Simulating the Movement of a Double Pendulum with Euler's Method

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1 Important note

This is a work in progress. It is not finished at this moment in time.

2 What are we trying to do here?

We want to spend some time thinking about double pendulums and numerical simulations. Our goal is, of course, to create a working simulation of a double pendulum. What has to be done to get there? What kind of math is necessary? Let's start our journey by breaking down the problem into smaller pieces.

- Defining and labeling of the mathematical double pendulum model
- Deriving the Differential Equation
- Setting up the simulation using the Differential Equation

That's better. The problem doesn't look all that daunting anymore.

3 defining the double pendulum

4 Getting results with Euler's Method

In this section we'll try to get ourselfes acquainted with Euler's Method. We want to get some intuition as to how it is applied and what exactly it does. Since we've never used this thing before, we're also curious to see it in action to confirm it's value as a mathematical tool.

$$\dot{y} = f(y,t) \tag{4.1}$$
$$y(t_0) = y_0 \tag{4.2}$$

Given an ordinary differential equation in the form of (4.1) and with the initial conditions as shown in (4.2), Euler's Method states that

$$t_1 = t_0 + \epsilon \tag{4.3}$$
$$y_1 = y_0 + \epsilon f(y_0, t_0) \tag{4.4}$$

Here, ϵ is a very small number. Though being small ϵ is not infinetely small at all. Since we're planning on running this simulation on a computer, there are very real constraints to how small we are able to make this number. Note how (4.3) gives us the recipe for the first value pair t_1 and y_1. Any further value pairs t_n and y_n are calculated following the same pattern. Note also how we started with a differential equation equalling a derivative $\dot{y}$ to a function of y and t, but the result we get is y. Speaking physically, if we started with a differential equation equalling the acceleration to a function of time and velocity, then Eulers Method would give us velocity, time pairs as a result.

After spending some time thinking about Euler's Method, we think that we got some intuition about how to use it and what results to expect. But we still haven't used the damn thing. So let's give it a roll.

$$\frac{d}{dt} v_y(t) = g - \frac{k}{m} \cdot v_y(t)^2 \tag{4.5}$$

Here's a first order differential equation. It's

5 CONSTRUCTING THE DIFFERENTIAL EQUATION

In order to get the simulation up and running, we need equations (14) and (19) from scienceworld's Double Pendulum page[1]. Here they are.

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2) \sin\theta_1 = 0 \tag{5.1}$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \tag{5.2}$$

Those two equations are both second order. In preparation for using Euler's Method on them, we need to split each of them into first order differential equations. But how does one split them? We do this by introducting λ_1 and λ_2 which are defined as follows.

$$\lambda_1 = \dot{\theta}_1 \tag{5.3}$$
$$\dot{\lambda}_1 = \ddot{\theta}_1 \tag{5.4}$$
$$\lambda_2 = \dot{\theta}_2 \tag{5.5}$$
$$\dot{\lambda}_2 = \ddot{\theta}_2 \tag{5.6}$$

[1] http://scienceworld.wolfram.com/physics/DoublePendulum.html

Putting these lambdas into (5.1) yields

$$(m_1 + m_2) l_1 \dot{\lambda}_1 + m_2 l_2 \dot{\lambda}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \lambda_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2) \sin\theta_1 = 0. \tag{5.7}$$

And doing the same thing to (5.2) yields

$$m_2 l_2 \dot{\lambda}_2 + m_2 l_1 \dot{\lambda}_1 cos(\theta_1 - \theta_2) - m_2 l_1 \lambda_1^2 sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0. \tag{5.8}$$

We are in the process of preparing our differential equations for Euler's Method and the next crucial step is to rewrite (5.7) and (5.8) in such a way that both have a single $\dot{\lambda}$ on the left side of the equal sign. For (5.7) we get

$$\dot{\lambda}_1 = \frac{-m_2 l_2 \dot{\lambda}_2 \cos(\theta_1 - \theta_2) - m_2 l_2 \lambda_2^2 \sin(\theta_1 - \theta_2) g(m_1 + m_2) \sin\theta_1}{(m_1 + m_2) l_1}. \tag{5.9}$$