SYNCHRONIZATION FOR KPZ

TOMMASO CORNELIS ROSATI

Abstract. We study the longtime behavior of KPZ-like equations:

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \eta(t,x), \qquad h(0,x) = h_0(x), \qquad (t,x) \in (0,\infty) \times \mathbb{T}^d,$$

on the $d-$dimensional torus $\mathbb{T}^d$ driven by an ergodic noise η (e.g. space-time white in $d=1$). The analysis builds on infinite-dimensional extensions of similar results for positive random matrices. We establish a one force, one solution principle and derive almost sure synchronization with exponential deterministic speed in appropriate Hölder spaces.

Introduction

The present article concerns the study of stochastic partial differential equations (SPDEs) of the form:

$$(\partial_t - \Delta_x) h(t,x) = |\nabla_x h|^2(t,x) + \eta(t,x), \qquad h(0,x) = h_0(x), \qquad (t,x) \in (0,\infty) \times \mathbb{T}^d, \tag{1}$$

where η is a random forcing and $\mathbb{T}^d$ the d-dimensional torus. In dimension $d=1$ with η space-time white noise the above is known as the Kardar-Parisi-Zhang (KPZ) equation, and is a renowned model in the study of growing interfaces. It is related to a universality conjecture [32], according to which many asymmetric growth models are described, on large spatial and temporal scales, by a common universal object: the so-called KPZ fixed point. The KPZ equation is itself the scaling limit of many such processes in a *weakly asymmetric* regime and it is expected to converge to the mentioned fixed point on large scales.

This picture forms a partial motivation for the mathematical study of equations of type (1) and especially their longtime behavior. In the case of the original KPZ equation, wellposedness was a mathematical milestone [23, 24, 20]. Preceding these results there was no clear understanding of the quadratic nonlinearity in Equation (1), yet the equation could be studied through the Cole-Hopf transform, by imposing that $u = \exp(h)$ solves the linear heat equation (SHE) with multiplicative noise (a step that can be made rigorous for smooth η but requires particular care and the introduction of renormalization constants if η is space-time white noise):

$$(\partial_t - \Delta_x) u(t,x) = \eta(t,x) u(t,x), \qquad u(0,x) = u_0(x), \qquad (t,x) \in (0,\infty) \times \mathbb{T}^d. \tag{2}$$

In addition to proving well-posedness of the KPZ equation, [24] introduces the notion of *subcriticality* [24], which provides a formal condition on η under which Equations (1) and (2) remain well-posed. That such condition is sufficient has been proven for a wide class of equations [7, 6, 12].

In consideration of these results, the present article will not discuss questions regarding the wellposedness of (1). Instead, the aim is to prove results concerning the longtime

Humboldt-Universität zu Berlin

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behavior of solutions, under the condition that a solution map to the SHE (2) is given and satisfies some natural properties that will be introduced below.

Indeed, the linearization of (1) provides additional structure to the equation that allows to prove particularly strong ergodic properties. The first simple observation is that Equation (1) is shift invariant, meaning that any ergodic property will be proved either "modulo constants", that is identifying two function $h, h' : [0, \infty) \times \mathbb{T}^d \to \mathbb{R}$ if there exists a $c \colon [0, \infty) \to \mathbb{R}$ such that $h(t, x) - h^{'}(t, x) = c(t), \forall t, x$, or for the gradient $v = \nabla_x h$, which satisfies the Burgers'-like equation:

$$(3) \quad (\partial_t - \Delta_x) v(t, x) = \nabla_x |v|^2(t, x) + \nabla_x \eta(t, x), \quad v(0, x) = v_0(x), \quad (t, x) \in (0, \infty) \times \mathbb{T}^d.$$

On one hand, unique ergodicity of the KPZ Equation "modulo constants" was first established by Hairer and Mattingly [25] as a consequence of a strong Feller property that holds for a wide class of SPDEs. Moreover, for KPZ Funaki and Quastel show that the invariant measure is the Brownian bridge [19] (in fact, the latter result considers the equation on the entire line, a more complicated setting), and Gubinelli and Perkowski proved a spectral gap for Burgers' equation [22], implying exponential convergence to the invariant measure, although the article considers initial conditions that are "near-stationary".

On the other hand, in a seminal work Sinai [33] considered $\eta(t, x) = V(x) \partial_t \beta(t)$ for $V \in C^\infty(\mathbb{T}^d)$ and a Brownian motion β. The author proved that there exists a function $\overline{v}(t, x)$ defined for all $t \in \mathbb{R}$ such that almost surely, for a certain class of initial conditions v_0, with v solving (3), and any $x \in \mathbb{T}^d$:

$$\lim_{t \to \infty} v(t, x) - \overline{v}(t, x) = 0.$$

This property is referred to as *synchronization* for Equation (3). In addition, if one starts Burgers' equation at time $-n$ with $v^{-n}(-n, x) = v_0(x)$:

$$\lim_{n \to \infty} v^{-n}(t, x) = \overline{v}(t, x), \qquad \forall (t, x) \in (-\infty, \infty) \times \mathbb{T}^d.$$

This property is called a *one force, one solution* principle (1F1S) and it implies that $\overline{v}$ is the unique (ergodic) solution to (3) on $\mathbb{R}$. Results of this kind have subsequently been generalized in many directions, most notably to the inviscid case [35] or to infinite volume, for example in [5] and recently in [16], all for specific classes of noises.

The current paper attempts to understand and extend the results by Sinai through an application of the theory of random dynamical systems. The power of this approach lies in the capacity of treating any forcing η such that:

(1) The noise η is ergodic (see Proposition 5.12 for a classical condition if η is Gaussian).
(2) Equation (2) is almost surely well-posed (there exists a unique, global in time solution for every $u_0 \in C(\mathbb{T}^d)$), the solution map being a linear, compact, strictly positive operator on $C(\mathbb{T}^d)$.

In particular, the first novelty of this work is that η can be chosen to be for example space-time white noise or a noise that is fractional in time.

In the original work by Sinai, the solution u to (2) evaluated at time n is represented by $u(n, x) = A^n u_0(x)$ for a compact strictly positive operator A^n. The proof of the result makes use in turn of the explicit representation of the operator A^n via the Feynman-Kac formula. Such representation becomes more technical when the noise η is not smooth and requires some understanding of random polymers (cf. [10, 15] for the case of space-time white noise). In this work we will avoid the language of random polymers, as we will try to explain in the following.

If η were a time-independent (static) noise, the synchronization of the solution v to (3) would amount to the convergence, upon rescaling, of u to the (random) eigenfunction of A^1 associated to its largest eigenvalue: an instance of the Krein-Rutman Theorem. We will extend this argument to the non-static case with an application of the theory of random dynamical systems. The key point is a contraction principle for positive operators in projective spaces under Hilbert's projective metric (see [8] for an overview). Such method was already developed by Arnold, Demetrius and Gundlach [3] and later refined by Hennion [26] for random matrices. Their proofs naturally extend to the infinite-dimensional case, giving rise to an ergodic version of the Krein-Rutman theorem (see Theorem 3.2). In this way one obtains synchronization and 1F1S "modulo constants" for the KPZ equation (in an example we show that the constants can be chosen time-independent, a fact that is expected to hold in general).

The last point of this article is to obtain convergences in appropriate Hölder spaces, depending on the regularity of the driving noise (see Theorem 4.3). This step requires (see the last assumption in 4.2) a bound on the average:

$$\mathbb{E} \sup_{x \in \mathbb{T}} |h(t,x)|,$$

for fixed $t > 0$. In concrete examples it is shown how a control on this term can be obtained from a quantitative version of a strong maximum principle for (2). These discussions as well as the moment condition seem to be new, although they are connected to similar conditions in [3].

The examples we treat are the original KPZ equation, namely the case of η being space-time white noise in $d = 1$ and a non-Markovian case, with $\eta(t,x) = V(x)\,\mathrm{d}\beta^H_t$ for β^H a fractional Brownian motion of Hurst parameter $H > \frac{1}{2}$ and $V \in C^\infty(\mathbb{T})$. In the latter case the solution is not Markovian, and ergodic results are rare, see for example a work by Maslowski and Pospíšil [28] for ergodicity of linear SPDEs with additive fractional noise.

Finally, let us remark that there are several instances of applications of the theory of random dynamical systems to stochastic PDEs. Particularly related to our work is the study of order-preserving systems which admit some random attractor, first addressed by Arnold and Chueshov [2], then by Flandoli, Gess and Scheutzow [17] and recently by Butkovsky and Scheutzow [9]. The spirit of these results is similar to ours, but although the linearity of (2) on one hand guarantees order preservation, on the other hand it does not allow the existence of a random attractor. In this sense our, essentially linear, case appears to be a degenerate example of the synchronization addressed in the just quoted works.

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1. Notations

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, +\infty)$ and $\iota = \sqrt{-1}$. Furthermore, for $d \in \mathbb{N}$ let $\mathbb{T}^d$ be the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ (here $\mathbb{Z}^d$ acts by translation on $\mathbb{R}^d$). The case $d = 1$ is of particular interest, so we write $\mathbb{T} = \mathbb{T}^1$.

For a general set $\mathcal{X}$ and functions $f, g \colon \mathcal{X} \to \mathbb{R}$ write $f \lesssim g$ if $f(x) \leqslant Cg(x)$ for all $x \in \mathcal{X}$ and a constant $C > 0$ independent of x. To clarify on which parameters C is allowed to depend we might add them as subscripts to the "$\lesssim$" sign.

For $\alpha > 0$ let $\lfloor \alpha \rfloor$ be the smallest integer beneath α and for a multiindex $k \in \mathbb{N}^d$ write $|k| = \sum_{i=1}^d k_i$. Denote with $C(\mathbb{T}^d)$ the space of continuous real-valued functions on $\mathbb{T}^d$, and, for $\alpha > 0$, with $C^\alpha(\mathbb{T})$ the space of $\lfloor \alpha \rfloor-$ differentiable functions f such that $\partial^k f$ is $(\alpha - \lfloor \alpha \rfloor)-$Hölder continuous for every multiindex $k \in \mathbb{N}^d$ such that $|k| = \lfloor \alpha \rfloor$, if $\alpha - \lfloor \alpha \rfloor > 0$, or simply continuous if $\alpha \in \mathbb{N}_0$. For $\alpha \in \mathbb{R}_+$ we obtain the following seminorms on $C^\alpha(\mathbb{T}^d)$:

$$[f]_\alpha = \max_{|k| = \lfloor \alpha \rfloor} \quad \|\partial^k f\|_\infty 1_{\{|k|>0\}} + \sup_{x,y \in \mathbb{T}^d} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^{\alpha - \lfloor \alpha \rfloor}}.$$

Now, let X be a Banach space. We denote with $\mathcal{B}(X)$ the Borel $\sigma-$algebra on X. Let $[a,b] \subseteq \mathbb{R}$ be an interval, then define $C([a,b];X)$ the space of continuous functions $f\colon [a,b] \to X$. For any $O \subseteq \mathbb{R}$, we write $C_{\mathrm{loc}}(O;X)$ for the space of continuous functions with the topology of uniform convergence on all compact subsets of O. Given two Banach spaces X, Y denote with $\mathcal{L}(X;Y)$ the space of linear bounded operators $A\colon X \to Y$ with the classical operator norm. If $X = Y$ we write simply $\mathcal{L}(X)$.

Next we introduce Besov spaces. Following [4, Section 2.2] choose a smooth dyadic partition of the unity on $\mathbb{R}^d$ (resp. $\mathbb{R}^{d+1}$) $(\chi, \{\varrho_j\}_{j \geqslant 0})$ and define $\varrho_{-1} = \chi$ and define the Fourier transforms for $f\colon \mathbb{T}^d \to \mathbb{R}$ and $g\colon \mathbb{R}\times\mathbb{T}^d \to \mathbb{R}$:

$$\mathcal{F}_{\mathbb{T}^d} f(k) = \int_{\mathbb{T}^d} e^{-2\pi\iota\langle k,x\rangle} f(x)\,\mathrm{d}x, \qquad k \in \mathbb{Z}^d,$$
$$\mathcal{F}_{\mathbb{R}\times\mathbb{T}^d} g(\tau,k) = \int_{\mathbb{R}\times\mathbb{T}^d} e^{-2\pi\iota(\tau t+\langle k,x\rangle)} g(t,x)\,\mathrm{d}t\,\mathrm{d}x, \qquad (\tau,k) \in \mathbb{R}\times\mathbb{Z}^d.$$

These definitions extend naturally to spatial (resp. space-time) tempered distributions $\mathcal{S}'(\mathbb{T}^d)$ (resp. $\mathcal{S}'(\mathbb{R}\times\mathbb{T}^d)$). Similarly one defines the respective inverse Fourier transforms $\mathcal{F}^{-1}_{\mathbb{T}^d}$ and $\mathcal{F}^{-1}_{\mathbb{R}\times\mathbb{T}^d}$. Then define the spatial (resp. space-time) Paley blocks:

$$\Delta_j f(x) = \mathcal{F}^{-1}_{\mathbb{T}^d}[\varrho_j \cdot \mathcal{F}_{\mathbb{T}^d} f](x), \qquad \Delta_j g(t,x) = \mathcal{F}^{-1}_{\mathbb{R}\times\mathbb{T}^d}[\varrho_j \cdot \mathcal{F}_{\mathbb{R}\times\mathbb{T}^d} g](t,x).$$

Eventually one defines the spaces $B^{\alpha}_{p,q}(\mathbb{T}^d)$ and $B^{\alpha,a}_{p,q}(\mathbb{R}\times\mathbb{T}^d)$ as the set of tempered distributions such that, respectively, the following norms are finite:

$$\|f\|_{B^{\alpha}_{p,q}(\mathbb{T}^d)} = \|(2^{j\alpha}\|\Delta_j f\|_{L^p(\mathbb{T})})_{j\geqslant -1}\|_{\ell^q},$$
$$\|g\|_{B^{\alpha,a}_{p,q}(\mathbb{R}\times\mathbb{T}^d)} = \|(2^{j\alpha}\|\Delta_j f(\cdot)/\langle\cdot\rangle^a\|_{L^p(\mathbb{R}\times\mathbb{T}^d)})_{j\geqslant -1}\|_{\ell^q},$$

where we denote with $\langle (t,x)\rangle$ the weight $\langle (t,x)\rangle = 1+|t|$. For $p = q = 2$ one obtains the Hilbert spaces $H^{\alpha}(\mathbb{T}^d) = B^{\alpha}_{2,2}(\mathbb{T}^d)$ and $H^{\alpha}_a(\mathbb{R}\times\mathbb{T}^d) = B^{\alpha,a}_{2,2}(\mathbb{R}\times\mathbb{T}^d)$. Finally, recall that for $p = q = \infty$ and $\alpha \in \mathbb{R}_+\backslash\mathbb{N}_0$: $B^{\alpha}_{\infty,\infty}(\mathbb{T}^d) = C^{\alpha}(\mathbb{T}^d)$ (see e.g. [34, Chapter 2]).

2. Setting

This section introduces the projective space of positive continuous functions, and a related contraction principle for strictly positive operators and is based on the work by Bushell [8]. Let X be a Banach space and $K \subseteq X$ a closed cone such that $K \cap (-K) = \{0\}$. Denote with $\mathring{K}$ the interior of K and write $K^+ = K\backslash\{0\}$. Such cone induces a partial order in X by defining for $x, y \in X$:

$$x \leqslant y \Leftrightarrow y - x \in K \quad \text{and} \quad x < y \Leftrightarrow y - x \in \mathring{K}.$$

Consider for $x, y \in K^+$:

$$M(x,y) = \inf\{\lambda \geqslant 0\colon x \leqslant \lambda y\}, \qquad m(x,y) = \sup\{\mu \geqslant 0\colon \mu y \leqslant x\},$$

with the convention $\inf\varnothing = \infty$. Then $M(x,y) \in (0,\infty]$ and $m(x,y) \in [0,\infty)$ so that one can define Hilbert's projective distance:

$$d_H(x,y) = \log(M(x,y)) - \log(m(x,y)) \in [0,\infty], \qquad \forall\, x, y \in K^+.$$

This metric is only semidefinite positive on K^+, and may be infinite. A remedy for the first issue is to consider an affine space $U \subseteq X$ which intersects transversely K^+, that is:

$$\forall x \in K^+, \quad \exists! \lambda > 0 \quad \text{s.t.} \quad \lambda x \in U.$$

Write $\lambda(x)$ for the normalization constant above. As for the second issue, one can observe that the distance is finite on the interior of K, cf. [8, Theorem 2.1] and thus, defining $E = \mathring{K} \cap U$ one has that (E, d_H) is a metric space.

Consider now $\mathcal{L}(X)$ the set of linear bounded operators on X, and for an operator $A \in \mathcal{L}(X)$ the following conventions define different concepts of positivity:

$$\begin{aligned} A(K) &\subseteq K && \Rightarrow && A \text{ nonnegative.} \\ A(\mathring{K}) &\subseteq \mathring{K} && \Rightarrow && A \text{ positive.} \\ A(K^+) &\subseteq \mathring{K} && \Rightarrow && A \text{ strictly positive.} \end{aligned}$$

The projective action of a positive operator A on X is then defined by: $A \cdot x = Ax/\lambda(Ax)$, an one denotes with $\tau(A)$ the projective norm associated to A:

$$\tau(A) = \sup_{\substack{x,y \in E \\ x \neq y}} \frac{d_H(A \cdot x, A \cdot y)}{d_H(x,y)}. \tag{4}$$

The backbone of our approach is Birkhoff's theorem for positive operators [8, Theorem 3.2], which we state from clarity.

Theorem 2.1. *Let $\Delta(F)$ denote the diameter of a set $F \subseteq E$:*

$$\Delta(F) = \sup_{x,y \in F} \{d_H(x,y)\}.$$

The following identity holds:

$$\tau(A) = \tanh\Big(\frac{1}{4}\Delta(A \cdot E)\Big) \leqslant 1.$$

Then denote with $\mathcal{L}_{\mathrm{cp}}(X)$ the space of positive operators A which are contractive in (E, d_H):

$$A \in \mathcal{L}_{\mathrm{cp}}(X) \quad \Leftrightarrow \quad \tau(A) < 1.$$

The only example considered in this work is $X = C(\mathbb{T}^d)$ the space of real-valued continuous functions on the torus, with the cone of positive functions. Here the following holds.

Lemma 2.2. *Let $X = C(\mathbb{T}^d)$ and $K = \{f \in X \ : \ f(x) \geqslant 0, \quad \forall x \in \mathbb{T}^d\}$, and consider:*

$$U = \Big\{ f \in X \colon \int_{\mathbb{T}^d} f(x)\,\mathrm{d}x = 1 \Big\}.$$

For the associated metric space (E, d_H) the following inequality holds:

$$\|\log(f) - \log(g)\|_\infty \leqslant d_H(f,g) \leqslant 2\|\log(f) - \log(g)\|_\infty, \qquad \forall f,g \in E. \tag{5}$$

In particular, (E, d_H) is a complete metric space. In addition, if a strictly positive operator A can be represented by a kernel, i.e. there exists $K \in C(\mathbb{T}^d \times \mathbb{T}^d)$ such that:

$$A(f)(x) = \int_{\mathbb{T}^d} K(x,y) f(y) \ \mathrm{d}y, \quad \forall x \in \mathbb{T}^d$$

and there exits constants $0 < \alpha \leqslant \beta < \infty$ such that

$$\alpha \leqslant K(x,y) \leqslant \beta, \quad \forall x,y \in \mathbb{T}^d,$$

then A is contractive, i.e. $A \in \mathcal{L}_{\mathrm{cp}}(X)$.

Proof. As for the inequality, since $f, g \in U$ (and hence $\int f(x)\,\mathrm{d}x = \int g(x)\,\mathrm{d}x = 1$), there exists a point x_0 such that $f(x_0) = g(x_0)$. In particular in the sum

$$\max(\log(f/g)) - \min(\log(f/g)) = \max(\log(f/g)) + \max(\log(g/f))$$

both terms are positive an bounded by $\|\log(f) - \log(g)\|_\infty$. Viceversa we have that:

$$\|\log(f) - \log(g)\|_\infty \leqslant \max(\log(f) - \log(g)) + \max(\log(g) - \log(f)).$$

Completeness of (E, d_H) is a consequence of Inequality (5): for a given Cauchy sequence $f_n \in E$ the sequence $\log(f_n)$ is a Cauchy sequence in $C(\mathbb{T}^d)$. By completeness of the latter there exists a $g \in C(\mathbb{T}^d)$ such that $\log(f_n) \to g$. By dominated convergence $\exp(g) \in E$, and hence $f_n \to \exp(g)$ in E.

The result regarding the kernel can be found in [8, Section 6].

□

Remark 2.3. *For the sake of simplicity we did not address the general question of completeness of the space (E, d_H), since in the case of interest to us it follows from Equation* (5)*. There are known criteria for completeness, cf.* [8, Section 4] *and the references therein.*

Remark 2.4. *In view of* (4)*, an application of Banach's fixed point theorem in (E, d_H) to operators satisfying the conditions of Lemma 2.2 delivers the existence of a unique positive eigenfunction. This is a variant of the Krein-Rutman theorem. The formulation we propose here is convenient because of its natural extension to random dynamical systems.*

3. A Random Krein-Rutman Theorem

In this section we reformulate the results of [3, 26] for positive operators on Banach spaces.

An *invertible metric discrete dynamical system* (IDS) $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a measurable map $\vartheta \colon \mathbb{Z} \times \Omega \to \Omega$ such that $\vartheta(z+z', \cdot) = \vartheta(z, \vartheta(z', \cdot))$ and $\vartheta(0, \omega) = \omega$ for all $\omega \in \Omega$, and such that $\mathbb{P}$ is invariant under $\vartheta(z, \cdot)$ for $z \in \mathbb{Z}$. For brevity we write $\vartheta^z(\cdot)$ for the map $\vartheta(z, \cdot)$. A set $\widetilde{\Omega} \subseteq \Omega$ is said to be *invariant* for ϑ if $\vartheta^z \widetilde{\Omega} = \widetilde{\Omega}$, for all $z \in \mathbb{Z}$ and an IDS is said to be *ergodic* if any invariant set $\widetilde{\Omega}$ satisfies $\mathbb{P}(\widetilde{\Omega}) \in \{0, 1\}$ (cf. [1, Appendix A]).

Consider X, E as in the previous section and, for a given IDS, a random variable $A \colon \Omega \to \mathcal{L}(X)$. This generates a measurable, linear, discrete random dynamical system (RDS) (see [1, Definition 1.1.1]) φ on X by defining:

$$\varphi_n(\omega)x = A(\vartheta^n\omega) \cdots A(\omega)x, \qquad n \in \mathbb{N}_0.$$

If $A(\omega)$ is in addition positive for every $\omega \in \Omega$ (we then simply say that A is positive), we can interpret φ as an RDS on E via the projective action:

$$\varphi_n(\omega) \cdot x = A(\vartheta^n\omega) \cdot \ldots \cdot A(\omega) \cdot x, \qquad n \in \mathbb{N}_0.$$

Assumption 3.1. *Assume we are given X, K, U, E as in the previous section and that (E, d_H) is a* complete *metric space. Assume in addition that there exists an* ergodic *IDS ϑ. Let φ_n be a RDS defined via a random* positive *operator A as above, such that:*

$$\mathbb{P}\Big(A \in \mathcal{L}_{cp}(X)\Big) > 0.$$

In this setting the following is a random version of the Krein-Rutman theorem.

Theorem 3.2. *Under Assumption 3.1 there exists a $\vartheta-$invariant set $\widetilde{\Omega} \subseteq \Omega$ of full $\mathbb{P}-$measure and a random variable $u \colon \Omega \to E$ such that:*

(1) *For all $\omega \in \widetilde{\Omega}$ and $f, g \in E$:*

$$\limsup_{n\to\infty} \left[\frac{1}{n} \sup_{f,g\in E} \Big(\log d_H(\varphi_n(\omega) \cdot f, \varphi_n(\omega) \cdot g)\Big)\right] \leqslant \mathbb{E} \log \big(\tau(A)\big) < 0.$$

(2) *u is measurable w.r.t. to the $\sigma-$field $\mathcal{F}^- = \sigma((A(\vartheta^{-n} \cdot))_{n\in\mathbb{N}})$ and:*

$$\varphi_n(\omega) \cdot u(\omega) = u(\vartheta^n\omega).$$

(3) *For all $\omega \in \widetilde{\Omega}$:*

$$\limsup_{n\to\infty} \left[\frac{1}{n} \sup_{f\in E} \Big(\log d_H(\varphi_n(\vartheta^{-n}\omega) \cdot f, u(\omega))\Big)\right] \leqslant \mathbb{E} \log \big(\tau(A)\big) < 0$$

as well as:

$$\limsup_{n\to\infty}\left[\frac{1}{n}\sup_{f\in E}\Big(\log d_H(\varphi_n(\omega)\cdot f, u(\vartheta^n\omega))\Big)\right]\leqslant \mathbb{E}\log\big(\tau(A)\big)<0.$$

(4) *The measure* $\delta_{u(\omega)}$ *on* E *is the unique invariant measure for the RDS* φ *on* E.

Notation 3.3. *We refer to the first property as* asymptotic synchronization *and to the third property as* one force, one solution *principle.*

Proof. As for the first property, compute:

$$\begin{aligned}d_H(\varphi_n(\omega)\cdot f,\varphi_n(\omega)\cdot g)\leqslant&\tau(A_n(\omega))d_H(\varphi_{n-1}(\omega)\cdot f,\varphi_{n-1}(\omega)\cdot g)\\ \leqslant&\prod_{i=0}^{n}\tau(A(\vartheta^i\omega))d_H(f,g).\end{aligned}$$

Then, applying the logarithm and Birkhoff's ergodic theorem we find:

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\tau(\varphi_n(\omega))\right)\leqslant\mathbb{E}\log\left(\tau(A)\right)<0.$$

If $\mathbb{E}\log\big(\tau(A)\big)=-\infty$ we can instead follow the previous computation with $\tau(A(\vartheta^i\omega))$ replaced by $\tau(A(\vartheta^i\omega))\vee e^{-M}$ and eventually pass to the limit $M\to\infty$. To obtain the result uniformly over f,g first observe that via Theorem 2.1:

$$\sup_{f,g\in E}\Big(\log d_H(\varphi_n(\omega)\cdot f,\varphi_n(\omega)\cdot g)\Big)=\log\Big(\Delta(\varphi_n(\omega)\cdot E)\Big)=\log\Big(4\ \mathrm{arctanh}(\tau(\varphi_n(\omega)))\Big),$$

and by a Taylor approximation, since $\lim_{n\to\infty}\tau(\varphi_n(\omega))=0$, there exists a constant $c(\omega)>0$ such that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\Big(4\ \mathrm{arctanh}(\tau(\varphi_n(\omega)))\Big)&\leqslant\limsup_{n\to\infty}\frac{1}{n}\log\Big((1+c(\omega))\tau(\varphi_n(\omega))\Big)\\ &=\limsup_{n\to\infty}\frac{1}{n}\log\Big(\tau(\varphi_n(\omega))\Big)\leqslant\mathbb{E}\log\tau(A).\end{aligned}$$

The second point as well as the first property of (3) follow from Lemma 3.4 below. The invariant sets in all points can be chosen to be equal to the same $\widetilde{\Omega}$ up to taking intersections of invariant sets, which are still invariant. Point (4) and the last point of (3) follow the from the previous properties.

□

Lemma 3.4. *There exists a* $\vartheta-$*invariant set* $\widetilde{\Omega}\subseteq\Omega$ *of full* $\mathbb{P}-$*measure and an* $\mathcal{F}^{-}-$*adapted random variable* $u:\Omega\to E$ *such that:*

$$\varphi_n(\omega)u(\omega)=u(\vartheta^n\omega),\qquad\forall\omega\in\widetilde{\Omega},n\in\mathbb{N}.$$

Moreover for all $\omega\in\widetilde{\Omega}$*:*

$$\limsup_{n\to\infty}\left[\frac{1}{n}\sup_{f\in E}\Big(\log d_H(\varphi_n(\vartheta^{-n}\omega)\cdot f,u(\omega))\Big)\right]\leqslant\mathbb{E}\log\big(\tau(A)\big)\in[-\infty,0).$$

Proof. We start by observing (as in [26, Proof of Lemma 3.3]) that the sequence of sets $F_n(\omega)=\varphi_n(\vartheta^{-n}\omega)\cdot E$ is decreasing, i.e. $F_{n+1}\subseteq F_n$. Let us write $F(\omega)=\bigcap_{n\geqslant 1}F_n(\omega)$. Hence by Theorem 2.1:

$$\Delta(F)\leqslant\lim_{n\to\infty}\Delta(F_n)=\lim_{n\to\infty}4\ \mathrm{arctanh}\ (\tau(\varphi_n(\vartheta^{-n}\omega))).$$

Now, by the ergodic theorem and Assumption 3.1 there exists a $\vartheta-$invariant set $\widetilde{\Omega}$ of full $\mathbb{P}-$measure such that for all $\omega\in\widetilde{\Omega}$:

$$\lim_{n\to\infty}\frac{1}{n}\log\Big(\tau(\varphi_n(\vartheta^{-n}\omega))\Big)\leqslant\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\log\tau(A(\vartheta^{-i}\omega))=\mathbb{E}\log\big(\tau(A)\big)<0.\tag{6}$$

In particular $\Delta(F) = 0$. By completeness of E it follows that F is a singleton. Let us write $F(\omega) = \{u(\omega)\}$ and extend u trivially outside of $\widetilde{\Omega}$: it is clear that u is adapted to $\mathcal{F}^-$. Since for $k \in \mathbb{N}$ and $n \geqslant k$

$$\varphi_n(\vartheta^{-n}\vartheta^k\omega) = \varphi_k(\omega) \cdot \varphi_{n-k}(\vartheta^{-(n-k)}\omega),$$

passing to the limit with $n \to \infty$ we have: $u(\vartheta^k\omega) = \varphi_k(\omega)u(\omega)$.

Finally, as in the former result, a Taylor expansion guarantees that there exists a constant $c(\omega) > 0$ such that:

$$\Delta(\varphi_n(\vartheta^{-n}\omega) \cdot E) = 4 \text{ arctanh } (\tau(\varphi_n(\vartheta^{-n}\omega))) \leqslant 4(1 + c(\omega))\tau(\varphi_n(\vartheta^{-n}\omega)).$$

This estimate, combined with the fact that

$$\sup_{f \in E} d_H(\varphi_n(\vartheta^{-n}\omega) \cdot f, u(\omega)) = \sup_{f \in E} d_H(\varphi_n(\vartheta^{-n}) \cdot f, \varphi_n(\vartheta^{-n}\omega) \cdot u(\vartheta^{-n}\omega)) \leqslant \Delta(\varphi_n(\vartheta^{-n}\omega) \cdot E)$$

and (6) provides the required convergence result.

□

4. Application to SPDEs

In this section we discuss how to apply the previous results to stochastic PDEs. Concrete examples will be covered in the next section. For clarity, nonetheless, the reader should keep in mind that we want to study ergodic properties of solutions to Equation (1). Since the associated heat equation with multiplicate noise (2) is linear and the solution map is expected to be strictly positive, we may assume that the solution map generates a continuous random dynamical system φ.

In general it is not trivial to prove that a stochastic system generates a continuous-time random dynamical system over a *continuous* dynamical system ϑ (the problem is known as perfection, cf. [1, Section 1.3]). Since we want to use the solution theory for (2) with η space-time white noise as a black box, we want to assume only that we are guaranteed the existence of a solution outside of a null-set N_0 depending on the initial time $t = 0$. In this case the system satisfies only what is known as a *crude* cocycle property. To avoid the issue of continuous perfection we then restrict our attention to *continuous* RDS over *discrete* IDS ϑ, as described below.

Definition 4.1. *A* continuous *RDS over a* discrete *IDS* $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ *and on a measure space* $(X, \mathcal{B})$ *is a map*

$$\varphi\colon \mathbb{R}_+ \times \Omega \times X \to X$$

such that the following two properties hold:

(1) Measurability: φ *is* $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}-$ *measurable.*
(2) Cocycle property: $\varphi(0, \omega) = \mathrm{Id}_X$, *for all* $\omega \in \Omega$ *and:*

$$\varphi(t+n, \omega) = \varphi(t, \vartheta^n\omega) \circ \varphi(n, \omega), \qquad \forall t \in \mathbb{R}_+, n \in \mathbb{N}_0, \omega \in \Omega.$$

We then formulate the following assumptions, under which our main result will hold.

Assumption 4.2. *Let* $d \in \mathbb{N}$ *and* $\beta > 0$. *Let* $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P}, \vartheta)$ *be a discrete ergodic IDS, over which is defined a continuous RDS* φ*:*

$$\varphi : \mathbb{R}_+ \times \Omega_{\mathrm{kpz}} \to \mathcal{L}(C(\mathbb{T}^d)).$$

There exists a $\vartheta-$*invariant set* $\widetilde{\Omega} \subseteq \Omega_{\mathrm{kpz}}$ *of full* $\mathbb{P}-$*measure such that the following properties are satisfied for all* $\omega \in \widetilde{\Omega}$ *and any* $T > S > 0$*:*

(1) *There exists a kernel* $K : \Omega_{\mathrm{kpz}} \to C_{\mathrm{loc}}((0, \infty); C(\mathbb{T}^d \times \mathbb{T}^d))$ *such that for all* $S \leqslant t \leqslant T$*:*

$$\varphi_t(\omega)f(x) = \int_{\mathbb{T}^d} K(\omega, t, x, y)f(y)\,\mathrm{d}y, \qquad \forall f \in C(\mathbb{T}^d), x \in \mathbb{T}^d.$$

(2) *There exist* $0 < \gamma(\omega, S, T) \leqslant \delta(\omega, S, T)$ *such that:*

$$\gamma(\omega, S, T) \leqslant K(\omega, t, x, y) \leqslant \delta(\omega, S, T), \qquad \forall x, y \in \mathbb{T}^d, \ S \leqslant t \leqslant T,$$

which implies that $\mathbb{P}\big(\varphi_t \in \mathcal{L}_{\mathrm{cp}}(C(\mathbb{T}^d)), \forall t \in (0, \infty)\big) = 1$.

(3) *There exists a constant* $C(\beta, \omega, S, T)$ *such that:*

$$\|\varphi_t f\|_\beta \leqslant C(\beta, \omega, S, T)\|f\|_\infty, \qquad \forall f \in C(\mathbb{T}^d), \ S \leqslant t \leqslant T.$$

(4) *The following moment estimates are satisfied for any* $f \in C(\mathbb{T}^d)$*:*

$$\mathbb{E} \log \big(C(\beta, S, T)\big) + \mathbb{E} \sup_{S \leqslant t \leqslant T} d_H(\varphi_t \cdot f, f) < +\infty.$$

The first two assumptions essentially imply that we can use the results from the previous section. The last two assert how much the solution map is regularizing (that there is a regularizing effect is already encoded in the continuity of K). We now state the main result of this section. In view of the motivating example and in the setting of the previous assumption, we say that for $z \in \mathbb{Z}$ the map

$$[z, +\infty) \times \mathbb{T}^d \ni (t, x) \mapsto h^z(\omega, t, x), \qquad h^z(\omega, z, x) = h_0(x)$$

solves Equation (1) if $h^z(\omega, t) = \log\big(\varphi_t(\vartheta^z\omega)\exp(h_0)\big)$ for φ_t as in the previous assumption.

Theorem 4.3. *Under Assumption 4.2, for* $i = 1, 2$, $h_0^i \in C(\mathbb{T}^d)$ *and* $n \in \mathbb{N}$, *let* $h_i(t) \in C(\mathbb{T}^d)$ *be the random solution to Equation* (1) *started at time* 0 *with initial data* h_0^i *and evaluated at time* $t \geqslant 0$. *Similarly, let* $h_i^{-n}(t) \in C(\mathbb{T}^d)$ *be the solution started in* $-n$ *with initial data* h_0^i *and evaluated at time* $t \geqslant -n$. *There exists an invariant set* $\widetilde{\Omega} \subseteq \Omega_{\mathrm{kpz}}$ *of full* $\mathbb{P}-$*measure such that for any* $\alpha < \beta$, $\alpha \notin \mathbb{N}$, *for any* $T > 0$ *and any* $\omega \in \widetilde{\Omega}$*:*

(1) *There exists a map* $c(h_0^1, h_0^2) \colon \Omega_{\mathrm{kpz}} \times \mathbb{R}_+ \to \mathbb{R}$ *such that for any* $T > 0, \omega \in \widetilde{\Omega}$*:*

$$\limsup_{n\to\infty} \left[\frac{1}{n} \log\Big(\sup_{t \in [n, n+T]} \|h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)\|_{C^\alpha(\mathbb{T}^d)}\Big)\right] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big) < 0,$$

as well as:

$$\limsup_{n\to\infty} \left[\frac{1}{n} \log\Big(\sup_{t \in [n, n+T]} [h_i(\omega, t)]_\alpha\Big)\right] \leqslant 0.$$

And uniformly over h_0^i*:*

$$\limsup_{n\to\infty} \left[\frac{1}{n} \log\Big(\sup_{\substack{h_0^i \in C(\mathbb{T}^d),\\ t \in [n, n+T]}} \|h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)\|_\infty\Big)\right] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big) < 0,$$

(2) *There exists a random function* $h_\infty \colon \Omega_{\mathrm{kpz}} \to C_{\mathrm{loc}}((-\infty, \infty); C^\alpha(\mathbb{T}^d))$ *and a sequence of maps* $c^{-n}(h_0^1) \colon \Omega_{\mathrm{kpz}} \times \mathbb{R}_+ \to \mathbb{R}$ *for which:*

$$\limsup_{n\to\infty} \left[\frac{1}{n} \log\Big(\sup_{\substack{h_0^1 \in C(\mathbb{T}^d),\\ t \in [(-T) \vee (-n), T]}} \|h_1^{-n}(\omega, t) - h_\infty(\omega, t) - c^{-n}(\omega, t, h_0^1)\|_{C^\alpha(\mathbb{T}^d)}\Big)\right] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big) < 0.$$

Passing to the gradient one can omit all constants and find the following principles for Burgers' Equation (3).

Corollary 4.4. *In the same setting as before, it immediately follows that also:*

$$\limsup_{n\to\infty} \left[\frac{1}{n} \log\Big(\sup_{t \in [n, n+T]} \|\nabla_x h_1(\omega, t) - \nabla_x h_2(\omega, t)\|_{C^{\alpha-1}(\mathbb{T}^d)}\Big)\right] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big) < 0,$$

$$\limsup_{n\to\infty} \left[\frac{1}{n} \log\Big(\sup_{\substack{h_0^1 \in C(\mathbb{T}^d),\\ t \in [(-T) \vee (-n), T]}} \|\nabla_x h_1^{-n}(\omega, t) - \nabla_x h_\infty(\omega, t)\|_{C^{\alpha-1}(\mathbb{T}^d)}\Big)\right] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big) < 0,$$

where the space $C^{\alpha-1}(\mathbb{T}^d)$ is understood as the Besov space $B^{\alpha-1}_{\infty,\infty}(\mathbb{T}^d)$ for $\alpha \in (0,1)$.

Proof of Theorem 4.3. Step 1. Define:

$$u_0^i = \exp(h_0^i)/\|\exp(h_0^i)\|_{L^1} \in E,$$

so that $h_i(\omega,t) = \log\Big(\varphi_t(\omega)\cdot u_0^i\Big) + c_i(\omega,t)$, where $c_i(\omega,t) \in \mathbb{R}$ is the normalization constant:

$$c_i(\omega,t) = \log\bigg(\int_{\mathbb{T}^d}\Big(\varphi_t(\omega)u_0^i\Big)(x)\,\mathrm{d}x\bigg) + \log\bigg(\int_{\mathbb{T}^d}\exp(h_0^i)(x)\,\mathrm{d}x\bigg).$$

Let us write $c(\omega,t,h_0^1,h_0^2) = c_1(\omega,t) - c_2(\omega,t)$. Similarly, for $-n \leqslant t \leqslant 0$ one has:

$$h_i^{-n}(\omega,t) = \log\Big(\varphi_{n+t}(\vartheta^{-n}\omega)\cdot u_0^i\Big) + c_i^{-n}(\omega,t) = h_i(\vartheta^{-n}\omega, n+t),$$

where $c_i^{-n}(\omega,t) = c_i(\vartheta^{-n}\omega, n+t)$. Also, write $c^{-n}(\omega,t,h_0^1,h_0^2) = c_1^{-n}(\omega,t) - c_2^{-n}(\omega,t)$. Now we prove the following simpler version of the required result:

$$\limsup_{n\to\infty}\bigg[\frac{1}{n}\log\Big(\sup_{\substack{h_0^i\in C(\mathbb{T}^d),\\ t\in[n,n+T]}}\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\Big)\bigg] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big),$$

$$\limsup_{n\to\infty}\bigg[\frac{1}{n}\log\Big(\sup_{\substack{h_0^i\in C(\mathbb{T}^d)\\ t\in[(-T)\vee(-n),T]}}\|h_1^{-n}(\omega,t)-h_2^{-n}(\omega,t)-c^{-n}(\omega,t,h_0^1,h_0^2)\|_\infty\Big)\bigg] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big). \tag{7}$$

First we eliminate the time supremum, since in view of Inequality (5):

$$\begin{aligned}
&\limsup_{n\to\infty}\bigg[\frac{1}{n}\log\Big(\sup_{\substack{h_0^i\in C(\mathbb{T}^d)\\ t\in[n,n+T]}}\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\Big)\bigg]\\
&\leqslant \limsup_{n\to\infty}\bigg[\frac{1}{n}\log\Big(\sup_{\substack{h_0^i\in C(\mathbb{T}^d)\\ t\in[n,n+T]}} d_H(\varphi_t(\omega)\cdot u_0^1, \varphi_t(\omega)\cdot u_0^2)\Big)\bigg]\\
&\leqslant \limsup_{n\to\infty}\bigg[\frac{1}{n}\log\Big(\sup_{h_0^i\in C(\mathbb{T}^d)} d_H(\varphi_n(\omega)\cdot u_0^1, \varphi_n(\omega)\cdot u_0^2)\Big)\bigg]
\end{aligned}$$

where we estimated:

$$\sup_{t\in[n,n+T]} d_H(\varphi_t(\omega)\cdot u_0^1, \varphi_t(\omega)\cdot u_0^2) \leqslant \sup_{t\in[n,n+T]}\tau(\varphi_{t-n}(\vartheta^n\omega))\cdot d_H(\varphi_n(\omega)\cdot u_0^1, \varphi_n(\omega)\cdot u_0^2)$$

and $\tau(\varphi_{t-n}(\vartheta^n\omega)) \leqslant 1$. Similarly, also for the backwards case. At this point, in view of Assumption 4.2, we can apply Theorem 3.2 in the setting of Lemma 2.2 with $A(\omega) = \varphi_1(\omega)$ to see that there exists a $u_\infty = \exp(h_\infty)\colon \Omega_{\mathrm{kpz}} \to C(\mathbb{T}^d)$ such that:

$$\limsup_{n\to\infty}\bigg[\frac{1}{n}\log\Big(\sup_{u_0^i\in E} d_H(\varphi_n(\omega)\cdot u_0^1, \varphi_n(\omega)\cdot u_0^2)\Big)\bigg] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big),$$

$$\limsup_{n\to\infty}\bigg[\frac{1}{n}\log\Big(\sup_{u_0^1\in E} d_H(\varphi_n(\vartheta^{-n}\omega)\cdot u_0^1, u_\infty(\omega))\Big)\bigg] \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big),$$

which via the previous calculation implies (7). In particular, this also proves the bound uniformly over h_0^i at point (1) of the theorem.

Step 2. We pass to prove convergence in $C^\alpha(\mathbb{T}^d)$ for $\alpha < \beta, \alpha \notin \mathbb{N}$. Since the same arguments extend to the more general case, assume that $\beta \in (0,1)$. Thus fix α and define $\theta \in (0,1)$ by $\alpha = \beta\theta$. As convergence in $\|\cdot\|_\infty$ is already established, to prove convergence in $C^\alpha(\mathbb{T}^d)$ one has to control the $\alpha-$seminorm $[\cdot]_\alpha$. We treat the forwards and backwards in time cases differently, starting with the first case. One can bound the Hölder seminorm via:

$$\begin{aligned}[h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)]_\alpha \leqslant \Big(2\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\Big)^{1-\theta}\cdot \\ \cdot\Big([\log(\varphi_t(\omega)\cdot u_0^1)]_\beta+[\log(\varphi_t(\omega)\cdot u_0^2)]_\beta\Big)^\theta.\end{aligned} \tag{8}$$

Then fix n, T and $t\in[n,n+T]$, denote $t=n-1+\tau$ and rewrite the last terms as:

$$\begin{aligned}[h_1(\omega,t)]_\beta &= [\log(\varphi_t(\omega)\cdot u_0^1)]_\beta \\ &\leqslant \frac{1}{m(\varphi_t(\omega)\cdot u_0^1)}[\varphi_t(\omega)\cdot u_0^1]_\beta \\ &= \frac{1}{m(\varphi_t(\omega)\cdot u_0^1)}[\varphi_\tau(\vartheta^{n-1}\omega)\varphi_{n-1}(\omega)\cdot u_0^1]_\beta \\ &\leqslant \frac{C(\beta,\vartheta^{n-1}\omega,1,T+1)}{m(\varphi_t(\omega)\cdot u_0^1)}\|\varphi_{n-1}(\omega)\cdot u_0^1\|_\infty\end{aligned}$$

where $m(\cdot)$ indicates the minimum of a function. Then estimate:

$$\begin{aligned}\log\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\alpha \leqslant &(1-\theta)\log 2\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty \\ &+\theta\log\Big(\sum_{i=1,2}\frac{C(\beta,\vartheta^{n-1}\omega,1,T+1)}{m(\varphi_t(\omega)\cdot u_0^i)}\|\varphi_{n-1}(\omega)\cdot u_0^i\|_\infty\Big).\end{aligned}$$

To conclude, in view of the result from the previous step, we have to prove that:

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\left(\sup_{n\leqslant t\leqslant n+T}\Big(\sum_{i=1,2}\frac{C(\beta,\vartheta^{n-1}\omega,1,T+1)}{m(\varphi_t(\omega)\cdot u_0^i)}\|\varphi_{n-1}(\omega)\cdot u_0^i\|_\infty\Big)\right)\right]\leqslant 0.$$

In particular, the latter inequality also implies the $\alpha-$Hölder norm of h_i at point (1) of the theorem. By the means of considerations on the line of $\log\max_i x_i = \max_i \log x_i$ it is sufficient to prove that for any $f\in C(\mathbb{T}^d)$:

$$\begin{aligned}\limsup_{n\to\infty}\Bigg[\frac{1}{n}\sup_{n\leqslant t\leqslant n+T}\Bigg(&\log\Big(C(\beta,\vartheta^{n-1}\omega,1,T+1)\Big) \\ &+\log\Big(\|\varphi_{n-1}(\omega)\cdot f\|_\infty\Big)-\log\Big(m\big(\varphi_t(\omega)\cdot f\big)\Big)\Bigg)\Bigg]\leqslant 0,\end{aligned}$$

which is once more equivalent to the following:

$$\limsup_{n\to\infty}\frac{1}{n}\left[\log\Big(C(\beta,\vartheta^{n-1}\omega,1,T+1)\Big)+\sup_{1\leqslant\tau\leqslant T+1}d_H(\varphi_{n-1+\tau}(\omega)f,f)\right]\leqslant 0. \tag{9}$$

Let us start with the last term and bound:

$$\begin{aligned}&d_H(\varphi_{n-1+\tau}(\omega)\cdot f,f) \\ &\quad\leqslant \tau(\varphi_\tau(\vartheta^{n-1}\omega))d_H(\varphi_{n-1}(\omega)\cdot f,f)+d_H(\varphi_\tau(\vartheta^{n-1}\omega)\cdot f,f) \\ &\quad\leqslant \sum_{i=0}^{n-1}\prod_{j=i+1}^{n-1}\tau(\varphi_1(\vartheta^j\omega))d_H(\varphi_1(\vartheta^i\omega)\cdot f,f)+\sup_{1\leqslant\tau\leqslant T+1}d_H(\varphi_\tau(\vartheta^{n-1}\omega)\cdot f,f).\end{aligned}$$

By Assumption 4.2 $\sup_{1\leqslant\tau\leqslant T+1}d_H(\varphi_\tau(\omega)\cdot f,f)\in L^1$, hence:

$$\lim_{n\to\infty}\frac{1}{n}\sup_{1\leqslant\tau\leqslant T+1}d_H(\varphi_\tau(\vartheta^{n-1}\omega)\cdot f,f)=0$$

by the ergodic theorem (since the series $\sum_n \frac{1}{n} \sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi_\tau(\vartheta^{n-1}\omega)\cdot f, f)$ is summable). Now observe that by Lebesgue dominated convergence as well as the ergodic theorem, since $d_H(\varphi_1(\omega)\cdot f, f) \in L^1$, it holds that:

$$\lim_{c\to\infty} \mathbb{E}\bigg[\prod_{j=1}^{c} \tau(\varphi_1(\vartheta^j\omega)) d_H(\varphi_1(\omega)\cdot f, f)\bigg] = 0.$$

Hence fix $\varepsilon > 0$ and choose $c \in \mathbb{N}$ so that the average above is bounded by ε. We can then estimate once more via the ergodic theorem:

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} \sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi_{n-1+\tau}(\omega)\cdot f, f)\\ &\qquad\leqslant \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1-c} \prod_{j=i+1}^{i+c} \tau(\varphi_1(\vartheta^j\omega)) d_H(\varphi_1(\vartheta^i\omega)\cdot f, f) \leqslant \varepsilon.\end{aligned}$$

As ε is arbitrarily small, this delivers the required result. To deduce (9) we are left with the term containing $C(\beta, \vartheta^n\omega)$. Once more the ergodic theorem and Assumption 4.2 imply that one has:

$$\lim_{n\to\infty} \frac{1}{n} \log C(\beta, \vartheta^n\omega, 1, T+1) = 0.$$

This concludes the proof of (9).

We pass to the bound backwards in time. Up to replacing T with $\lceil T\rceil$ assume $T \in \mathbb{N}$. Then, for $T < n-1$ and $-T \leqslant t \leqslant T$ so that $t = -T-1+\tau$ with $1 \leqslant \tau \leqslant 2T+1$:

$$\begin{aligned}&[h_1^{-n}(\omega,t) - h_\infty(\omega,t) - c^{-n}(\omega,t,h_0^1,h_0^2)]_\alpha =\\ &\quad= \Big[\log\Big(\varphi_\tau(\vartheta^{-T-1}\omega)\big(\varphi_{n-T-1}(\vartheta^{-n}\omega)\cdot u_0^1\big)\Big) - \log\Big(\varphi_\tau(\vartheta^{-T-1}\omega)\big(u_\infty(\omega,-T-1)\big)\Big)\Big]_\alpha.\end{aligned}$$

Now since $\varphi_{n-T-1}(\vartheta^{-n}\omega)\cdot u_0^1 \to u_\infty(\omega,-T-1)$ in $C(\mathbb{T}^d)$ uniformly over u_0^1 there exists a $c > 0$ such that $\big(\varphi_n(\omega)\cdot u_0^1\big)(x) \geqslant c,\ \forall x \in \mathbb{T}^d, n \in \mathbb{N}, u_0^1 \in C(\mathbb{T}^d)$. By Assumption 4.2 this implies that:

$$\inf_{u_0^1\in C(\mathbb{T}^d)} \inf_{n>T+1} \inf_{\substack{1\leqslant\tau\leqslant T+1\\ x\in\mathbb{T}^d}} \varphi_\tau(\vartheta^{-T}\omega)\big(\varphi_{n-1}(\vartheta^{-n}\omega)\cdot u_0^1\big)(x) \geqslant c\gamma(\vartheta^{-T}\omega, 1, 2T+1).$$

With the above estimate we can follow the interpolation bound (8) and a simpler version of the steps that follow (indeed, we do not need to apply the ergodic theorem in this case) to conclude the proof.

□

5. Examples

We treat two prototypical examples, which show the range of applicability of the previous results. First, we consider the KPZ equation driven by a noise that is fractional in time but smooth in space. In a second example, we consider the KPZ equation driven by space-time white noise.

5.1. **KPZ driven by fractional noise.** Fix a Hurst parameter $H \in \big(\frac{1}{2}, 1\big)$. The reason for such choice is that the case $H = \frac{1}{2}$ is identical to the setting in [33], while for $H < \frac{1}{2}$ one encounters difficulties with fractional stochastic calculus that lie beyond the scopes of this work. The noise under consideration is given by $\eta(t,x) = \xi^H(t)V(x)$ for some $V \in C^\infty(\mathbb{T})$ and $\xi^H(t) = \partial_t\beta^H(t)$ for a fractional Brownian motion of Hurst parameter H. For convenience, we rigorously define the noise via its spectral covariance function, see

[31, Section 3], namely ξ^H is a Gaussian process indexed by functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |\tau|^{1-2H} |\hat{f}(\tau)|^2 \, \mathrm{d}\tau < \infty$ (with $\hat{f}$ being the temporal Fourier transform) with covariance:

$$\mathbb{E}\Big[\xi^H(f)\xi^H(g)\Big] = c_H \int_{\mathbb{R}} |\tau|^{1-2H} \hat{f}(\tau)\overline{\hat{g}(\tau)} \, \mathrm{d}\tau, \qquad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{10}$$

Lemma 5.1. *Fix any $\alpha < H - 1, a > \frac{1}{2}$. Consider ξ^H a Gaussian process as in* (10). *Almost surely, ξ^H takes values in $H^\alpha_a(\mathbb{R})$. Then, define $\Omega_{\mathrm{kpz}} = H^\alpha_a(\mathbb{R})$ and $\mathcal{F} = \mathcal{B}(H^\alpha_a(\mathbb{R}))$ and let $\mathbb{P}$ be the law of ξ^H in Ω_{kpz}. The space $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P})$ is extended to an* ergodic *IDS via the integer translation group (in the sense of distributions) $\{\vartheta^z\}_{z\in\mathbb{Z}}$, which acts by:*

$$\vartheta^z\omega(t) = \omega(t+z), \qquad \forall \omega \in \Omega_{\mathrm{kpz}}, \quad t \in \mathbb{R}.$$

Consider hence ξ^H as the canonical process on Ω_{kpz} under $\mathbb{P}$. There exists a $\vartheta-$invariant null-set N_0 such that for any $\omega \notin N_0$ there exists a $\beta^H(\omega) \in C^{\alpha+1}_{\mathrm{loc}}(\mathbb{R})$ with:

$$\xi^H(\omega) = \partial_t \beta^H(\omega) \ \ \textit{in the sense of distributions}, \qquad \beta^H_0(\omega) = 0.$$

Moreover, $(\beta^H_t)_{t\geqslant 0}$ has the law of a fractional Brownian motion of parameter H.

Proof. To show that ξ^H takes values in H^α_a almost surely, calculate:

$$\begin{aligned} \mathbb{E}\Big[\|\Delta_j \xi^H(\cdot)/\langle\cdot\rangle^a\|^2_{L^2}\Big] &= \int_{\mathbb{R}} \frac{1}{(1+|t|)^{2a}} \mathbb{E}\big[|\Delta_j\xi^H(t)|^2\big] \, \mathrm{d}t \lesssim_a \sup_{t\in\mathbb{R}} \mathbb{E}\big[|\Delta_j\xi^H(t)|^2\big] \\ &= c_H \int_{\mathbb{R}} |\tau|^{1-2H} \varrho_j^2(\tau) \, \mathrm{d}\tau \lesssim 2^{j2(1-H)}, \end{aligned}$$

where in the first line we used that $2a > 1$. In the second line, we used that for $j \geqslant 0$ $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ for a function ϱ with support in an annulus. This provides the required regularity estimate.

The ergodicity is a consequence of the criterion in Proposition 5.12. In fact the condition on the covariances is satisfied, as we have for $\varphi, \varphi' \in C^\infty_c(\mathbb{R})$:

$$\begin{aligned} \mathrm{Cov}(\langle\xi^H, \varphi\rangle, \langle\vartheta^n\xi^H, \varphi'\rangle) &\simeq \int_{\mathbb{R}} |\tau|^{1-2H} e^{in\tau} \hat{\varphi}(\tau)\hat{\varphi}'(\tau) \, \mathrm{d}\tau \\ &= (\psi^H * \varphi * \varphi')(n) \to 0, \qquad \text{for } n \to \infty. \end{aligned}$$

with $\psi^H(t) = (\mathcal{F}^{-1}[|\cdot|^{1-2H}])(t)$.

Now, one can define the primitive $\beta^H(\omega)$ through the formula:

$$\langle\beta^H(\omega), \varphi\rangle = \langle\xi^H(\omega), \int_\cdot^\infty \varphi(r) \, \mathrm{d}r\rangle, \qquad \forall \varphi \in C^\infty_c(\mathbb{R}),$$

so that automatically in the sense of distributions $\xi^H(\omega) = \partial_t\beta^H(\omega)$. For fixed $t \geqslant 0$, taking an approximate Dirac δ sequence $\varphi_n \in C^\infty_c, \varphi_n \to \delta_t$ one then obtains:

$$\beta^H_t = \xi^H(1_{[0,t]}), \ \text{ in } \ L^2(\mathbb{P}),$$

so that following [31, Section 3] β^H has the law of a fractional Brownian motion. In particular, almost surely, the process $\beta^H_t(\omega)$ has the required regularity. The null-set $\overline{N}_0$ on which the result does not hold can be chosen to be $\vartheta-$invariant, by defining $N_0 = \bigcup_{z\in\mathbb{Z}} \vartheta^z \overline{N}_0$.

□

The next step is to show well-posedness of the SPDE:

$$(\partial_t - \partial_x^2)u(t,x) = \xi^H(t)V(x)u(t,x), \qquad u(0,x) = u_0(x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}. \tag{11}$$

We consider mild solutions to this equation, namely u such that:

$$u(\omega,t,x) = P_t u_0 + \int_0^t P_{t-s}[u(\omega,s,\cdot)V(\cdot)](x)\xi^H(\omega, \mathrm{d}s),$$

where P_t is the periodic heat semigroup: $P_t f(x) = \sum_{z\in\mathbb{Z}^d}(4\pi t)^{-\frac{d}{2}}\int_{\mathbb{T}^d} f(y)e^{-\frac{|x-y-z|^2}{4t}}\,\mathrm{d}y$. In particular, we can prove the following result:

Lemma 5.2. *Consider $H > \frac{1}{2}$. For almost all $\omega \in \Omega_{\mathrm{kpz}}$, for every $u_0 \in C(\mathbb{T})$ there exists a unique mild solution u to Equation (11) such that for any $\alpha < H, k \in \mathbb{N}, 0 < S < T < \infty$:*

$$(t,x) \mapsto \partial_x^k u(\omega,t,x) \in C^\alpha([S,T]\times\mathbb{T}).$$

There exists a continuous RDS φ taking values in $\mathcal{L}(C(\mathbb{T}))$ such that:

$$\mathbb{P}\Big(\varphi_t u_0 = u_t \quad in \quad C(\mathbb{T}), \quad \forall t \geqslant 0, u_0 \in C(\mathbb{T})\Big) = 1.$$

Proof. Fix ω outside the null-set N_0 of Lemma 5.1. To solve the equation above, consider the solution $X(\omega,t,x)$ to:

$$(\partial_t - \partial_x^2)X(\omega,t,x) = \xi^H(\omega,t)V(x), \qquad X(\omega,0,x) = 0, \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{T}.$$

Note that $(s,x) \mapsto P_{t-s}V(x) \in C^\infty([0,t]\times\mathbb{T})$, since V is smooth. Hence we can write X explicitly by integration by parts:

$$\begin{aligned} X(t,x) &= \int_0^t P_{t-s}V(x)\xi^H(\,\mathrm{d}s) \\ &= \int_0^t \beta_s^H(P_{t-s}\partial_x^2 V)(x)\,\mathrm{d}s + V(x)\beta_t^H - (P_t V)(x)\beta_0^H. \end{aligned} \tag{12}$$

Taking spatial derivatives in the above representation and by the previous lemma, one has that:

$$(t,x) \mapsto \partial_x^k[X(\omega,t,x) - V(x)\beta_t^H(\omega)] \in C^1([0,T]\times\mathbb{T})$$

for any $T > 0, k \in \mathbb{N}_0$ (the latter is the spaces of differentiable functions on $(0,T)\times\mathbb{T}$, with uniformly bounded derivatives). Now consider w the solution to:

$$\begin{aligned} (\partial_t - \partial_x^2)w(\omega,t,x) &= 2\partial_x X(\omega,t,x)\partial_x w(\omega,t,x) + (\partial_x X)^2(\omega,t,x)w(\omega,t,x), \\ w(\omega,0,x) &= u_0(x). \end{aligned}$$

As a consequence of Lemma A.1 there exists a unique mild solution to this equation and the same result implies that the solution w satisfies:

$$(t,x) \mapsto \partial_x^k w(\omega,t,x) \in C^1((0,T)\times\mathbb{T}),$$

for any $T > 0, k \in \mathbb{N}_0$. Here $C^1((0,T)\times\mathbb{T})$ is the space of differentiable functions with derivatives bounded on any compact subset of $(0,T)\times\mathbb{T}$.

Now, define u as $u = e^X w$. Since $H > \frac{1}{2}$ an application of the chain rule (that is, the Itô formula for β^H) guarantees that u is a mild solution to Equation (11), and conversely every solution can be represented in this way.

Finally, Lemma A.1 also implies that the solution map is, for fixed $t \geqslant 0$ an element of $\mathcal{L}(C(\mathbb{T}))$. Since we solved the equation outside of a $\vartheta-$invariant null-set, one can define the RDS φ as required by defining it as the identity on N_0. □

We can now prove that Equation (11) falls in the framework of the theory in the previous sections.

Proposition 5.3. *The RDS φ introduced in Lemma 5.2 satisfies, for any $\beta > 0$, Assumption 4.2. In particular, for almost all $\omega \in \Omega_{\mathrm{kpz}}$, for any $u_0 \in C(\mathbb{T}), u_0 > 0$, the function $t \mapsto \log(\varphi_t(\omega)u_0) =: h_t(\omega)$ is the unique function that satisfies for any $\alpha < H, k \in \mathbb{N}, 0 < S < T < \infty$:*

$$(t,x) \mapsto \partial_x^k h(t,x) \in C^\alpha((S,T)\times\mathbb{T})$$

and is additionally a mild solution to:

$$(\partial_t - \partial_x^2)h(t,x) = (\partial_x h(t,x))^2 + V(x)\xi^H(t), \qquad h(0,x) = \log(u_0(x)). \tag{13}$$

Such solution satisfies all the results of Theorem 4.3.

Proof. Fix $\omega \in \Omega_{\mathrm{kpz}} \bigcap N_0^c$, the latter being the same null-set as in Lemma 5.1. The first step is to prove that for such ω, points 1, 2 and 3 of Assumption 4.2 are satisfied. Let us start with the kernel representation. Formally, one can write:

$$K(t,x,y) = \varphi_t(\delta_y)(x). \tag{14}$$

This can be made rigorous, if one can start Equation (11) in δ_y. In Lemma A.2 we show that that for any $\gamma > 0$, $\{\delta_y\}_{y\in\mathbb{T}} \subseteq B^{-\gamma}_{1,\infty}$, and $\|\delta_x - \delta_y\|_{B^{-\gamma}_{1,\infty}} \lesssim |x-y|^\gamma$. In addition, following the proof of Lemma 5.2, the solution $\varphi_t u_0 = e^{X_t} w_t$, where the latter is the solution to:

$$(\partial_t - \partial_x^2) w = 2\partial_x X \partial_x w + (\partial_x X)^2 w, \qquad w(0) = u_0.$$

Since the coefficients $(\partial_x X)^2$ and $\partial_x X$ are smooth, Lemma A.1 implies that the equation for w can be started also in $u_0 = \delta_y$. Hence K in Equation (14) is rigorously defined as $K(t,x,y) = e^{X_t} w_t$, where the latter is started in $w(0) = \delta_y$. In particular, the regularity result and the continuous dependence on the initial condition in Lemma A.2 imply that for any $t > 0, K(t) \in C(\mathbb{T}\times\mathbb{T})$. That K is a fundamental solution for the PDE follows by linearity, thus concluding the proof of (1) in Assumption 4.2. The fact that K is strictly positive, as required in point (2) of the assumptions is instead the consequence of a strong maximum principle (cf. [27, Theorem 2.7]). The smoothing effect of point (3) in Assumption 4.2 follows again from the representation $\varphi_t u_0 = e^{X_t} w_t$ and spatial smoothness of both X and w we already showed in the proof of Lemma 5.2. In particular, the smoothing effect can be made quantitative, via the estimate of Lemma A.1, to obtain that for $0 < S < T < \infty$ there exists constants $C(S,T), q \geqslant 0$ such that for $\omega \in N_0^c$:

$$\sup_{S\leqslant t\leqslant T} \|\varphi_t(\omega)u_0\|_{C^\beta} \leqslant \|u_0\|_\infty e^{C(S,T)(1+\sum_{k=0}^{\lfloor\beta\rfloor+1} \sup_{0\leqslant t\leqslant T}\|\partial_x^k X_t(\omega)\|_\infty)^q}.$$

Note that at first Lemma A.1 allows to regularize at most by $\beta < 2$, but splitting the interval $[0,S]$ into small pieces and applying iteratively the result on every piece provides the result for arbitrary β. Now observe that in view of Equation (12):

$$\sum_{k=1}^{\lfloor\beta\rfloor+1} \mathbb{E} \sup_{0\leqslant t\leqslant T} \|\partial_x^k X_t\|_\infty^q < \infty,$$

thus proving the first average bound of point (4) in Assumption 4.2. As for the second bound, we have to take more care. In view of Lemma 2.2, one has:

$$\begin{aligned} d_H(\varphi_t(\omega)\cdot f, f) &\lesssim \|\log(\varphi_t(\omega)f) - \log\int_{\mathbb{T}} (\varphi_t(\omega)f)(x)\,\mathrm{d}x\|_\infty + \|\log f - \log\int_{\mathbb{T}} f(x)\,\mathrm{d}x\|_\infty \\ &\lesssim \|\log(\varphi_t(\omega)f)\|_\infty + \|\log f\|_\infty, \end{aligned}$$

so that our aim is to bound

$$\mathbb{E}\sup_{S\leqslant t\leqslant T} \|\log(\varphi_t f)\|_\infty.$$

On one side, one has the trivial upper bound:

$$\log(\varphi_t(\omega)f) \leqslant \log\|\varphi_t(\omega)f\|_\infty \lesssim_{S,T} \log\|f\|_\infty + \Big(1 + \sum_{k=0}^{1} \sup_{0\leqslant t\leqslant T} \|\partial_x^k X_t(\omega)\|_\infty\Big)^q,$$

which is integrable. As for the lower bound, observe that $\log(\varphi_t(\omega)f) = X_t(\omega) + \log w_t(\omega)$. One can check that $v_t(\omega) = \log w_t(\omega)$ is a mild solution to the equation:

$$(\partial_t - \partial_x^2) v = 2(\partial_x X)\partial_x v + (\partial_x X)^2 + (\partial_x v)^2, \qquad v(0) = \log f. \tag{15}$$

By comparison (cf. [27, Theorem 2.7]), one has: $v(t,x) \geqslant -\|f\|_\infty, \forall t \geqslant 0, x \in \mathbb{T}$. So assuming that $q \geqslant 1$, one has overall:

$$\|\log(\varphi_t(\omega)f)\|_\infty \lesssim \log\|f\|_\infty + \Big(1 + \sum_{k=0}^{1} \sup_{0 \leqslant t \leqslant T} \|\partial_x^k X_t(\omega)\|_\infty\Big)^q,$$

which is once again integrable. Hence the required assumptions are satisfied and we can apply Theorem 4.3.

Finally, that h_t satisfies the smootheness assumption and is a mild solution to the KPZ equation driven by fractional noise is a consequence of the regularity of $\varphi_t(\omega)u_0$ and Equation (15).

□

Remark 5.4. *In the same setting as in Proposition 5.3, for any $h_0^1, h_0^2 \in C(\mathbb{T})$ the constant $c(\omega, t, h_0^1, h_0^2)$ in the Theorem 4.3 can be chosen independent of time. Similarly, $h_\infty(\omega)$ is the unique solution to Equation* (13) *for all times $t \in \mathbb{R}$, up to a constant (random, but time-independent) shift.*

Proof. Observe that it is sufficient to prove that there exists a constant $\bar{c}(\omega, h_0^1, h_0^2)$ such that for every $\omega \in \widetilde{\Omega}$ (for an invariant set $\widetilde{\Omega}$ of full $\mathbb{P}-$measure) and any $T > 0$:

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{t\in[n,n+T]} |c(\omega,t,h_0^1,h_0^2) - \bar{c}(\omega,h_0^1,h_0^2)| \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big).$$

As a simple consequence of Theorem 4.3 one has:

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{t\in[n,n+T]} \log\|\Pi_\times(h_1(\omega,t) - h_2(\omega,t))\|_\alpha \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big),$$

for any $\alpha > 0$, where $\Pi_\times$ is defined for $f \in C(\mathbb{T})$ as $\Pi_\times f = f - \int_{\mathbb{T}} f(x)\,\mathrm{d}x$, and without loss of generality one can choose the constants to be defined as:

$$c(\omega,t,h_0^1,h_0^2) = \int_{\mathbb{T}} h_1(\omega,t,x) - h_2(\omega,t,x)\,\mathrm{d}x.$$

Since by Proposition 5.3, h_i is a solution to the KPZ Equation one has that:

$$\partial_t c(\omega,t,h_0^1,h_0^2) = \int_{\mathbb{T}} \partial_x(h_1 - h_2)\partial_x(h_1 + h_2)(\omega,t,x)\,\mathrm{d}x. \tag{16}$$

Now, in view of (1) of Theorem 4.3 one has:

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{t\in[n,n+1]} \log|\partial_t c(\omega,t,h_0^1,h_0^2)| \leqslant \mathbb{E}\log\big(\tau(\varphi_1)\big).$$

In particular this implies that there exists a constant $\bar{c}(\omega,h_0^1,h_0^2) := \lim_{t\to\infty} c(\omega,t,h_0^1,h_0^2)$ and in addition

$$|\bar{c}(\omega,h_0^1,h_0^2) - c(\omega,t,h_0^1,h_0^2)| \leqslant \int_t^\infty |\partial_s c(\omega,s,h_0^1,h_0^2)|\,\mathrm{d}s \lesssim e^{-ct},$$

for any $0 < c < -\mathbb{E}\log\big(\tau(\varphi_1)\big)$, which proves the required result.

We pass to the last statement. Let h_∞ be the solution to the KPZ equation for $t \in \mathbb{R}$ obtained from Theorem 4.3. Suppose $\overline{h}_\infty$ is a second solution. The uniform dependence on h_0^1 in point (2) of Theorem 4.3 guarantees that there exists $c_\infty(\omega,t)$ such that $h_\infty(\omega,t,x) - \overline{h}_\infty(\omega,t,x) = c_\infty(\omega,t)$. Since $t \mapsto c_\infty(\omega,t)$ solves the same ODE as above with h_1, h_2 replaced by $h_\infty, \overline{h}_\infty$ we have that $c(\omega,t) \equiv c(\omega)$ is time-independent.

□

5.2. **KPZ driven by space-time white noise.** In this section we consider the random force η in (1) to be space-time white noise ξ in one spatial dimension. That is, a Gaussian processes indexed by functions in $L^2(\mathbb{R}\times\mathbb{T})$ such that:

$$\mathbb{E}\Big[\xi(f)\xi(g)\Big] = \int_{\mathbb{R}\times\mathbb{T}} f(t,x)g(t,x)\,\mathrm{d}t\,\mathrm{d}x. \tag{17}$$

Lemma 5.5. *Fix any $\alpha < -1, a > \frac{1}{2}$. Consider ξ a Gaussian process as in* (17)*. Almost surely, ξ takes values in $H^\alpha_a(\mathbb{R}\times\mathbb{T})$. Then, define $\Omega_{\mathrm{kpz}} = H^\alpha_a(\mathbb{R}\times\mathbb{T}), \mathcal{F} = \mathcal{B}(H^\alpha_a(\mathbb{R}\times\mathbb{T}))$ and let $\mathbb{P}$ be the law of ξ on Ω_{kpz}. The space $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P})$ is extended to an ergodic IDS via the integer translation group (in the sense of distributions) $\{\vartheta^z\}_{z\in\mathbb{Z}}$, which acts by:*

$$\vartheta^z\omega(t,x) = \omega(t+z,x), \qquad \forall\omega\in\Omega_{\mathrm{kpz}}, \quad (t,x)\in\mathbb{R}\times\mathbb{T}.$$

We will denote with ξ the canonical process on Ω_{kpz} under $\mathbb{P}$.

Proof. We start by showing that ξ takes values in $H^\alpha_a(\mathbb{R}\times\mathbb{T})$ almost surely. Once can calculate:

$$\mathbb{E}\|\xi\|^2_{H^\alpha_a} = \sum_{j\geqslant -1} 2^{2\alpha j}\mathbb{E}\|\Delta_j\xi(\cdot)/\langle\cdot\rangle^a\|_{L^2},$$

and for the latter one has:

$$\begin{aligned}\mathbb{E}\Big[\|\Delta_j\xi(\cdot)/\langle\cdot\rangle^a\|^2_{L^2}\Big] &= \int_{\mathbb{R}\times\mathbb{T}} \frac{1}{(1+|t|)^{2a}}\mathbb{E}\big[|\Delta_j\xi(t,x)|^2\big]\,\mathrm{d}t\,\mathrm{d}x \lesssim_a \sup_{(t,x)\in\mathbb{R}\times\mathbb{T}} \mathbb{E}\big[|\Delta_j\xi(t,x)|^2\big] \\ &= \int_{\mathbb{R}\times\mathbb{T}} |\mathcal{F}^{-1}_{\mathbb{R}\times\mathbb{T}}\varrho_j|^2(t,x)\,\mathrm{d}t\,\mathrm{d}x \simeq \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \varrho_j^2(k,\tau)\,\mathrm{d}\tau \lesssim 2^{2j},\end{aligned}$$

where we used that $2a > 1$ and that for $j \geqslant 0$ $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ for a function ϱ with support in an annulus. We are left with proving the ergodicity. Here we apply Proposition 5.12, so we have to check the asymptotic decorrelation. For $\varphi, \varphi' \in C^\infty_c(\mathbb{R}\times\mathbb{T})$:

$$\mathrm{Cov}(\langle\xi,\varphi\rangle, \langle\vartheta^n\xi,\varphi'\rangle) = \int_{\mathbb{R}\times\mathbb{T}} \varphi(t,x)\varphi'(t-n,x)\,\mathrm{d}t\,\mathrm{d}x = 0,$$

for n sufficiently large, depending on the supports of the functions φ, φ'. This concludes the proof. □

Now we will consider h, u the respective solutions to the KPZ and stochastic heat equation driven by space-time white noise:

$$(\partial_t - \partial_x^2)h = (\partial_x h)^2 + \xi - \infty, \qquad h(0,x) = h_0(x), \qquad (t,x)\in\mathbb{R}_+\times\mathbb{T}, \tag{18}$$

$$(\partial_t - \partial_x^2)u = u(\xi - \infty), \qquad u(0,x) = u_0(x), \qquad (t,x)\in\mathbb{R}_+\times\mathbb{T}. \tag{19}$$

Here the presence of the infinity "∞" indicates the necessity of renormalization to make sense of the solution. Well-posedness of the stochastic heat equation (19) can be proven also with martingale techniques (which do not provide a solution theory for the KPZ equation, though). Instead, here we make use of path-whise approaches to solving the above equations [23, 24, 20], that require tools such as regularity structures or paracontrolled distributions. Such theories consider smooth approximations ξ_ε of the noise ξ, for which the equations are well-posed, and study the convergence of the solutions as $\varepsilon \to 0$. The renormalization can then be understood also as a Stratonovich-Itô correction term.

Remark 5.6 (On the path-whise solution theory for (18) and (19))**.** *The solution u to Equation* (19) *is described by an expansion similar to the one in the proof of Lemma 5.2, namely given Y that solves $(\partial_t - \partial_x^2)Y = \xi$, the solution can be represented by $u = e^Y w$, where w solves:*

$$(\partial_t - \partial_x^2)w = 2\partial_x Y\partial_x w + ((\partial_x Y)^2 - \infty)w, \qquad w(0) = u_0.$$

Now the "∞" should be interpreted as the projection on the second Wiener-Itô chaos of the square $(\partial_x Y_t)^2$ *(note that* Y_t *lies in* $H^{1-\varepsilon}(\mathbb{T})$ *for any* $\varepsilon > 0$, *so* $(\partial_x Y)^2$ *is ill-defined). Yet this expansion is still not sufficient. Eventually, one will decompose* $u = e^{Y+Y^{\vee}+Y^{\vee\!\!\vee}} w^P$ *with* $(\partial_t - \partial_x^2)Y^{\vee} = (\partial_x Y)^2 - \infty$ *and* $(\partial_t - \partial_x^2)Y^{\vee\!\!\vee} = 2\partial_x Y \partial_x Y^{\vee}$ *(all these terms are defined via stochastic estimates since the products that appear are a-priori ill-posed and together with other similar terms compose the enhanced noise* $\mathbb{Y}$, *cf.* [21, Definition 4.1]*). The final equation for* w^P *remains ill-posed in terms of regularity, and an additional (para-)controlled structure is required to make sense of it (see e.g. the discussion in* [21, Chapter 3]*, where also the reason behind the tree notation is explained).*

Similarly, for Equation (18) *the solution has the form* $h = Y + Y^{\vee} + Y^{\vee\!\!\vee} + h^P$, *with* $h^P = \log w^P$ *being controlled in an appropriate sense.*

We shall use this theory as a black box, via the following lemma.

Lemma 5.7. *Consider* $(\Omega_{\mathrm{kpz}}, \mathcal{F}, \mathbb{P})$ *as in Lemma 5.5. There exists a* $\vartheta-$*invariant null-set* $N_0 \subseteq \Omega_{\mathrm{kpz}}$ *and a continuous RDS* φ *taking values in* $\mathcal{L}(C(\mathbb{T}^d))$, *such that:*

$$\forall \omega \in \Omega_{\mathrm{kpz}} \backslash N_0 : \quad \varphi_t(\omega) u_0 = u_t(\omega), \quad \forall t \geqslant 0, u_0 \in C(\mathbb{T}),$$

where u_t *is the unique solution to Equation* (19) *started in* u_0, *as in* [21, Theorem 6.15].

Proof. The quoted existence result [21, Theorem 6.15] builds a unique solution that depends continuously on the initial condition and a driving signal in a space $\mathcal{Y}_{\mathrm{kpz}}$ of the so-called enhanced noise (that includes among others the terms X, Y, Z from the discussion in Remark 5.6). In [21, Section 9] it is proven that outside a null-set $\overline{N}_0$ the enhancement of space-time white noise lives in $\mathcal{Y}_{\mathrm{kpz}}$. Such null-set can be chosen to be $\vartheta-$invariant by setting $N_0 = \bigcup_{z \in \mathbb{Z}} \vartheta^z \overline{N}_0$. We can set φ to be the identity outside of N_0, thus concluding the proof. □

The RDS φ introduced in the previous lemma falls into the framework of this article.

Proposition 5.8. *The RDS* φ *be defined as in Lemma 5.7 satisfies Assumption 4.2 for any* $\beta < \frac{1}{2}$. *In particular, all the results of Theorem 4.3 apply.*

Remark 5.9. *The proof of the proposition above follows the same idea of the proof of Proposition 5.3. Yet certain technical parts rely on results concerning the solution theory, to which we will promptly refer.*

Proof. Fix $\omega \in \Omega_{\mathrm{kpz}} \backslash N_0$, the latter being the same null-set as in Lemma 5.7.

Let us start by checking the first property of Assumption 4.2. We can formally define the kernel by $K(\omega, t, x, y) = \varphi_t(\omega)(\delta_y)(x)$, where δ_y indicates a Dirac δ centered at y. Indeed, let us define $\varphi_t(\omega)(\delta_y)$ as the solution to (19) with $u_0 = \delta_y$. This solution exists (for ω outside of the null-set N_0), in view of [21, Theorem 6.15]. In fact, this result shows that for any $0 < \beta, \zeta < \frac{1}{2}$, and any $p \in [1, \infty]$ the function $\varphi(\omega)$ can be extended to a map

$$\varphi \in C_{\mathrm{loc}}((0, \infty); \mathcal{L}(B^{-\gamma}_{p,\infty}; B^{\beta}_{p,\infty})),$$

where we used that, in the language of the quoted article, the space $\mathcal{D}^{\exp,\delta}_{\mathrm{rhe}}$ of paracontrolled distributions embeds in $C_{\mathrm{loc}}((0, \infty); B^{\beta}_{p,\infty})$, for suitable values of δ (as described in the quoted theorem). Near $t = 0$ one expects that $\|\varphi_t(\omega) u_0\|_{B^{\beta}_{p,\infty}}$ blows up, if $u_0 \in B^{-\zeta}_{p,\infty}$. The exact speed of this blow-up is provided as well in the quoted theorem, but since we are not interested in quantifying the blow-up, we can exploit the result we wrote to deduce the apparently stronger:

$$\varphi \in C_{\mathrm{loc}}((0, \infty); \mathcal{L}(B^{-\gamma}_{p,\infty}; C^{\beta})). \tag{20}$$

This follows by Besov embeddings (for $p \leqslant q$: $B^{\alpha}_{p,\infty}(\mathbb{T}^d) \subseteq B^{\alpha - d(\frac{1}{p} - \frac{1}{q})}_{q,\infty}(\mathbb{T}^d)$), as, assuming without loss of generality that $\beta, \zeta > \frac{1}{4}$, unifomrly over $0 < S \leqslant t \leqslant T < \infty$ one can bound:

$$\|\varphi_t u_0\|_{C^\beta} \lesssim \|\varphi_{S/2} u_0\|_{C^{-\zeta}} \lesssim \|\varphi_{S/2} u_0\|_{B^{\beta}_{2,\infty}} \lesssim \|\varphi_{S/4} u_0\|_{B^{-\zeta}_{2,\infty}} \lesssim \|\varphi_{S/4} u_0\|_{B^{\beta}_{1,\infty}} \lesssim \|u_0\|_{B^{-\zeta}_{p,\infty}}.$$

So overall we obtain (20), and in particular:

$$\sup_{S \leqslant t \leqslant T} \|\varphi_t(\omega) u_0\|_{C^\beta} \leqslant C(\omega, \beta, \zeta, p, S, T) \|u_0\|_{B^{-\zeta}_{p,\infty}}, \quad \text{for any} \quad 0 < S < T < \infty. \tag{21}$$

Now since $\{\delta_y\}_{y \in \mathbb{T}} \subseteq B^{-\zeta}_1$ for any $\zeta > 0$, as proven in Lemma A.2, the kernel $K(\omega, t, x, y)$ is well-defined. The continuity in t, x follows from the previous estimates, while the continuity in y follows from (21) together with Lemma A.2.

We can pass to the second property of Assumption 4.2. The upper bound $\delta(\omega, S, T)$ is a simple consequence of the continuity of the kernel K. The lower bound $\gamma(\omega, S, T)$ is instead a consequence of a strong maximum principle which, implies that $K(\omega, t, x, y) > 0, \forall t > 0, x, y \in TT$. In this path-whise setting, the strong maximum principle is proven in [11, Theorem 5.1] (it was previously established in [29] with probabilistic techniques).

The third property is a consequence of Equation (21), by defining $C(\omega, \beta, S, T) := C(\omega, \beta, \frac{1}{4}, \infty, S, T)$, so we are left with only the last property to check. We start with the fact that

$$\mathbb{E} \log C(\beta, S, T) < \infty.$$

To see this, note that there exists a random valiable $\mathbb{Y}(\omega)$ taking values in a Banach space $\mathcal{Y}_{\mathrm{kpz}}$, which in the language of [21] we call enhancement of the noise $\xi(\omega)$ (see [21, Definition 4.1]) such that for some deterministic $A(\beta, S, T), q > 0$:

$$\sup_{t \in [S,T]} \|\varphi_t(\omega) f\|_\beta \leqslant e^{A(\beta,S,T)(1 + \|\mathbb{Y}(\omega)\|_{\mathcal{Y}_{\mathrm{kpz}}})^q} \|f\|_{C^{-\frac{1}{4}}},$$

that is:

$$C(\omega, \beta, S, T) = e^{A(\beta,S,T)(1 + \|\mathbb{Y}(\omega)\|_{\mathcal{Y}_{\mathrm{kpz}}})^q}.$$

This result is implicit in the proof of [21, Theorem 6.15], since the proof relies on a Picard iteration and a Gronwall argument. The bound can be found explicitly in [30, Theorem 5.5 and Section 5.2]: here the equation in set on the entire line $\mathbb{R}$, which is a more general setting, since one can always extend the noise periodically. Thus we have $\mathbb{E} \log C(\beta, S, T) \lesssim_{\beta, S, T} 1 + \mathbb{E}\big[\|\mathbb{Y}\|^q_{\mathcal{Y}_{\mathrm{kpz}}}\big]$, so that the result is proven if one shows that for any $q \geqslant 0$: $\mathbb{E}\|\mathbb{Y}\|^q_{\mathcal{Y}_{\mathrm{kpz}}} < \infty$. This is the content of [21, Theorem 9.3]: the different norms appearing in $\mathbb{Y}$ lie essentially in certain Wiener-Itô chaos levels and thus have bounded moments of any order.

We then pass to the second bound. Since by the triangle inequality the bound does not depend on the choice of f, set $f = 1$. It is thus enough to prove that:

$$\mathbb{E} \sup_{S \leqslant t \leqslant T} \|\log(\varphi_t(\omega) 1)\|_\infty < \infty.$$

We proceed as in the proof of Proposition 5.8. On one side one has the upper bound:

$$\log(\varphi_t(\omega) 1) \leqslant \log \|\varphi_t(\omega) 1\|_\infty \leqslant \log C(\omega, \beta, S, T),$$

which is integrable by the arguments we just presented. As for the lower bound, the approach of Proposition 5.8 has to be adapted to the present singular setting. This was already done in the proof of [30, Lemma 3.10], and we sketch again the argument here. The proof of this result (recall the notation in Remark 5.6) shows by comparison, that:

$$h^P_t(\omega) := \log(\varphi_t(\omega) 1) - Y - Y^{\vee} - Y^{\vee\vee} \geqslant -\log u_t(\omega),$$

with $u(\omega)$ solving a linear equation:

$$(\partial_t - \partial_x^2)u = b(\mathbb{Y})\partial_x u + c(\mathbb{Y})u, \qquad u(0) = e^{Y(0)}, \tag{22}$$

where b, c are two distributions for which the previous equation remains singular and is solved in a controlled sense. To be precise, to ease the comparison with [30] and following the notation of the same article, the coefficients are:

$$b(\mathbb{Y}) = \partial_x Y + \partial_x Y^{\vee} + \partial_x Y^{\vee\vee}$$

$$c(\mathbb{Y}) = -[(\partial_t - \partial_x^2)(Y^{\vee\vee\vee} + Y^{\vee\vee\vee\vee}) + 2(\partial_x Y \partial_x Y^{\vee\vee} - \partial_x Y \odot \partial_x Y^{\vee\vee}) + 2\partial_x Y^{\vee}\, \partial_x Y^{\vee\vee} + (\partial_x Y^{\vee\vee})^2].$$

Note that with respect to the equation in the proof of [30, Lemma 3.10] some factors 2 are out of place: this is because here we consider the operator ∂_x^2 instead of $\frac{1}{2}\partial_x^2$. Equation (22) then admits a solution as an application of [30, Proposition 5.6] and in particular, the last result implies that:

$$\sup_{S \leqslant t \leqslant T} \|u_t\|_\infty \leqslant e^{\overline{C}(S,T)(1+\|\mathbb{Y}\|_{\mathcal{Y}_{\mathrm{kpz}}})^{\overline{q}}},$$

for some $\overline{C}, \overline{q} \geqslant 1$. Since $\|Y\|_\infty + \|Y^{\vee}\|_\infty + \|Y^{\vee\vee}\|_\infty \lesssim \|\mathbb{Y}\|_{\mathcal{Y}_{\mathrm{kpz}}}$, one has overall that:

$$\log \varphi_t(\omega)1 \gtrsim_{S,T} -1 - \|\mathbb{Y}\|^{\overline{q}}_{\mathcal{Y}_{\mathrm{kpz}}}.$$

Together with the previous results and the moment bound on $\mathbb{E}\|\mathbb{Y}\|^{\overline{q}}$ we already recalled, this proves that:

$$\mathbb{E} \sup_{S \leqslant t \leqslant T} d_H(\varphi_t \cdot f, f) < \infty.$$

Hence the proof is complete. □

Remark 5.10. *In the previous proof the lower bound for the solution h to the KPZ equation is arguably the result that requires most attention. As an alternative to our approach, it seems also possible to prove the lower bound we obtained in the previous proof via an optimal control representation of h, see* [21, Theorem 7.13].

Remark 5.11. *In the previous proposition we have proven that we can apply Theorem 4.3. The latter guarantees synchronization up to subtracting time-dependent constants $c(\omega, t)$. In fact it is possible to choose $c(\omega, t) \equiv \overline{c}(\omega)$ for a time-independent $\overline{c}(\omega)$. For fractional noise we could show this in Remark 5.4, but in the argument we made use of the spatial smoothness of the noise to write an ODE for the constant $c(\omega, t)$: Equation* (16)*. Backwards in time the same approach guarantees that h_∞ is the unique solution to the KPZ Equation for $t \in \mathbb{R}$, up to constant (in time and space) shifts.*

The approach of Remark 5.4 can be lifted to the space-time white noise setting by defining the product which appears in the ODE for example in a paracontrolled way. To complete the argument one then needs to control the paracontrolled, and not only the Hölder norms in the convergences of Theorem 4.3. This appears feasible, but falls outside the aims of the present paper.

5.3. **Mixing of Gaussian fields.** Let us state a general criterion which ensures that a possibly infinite-dimensional Gaussian field is mixing (and hence ergodic). This is a simple generalization of a classical result for one-dimensional processes, cf. [13, Chapter 14]. We indicate with $\mathbf{B}^*$ the dual of a Banach space $\mathbf{B}$ and write $\langle \cdot, \cdot \rangle$ for the dual pairing.

Proposition 5.12. *Let $\mathbf{B}$ be a separable Banach space. Let μ be a Gaussian measure on $(\mathbf{B}, \mathcal{B}(\mathbf{B}))$ and $\vartheta \colon \mathbb{N}_0 \times \mathbf{B} \to \mathbf{B}$ a dynamical system which leaves μ invariant. Denote with ξ the canonical process on $\mathbf{B}$ under μ. The condition*

$$\lim_{n \to \infty} \mathrm{Cov}(\langle \xi, \varphi \rangle, \langle \vartheta^n \xi, \varphi' \rangle) = 0, \qquad \forall \varphi, \varphi' \in \mathbf{B}^*$$

implies that the system is mixing, that is for all $A, B \in \mathcal{B}(\mathbf{B})$:

$$\lim_{n\to\infty} \mu(A \cap \vartheta^{-n} B) = \mu(A)\mu(B).$$

Proof. First, we reduce ourselves to the finite-dimensional case. Indeed, note that the sequence $(\xi, \vartheta^n \xi)$ is tight in $\mathbf{B} \times \mathbf{B}$, because ϑ leaves μ invariant. Furthermore, tightness implies that the sequence is flatly concentrated (cf. [14, Definition 2.1]), that is for every $\varepsilon > 0$ there exists a finite-dimensional linear space $S^\varepsilon \subseteq \mathbf{B} \times \mathbf{B}$ such that:

$$\mathbb{P}\big((\xi, \vartheta^n \xi) \in S^\varepsilon\big) \geqslant 1 - \varepsilon.$$

Hence, it is sufficient to check the mixing property for $A, B \in \mathcal{B}(S^\varepsilon)$.

This means that there exists an $n \in \mathbb{N}$ and $\varphi_i \in \mathbf{B}^*$ for $i = 1, \dots, n$ such that we have to check the mixing property for the vector:

$$((\langle \xi, \varphi_i \rangle)_{i=1,\dots,n}, \quad (\langle \vartheta^n \xi, \varphi_i \rangle)_{i=1,\dots,n}).$$

In this setting and in view of our assumptions the result follows from [18, Theorem 2.3]. □

Appendix A.

Lemma A.1. *Let P_t be the heat semigroup. One can estimate, for $\alpha \in \mathbb{R}, \beta \in [0, 2), p \in [1, \infty]$ and any $T > 0$:*

$$\sup_{0 \leqslant t \leqslant T} t^{\frac{\beta}{2}} \|P_t f\|_{B^{\alpha+\beta}_{p,\infty}(\mathbb{T}^d)} \lesssim \|f\|_{B^\alpha_{p,\infty}(\mathbb{T}^d)}.$$

If one additionally chooses $b \in L^\infty([0,T]; B^\gamma_{\infty,\infty}(\mathbb{T}^d; \mathbb{R}^d)), c \in L^\infty([0,T]; B^\gamma_{\infty,\infty}(\mathbb{T}^d))$, such that:

$$\zeta := \gamma \wedge \alpha + \beta, \qquad \gamma + \zeta - 1 > 0, \qquad \beta \geqslant 1$$

there exists a unique mild solution w to:

$$(\partial_t - \Delta) w(t,x) = b(t,x) \cdot \nabla w(t,x) + c(t,x) w(t,x), \qquad w(0,x) = w_0(x).$$

Moreover, there exists a $q \geqslant 0$ and $C(T) \geqslant 0$ such that:

$$\sup_{0 \leqslant t \leqslant T} t^{\frac{\beta}{2}} \|w_t\|_{B^\zeta_{p,\infty}} \lesssim \|w_0\|_{B^\alpha_{p,\infty}} e^{C(T)(1 + \|b\|_{L^\infty([0,T]; B^\gamma_{\infty,\infty})} + \|c\|_{L^\infty([0,T]; B^\gamma_{\infty,\infty})})^q}.$$

Proof. The estimate regarding the heat kernel is classical. For a reference from the field of singular SPDEs see [20, Lemma A.7]. Let us pass to the PDE. Here consider any w such that $M := \sup_{0 \leqslant t \leqslant T} t^{\frac{\beta}{2}} \|w\|_{B^\zeta_{p,\infty}} < \infty$, and let $N := \sup_{0 \leqslant t \leqslant T} \Big\{ \|b_t\|_{B^\gamma_{\infty,\infty}(\mathbb{T}^d;\mathbb{R}^d)} + \|c_t\|_{B^\gamma_{\infty,\infty}(\mathbb{T}^d)} \Big\}$. Then consider:

$$\mathcal{I}(w)_t = P_t w_0 + \int_0^t P_{t-s}\Big[b_s \cdot \nabla w_s + c_s w_s\Big] \, \mathrm{d}s.$$

It follows from the smoothing effect of the heat kernel that:

$$\sup_{0 \leqslant t \leqslant T} t^{\frac{\beta}{2}} \|\mathcal{I}(w)_t\|_{B^\zeta_{p,\infty}} \lesssim \|w_0\|_{B^\alpha_{p,\infty}} + \sup_{0 \leqslant t \leqslant T} t^{\frac{\beta}{2}} \int_0^t (t-s)^{-\frac{\beta}{2}} \Big(\|b_s \cdot \nabla w_s\|_{B^{(\zeta-1)\wedge\gamma}_{p,\infty}} + \|c_s w_s\|_{B^{\zeta\wedge\gamma}_{p,\infty}} \Big) \, \mathrm{d}s$$

Now from our condition on the coefficient and estimates on products of distributions (see [4, Theorem 2.82 and 2.85]) the latter term can in turn be bounded by:

$$\begin{aligned} \sup_{0 \leqslant t \leqslant T} t^{\frac{\beta}{2}} \|\mathcal{I}(w)_t\|_{B^\zeta_{p,\infty}} &\lesssim |w_0\|_{B^\alpha_{p,\infty}} + MN \sup_{0 \leqslant t \leqslant T} t^{\frac{\beta}{2}} \int_0^t (t-s)^{-\frac{\beta}{2}} s^{-\frac{\beta}{2}} \, \mathrm{d}s \\ &\lesssim \|w_0\|_{B^\alpha_{p,\infty}} + MNT^{1-\frac{\beta}{2}}. \end{aligned}$$

It follows that for small $T > 0$ the map $\mathcal{I}$ is a contraction providing the existence of solutions for small times. By linearity and a Gronwall-type argument, this estimate also provides the required a-priori bound. $\square$

Lemma A.2. *For any $\gamma > 0$, the inclusion $\{\delta_y\}_{y\in\mathbb{T}^d} \subseteq B^{-\gamma}_{1,\infty}$ holds. Moreover, there exists an $L > 0$ such that:*

$$\|\delta_x - \delta_y\|_{B^{-\gamma}_{1,\infty}} \leqslant L|x-y|^\gamma.$$

Proof. We divide the proof in two steps. Recall that by definition we have to bound $\sup_{j\geqslant -1} 2^{-\gamma j}\|\Delta_j(\delta_x - \delta_y)\|_{L^1}$. Hence we choose j_0 as the smallest integer such that $2^{-j_0} \leqslant |x-y|$. We first look at small scales $j \geqslant j_0$ and then at large scales $j < j_0$. For small scales, by the Poisson summation formula, since $\varrho_j(k) = \varrho_0(2^{-j}k)$, and by defining $K_j(x) = \mathcal{F}^{-1}_{\mathbb{R}}\varrho_j(x) = 2^j K(2^j x)$ for some $K \in \mathcal{S}(\mathbb{R})$ (the space of tempered distributions):

$$\begin{aligned} 2^{-\gamma j}\big\|\Delta_j(\delta_x - \delta_y)\big\|_{L^1} &\leqslant |x-y|^\gamma \int_{\mathbb{R}} 2^j |K(2^j(z-x)) - K(2^j(z-y))|\,\mathrm{d}z \\ &\lesssim |x-y|^\gamma \int_{\mathbb{R}} 2^j|K(2^j z)|\,\mathrm{d}z \lesssim |x-y|^\gamma. \end{aligned}$$

While for large scales, since we have $|2^j(x-y)| \leqslant 1$, applying the Poisson summation formula, by the mean value theorem and since $K \in \mathcal{S}(\mathbb{R})$ (the Schwartz space of functions):

$$\begin{aligned} 2^{-\gamma j}\big\|\Delta_j(\delta_x - \delta_y)\big\|_{L^1} &\leqslant 2^{-\gamma j} \int |K(z) - K(z + 2^j(x-y))|\,\mathrm{d}z \\ &\leqslant |x-y|^\gamma \int \max_{|\xi - z|\leqslant 1} \frac{|K(\xi) - K(z)|}{|\xi - z|^\alpha}\,\mathrm{d}z \lesssim |x-y|^\gamma. \end{aligned}$$

Hence the result follows.

$\square$

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