \subseteq B^{-\gamma}_{1,\infty}$ for any $\gamma > 0$, together with the continuity: $\lim_{x\to y} \delta_x = \delta_y$ in $B^{-\gamma}_{1,\infty}(\mathbb{T})$. In particular, we shall prove the following:

$$\|\delta_x - \delta_y\|_{B^{-\gamma}_{1,\infty}(\mathbb{T})} \leqslant L|x-y|^\gamma, \qquad \forall x, y \in \mathbb{T},$$

for some $L > 0$. We divide the proof in two steps. Recall that by definition we have to bound $\sup_{j\geqslant -1} 2^{-\gamma j}\|\Delta_j(\delta_x - \delta_y)\|_{L^1}$. Hence we choose j_0 as the smallest integer such that $2^{-j_0} \leqslant |x-y|$. We first look at small scales $j \geqslant j_0$ and then at large scales $j < j_0$. For small scales, by the Poisson summation formula, since $\varrho_j(k) = \varrho_0(2^{-j}k)$, and by defining $K_j(x) = \mathcal{F}^{-1}_{\mathbb{R}}\varrho_j(x) = 2^j K(2^j x)$ for some $K \in \mathcal{S}(\mathbb{R})$ (the space of tempered distributions):

$$\begin{aligned}2^{-\gamma j}\big\|\Delta_j(\delta_x - \delta_y)\big\|_{L^1} &\leqslant |x-y|^\gamma \int_{\mathbb{R}} 2^j |K(2^j(z-x)) - K(2^j(z-y))|\,\mathrm{d}z \\ &\lesssim |x-y|^\gamma \int_{\mathbb{R}} 2^j |K(2^j z)|\,\mathrm{d}z \lesssim |x-y|^\gamma.\end{aligned}$$

While for large scales, since we have $|2^j(x-y)| \leqslant 1$, applying the Poisson summation formula, by the mean value theorem and since $K \in \mathcal{S}(\mathbb{R})$ (the Schwartz space of functions):

$$\begin{aligned}2^{-\gamma j}\big\|\Delta_j(\delta_x - \delta_y)\big\|_{L^1} &\leqslant 2^{-\gamma j} \int |K(z) - K(z + 2^j(x-y))|\,\mathrm{d}z \\ &\leqslant |x-y|^\gamma \int \max_{|\xi - z|\leqslant 1} \frac{|K(\xi) - K(z)|}{|\xi - z|^\alpha}\,\mathrm{d}z \lesssim |x-y|^\gamma.\end{aligned}$$

Hence the result follows.

Property 2. The upper bound $\delta(\omega, S, T)$ is a simple consequence of the continuity of the kernel K and Equation (16). The lower bound $\gamma(\omega, S, T)$ is instead a consequence of a strong maximum principle which is satisfied almost surely by the equation. Such property is the consequence of Müller's principle [22], see also [8, Theorem 5.1].

Property 3. This property is implied by Equation (16), observing that $C^\alpha(\mathbb{T}) = B^\alpha_{\infty,\infty}(\mathbb{T})$ for any $\alpha \in \mathbb{R}_\geqslant \setminus \mathbb{N}_0$.

Property 4. Let us start with the average bound for $C(\omega, \beta, S, T)$ (which without loss of generality we can consider equal to $C(\omega, \beta, -\frac{\varepsilon}{2}, \infty, S, T)$ in the notation of (16)). Note that such constant is derived from Schauder estimates by a Gronwall-type argument and is formally of the form

$$C(\omega, \beta, S, T) = A(\beta, S, T) e^{A(\beta,S,T)\|\xi(\omega)\|^q}$$

for some $q \geqslant 1$, some appropriate norm of ξ and a deterministic constant A. Since the equation is singular, the norm of ξ has to be replaced with the norm of an enhanced version of the noise, see [15, Definition 4.1]. We thus find (a rigorous proof is provided by [23, Theorem 5.5 and Section 5.2]) for some some $A(\beta, S, T), q > 0$:

$$\sup_{t \in [S,T]} \|\varphi_t(\omega) f\|_\beta \leqslant A(\beta, S, T) e^{A(\beta,S,T)\|\mathbb{Y}(\omega)\|^q_{\mathcal{Y}_{\mathrm{kpz}}}} \|f\|_\infty.$$

Then we have $\mathbb{E} \log \big(C(\beta, S, T) \big) \lesssim_{S,T} 1 + \mathbb{E}\big[\|\mathbb{Y}\|^q_{\mathcal{Y}_{\mathrm{kpz}}} \big] < \infty$. Indeed the different norms appearing in $\mathbb{Y}$ lie essentially in some Wiener-Itô chaos and thus have bounded polynomial moments of any order, cf. [15, Section 9].

We then pass to the second bound. Since by the triangle inequality the bound does not depend on the choice of f, set $f = 1$. It is thus enough to prove that:

$$\mathbb{E} \sup_{S \leqslant t \leqslant T} \| \log (\varphi_t(\omega) 1) \|_\infty < \infty.$$

Here the problem is the lower bound to the logarithm (an estimate for the upper bound follows as in the previous case). We already established that the norm above is almost surely finite via Property 1, where we used a strong maximum principle as in [22, 8], but these provide no quantitative lower bound for the solution, so we cannot use their results to derive the required conditions. Instead the result follows by [23, lemma 3.10]. Although there the equation is considered on the entire real line (and thus weights are present) the same arguments apply on the torus and it is possible to bound:

$$-\log (\varphi_t(\omega) 1) \leqslant Z(\omega, t) + \log (u(\omega, t)),$$

where $Z(\omega, t)$ is a process whose supremum norm can be bounded by a random variable which lives in some finite Wiener-Itô chaos (and thus has finite expectation) and u solves a linear equation similar to the stochastic heat equation. Thus the average bound follows as in the first part, rigorously by applying [23, Theorem 5.5] to u.

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