KILLED ROUGH SUPER-BROWNIAN MOTION

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Abstract. This note extends the results in [8] to construct the rough super-Brownian motion on finite volume with Dirichlet boundary conditions. The backbone of this study is the convergence of discrete approximations of the parabolic Anderson model (PAM) on a box.

1. Introduction

In [8] a superprocess on infinite volume is constructed (named rough super-Brownian motion, rSBM), as a scaling limit of a branching random walk in a static random environment (BRWRE). In the quoted work, the analysis of persistence of the superprocess relies on the existence of the same process on finite volume with Dirichlet boundary conditions, due to the spectral properties of the Anderson Hamiltonian. The construction of such process is the aim of the current work.

Such process is the scaling limit of the same branching particle system as in [8], where any particle is killed as soon as it leaves a box of size L. Morally, this scaling limit is simpler to treat than in the infinite volume case, since explosions are less likely to happen. Indeed the convergence of the empirical measure associated to the particle system is an application of the results in [8, Section 3].

On a more technical level, the construction in [8] relies on the tools of [7] for discrete approximations of the parabolic Anderson model (PAM) on infinite volume. In this work we extend the latter approach within the framework of [3] for paracontrolled analysis with Dirichlet boundary conditions, with the aim of proving the convergence of discrete approximations to PAM with Dirichlet boundary conditions. That is, we study the equation:

$$\begin{aligned} \partial_t w(t,x) &= \Delta w(t,x)+\xi(x)w(t,x)+f(t,x), && (t,x)\in(0,T]\times(0,L)^d, \\ w(0,x) &= w_0(x), \qquad w(t,x)=0, && (t,x)\in(0,T]\times\partial[0,L]^d, \end{aligned} \tag{1}$$

where ξ is space white noise. The details are explained in the next section.

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2. PAM with Dirichlet Boundary Conditions

Define $\mathbb{N}=\{1,2,\dots\},\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Fix $L\in\mathbb{N}$ and $N=2L$. Consider $n\in\mathbb{N}\cup\{\infty\}$ ($n=\infty$ refers to the continuous case, studied in [3]). Write $\mathbb{Z}^d_n$ for the lattice $\frac{1}{n}\mathbb{Z}^d$ (resp. $\mathbb{R}^d$ if $n=\infty$), Λ_n for the lattice $\frac{1}{n}(\mathbb{Z}^d\cap[0,Ln]^d)$ (resp. $[0,L]^d$), Θ_n for the lattice $\frac{1}{n}(\mathbb{Z}^d\cap[-\frac{Nn}{2},\frac{Nn}{2}]^d)/\!\sim$ with opposite boundaries identified (resp. $\mathbb{T}^d_N\colon=[-\frac{N}{2},\frac{N}{2}]^d/\!\sim$) and define the "dual lattice" $\Xi_n=\frac{1}{N}(\mathbb{Z}^d\cap[-\frac{Nn}{2},\frac{Nn}{2}]^d)/\!\sim$, (resp. $\frac{1}{N}\mathbb{Z}^d$) as well as $\Xi^+_n=\frac{1}{N}(\mathbb{Z}^d\cap[0,Ln]^d)$, (resp. $\frac{1}{N}\mathbb{N}^d_0$) and $\partial\Xi^+_n=\{k\in\Xi^+_n\ :\ k_i=0 \text{ for some } i\in\{1,\dots,d\}\}$. Write $A^n_{\mathfrak{d}}=\Xi^+_n\setminus\partial\Xi^+_n, A^n_{\mathfrak{n}}=\Xi^+_n$. Finally, for $p\geq1$ and any function $f\colon\Theta^n\to\mathbb{R}$, write $\|f\|_{L^p(\Theta^n)}=(n^{-d}\sum_{x\in\Theta^n}|f(x)|^p)^{\frac{1}{p}}$ (resp. the classical $L^p([-\frac{N}{2},\frac{N}{2}]^d)$ norm if $n=\infty$).

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2.1. **The Analytic Setting.** The idea of [3] in the case $n=\infty$ is to consider suitable even and odd extensions of functions on Λ_n to periodic functions on Θ_n, and then to work with the usual tools from periodic paracontrolled distributions on Θ_n. So for $u\colon \Lambda_n \to \mathbb{R}$ we define

$$\Pi_o u\colon \Theta_n \to \mathbb{R}, \quad \Pi_o u(\mathfrak{q}\circ x) = \prod \mathfrak{q}\cdot u(x), \qquad \Pi_e u\colon \Theta_n \to \mathbb{R}, \quad \Pi_e u(\mathfrak{q}\circ x) = u(x),$$

where $x \in \Lambda_n, \mathfrak{q} \in \{-1,1\}^d$ and we define the product $\mathfrak{q}\circ x = (\mathfrak{q}_i x_i)_{i=1,\dots,d}$ as well as $\prod \mathfrak{q} = \prod_{i=1}^d \mathfrak{q}_i$. We shall work with the discrete periodic Fourier transform, defined for $\varphi\colon \Theta_n \to \mathbb{R}$ by

$$\mathscr{F}_{\Theta_n}\varphi(k) = \frac{1}{n^d}\sum_{x\in\Theta_n}\varphi(x)e^{-2\pi\iota\langle x,k\rangle}, \quad k\in\Xi_n.$$

As in [3] we have a periodic, a Dirichlet and a Neumann basis, which we will denote with: $\{\mathfrak{e}_k\}_{k\in\Xi_n}, \{\mathfrak{d}_k\}_{k\in\Xi_n^+\setminus\partial\Xi_n^+}$ and , $\{\mathfrak{n}_k\}_{k\in\Xi_n^+}$ respectively. Here $\mathfrak{e}_k$ is the classical Fourier basis:

$$\mathfrak{e}_k(x) = \frac{e^{2\pi\iota\langle x,k\rangle}}{N^{\frac{d}{2}}}, \qquad \text{so that} \quad \mathscr{F}_{\Theta_n}\varphi(k) = N^{\frac{d}{2}}\langle\varphi,\mathfrak{e}_k\rangle, \quad k\in\Xi_n,$$

the Dirichlet and Neumann bases consists of sine and cosine functions respectively:

$$\mathfrak{d}_k(x) = \frac{1}{N^{\frac{d}{2}}}\prod_{i=1}^d 2\sin(2\pi k_i x_i),\ k\in A^n_{\mathfrak{d}} \quad \mathfrak{n}_k(x) = \frac{1}{N^{\frac{d}{2}}}\prod_{i=1}^d 2^{1-1_{\{k_i=0\}}/2}\cos(2\pi k_i x_i),\ k\in A^n_{\mathfrak{n}}.$$

To the previous explicit expressions we will prefer the following alternative characterization (with $\nu_k = 2^{-\#\{i: k_i=0\}/2}$):

$$\Pi_o\mathfrak{d}_k = \iota^d\sum_{\mathfrak{q}\in\{-1,1\}^d}\prod\mathfrak{q}\cdot\mathfrak{e}_{\mathfrak{q}\circ k}, \quad \forall k\in A^n_{\mathfrak{d}}, \qquad \Pi_e\mathfrak{n}_k = \nu_k\sum_{\mathfrak{q}\in\{-1,1\}^d}\mathfrak{e}_{\mathfrak{q}\circ k}, \quad \forall k\in A^n_{\mathfrak{n}}.$$

For $\mathfrak{l}\in\{\mathfrak{d},\mathfrak{n}\}$ and $n<\infty$ write $\mathscr{S}'_{\mathfrak{l}}(\Lambda_n) = \mathrm{span}\{\mathfrak{l}_k\}_{k\in A^n_{\mathfrak{l}}}$ for the space of discrete distributions. For $n=\infty$ we define distributions via formal Fourier series:

$$\mathscr{S}'_{\mathfrak{l}}([0,L]^d) = \Big\{\sum_{k\in A^\infty_{\mathfrak{l}}}\alpha_k\mathfrak{l}_k \ : \ |\alpha_k|\le C(1+|\kappa|^\gamma),\ \text{for some } C,\gamma\ge 0\Big\}.$$

Now let us introduce Littlewood-Paley theory on the lattice, in order to control products between distributions on Λ_n uniformly in n. Consider an *even* function $\sigma\colon\Xi_n\to\mathbb{R}$. Then for $\varphi\in\mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$ we define the *Fourier multiplier*:

$$\sigma(D)\varphi = \sum_{k\in A^n_{\mathfrak{l}}}\sigma(k)\langle\varphi,\mathfrak{l}_k\rangle\mathfrak{l}_k.$$

Upon extending φ in an even or odd fashion we recover the classical notion of Fourier multiplier (namely on a torus: $\sigma(D)\varphi = \mathscr{F}^{-1}_{\Theta_n}(\sigma\mathscr{F}_{\Theta_n}\varphi)$), since $\Pi_o\big(\sigma(D)\varphi\big) = \sigma(D)\Pi_o\varphi$ and verbatim for Π_e. Fix then a dyadic partition of the unity $\{\varrho_j\}_{j\ge -1}$ as in [7, Definition 2.4] and let $j_n = \min\{j\ge -1\colon \mathrm{supp}(\varrho_j)\not\subseteq(-\frac{Nn}{2},\frac{Nn}{2})^d\}$ ($j_n=\infty$ if $n=\infty$), so as to define for $\varphi\in\mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$:

$$\Delta^n_j\varphi = \varrho_j(D)\varphi \text{ for } j<j_n, \qquad \Delta^n_{j_n}\varphi = \Big(1-\sum_{-1\le j<j_n}\varrho_j(D)\Big)\varphi.$$

This allows us to define the *paraproduct* and the *resonant product* of two distributions respectively (for $n=\infty$ the latter is a-piori ill-posed):

$$\varphi\varolessthan\psi = \sum_{-1\le j\le j_n}\sum_{-1\le i\le j-1}\Delta^n_i\varphi\Delta^n_j\psi, \qquad \varphi\odot\psi = \sum_{|i-j|\le 1}\Delta^n_i\varphi\Delta^n_j\psi.$$

In view of the previous calculations this is coherent with the definition on the lattice in [7], in the sense that:

$$\Pi_o\big(\Delta^n_j\varphi\big) = \Delta^n_j\Pi_o\varphi, \qquad \Pi_e\big(\Delta^n_j\varphi\big) = \Delta^n_j\Pi_e\varphi, \quad -1\le j\le j_n.$$

We then define Dirichlet and Neumann Besov spaces via the following norms:

$$\|u\|_{B^{\mathfrak{d};\alpha}_{p,q}(\Lambda_n)} = \|\Pi_o u\|_{B^{\alpha}_{p,q}(\Theta_n)} = \|(2^{\alpha j}\|\Delta_j \Pi_o u\|_{L^p(\Theta_n)})_j\|_{\ell^q(\leq j_n)} \qquad u \in \mathscr{S}'_{\mathfrak{d}}(\Lambda_n)$$

and similarly for $\mathfrak{n}$ upon replacing Π_o with Π_e. For brevity we write $\mathscr{C}^{\alpha}_{\mathfrak{l},p}(\Lambda_n) = B^{\mathfrak{l},\alpha}_{p,\infty}(\Lambda_n)$ and $\mathscr{C}^{\alpha}_{\mathfrak{l}}(\Lambda_n) = B^{\mathfrak{l},\alpha}_{\infty,\infty}(\Lambda_n)$ for $\mathfrak{l} \in \{\mathfrak{n}, \mathfrak{d}\}$. We also write $\|u\|_{L^p_{\mathfrak{d}}(\Lambda_n)} = \|\Pi_o u\|_{L^p(\Theta_n)}$ and $\|u\|_{L^p_{\mathfrak{n}}(\Lambda_n)} = \|\Pi_e u\|_{L^p(\Theta_n)}$. Having introduced Besov spaces we can define the spaces of time-dependent functions $\mathscr{M}^{\gamma}\mathscr{C}^{\alpha}_{\mathfrak{l},p}$ and $\mathscr{L}^{\gamma,\alpha}_{\mathfrak{l},\alpha}$ for $\mathfrak{l} \in \{\mathfrak{d}, \mathfrak{n}\}$ as in [7, Definition 3.8] without the necessity of taking into account weights. The above spaces allow for a detailed analysis of products of distributions. The last ingredient in this sense are the following identities:

$$\Pi_e(\varphi\psi) = \Pi_e\varphi\Pi_e\psi, \qquad \Pi_o(\varphi\psi) = \Pi_o\varphi\Pi_e\psi. \tag{2}$$

To solve equations with Dirichlet boundary conditions, introduce the following Laplace operators for $n < \infty$ (let $\varphi \colon \Lambda_n \to \mathbb{R}$, $\psi \colon \Theta_n \to \mathbb{R}$):

$$\Delta^n \psi(x) = n^2 \sum_{|x-y|=n^{-1}} \psi(y) - \psi(x), \qquad \Delta^n_{\mathfrak{d}}\varphi = (\Delta^n \Pi_o \varphi)|_{\Lambda_n}, \qquad \Delta^n_{\mathfrak{n}}\varphi = (\Delta^n \Pi_e \varphi)|_{\Lambda_n}.$$

The latter two operators are defined only on the domain $\mathrm{Dom}(\Delta^n_{\mathfrak{l}}) = \mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$. A direct computation (cf. [7, Section 3]) then shows that we can represent both Laplacians as Fourier multipliers:

$$\Delta^n_{\mathfrak{l}}\mathfrak{l}_k = l^n(k)\mathfrak{l}_k, \qquad l^n(k) = \sum_{j=1}^{d} 2n^2\big(\cos(2\pi k_j/n) - 1\big), \text{ for } \mathfrak{l} \in \{\mathfrak{d}, \mathfrak{n}\}.$$

Note that l^n is an even function in k, so all the remarks from the previous discussion apply. For $n = \infty$ we use the classical Laplacian: the boundary condition is encoded in the domain. We write $\Delta_{\mathfrak{l}}$ for the Laplacian on $\mathscr{S}'_{\mathfrak{l}}([0, L]^d)$. We introduce Dirichlet and Neumann extension operators as follows:

$$\mathscr{E}^n_{\mathfrak{d}} u = \mathscr{E}^n(\Pi_o u)\big|_{[0,L]^d}, \qquad \mathscr{E}^n_{\mathfrak{n}} u = \mathscr{E}^n(\Pi_e u)\big|_{[0,L]^d}, \qquad \text{for } n < \infty,$$

where the periodic extension operator $\mathscr{E}^n$ is defined as in [7, Lemma 2.24]. These functions are well-defined since for fixed n the extension $\mathscr{E}_n(\cdot)$ is a smooth function. Moreover a simple calculation shows that

$$\Pi_o(\mathscr{E}^n_{\mathfrak{d}} u) = \mathscr{E}^n(\Pi_o u), \qquad \Pi_e(\mathscr{E}^n_{\mathfrak{n}} u) = \mathscr{E}^n(\Pi_e u). \tag{3}$$

2.2. **Solving the Equation.** We now study Equation (1) on a box. We start with the crucial probabilistic assumptions on the noise (cf. [8, Asumption 2.1]).

Assumption 2.1. *We assume that for every $n \in \mathbb{N}$, $\{\xi^n(x)\}_{x\in\mathbb{Z}^d_n}$ is a set of i.i.d random variables which satisfy:*

$$n^{-d/2}\xi^n(x) \sim \Phi, \tag{4}$$

for a probability distribution Φ on $\mathbb{R}$ with finite moments of every order and which satisfies

$$\mathbb{E}[\Phi] = 0, \quad \mathbb{E}[\Phi^2] = 1.$$

These probabilistic assumptions guarantee certain analytical properties which we highlight in the next lemma. In the remainder of this work we shift Λ_n to be centered around the origin and identify it with a subset of $[-L/2, L/2]^d$. This is convenient because later we want to interpret processes on Λ_n as "restrictions" of a processes on $\mathbb{Z}^d_n$ to (large) boxes centered around the origin. By this we mean that for $L \in 2\mathbb{N}$ we define $\Lambda_n = \{x \in \mathbb{Z}^d_n \ : \ x \in [-L/2, L/2]^d\}$ In the following let χ be the same cut-off function as in [7, Section 5.1] and in dimension $d = 2$ define the renormalization constant (note that this constant does not depend on L):

$$\kappa_n = \int_{\mathbb{T}^2_n} \mathrm{d}k \ \frac{\chi(k)}{l^n(k)} \sim log(n). \tag{5}$$

Lemma 2.2. *Let $\overline{\xi}^n$ be a sequence of random variable satisfying Assumption 2.1. There exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ supporting random variables ξ^n, ξ such that ξ is space white noise on $\mathbb{R}^d$ and $\xi^n = \overline{\xi}^n$ in distribution for every $n \in \mathbb{N}$.*

Such random variables satisfy the following requirements. Let $X^n_{\mathfrak{n}}$ be the (random) solution to the equation $-\Delta^n_{\mathfrak{n}} X^n_{\mathfrak{n}} = \chi(D)\xi^n$. For every $\omega \in \Omega$ and α satisfying

$$\alpha \in (1, \tfrac{3}{2}) \text{ in } d = 1, \qquad \alpha \in (\tfrac{2}{3}, 1) \text{ in } d = 2, \tag{6}$$

the following holds for all $L \in 2\mathbb{N}$:

(i) $\xi(\omega) \in \mathscr{C}^{\alpha-2}_{\mathfrak{n}}([-L/2, L/2]^d)$ as well as $\sup_n \|\xi^n(\omega)\|_{\mathscr{C}^{\alpha-2}_{\mathfrak{n}}(\Lambda_n)} < +\infty$ and $\mathscr{E}^n_{\mathfrak{n}}\xi^n(\omega) \to \xi(\omega)$ in $\mathscr{C}^{\alpha-2}_{\mathfrak{n}}([-L/2, L/2]^d)$.

(ii) For any $\varepsilon > 0$ (with $(\cdot)_+ = \max\{0, \cdot\}$):

$$\sup_n \|n^{-d/2}\xi^n_+(\omega)\|_{\mathscr{C}^{-\varepsilon}_{\mathfrak{n}}(\Lambda_n)} + \sup_n \|n^{-d/2}|\xi^n(\omega)|\|_{\mathscr{C}^{-\varepsilon}_{\mathfrak{n}}(\Lambda_n)} + \sup_n \|n^{-d/2}\xi^n_+(\omega)\|_{L^2_{\mathfrak{n}}(\Lambda_n)} < +\infty.$$

Moreover, there exists a $\nu \geq 0$ such that $\mathscr{E}^n_{\mathfrak{n}} n^{-d/2}\xi^n_+(\omega) \to \nu$, $\mathscr{E}^n_{\mathfrak{n}} n^{-d/2}|\xi^n(\omega)| \to 2\nu$ in $\mathscr{C}^{-\varepsilon}_{\mathfrak{n}}(\Lambda_n)$.

(iii) If $d = 2$ there exists a sequence $c_n(\omega) \in \mathbb{R}$ such that $n^{-d/2}c_n \to 0$ and distributions $X_{\mathfrak{n}}(\omega), X_{\mathfrak{n}} \diamond \xi(\omega)$ in $\mathscr{C}^{\alpha}_{\mathfrak{n}}([-L/2, L/2]^d)$ and $\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}([-L/2, L/2]^d)$ respectively, such that:

$$\sup_n \|X^n_{\mathfrak{n}}(\omega)\|_{\mathscr{C}^{\alpha}_{\mathfrak{n}}(\Lambda_n)} + \sup_n \|(X^n_{\mathfrak{n}} \odot \xi^n)(\omega) - c_n(\omega)\|_{\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}(\Lambda_n)} < +\infty$$

and $\mathscr{E}^n_{\mathfrak{n}} X^n_{\mathfrak{n}}(\omega) \to X_{\mathfrak{n}}(\omega)$ in $\mathscr{C}^{\alpha}_{\mathfrak{n}}([-L/2, L/2]^d)$, $\mathscr{E}^n_{\mathfrak{n}}\big((X^n_{\mathfrak{n}} \odot \xi^n)(\omega) - c_n(\omega)\big) \to X_{\mathfrak{n}} \diamond \xi(\omega)$ in $\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}([-L/2, L/2]^d)$.

Finally, $\mathbb{P}(c_n(\omega) = \kappa_n, \forall n \in \mathbb{N}$ and $\nu = \mathbb{E}\Phi_+) = 1$ and for all $\omega \in \Omega$, $\xi^n(\omega)$ is a deterministic environment satisfying [8, Assumption 2.3], *with the same renormalization constant $c_n(\omega)$ as above if $d = 2$.*

The proof of this lemma is postponed to the next subsection. We pass to the main analytic statement of this work.

Theorem 2.3. *Consider ξ^n as in Lemma 2.2 and α as in* (6)*, any $T > 0$, $p \in [1, +\infty], \gamma_0 \in [0, 1)$ and $\vartheta, \zeta, \alpha_0$ satisfying:*

$$\vartheta \in \begin{cases} (2-\alpha, \alpha), & d = 1, \\ (2-2\alpha, \alpha), & d = 2, \end{cases} \qquad \zeta > (\vartheta-2) \vee (-\alpha), \qquad \alpha_0 > (\vartheta-2) \vee (-\alpha), \tag{7}$$

and let $w^n_0 \in \mathscr{C}^{\zeta}_{\mathfrak{d},p}(\Lambda_n)$ and $f^n \in \mathscr{M}^{\gamma_0}\mathscr{C}^{\alpha_0}_{\mathfrak{d},p}(\Lambda_n)$ be such that

$$\mathscr{E}^n w^n_0 \to w_0 \text{ in } \mathscr{C}^{\zeta}_{\mathfrak{d},p}([-L/2, L/2]^d), \qquad \mathscr{E}^n f^n \to f \text{ in } \mathscr{M}^{\gamma_0}\mathscr{C}^{\alpha_0}_{\mathfrak{d},p}([-L/2, L/2]^d).$$

Let $w^n\colon [0, T] \times \Lambda_n \to \mathbb{R}$ be the unique solution to the finite-dimensional linear ODE:

$$\partial_t w^n = (\Delta^n_{\mathfrak{d}} + \xi^n - c_n 1_{\{d=2\}})w^n + f^n, \quad w^n(0) = w^n_0, \quad w(t, x) = 0 \ \ \forall (t, x) \in (0, T] \times \partial\Lambda_n. \tag{8}$$

There exist a unique (paracontrolled in the sense of [3] *or* [7] *in $d = 2$) solution w to the equation*

$$\partial_t w = (\Delta_{\mathfrak{d}} + \xi)w + f, \quad w(0) = w_0, \quad w(t, x) = 0 \ \ \forall (t, x) \in (0, T] \times \partial[-L/2, L/2]^d, \tag{9}$$

and for all $\gamma > (\vartheta-\zeta)_+/2 \vee \gamma_0$ the sequence w^n is uniformly bounded in $\mathscr{L}^{\gamma,\vartheta}_{\mathfrak{d},p}(\Lambda_n)$:

$$\sup_n \|w^n\|_{\mathscr{L}^{\gamma,\vartheta}_{\mathfrak{d},p}(\Lambda_n)} \lesssim \sup_n \|w^n_0\|_{\mathscr{C}^{\zeta}_{\mathfrak{d},p}(\Lambda_n)} + \sup_n \|f^n\|_{\mathscr{M}^{\gamma_0}\mathscr{C}^{\alpha_0}_{\mathfrak{d},p}(\Lambda_n)},$$

where the proportionality constant depends on the time horizon T and the magnitude of the norms in Lemma 2.2. Moreover,

$$\mathscr{E}^n w^n \to w \text{ in } \mathscr{L}^{\gamma,\vartheta}_{\mathfrak{d},p}([-L/2, L/2]^d).$$

Proof. Note that in view of (2) solving Equation (8) (resp. (9)) is equivalent to solving on the discrete (resp. continuous) torus Θ_n the equation:

$$\partial_t \tilde{w}^n = \Delta^n \tilde{w}^n + \Pi_e(\xi^n_e)\tilde{w}^n + \Pi_o f, \qquad \tilde{w}^n(0) = \Pi_o w_0,$$

and then restricting the solution to the cube Λ_n, i.e. $w^n = \tilde{w}^n|_{\Lambda_n}$, and $\tilde{w}^n = \Pi_o w^n$. Via the bounds in Lemma 2.2 this equation can be solved for all $\omega \in \Omega^p$ via Schauder estimates and (in dimension $d = 2$) paracontrolled theory following the arguments of [7] (without considering weights). From the arguments of the same article and Equation (3) we can also deduce the convergence of the extensions. Note that the solution theories in [3] and [7] coincide, although the latter concentrates on the construction of the Hamiltonian rather than the solutions to the parabolic equation (cf. [8, Proposition 3.1]). $\square$

For every $\omega \in \Omega$ it is also possible to define the Anderson Hamiltonian $\mathscr{H}^\omega_{\mathfrak{d},L}$ with Dirichlet boundary conditions. The domain and spectral decomposition for this operator are rigorously constructed in [3] with the help of the resolvent equation for $d = 2$ and [6] via Dirichlet forms in $d = 1$. We write $\mathscr{H}^{n,\omega}_{\mathfrak{d},L}, \mathscr{H}^\omega_{\mathfrak{d},L}$ for the operators $\Delta^n_{\mathfrak{d}} + \xi^n(\omega) - c_n(\omega)1_{\{d=2\}}$ and (formally) $\Delta_{\mathfrak{d}} + \xi(\omega) - \infty 1_{\{d=2\}}$ respectively. These operators generate semigroups $T^{n,\mathfrak{d},L,\omega}_t = e^{t\mathscr{H}^{n,\omega}_{\mathfrak{d},L}}$ and $T^{\mathfrak{d},L,\omega}_t = e^{t\mathscr{H}^\omega_{\mathfrak{d},L}}$. In particular, the following result is a simple consequence of the just quoted works.

Lemma 2.4. *For a given null-set $N_0 \subseteq \Omega$ and all $\omega \in N^c_0$, for all $L \in \mathbb{N}$ the operator $\mathscr{H}^\omega_{\mathfrak{d},L}$ has a discrete, bounded from above, spectrum and admits an eigenfunction $e_{\lambda(\omega,L)}$ associated to the largest eigenvalue $\lambda(\omega, L)$, such that $e_{\lambda(\omega,L)}(x) > 0$ for all $x \in (-\frac{L}{2}, \frac{L}{2})^d$.*

Proof. That the spectrum is discrete and bounded from above can be found in the works quoted above. For $\varphi, \psi \in L^2((-\frac{L}{2}, \frac{L}{2})^d)$ we write $\psi \geq \varphi$ if $\psi(x) - \varphi(x) \geq 0$ for Lebesgue-almost all x and we write $\psi \gg \varphi$ if $\psi(x) - \varphi(x) > 0$ for Lebesgue-almost all x. By the strong maximum principle of [1, Theorem 5.1] (which easily extends to our setting, see Remark 5.2 of the same paper) we know that for the semigroup $T^{\mathfrak{d},L,\omega}_t = e^{t\mathscr{H}^\omega_{\mathfrak{d},L}}$ of the PAM we have $T^{\mathfrak{d},L,\omega}_t \varphi \gg 0$ whenever $\varphi \geq 0$ and $\varphi \neq 0$; we even get $T^{\mathfrak{d},L,\omega}_t \varphi(x) > 0$ for all x in the interior $(-\frac{L}{2}, \frac{L}{2})^d$. So by a consequence of the Krein-Rutman theorem, see [5, Theorem 19.3], there exists an eigenfunction $e_{\lambda(\omega,L)} \gg 0$. And since $e_{\lambda(\omega,L)} = e^{-t\lambda(\omega,L)} T^{\mathfrak{d},L,\omega}_t e_{\lambda(\omega,L)}$, we have $e_{\lambda(\omega,L)}(x) > 0$ for all $x \in (-\frac{L}{2}, \frac{L}{2})^d$. $\square$

2.3. **Stochastic Estimates.** Here we prove Lemma 2.2. The following bounds are essentially an adaptation of [2, Section 4.2] to the Dirichlet boundary condition setting (see also [3] for the spatially continuous setting).

Proof of Lemma 2.2. Step 0. Let us write ξ^n instead of $\overline{\xi}^n$. Fix $L \in \mathbb{N}$ and take α, ε as in the statement of the lemma. Instead of proving the path-wise bounds and convergences of the lemma, it is sufficient to prove the bounds on average and the convergences in distribution. By this we mean that there exists space white noise ξ on $\mathbb{R}^d$ and (if $d = 2$) a random distribution $X_{\mathfrak{n}} \diamond \xi$ such that (all convergences being in distribution):

$$\sup_n \mathbb{E}[\|\xi^n\|^q_{\mathscr{C}^{\alpha-2}_{\mathfrak{n}}(\Lambda_n)}] < +\infty, \qquad \mathscr{E}^n_{\mathfrak{n}} \xi^n \to \xi \text{ in } \mathscr{C}^{\alpha-2}_{\mathfrak{n}}([0, L]^d), \tag{10}$$

as well as:

$$\sup_n \mathbb{E}[\|n^{-d/2}(\xi^n)_+\|_{\mathscr{C}^{-\varepsilon}_{\mathfrak{n}}(\Lambda_n)} + \|n^{-d/2}(\xi^n)_+\|_{L^2(\Lambda_n)}] < +\infty, \tag{11}$$

with $\mathscr{E}^n_{\mathfrak{n}} n^{-d/2}(\xi^n)_+ \to \nu$ in $\mathscr{C}^{-\varepsilon}_{\mathfrak{n}}([0, L]^d)$. Moreover, in dimension $d = 2$, we have (recall κ_n from (5)):

$$\sup_n \mathbb{E}[\|X^n_{\mathfrak{n}}\|_{\mathscr{C}^\alpha_{\mathfrak{n}}(\Lambda_n)} + \|(X^n_{\mathfrak{n}} \odot \xi^n) - \kappa_n\|_{\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}(\Lambda_n)}] < +\infty \tag{12}$$

as well as $\mathscr{E}^n_{\mathfrak{n}} X^n_{\mathfrak{n}} \to X_{\mathfrak{n}}$ in $\mathscr{C}^\alpha_{\mathfrak{n}}([0, L]^d)$, and $\mathscr{E}^n_{\mathfrak{n}}(X^n_{\mathfrak{n}} \odot \xi^n - \kappa_n) \to X_{\mathfrak{n}} \diamond \xi$ in $\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}([0, L]^d)$. Once these bounds and convergences are established, and in view of [8, Lemma 2.4], the Lemma follows from Skorohod's representation theorem. So far we have proven convergence a.s. for fixed L, α, ε. The extension to all L, α, ε follows as in Corollary 3.9. To find convergence for all ω we set all functions to zero on a null-set.

Step 1. We now observe that the bound and convergence from (10) as well as the bound and convergence for $X^n_{\mathfrak{n}}$ from (12) are similar to and simpler than the bound for $X^n_{\mathfrak{n}} \odot \xi^n$. Also, Equation (11) and the following convergences are analogous to [8, Appendix B]. We are left with proving the bound and convergence of $X^n_{\mathfrak{n}} \odot \xi^n$ from (12).

Step 2. First, we establish the uniform bounds. We will derive only bounds in spaces of the kind $B^{\mathfrak{n},\beta}_{p,p}(\Lambda_n)$ for appropriate β and any p sufficiently large. The results on the Hölder scale then follow by Besov embedding. In order to avoid confusion, we will omit the subindex $\mathfrak{n}$ in the noise terms. We write sums as discrete integrals against scaled measures with the following definitions:

$$\int_{\Theta_n} \mathrm{d}x \ f(x) = \sum_{x \in \Theta_n} \frac{f(x)}{n^d}, \quad \int_{\Xi_n} \mathrm{d}k \ f(k) = \sum_{k \in \Xi_n} \frac{f(k)}{N^d}, \quad \int_{\{-1,1\}^d} \mathrm{d}\mathfrak{q} \ f(\mathfrak{q}) = \sum_{\mathfrak{q} \in \{-1,1\}^d} f(\mathfrak{q}).$$

For $k_1, k_2 \in \Xi_n$ and $\mathfrak{q}_1, \mathfrak{q}_2 \in \{-1,1\}^d$ we moreover adopt the notation: $k_{[12]} = k_1 + k_2, \mathfrak{q}_{[12]} = \mathfrak{q}_1 + \mathfrak{q}_2$ and $(\mathfrak{q} \circ k)_{[12]} = \mathfrak{q}_1 \circ k_1 + \mathfrak{q}_2 \circ k_2$. We first compute:

$$\begin{aligned}
\Delta_j \Pi_e(\xi^n \odot X^n)(x) = & \int_{(\{-1,1\}^d \times \Xi_n^+)^2} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \ N^d \nu_{k_1} \nu_{k_2} e^{2\pi\iota \langle x, (\mathfrak{q} \circ k)_{[12]} \rangle} \cdot \\
& \cdot \varrho_j((\mathfrak{q} \circ k)_{[12]}) \psi_0(k_1, k_2) \frac{\chi(k_2)}{l^n(k_2)} \langle \xi^n, \mathfrak{n}_{k_1} \rangle \langle \xi^n, \mathfrak{n}_{k_2} \rangle \\
= & \int_{(\{-1,1\}^d \times \Xi_n^+)^2} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \ 1_{\{k_1 \neq k_2\}} N^d \nu_{k_1} \nu_{k_2} e^{2\pi\iota \langle x, (\mathfrak{q} \circ k)_{[12]} \rangle} \cdot \\
& \cdot \varrho_j((\mathfrak{q} \circ k)_{[12]}) \psi_0(k_1, k_2) \frac{\chi(k_2)}{l^n(k_2)} \langle \xi^n, \mathfrak{n}_{k_1} \rangle \langle \xi^n, \mathfrak{n}_{k_2} \rangle + \mathrm{Diag}
\end{aligned}$$

where Diag indicates the integral over the set $\{k_1 = k_2\}$. The first term can be bounded by a generalized discrete BDG inequality for multiple discrete stochastic integrals, see [2, Proposition 4.3]. We can thus bound for arbitrary $\ell \in \mathbb{N}$:

$$\begin{aligned}
& \mathbb{E}[|\Delta_j(\Pi_e(\xi^n \odot X^n)(x) - \kappa_n)|^p] \\
& \lesssim \left[\int \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \left| \varrho_j((\mathfrak{q} \circ k)_{[12]}) \psi_0(k_1, k_2) \frac{\chi(k_2)}{l^n(k_2)} \right|^2 \right]^{\frac{p}{2}} \mathbb{E}[\langle \xi^n, \mathfrak{n}_\ell \rangle^p]^2 + \mathbb{E}[|\mathrm{Diag} - 1_{\{j=-1\}} \kappa_n|^p].
\end{aligned}$$

For the first term on the right hand side we have:

$$\begin{aligned}
& \int_{(\{-1,1\}^d \times \Xi_n^+)^2} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \left| \varrho_j((\mathfrak{q} \circ k)_{[12]}) \psi_0(k_1, k_2) \frac{\chi(k_2)}{l^n(k_2)} \right|^2 = \int_{\Xi_n^2} \mathrm{d}k_{12} \left| \varrho_j(k_{[12]}) \psi_0(k_1, k_2) \frac{\chi(k_2)}{l^n(k_2)} \right|^2 \\
& \lesssim \sum_{i \geq j - \ell} \int_{\Xi_n^2} \mathrm{d}k_{12} \ 1_{\{|k_1 + k_2| \sim 2^j\}} 1_{\{|k_2| \sim 2^i\}} 2^{-4i} \lesssim \sum_{i \geq j - \ell} 2^{jd} 2^{i(d-4)} \lesssim 2^{2j(d-2)},
\end{aligned}$$

which is of the required order (and we used that $d < 4$). Let us pass to the diagonal term. We first smuggle in the expectation of Diag:

$$\mathbb{E}[|\mathrm{Diag} - \mathbb{E}[\mathrm{Diag}]|^p] = \mathbb{E}\left[\left| \int_{\Xi_n^+ \times (\{-1,1\}^d)^2} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k \ \nu_k^2 e^{2\pi\iota \langle x, \mathfrak{q}_{[12]} \circ k \rangle} \varrho_j(\mathfrak{q}_{[12]} \circ k) \frac{\chi(k)}{l^n(k)} \eta(k) \right|^p \right],$$

where we have lost the factor N^d due to the normalization of the integral in k and $\eta(k) = \langle \xi^n, \mathfrak{n}_k \rangle^2 - \mathbb{E}[\langle \xi^n, \mathfrak{n}_k \rangle^2] = \langle \xi^n, \mathfrak{n}_k \rangle^2 - 1$ is sequence of centered i.i.d random variables. Therefore, we can use the same martingale argument as above to bound the integral by:

$$\mathbb{E}[|\operatorname{Diag}-\mathbb{E}[\operatorname{Diag}]|^p] \lesssim \bigg(\int_{\Xi_n^+}\mathrm{d}k\bigg|\int_{(\{-1,1\}^d)^2}\mathrm{d}\mathfrak{q}_{12}\ \varrho_j(\mathfrak{q}_{[12]}\circ k)\bigg|^2\bigg|\frac{\chi(k)}{l^n(k)}\bigg|^2\mathbb{E}[|\eta(k)|^p]^{\frac{2}{p}}\bigg)^{\frac{p}{2}}$$
$$\lesssim \bigg(\int_{x\in\mathbb{R}^d:|x|\gtrsim 2^j}\frac{1}{|x|^4}\,\mathrm{d}x\bigg)^{p/2} \lesssim 2^{j(d-4)} = 2^{j(\frac{d}{2}-2)}$$

whenever $d<4$, which is even better than the bound for the off-diagonal terms. We are hence left with a last, deterministic term:

$$\int_{\Xi_n^+\times(\{-1,1\}^d)^2}\mathrm{d}\mathfrak{q}_{12}\,\mathrm{d}k\ \nu_k^2 e^{2\pi\iota\langle x,\mathfrak{q}_{[12]}\circ k\rangle}\varrho_j(\mathfrak{q}_{[12]}\circ k)\frac{\chi(k)}{l^n(k)} \quad - \quad 1_{\{j=-1\}}\kappa_n.$$

We split up this sum in different terms according to the relative value of $\mathfrak{q}_1,\mathfrak{q}_2$. If $\mathfrak{q}_1=-\mathfrak{q}_2$ (there are 2^d such terms) the sum does not depend on x and it disappears for $j\geq 0$. Let us assume $j=-1$. We are then left with the constant:

$$2^d\int_{\Xi_n^+}\mathrm{d}k\ \nu_k^2\frac{\chi(k)}{l^n(k)}-\kappa_n=\int_{\Xi_n}\mathrm{d}k\ \frac{\chi(k)}{l^n(k)}-\kappa_n.$$

Note that the sum on the left-hand side diverges logarithmically in n and we now show how to renormalize with κ_n. To clarify our computation let us also introduce an auxiliary constant $\bar{\kappa}_n=\int_{\Xi_n}\mathrm{d}k\ \bar{\nu}_k^2\frac{\chi(k)}{l^n(k)}$, where $\bar{\nu}_k=2^{-\#\{i\colon k_i=\pm n\}/2}$. For $x\in\mathbb{R}^d, r\geq 0$, let us indicate with $Q_r^n(x)\subseteq\mathbb{T}_n^d$ the box $Q_r^n(x)=\{y\in\mathbb{T}_n^d\colon |y-x|_\infty\leq r/2\}$ ($|\cdot|_\infty$ being the maximum of the component-wise distances in $\mathbb{T}_n^d$). Then note that we can bound uniformly over n and N:

$$|\kappa_n-\bar{\kappa}_n| = \bigg|\int_{\mathbb{T}_n^d}\mathrm{d}k\ \frac{\chi(k)}{l^n(k)}-\int_{\Xi_n}\mathrm{d}k\ \bar{\nu}_k^2\frac{\chi(k)}{l^n(k)}\bigg| = \bigg|\sum_{k\in\Xi_n}\int_{Q^n_{\frac{1}{N}}(k)}\mathrm{d}k'\ \frac{\chi(k+k')}{l^n(k+k')}-\frac{\chi(k)}{l^n(k)}\bigg|$$
$$\lesssim\frac{1}{N}\bigg(1+\frac{1}{N^d}\sum_{k\in\Xi_n}\sup_{\vartheta\in Q_{\frac{1}{N}}(k)}\frac{\chi(k)}{(l^n(\vartheta))^2}|\nabla l^n(\vartheta)|\bigg)\lesssim\frac{1}{N}\bigg(1+\frac{1}{N^d}\sum_{k\in\frac{1}{N}\mathbb{Z}^d}\frac{\chi(k)}{|k|^3}\bigg)\lesssim\frac{1}{N},$$

where we have used that $d=2$, $|l^n(\vartheta)|\gtrsim|\vartheta|^2$ on $[-n/2,n/2]^d$ as well as $|\nabla l^n(\vartheta)|\lesssim|\theta|$ on $[-n/2,n/2]^d$. Similar calculations show that the difference converges: $\lim_{n\to\infty}\kappa_n-\bar{\kappa}_n\ \in\mathbb{R}$. We are now able to estimate:

$$\bigg|\int_{\Xi_n}\mathrm{d}k\ \frac{\chi(k)}{l^n(k)}-\kappa_n\bigg|\lesssim 1+|\bar{\kappa}_n-\kappa_n|\lesssim 1$$

where we used that the sum on the boundary $\partial\Xi_n$ converges to zero and is thus uniformly bounded in n. For the same reason, the above difference converges to the limit $\lim_{n\to\infty}\bar{\kappa}_n-\kappa_n\in\mathbb{R}$.

For all other possibilities of $\mathfrak{q}_1,\mathfrak{q}_2$ we will show boundedness in a distributional sense. If $\mathfrak{q}_1=\mathfrak{q}_2$ we have:

$$\bigg|\int_{\Xi_n^+}\mathrm{d}k\ \nu_k^2 e^{2\pi\iota\langle x,2\mathfrak{q}_1 k\rangle}\varrho_j(2k)\frac{\chi(k)}{l^n(k)}\bigg|\lesssim 2^{j(d-2)}.$$

Finally, if only one of the two components of $\mathfrak{q}_1,\mathfrak{q}_2$ differs (let us suppose it is the first one) we find (with $x=(x_1,x_2)$ and $k=(k_1,k_2)$):

$$\bigg|\int_{\Xi_n^+}\mathrm{d}k\ \nu_k^2 e^{2\pi\iota 2x_2k_2}\varrho_j(2k_2)\frac{\chi(k)}{l^n(k)}\bigg|\lesssim\bigg(\sum_{k_1\geq 1}\frac{1}{|k_1|^{2\theta}}\bigg)\bigg(\sum_{k_2\geq 1}\frac{\varrho_j(2k_2)}{|k_2|^{2(1-\theta)}}\bigg)\lesssim 2^{j\varepsilon}$$

for any $\varepsilon>0$, up to choosing $\theta\in(1/2,1)$ sufficiently close to $1/2$.

Step 3. Now we briefly address the convergence in distribution. Clearly the previous calculations and compact embeddings of Hölder-Besov spaces guarantee tightness of the sequence $X_{\mathfrak{n}}^n\odot\xi^n-\kappa_n$ in the required Hölder spaces for any $\alpha<2-d/2$. We have to uniquely identify the distribution of any limit point. Whereas for $\xi, X_{\mathfrak{n}}^n$ the limit points are Gaussian and uniquely identified as white noise ξ and $\Delta_{\mathfrak{n}}^{-1}\chi(D)\xi$ respectively, the resonant product requires more care, but we can use the same arguments as in [7, Section 5.1] for higher order Gaussian chaoses. $\square$

3. Killed rSBM

In this last section we briefly introduce a killed version of the rSBM described in [8]. This process arises as a scaling limit of a branching random walk in a random environment in which a walker is killed once he leaves a box of size $L \in 2\mathbb{N}$. Recall that we consider the lattice approximation $\Lambda_n^L = \{x \in \mathbb{Z}_n^d \ : \ x \in [-L/2, L/2]^d\}$ (we explicitly write the dependence on L because we will let L vary). Define in addition the space of functions $E^L = \{\eta \in \mathbb{N}_0^{\Lambda_n^L} \ : \ \eta(x) = 0, \forall x \in \partial\Lambda_n^L\}$. Recall that the last point of Lemma 2.2 allows us to apply the results of [8]. We work in the following framework.

Assumption 3.1. *Let ξ^n be the sequence of random variables on Ω constructed in Lemma 2.2 and write:*

$$\xi_e^n(\omega, x) = \xi^n(\omega, x) - c_n(\omega)1_{\{d=2\}}.$$

Fix $\varrho = d/2$, let $u^n(\omega, t, x)$ be the process constructed in [8, Definition 2.6] *and let $\mu^n(\omega, t)$ be the measure associated to it. Such process lives on a probability space:*

$$\big(\Omega \times \overline{\Omega}, \mathscr{F}, \mathbb{P} \ltimes \mathbb{P}^{\omega,n}\big),$$

where $\mathbb{P}^\omega$ is the quenched law of u^n, conditional on the environment $\xi^n(\omega)$, for $\omega \in \Omega$.

The BRWRE u^n does not keep track of the individual particles (all particles are identical, only their position matters, cf [8, Appendix A]). We shall also consider the labelled process, which distinguishes individual particles and kill all particles which leave a given box. We thus introduce the space $E_{\text{lab}} = \bigsqcup_{m\in\mathbb{N}}(\frac{1}{n}\mathbb{Z}^d \cup \{\Delta\})^m$, where $\bigsqcup$ denotes the disjoint union, endowed with the discrete topology. Here Δ is a cemetery state. For $\eta \in E_{\text{lab}}^n$ we write $\dim(\eta) = m$ if $\eta \in (\frac{1}{n}\mathbb{Z}^d \cup \{\Delta\})^m$. A rigorous construction of the process below follows a in [8, Appendix A].

Definition 3.2. *Fix $\omega \in \Omega$ and $X_0^n \in E_{\text{lab}}^n$ with $\dim(X_0^n) = \lfloor n^\varrho \rfloor, (X_0^n)_i = 0, i = 1 \dots \lfloor n^\varrho \rfloor$. Construct the Markov jump process $X^n(\omega)$ on E_{lab}^n via $X^n(0) = X_0^n$ with generator:*

$$\begin{aligned}\mathscr{L}_{\text{lab}}^\omega(F)(\eta) = \sum_{i=1}^{\dim(\eta)} 1_{\{\frac{1}{n}\mathbb{Z}^d\}}(\eta_i)\Big[&\sum_{|y-\eta_i|=n^{-1}} \big(F(\eta^{i\mapsto y}) - F(\eta)\big) \\ &+ (\xi^n)_+(\omega, \eta_i)\big(F(\eta^{i,+}) - F(\eta)\big) + (\xi^n)_-(\omega, \eta_i)\big(F(\eta^{i,-}) - F(\eta)\big)\Big],\end{aligned}$$

where $\eta_j^{i\mapsto y} = \eta_j(1 - 1_{\{i\}}(j)) + y1_{\{i\}}(j)$ and $\eta_j^{i,+} = \eta_j 1_{[0,\dim(\eta)]}(j) + \eta_i 1_{\{\dim(\eta)+1\}}(j)$ as well as $\eta_j^{i,-} = \eta_j(1 - 1_{\{i\}}(j)) + \Delta 1_{\{i\}}(j)$, on the domain $\mathscr{D}(\mathscr{L}_{\text{lab}}^\omega)$ of functions F is such that the right hand-side is bounded. We can then redefine the process

$$u^n(\omega, t, x) = \#\{i \in \{i, \dots, \dim(X^n(\omega, t))\} \ : \ X_i(\omega, t) = x\}$$

which has the same quenched law $\mathbb{P}^\omega$ as the process above.

Similarly, for $i \in \mathbb{N}$ consider $\tau_i^{n,L}(\omega) = \inf\{t \geq 0 \colon \dim(X^n(\omega, t)) \geq i$ and $X_i^n(t) \in \partial\Lambda_n^L\}$. Define $X^{n,L}(\omega, t) \in E_{\text{lab}}^n$ by $\dim(X^{n,L}(\omega, t)) = \dim(X^n(\omega, t))$ and $X_i^{n,L}(\omega, t) = X_i^n(\omega, t)1_{\{t < \tau_i^{n,L}(\omega)\}} + \Delta 1_{\{\tau_i^{n,L}(\omega) \leq t\}}$.

Define $u^{n,L}$ taking values in E^L by

$$u^{n,L}(\omega, t, x) = \#\{i \in \{1, \dots, \dim(X^{n,L}(\omega, t))\} \ : \ X_i^{n,L}(\omega, t) = x\}.$$

Write $\mathscr{M}((-L/2, L/2)^d)$ for the set of all finite positive measures on $(-L/2, L/2)^d$ and for μ, ν in this space we say $\mu \geq \nu$ if also $\mu - \nu$ is a *positive* measure. The following result is now easy to verify (cf. [8, Appendix A]).

Lemma 3.3. *For any $\omega \in \Omega$ the process $t \mapsto u^{n,L}(\omega,t,\cdot)$ is a Markov process with paths in $\mathbb{D}([0,+\infty);E^L)$, associated to the generator $\mathscr{L}^{n,\omega}_L\colon C_b(E^L) \to C_b(E^L)$ defined via:*

$$\mathscr{L}^{n,\omega}_L(F)(\eta) = \sum_{x\in\Lambda^L_n\setminus\partial\Lambda^L_n} \eta_x \cdot \Bigg[\sum_{x\sim y} n^2(F(\eta^{x\mapsto y}){-}F(\eta)) \\ +(\xi^n_e)_+(\omega,x)[F(\eta^{x+}){-}F(\eta)] + (\xi^n_e)_-(\omega,x)[F(\eta^{x-}){-}F(\eta)]\Bigg],$$

where for $\eta \in E^L$ we define $\eta^{x\mapsto y}(z) = (\eta(z){-}1_{\{z=x\}}{+}1_{\{z=y,\ y\notin\partial\Lambda^L_n\}})_+$ and $\eta^{x\pm}(z) = (\eta(z) \pm 1_{\{z=x\}})_+$. We associate to $u^{n,L}(\omega,t)$ a measure:

$$\mu^{n,L}(\omega,t)(\varphi) = \sum_{x\in\Lambda^L_n} \lfloor n^{-\varrho}\rfloor u^{n,L}(\omega,t,x)\varphi(x), \qquad \forall\varphi \in C((-L/2,L/2)^d). \tag{13}$$

Finally:

$$\mu^{n,L}(\omega,t) \le \mu^{n,L+2}(\omega,t) \le \cdots \le \mu^n(\omega,t) \qquad \forall\omega\in\Omega, t\ge 0. \tag{14}$$

When studying the convergence of the process $\mu^{n,L}$, special care has to be taken with regard to what happens on the boundary of the box. Indeed a function $\varphi \in C^\infty([-L/2,L/2]^d)$ (i.e. smooth in the interior with all derivatives continuous on the entire box) is not smooth in the scale of spaces $B^{\mathfrak{l},\alpha}_{p,q}$ for $\mathfrak{l} \in \{\mathfrak{d},\mathfrak{n}\}$, since it does not satisfy the required boundary conditions: a priori it only lies in the above space for $\alpha = 0$ and any value of p,q. For this reason we consider a weaker kind of convergence for the processes $\mu^{n,L}$ than one might expect. We write

$$\mathscr{M}^L_0 = \big(\mathscr{M}((-L/2,L/2)^d),\tau_v\big)$$

of finite positive measures on $(-L/2,L/2)^d$ endowed with the vague topology τ_v (cf. [4, Section 3]), i.e. $\mu^n \to \mu$ in $\mathscr{M}^L_0$ if $\mu^n(\varphi) \to \mu(\varphi)$, for all $\varphi \in X$, where X can be chosen to be either the space $C^\infty_c((-L/2,L/2)^d)$ of smooth functions with compact support or the space $C_0((-L/2,L/2)^d)$ of continuous functions which vanish on the boundary of the box (the latter is a Banach space, when endowed with the uniform norm). The reason why this topology is convenient is that sets of the form $K_R \subset \mathscr{M}^L_0$, with $K_R = \{\mu \in \mathscr{M}^L_0 \ :\ \mu(1) \le R\}$ are compact. In this setting it is also important to remark the following embedding, which follows from a short calculation.

Remark 3.4. *For $\alpha > 0$ there is a continuous (in the sense of Banach spaces) embedding*

$$\mathscr{C}^\alpha_{\mathfrak{d}}([-L/2,L/2]^d) \hookrightarrow C_0((-L/2,L/2)^d).$$

Now we can pass to study the convergence of the killed process.

Lemma 3.5. *We can bound the mass of the killed process locally uniformly in time. Namely, for any $\omega\in\Omega$:*

$$\lim_{R\to\infty}\sup_n \mathbb{P}^{\omega,n}\Big(\sup_{t\in[0,T]}\mu^{n,L}(\omega,t)(1) \ge R\Big) = 0, \qquad \sup_n\sup_{t\in[0,T]}\|T^{n,\mathfrak{d},L,\omega}_t 1\|_\infty < +\infty.$$

Proof. The first bound follows from comparison with the process on the whole real line (i.e. Equation (14)), see [8, Corollary 4.3]. The second bound follows from Theorem 2.3 because the antisymmetric extension of 1 is in L^∞: we have $|\Pi_o 1(\cdot)| \equiv 1$. □

Lemma 3.6. *For every $\omega\in\Omega$ the sequence $\{t\mapsto\mu^{n,L}(\omega,t)\}_{n\in\mathbb{N}}$ is tight in the space $\mathbb{D}(\mathbb{R}_{\ge0};\mathscr{M}^L_0)$. Any limit point $\mu^L(\omega)$ lies in $C(\mathbb{R}_{\ge0};\mathscr{M}^L_0)$.*

Proof. We want to apply Jakubowski's tightness criterion [4, Theorem 3.6.4]. The sequence $\mu^{n,L}$ satisfies the compact containment condition in view of Lemma 3.5. The tightness thus follows if we prove that the sequence $\{t\mapsto\mu^n(t)(\varphi)\}_{n\in\mathbb{N}}$ is tight in $\mathbb{D}([0,T];\mathbb{R})$ for any $\varphi \in C^\infty_c((-L/2,L/2)^d)$. Here we can follow the calculation of [8, Lemma 4.2] (only simpler, since we do not need weights), using the results from Theorem 2.3. The continuity of the limit points is shown as in [8, Lemma 4.4].

□

We will characterize the limit points of $\{\mu^{n,L}\}_{n\in\mathbb{N}}$ in a similar way as the rough super-Brownian motion, and for that purpose we need to solve the following equation (for any $\omega\in\Omega, L\in 2\mathbb{N}$):

$$\partial_t\varphi=\mathscr{H}^{\omega}_{\mathfrak{d},L}\varphi-\nu\varphi^2,\ \ \varphi(0)=\varphi_0,\ \ \varphi(t,x)=0,\ \forall(t,x)\in(0,T]\times\partial[-L/2,L/2]^d, \tag{15}$$

where we define φ a solution to (15) if

$$\varphi(t)=T^{\mathfrak{d},L,\omega}_t\varphi_0-\nu\int_0^t T^{\mathfrak{d},L,\omega}_{t-s}[\varphi^2(s)]\,\mathrm{d}s.$$

Lemma 3.7. *Fix $\omega\in\Omega, L\in 2\mathbb{N}$. For $T>0$ and $\varphi_0\in C^\infty_c((-L/2,L/2)^d)$ with $\varphi_0\geq 0$ and ϑ as in Theorem 2.3, there exists a unique (paracontrolled in $d=2$) solution $\varphi\in\mathscr{L}^{\vartheta}_{\mathfrak{d}}([-L/2,L/2]^d)$ to* (15) *and the following bounds hold:*

$$0\leq\varphi(t)\leq T^{\mathfrak{d},L,\omega}_t\varphi_0,\qquad \|\varphi\|_{\mathscr{L}^{\theta}_{\mathfrak{d}}([-L/2,L/2]^d)}\lesssim e^{C\|\{T^{\mathfrak{d},L,\omega}_t\varphi_0\}_{t\in[0,T]}\|_{CL^\infty([-L/2,L/2]^d)}}.$$

The proof is analogous to the one of [8, Proposition 4.5]. We thus arrive at the following description of the limit points of $\{\mu^{n,L}\}_{n\in\mathbb{N}}$.

Theorem 3.8. *For any $\omega\in\Omega$ and $L\in 2\mathbb{N}$, under Assumption 3.1, there exists $\mu^L(\omega)\in C(\mathbb{R}_{\geq 0};\mathscr{M}^L_0)$ such that $\mu^{n,L}(\omega)\to\mu^L(\omega)$ in distribution in $\mathbb{D}(\mathbb{R}_{\geq 0};\mathscr{M}^L_0)$. The process $\mu^L(\omega)$ is the unique (in law) process in $C(\mathbb{R}_{\geq 0};\mathscr{M}^L_0)$ which satisfies one (and then all) of the following equivalent properties with $\mathscr{F}^\omega=\{\mathscr{F}^\omega_t\}_{t\geq 0}$ being the usual augmentation of the filtration generated by $\mu^L(\omega)$.*

(i) For any $t\geq 0$ and $\varphi_0\in C^\infty_c((-L/2,L/2)^d),\varphi_0\geq 0$ and for $U^{\mathfrak{d},L,\omega}_t\varphi_0$ the solution to Equation (15) *with initial condition φ_0 the process*

$$N^{\varphi_0}_t(s)=e^{-\langle\mu^L(\omega,s),U^{\mathfrak{d},L,\omega}_{t-s}\varphi_0\rangle},\qquad s\in[0,t]$$

is a bounded continuous $\mathscr{F}^\omega-$martingale.

(ii) For any $\varphi\in\mathscr{D}_{\mathscr{H}^{\omega}_{\mathfrak{d},L}}$ the process:

$$K^{\varphi}(t)=\langle\mu^L(\omega,t),\varphi\rangle-\langle\delta_0,\varphi\rangle-\int_0^t\mathrm{d}r\ \langle\mu^L(\omega,r),\mathscr{H}^{\omega}_{\mathfrak{d},L}\varphi\rangle,\qquad t\in[0,T]$$

is a continuous $\mathscr{F}^\omega-$martingale, square-integrable on $[0,T]$ for all $T>0$, with quadratic variation

$$\langle K^{\varphi}\rangle_t=2\nu\int_0^t\mathrm{d}r\ \langle\mu^L(\omega,r),\varphi^2\rangle.$$

Proof. The proof is almost identical to the one of [8, Theorem 2.13]. The main difference is that here we only test against functions with zero boundary conditions and thus use the results from Section 2. □

We call the above process the killed rSBM on $(-\frac{L}{2},\frac{L}{2})^d$. Note that we can interpret the killed rSBM as an element of $C(\mathbb{R}_{\geq 0};\mathscr{M}(\mathbb{R}^d))$ by extending it by zero, i.e. $\mu^L(\omega,t,A)=\mu^L(\omega,t,A\cap(-L/2,L/2)^d)$ for any measurable $A\subset\mathbb{R}^d$. This allows us to couple infinitely many killed rSBMs with a rSBM on $\mathbb{R}^d$ so that they are ordered in the natural way.

Corollary 3.9. *For any $\omega\in\Omega$, under Assumption 3.1, there exists a process*

$$(\mu(\omega,\cdot),\mu^2(\omega,\cdot),\mu^4(\omega,\cdot),\dots)$$

taking values in $C(\mathbb{R}_{\geq 0}; \mathscr{M}(\mathbb{R}^d))^{\mathbb{N}}$ *(equipped with the product topology) such that* μ *is an rSBM and* μ^L *is a killed rSBM for all* $L \in 2\mathbb{N}$ *(all associated to the environment* $\{\xi^n\}_{n\in\mathbb{N}}$*), and such that:*

$$\mu^2(\omega, t, A) \leq \mu^4(\omega, t, A) \leq \cdots \leq \mu(\omega, t, A) \tag{16}$$

for all $t \geq 0$ *and all Borel sets* $A \subset \mathbb{R}^d$.

Proof. The construction (13) of μ^n and $\mu^{n,L}$ based on the labelled particle system gives us a coupling $(\mu^n, \mu^{n,2}, \mu^{n,4}, \dots)$ such that for all $\omega \in \Omega$

$$\mu^{n,2}(\omega, t, A) \leq \mu^{n,4}(\omega, t, A) \leq \cdots \leq \mu^n(\omega, t, A)$$

for all $t \geq 0$ and all Borel sets $A \subset \mathbb{R}^d$, where as above we extend $\mu^{n,L}$ to $\mathbb{R}^d$ by setting it to zero outside of $(-\frac{L}{2}, \frac{L}{2})^d$ (cf. Equation (14)). By [8, Theorem 2.13] and Theorem 3.8 we get tightness of the finite-dimensional projections $(\mu^n, \mu^{n,2}, \dots, \mu^{n,L})$ for $L \in 2\mathbb{N}$, and this gives us tightness of the whole sequence in the product topology. Moreover, for any subsequential limit $(\mu, \mu^2, \mu^4, \dots)$ we know that μ is an rSBM and μ^L is a killed rSBM on $(-\frac{L}{2}, \frac{L}{2})^d$. It is however a little subtle to obtain the ordering (16), because we only showed tightness in the vague topology on $\mathscr{M}_0^L$ for the $\mu^{n,L}$ component. So we introduce suitable cut-off functions to show that the ordering is preserved along any (subsequential) limit: Let $\chi^m \in C_c^\infty((-L/2, L/2)^d)$, $\chi^m \geq 0$ such that $\chi^m = 1$ on a sequence of compact sets K^m which increase to $(-L/2, L/2)^d$ as $m \to \infty$. Note that on compact sets the sequence $\mu^{n,L}$ converges weakly (and not just vaguely). We then estimate (in view of Equation (14)) for $\varphi \in C_b(\mathbb{R}^d)$ with $\varphi \geq 0$:

$$\begin{aligned} \langle \mu^L(t), \varphi \rangle = \lim_{m\to\infty} \langle \mu^L(t), \varphi \cdot \chi^m \rangle &= \lim_{m\to\infty} \lim_{n\to\infty} \langle \mu^{n,L}(t), \varphi \cdot \chi^m \rangle \\ &\leq \lim_{m\to\infty} \langle \mu(t), \varphi \cdot \chi^m \rangle = \langle \mu(t), \varphi \rangle, \end{aligned}$$

and similarly we get $\langle \mu^L(t), \varphi \rangle \leq \langle \mu^{L'}(t), \varphi \rangle$ for $L \leq L'$. Since a signed measure that has a positive integral against every positive continuous function must be positive, our claim follows. □

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