Simulating the Movement of a Double Pendulum with Euler's Method

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March 10, 2019

1 Important note

This is a work in progress. It is not finished at this moment in time.

2 What are we trying to do here?

We want to spend some time thinking about double pendulums and numerical simulations. Our goal is, of course, to create a working simulation of a double pendulum. What has to be done to get there? What kind of math is necessary? Let's start our journey by breaking down the problem into smaller pieces.

- Defining and labeling of the mathematical double pendulum model
- Deriving the Differential Equation
- Setting up the simulation using the Differential Equation

That's better. The problem doesn't look all that daunting anymore.

3 defining the double pendulum

4 Getting results with Euler's Method

In this section we'll try to get ourselfes acquainted with Euler's Method. We want to get some intuition as to how it is applied and what exactly it does. Since we've never used this thing before, we're also curious to see it in action to confirm it's value as a mathematical tool.

$$\dot{y} = f(y, t) \tag{4.1}$$

$$y(t_0) = y_0 \tag{4.2}$$

Given an ordinary differential equation in the form of (4.1) and with the initial conditions as shown in (4.2), Euler's Method states that

$$t_1 = t_0 + \epsilon \tag{4.3}$$

$$y_1 = y_0 + \epsilon f(y_0, t_0) \tag{4.4}$$

Here, ϵ is a very small number. Though being small ϵ is not infinetely small at all. Since we're planning on running this simulation on a computer, there are very real constraints to how small we are able to make this number. Note how (4.18) gives us the recipe for the first value pair t_1 and y_1. Any further value pairs t_n and y_n are calculated following the same pattern. Note also how we started with a differential equation equalling a derivative $\dot{y}$ to a function of y and t, but the result we get is y. Speaking physically, if we started with a differential equation equalling the acceleration to a function of time and velocity, then Eulers Method would give us velocity, time pairs as a result.

After spending some time thinking about Euler's Method, we are now confident that we got some intuition about how to use it and what results to expect. But we still haven't used the damn thing. So let's give it a roll.

$$\frac{d}{dt} v(t) = g - \frac{k}{m} \cdot v(t)^2 \tag{4.5}$$

Here's a first order differential equation. And it's quite a special one, for it is a differential equation whose solution can be determined analytically. This is quite rare. The differential equation modeling the double pendulum isn't quite as tame - There are no mathematical tools which could lead us to an analytically derived solution. But back to (4.5). This equation models a falling object experiencing air resistance. The air resistance in this specific case is modeled to be proportional to the velocity squared. There are also quite a few constants in the mix. Let's go through them one by one.
k is the air resistance coefficient and it's magnitude is

$$k = 7.757 \cdot 10^{-6} [\mathrm{kg\,m^{-1}}]. \tag{4.6}$$

g is the acceleration of gravity.

$$g = 9.81 [\mathrm{m\,s^{-2}}] \tag{4.7}$$

and m is the mass of the falling object.

$$m = 70 [\mathrm{mg}] \tag{4.8}$$

Furthermore we know that the solution to (4.5) is

$$v(t) = \frac{1}{\beta} \cdot \frac{e^{2g\beta t} - 1}{e^{2g\beta t} + 1}. \tag{4.9}$$

Where β is defined as

$$\begin{aligned}
\beta &= \sqrt{\frac{k}{gm}}. && (4.10)\\
&= \sqrt{\frac{7.757 \cdot 10^{-6}[\mathrm{kg\,m^{-1}}]}{9.81[\mathrm{m\,s^{-2}}] \cdot 7 \cdot 10^{-5}[\mathrm{kg}]}} && (4.11)\\
&= 0.106[\mathrm{m^{-1}\,s}]. && (4.12)
\end{aligned}$$

Knowing all this allows us to put Euler's Method to the test. How will our numerically computed value for, for instance, $t_p = 10\mathrm{s}$ measure up to the exact value computed by inserting $t_p = 10\mathrm{s}$ into (4.9)? We might use $v_a(t)$ and $v_n(t)$ to differentiate between results gained from the analytical equation as opposed to results computed by the numerical approach.

$$\begin{aligned}
v_a(t_p = 10\mathrm{s}) &= \frac{1}{\beta} \cdot \frac{e^{2g\beta t_p} - 1}{e^{2g\beta t_p} + 1}. && (4.13)\\
&= 9.434\mathrm{m\,s^{-1}}. && (4.14)
\end{aligned}$$

Here we go. The exact value for our falling object after 10 seconds - assuming the initial velocity is 0 - amounts to $v_a(t_p = 10\mathrm{s}) = 9.434\mathrm{m\,s^{-1}}$.
Now we need to get our simulation going. From (4.1) and (4.5) we know that

$$\begin{aligned}
\dot{y} &= f(y, t) && (4.15)\\
&= \frac{d}{dt} y(t) && (4.16)\\
&= g - \frac{k}{m} \cdot y(t)^2 && (4.17)
\end{aligned}$$

Inserting (4.17) into (4.18), we get

$$\begin{aligned}
t_1 &= t_0 + \epsilon && (4.18)\\
y_1 &= y_0 + \epsilon\left(g - \frac{k}{m} \cdot y_0^2\right). && (4.19)
\end{aligned}$$

And that's it. Now all we need to do is choose some small number for ϵ (we'll go with 0.1 for now), insert 0 for both t_0 and y_0, and we're ready to run the simulation. Running it, we get $v_n(t_p = 10\mathrm{s}) = 9.408\mathrm{m\,s^{-1}}$ - which isn't that far off the exact value.

5 CONSTRUCTING THE DIFFERENTIAL EQUATION

In order to get the simulation up and running, we need equations (14) and (19) from science-world's Double Pendulum page[1]. Here they are.

$$(m_1+m_2)l_1\ddot{\theta}_1+m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)+g(m_1+m_2)\sin\theta_1=0 \tag{5.1}$$

$$m_2l_2\ddot{\theta}_2+m_2l_1\ddot{\theta}_1 cos(\theta_1-\theta_2)-m_2l_1\dot{\theta}_1^2 sin(\theta_1-\theta_2)+m_2g\sin\theta_2=0 \tag{5.2}$$

Those two equations are both second order. In preparation for using Euler's Method on them, we need to split each of them into two first order differential equations. But how does one split them? We do this by introducting λ_1 and λ_2 which we define as follows.

$$\lambda_1=\dot{\theta}_1 \tag{5.3}$$
$$\dot{\lambda}_1=\ddot{\theta}_1 \tag{5.4}$$
$$\lambda_2=\dot{\theta}_2 \tag{5.5}$$
$$\dot{\lambda}_2=\ddot{\theta}_2 \tag{5.6}$$

Putting these lambdas into (5.1) yields

$$(m_1+m_2)l_1\dot{\lambda}_1+m_2l_2\dot{\lambda}_2\cos(\theta_1-\theta_2)+m_2l_2\lambda_2^2\sin(\theta_1-\theta_2)+g(m_1+m_2)\sin\theta_1=0. \tag{5.7}$$

And doing the same thing to (5.2) yields

$$m_2l_2\dot{\lambda}_2+m_2l_1\dot{\lambda}_1 cos(\theta_1-\theta_2)-m_2l_1\lambda_1^2 sin(\theta_1-\theta_2)+m_2g\sin\theta_2=0. \tag{5.8}$$

We are in the process of preparing our differential equations for Euler's Method and the next crucial step is to rewrite (5.7) and (5.8) in such a way that both have a single $\dot{\lambda}$ on the left side of the equal sign. For (5.7) we get

$$\dot{\lambda}_1=\frac{-m_2l_2\dot{\lambda}_2\cos(\theta_1-\theta_2)-m_2l_2\lambda_2^2\sin(\theta_1-\theta_2)g(m_1+m_2)\sin\theta_1}{(m_1+m_2)l_1}. \tag{5.9}$$

And now we intended to do the same for (5.8), but after pondering for a while over (5.9) it dawns on us that something isn't quite right. We planned to use Euler's Method on

$$\dot{\lambda}_1=\frac{-m_2l_2\dot{\lambda}_2\cos(\theta_1-\theta_2)-m_2l_2\lambda_2^2\sin(\theta_1-\theta_2)g(m_1+m_2)\sin\theta_1}{(m_1+m_2)l_1} \tag{5.9}$$

to obtain λ_1, and then one more time on

$$\dot{\theta}_1=\lambda_1 \tag{5.10}$$

to obtain θ_1 and then applying Euler's Method in a similar way to also get λ_2 and θ_2. Here's the problem: There's a $\dot{\lambda}_2$ in equation (5.9) and there's no specific value we can plug in for it.

[1] http://scienceworld.wolfram.com/physics/DoublePendulum.html

Here's what needs to be done: We need to solve (5.8) for $\dot{\lambda}_2$, put the result into (5.7) and then solve for $\dot{\lambda}_1$. That should get us back on track. Solving (5.8) for $\dot{\lambda}_2$, we get

$$\dot{\lambda}_2 = \frac{m_2 l_1 \lambda_1^2 sin(\theta_1-\theta_2) - m_2 l_1 \dot{\lambda}_1 cos(\theta_1-\theta_2) - m_2 g \sin\theta_2}{m_2 l_2}. \tag{5.11}$$

Now we insert (5.11) into (5.7)

$$\begin{aligned}(m_1+m_2) l_1 \dot{\lambda}_1 + m_2 l_2 \frac{m_2 l_1 \lambda_1^2 sin(\theta_1-\theta_2) - m_2 l_1 \dot{\lambda}_1 cos(\theta_1-\theta_2) - m_2 g \sin\theta_2}{m_2 l_2} \cos(\theta_1-\theta_2) \\ + m_2 l_2 \lambda_2^2 \sin(\theta_1-\theta_2) + g(m_1+m_2)\sin\theta_1 = 0.\end{aligned} \tag{5.12}$$

clean up a little

$$\begin{aligned}(m_1+m_2) l_1 \dot{\lambda}_1 + \big(m_2 l_1 \lambda_1^2 sin(\theta_1-\theta_2) - m_2 l_1 \dot{\lambda}_1 cos(\theta_1-\theta_2) - m_2 g \sin\theta_2\big)\cos(\theta_1-\theta_2) \\ + m_2 l_2 \lambda_2^2 \sin(\theta_1-\theta_2) + g(m_1+m_2)\sin\theta_1 = 0.\end{aligned} \tag{5.13}$$

multiply out all the terms

$$\begin{aligned}(m_1+m_2) l_1 \dot{\lambda}_1 + m_2 l_1 \lambda_1^2 sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) - m_2 l_1 \dot{\lambda}_1 cos(\theta_1-\theta_2)\cos(\theta_1-\theta_2) \\ - m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) + m_2 l_2 \lambda_2^2 \sin(\theta_1-\theta_2) + g(m_1+m_2)\sin\theta_1 = 0.\end{aligned} \tag{5.14}$$

rearrange the terms in such a way that there is only one $\dot{\lambda}_1$

$$\begin{aligned}\dot{\lambda}_1 \{(m_1+m_2) l_1 - m_2 l_1 cos^2(\theta_1-\theta_2)\} + m_2 l_1 \lambda_1^2 sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) \\ - m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) + m_2 l_2 \lambda_2^2 \sin(\theta_1-\theta_2) + g(m_1+m_2)\sin\theta_1 = 0.\end{aligned} \tag{5.15}$$

and finally solve for $\dot{\lambda}_1$ by isolating it on the left side.

$$\begin{aligned}&\dot{\lambda}_1 \{(m_1+m_2) l_1 - m_2 l_1 cos^2(\theta_1-\theta_2)\} \\ &= m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) - m_2 l_1 \lambda_1^2 sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) \\ &- m_2 l_2 \lambda_2^2 \sin(\theta_1-\theta_2) - g(m_1+m_2)\sin\theta_1.\end{aligned} \tag{5.16}$$

$$\dot{\lambda}_1 = \tag{5.17}$$

$$\frac{m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) - m_2 l_1 \lambda_1^2 sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) - m_2 l_2 \lambda_2^2 \sin(\theta_1-\theta_2) - g(m_1+m_2)\sin\theta_1}{(m_1+m_2) l_1 - m_2 l_1 cos^2(\theta_1-\theta_2)} \tag{5.18}$$

The equation fits barely on the screen. But now we finally got what we need to get our double pendulum swinging. Or at least the first half of it. We still need to solve (5.7) for $\dot{\lambda}_1$, put the result into (5.8) and then solve for $\dot{\lambda}_2$. So let's get it over with.

Luckily, we already solved (5.7) for $\dot{\lambda}_1$ in our first failed attempt of getting all the equations ready. Here it is again:

$$\dot{\lambda}_1 = \frac{-m_2 l_2 \dot{\lambda}_2 \cos(\theta_1-\theta_2) - m_2 l_2 \lambda_2^2 \sin(\theta_1-\theta_2) g(m_1+m_2)\sin\theta_1}{(m_1+m_2) l_1} \tag{5.9}$$

(5.9) needs to be inserted into (5.8). Some quick copy and paste action saves us from scrolling all the way up to look at (5.8).

$$m_2 l_2 \dot{\lambda}_2 + m_2 l_1 \dot{\lambda}_1 cos(\theta_1 - \theta_2) - m_2 l_1 \lambda_1^2 sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0. \tag{5.8}$$

inserting (5.9) into (5.8) we get

$$\begin{aligned} &m_2 l_2 \dot{\lambda}_2 + m_2 l_1 \frac{-m_2 l_2 \dot{\lambda}_2 \cos(\theta_1 - \theta_2) - m_2 l_2 \lambda_2^2 \sin(\theta_1 - \theta_2) g(m_1 + m_2) \sin\theta_1}{(m_1 + m_2) l_1} cos(\theta_1 - \theta_2) \\ &- m_2 l_1 \lambda_1^2 sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0. \end{aligned} \tag{5.19}$$