ON THE COHOMOLOGY GROUPS OF REAL LAGRANGIANS: DOCUMENTATION

Abstract. We describe the calculation made in `coarse_cover_computation.m`. This script computes the Čech differential used in §7 of the attached article to compute the cohomology groups $\check{H}^0(\mathcal{U}, \mathcal{H}^1_{\mathbb{Z}})$ and $\check{H}^1(\mathcal{U}, \mathcal{H}^1_{\mathbb{Z}})$.

We utilize the computer algebra system Magma to compute the Čech complex used in §7 to obtain information on the integral cohomology groups of $\breve{\mathcal{L}}_{\mathbb{R}}$. The calculation may be implemented by running the Magma script file `coarse_cover_computation.m` contained in the supplementary material.

We briefly recall the structure of the 7-to-1 branched covering map $\breve{\pi}\colon \breve{L}_{\mathbb{R}} \to B$. We recall that in §5 we endowed the integral affine manifold $B = \partial\Delta_{\mathbb{P}^4}$ with an open cover $\mathcal{U}$. This cover associates an open set $\mathcal{U}_{\sigma_i^d}$ to each face σ_i^d, where d records the dimension of σ_i^d and i records the index of this face. The statement of Theorem 1.2 requires the computation of seven non-zero Čech cohomology groups $\check{H}^i(\mathcal{U}, \mathcal{H}^j_{\mathbb{Z}})$, with $i, j \in \mathbb{Z}_{\geq 0}$, where $\mathcal{H}^j_{\mathbb{Z}}$ is the pre-sheaf

$$\mathcal{H}^j_{\mathbb{Z}}\colon U \mapsto H^j(\breve{\pi}^{-1}(U), \mathbb{Z}).$$

In this document we describe the computation of $\check{H}^i(\mathcal{U}, \mathcal{H}^1_{\mathbb{Z}})$, for $i \in \{0, 1\}$. These groups are computed from the two-term Čech complex

$$\check{C}^0(\mathcal{U}, \mathcal{H}^1_{\mathbb{Z}}) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{H}^1_{\mathbb{Z}}).$$

Each of these cochain groups is the direct sum of the first cohomology groups of the spaces $\breve{\pi}^{-1}(\mathcal{U}_{\sigma_i^d})$ and $\breve{\pi}^{-1}(\mathcal{U}_{\sigma_i^d} \cap \mathcal{U}_{\sigma_j^e})$ respectively.

To compute these cohomology groups, we first note that $H^1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma_i^d})) = \{0\}$ if $d = 0$ or $d = 3$. Moreover, the only intersections of open sets in $\mathcal{U}$ whose preimages under $\breve{\pi}$ have non-trivial first cohomology group are those of the form $\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1}$, where σ^1 is an edge of a triangular face $\sigma^2 \subset \Delta_{\mathbb{P}^4}$. Given an edge σ^1 of a two-dimensional face σ^2 we have that:

- $H^1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2})) \cong \mathbb{Z}^{12}$,
- $H^1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})) \cong \mathbb{Z}^{12}$,
- $H^1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1} \cap \mathcal{U}_{\sigma^2})) \cong \mathbb{Z}^{8}$.

There are three connected components of $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})$ which have non-trivial first homology, each of which deformation retracts onto a wedge union of 4 circles. Similarly, there are two connected components of $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1} \cap \mathcal{U}_{\sigma^2})$ which retract onto the wedge union of four circles. The group $H^1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}))$ is generated by 12 elements. In particular, the preimage of each

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segment γ_j^i, displayed in Figure 2, consists of a circle and three (disjoint) line segments. It follows from the proof of Lemma 5.3 that orientations of these 12 circles generate the group

$$H_1^{tf}(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2})) = H^1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}))^\star.$$

We let $\widetilde{\gamma}_j^i$ denote the unique S^1 component in $\breve{\pi}^{-1}(\gamma_j^i)$. Combining the above observations, we observe that $\breve{C}^i(\mathcal{U}, \mathcal{H}^1_{\mathbb{Z}}) \cong \mathbb{Z}^{240}$ for each $i \in \{0, 1\}$; see §7 of the associated article for further details.

The 240 columns of $[\delta]$ divide into ten blocks of twelve columns, such that each block corresponds to a 2-dimensional face of $\Delta_{\mathbb{P}^4}$ and ten further blocks of twelve columns, such that each block corresponds to an edge of $\Delta_{\mathbb{P}^4}$. The rows of $[\delta]$ are divided into thirty blocks, each containing eight rows. Each of these thirty blocks is indexed by a pair (f, e) where $f \in \{1, \ldots, 10\}$ is the index of a two-dimensional face $\sigma^2 := \sigma^2_f$ of $\Delta_{\mathbb{P}^4}$, and $e \in \{1, \ldots, 10\}$ is the index of an edge $\sigma^1 := \sigma^1_e$ contained in f.

We order blocks of columns of $[\delta]$ by fixing an ordering of the faces and an ordering of the edges of $\Delta_{\mathbb{P}^4}$. We order blocks of rows by ordering double intersections lexicographically, using the indices of the faces and edges associated with each intersection. Part of this block decomposition is shown below. Note that each block of columns contains three non-zero sub-blocks, as each triangular face has three edges and each edge is contained in three triangular faces. Moreover, the collection of rows corresponding to a pair (f, e) contains a pair of non-zero sub-blocks, as each intersection of open sets is the intersection of the unique open set corresponding to the edge σ^1 with index e and the unique open set corresponding to the triangular face σ^2 with index f.

	face					edge		
(face, edge)	1	$\cdots$	7	$\cdots$	10	1	$\cdots$	10
$(1, 1)$	$\star$	$\cdots$	0	$\cdots$	0	$\star$	$\cdots$	0
$(7, 1)$	0	$\cdots$	$\star$	$\cdots$	0	$\star$	$\cdots$	0
$(10, 1)$	0	$\cdots$	0	$\cdots$	$\star$	$\star$	$\cdots$	0
$\cdots$			$\cdots$				$\cdots$	
$(10, 10)$	0	$\cdots$	0	$\cdots$	$\star$	0	$\cdots$	$\star$

We construct the matrix $[\delta]$ one block of columns at a time, using

```
Meets_23 := [[f,e] :  f in [1..10], e in [1..10] | e in edges_of_fs[f]];
```

to determine which submatrices are non-zero. We give the listing for this construction in Listing 1.

Listing 1. Constructing the matrix δ

```
// We construct the left hand block matrix one block of columns at a time.
```

```
for f in [1..10] do

   // This is the column we are currently working on. Initially this column
       contains no rows.
   curr_col := Matrix(Z,0,12,[]);

   // Meets_23 stores pairs [f,e] where e is the index of an edge contained in
       the
   // two-dimensional face with index f.
   for m in Meets_23 do

      // If m contains the 2-face corresponding to the current column we insert
          a non-zero block.
      if f eq m[1] then

         curr_col := VerticalJoin(curr_col,<<Insert Block>>);
      else

         // If m does not contain the index f of the given two-dimensional face,
             we append a zero block.
         curr_col := VerticalJoin(curr_col,Matrix(Z,8,12,[]));
      end if;
   end for;

   // Now we join the current column to the the left half of the differential.
   LH := HorizontalJoin(LH,curr_col);

end for;


// We construct the right hand block matrix RH. We recall that this describes
// the restriction map from neighbourhoods of edges U_e, to intersections of
// such open sets with open sets U_f corresponding to two-dimensional faces.

RH := Matrix(Z,240,0,[]);

for e in [1..10] do
   curr_col := Matrix(Z,0,12,[]);

   for m in Meets_23 do
      if e eq m[2] then
```

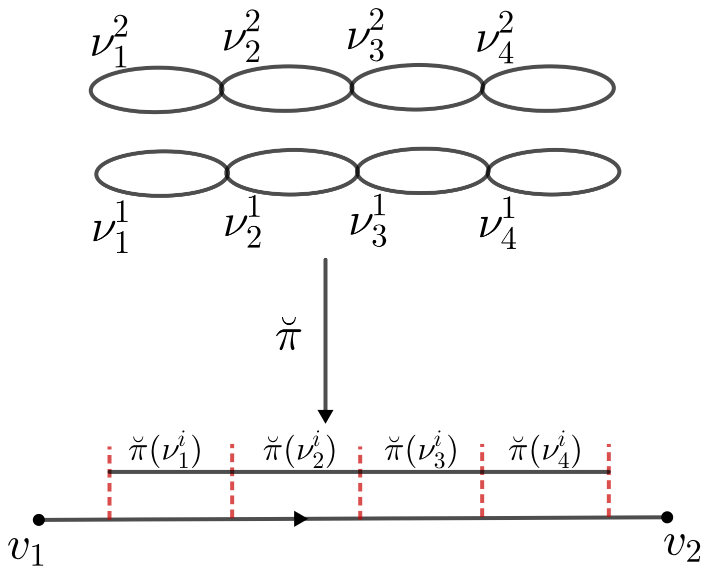

FIGURE 1. Describing cycles ν_j^i.

```
      curr_col := VerticalJoin(curr_col,<<Insert Block>>);
    else
      curr_col := VerticalJoin(curr_col,Matrix(Z,8,12,[]));
    end if;
  end for;

  RH := HorizontalJoin(RH,curr_col);
end for;

// We form the Cech differential by joining the left and right halves, together
   with
// the sign change (although we note that this will not affect the column span
   of this matrix).
Cech_H1 := HorizontalJoin(LH,ScalarMatrix(240,-1)*RH);
```

The most delicate part of this calculation is the definition of the sub-matrices of $[\delta]$. To construct these we choose oriented cycles defining bases of $H_1^{tf}(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}))$, and $H_1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}\cap\mathcal{U}_{\sigma^1}))$, where $\sigma^1 \subset \sigma^2$ are an edge and face of $\Delta_{\mathbb{P}^4}$, respectively. We recall that $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}\cap\mathcal{U}_{\sigma^1})$ is homotopy equivalent to a union of eight circles (two disjoint copies of the wedge union of four circles) and 3 points. We let ν_j^i denote these eight circles, where $i \in \{1,2\}$ denotes the connected component of ν_j^i in $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}\cap\mathcal{U}_{\sigma^1})$, and $j \in \{1,\dots,4\}$ records the position of the circle along σ^1, starting from the vertex of σ^1 with the lowest index, see Figure 1.

The two connected components are ordered by first fixing an ordering of points in the fibre of $\breve{\pi}$ over a basepoint near σ^1 and σ^2 (we take this basepoint to lie in the tetrahedron with index `ts_of_fs[f][1]` containing σ^2). That is, we fix a bijection between the fibre of $\breve{\pi}$ over this basepoint and the set $\{1, \dots, 7\}$. Note that each circle ν^i_j consists of two arcs, each of which forms a section of $\breve{\pi}$ over a segment in σ^2. Extending the enumeration of a fibre of $\breve{\pi}$ described above to a local trivialisation of $\breve{\pi}$, each of these arcs can be identified with a number in $\{1, \dots, 7\}$. These indices are stored in `sheets_used`.

We orient the edge σ^1 by ordering its vertices (from lower index to higher), and fix an orientation for each of the eight circles ν^i_j by orienting the arc in each circle with the *lowest* index sheet in the direction of σ^1, and endowing the remaining arcs with the reverse orientation. We orient σ^2 via the cyclic ordering of its vertices $\{v_1, v_2, v_3\}$ from lowest index to highest. Note that this orientation of σ^2 induces an orientation of each edge σ^1 of σ^2. This orientation agrees with our convention for the orientation of σ^1, except when the vertices of σ^1 are v_1 and v_3, when the orientation is reversed.

We orient the circles $\widetilde{\gamma}^i_j \subset \breve{\pi}^{-1}(\gamma^i_j)$ by noting that these circles each consist of a pair of arcs, each of which is labelled by the *pair* of sheets of $\breve{\pi}$ which are identified in $\breve{\pi}^{-1}(\gamma^i_j)$. The orientation of $\widetilde{\gamma}^i_j$ is fixed by orienting the arc corresponding to the lowest index sheet of $\breve{\pi}$ anti-clockwise, see Figure 2, and endowing the remaining arc with the reverse orientation. Our aim is to express the homology classes of cycles ν^i_j in $H^{tf}_1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}))$ in terms of the classes of circles $\widetilde{\gamma}^a_b$ for $a \in \{1, 2, 3\}$ and $b \in \{1, 2, 3, 4\}$. To achieve this we note that if γ^i_j is the edge of one of the six hexagonal regions in σ^2 formed by Δ, the circle $\widetilde{\gamma}^i_j$ represents the same class in $H^{tf}_1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}))$ as the class of the (unique) circle in the pre-image of the opposite edge of this hexagonal region (orienting this edge in the same direction as γ^i_j).

Example 1. Consider the pre-image of segment γ, displayed in Figure 2, between γ^1_1 and γ^1_2. The pre-image of this segment under $\breve{\pi}$ contains a single circle $\widetilde{\gamma}$, and we orient this circle as above (that is, by orienting the arc with the minimal index in the direction of the edge from v_1 to v_2), we observe that the homology class $[\widetilde{\gamma}] \in H^{tf}_1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2}))$ is equal to $-[\widetilde{\gamma}^2_4]$. The negative sign here is due to the difference in orientation of the arc with minimal index, compared with that of the circle $\widetilde{\gamma}^2_4$, the orientation of which is determined by orienting the lowest index arc in the direction from v_2 to v_3.

The left hand block of δ consists of 10 blocks of 12 columns, indexed by triangular faces σ^2 of $\Delta_{\mathbb{P}^4}$. Let $\delta^L_{\sigma^2,\sigma^1}$ denote the 8×12 submatrix δ in the block of columns indexed by the index $f \in \{1, \dots, 10\}$ of σ^2 and block of rows indexed by (f, e) (where $e \in \{1, \dots, 10\}$ denotes the index of the edge σ^1 of σ^2). The construction of $\delta^L_{\sigma^2,\sigma^1}$ consists of two parts: identifying the matrix up to sign, and determining signs. Note that $\delta^L_{\sigma^2,\sigma^1}$ itself has a block

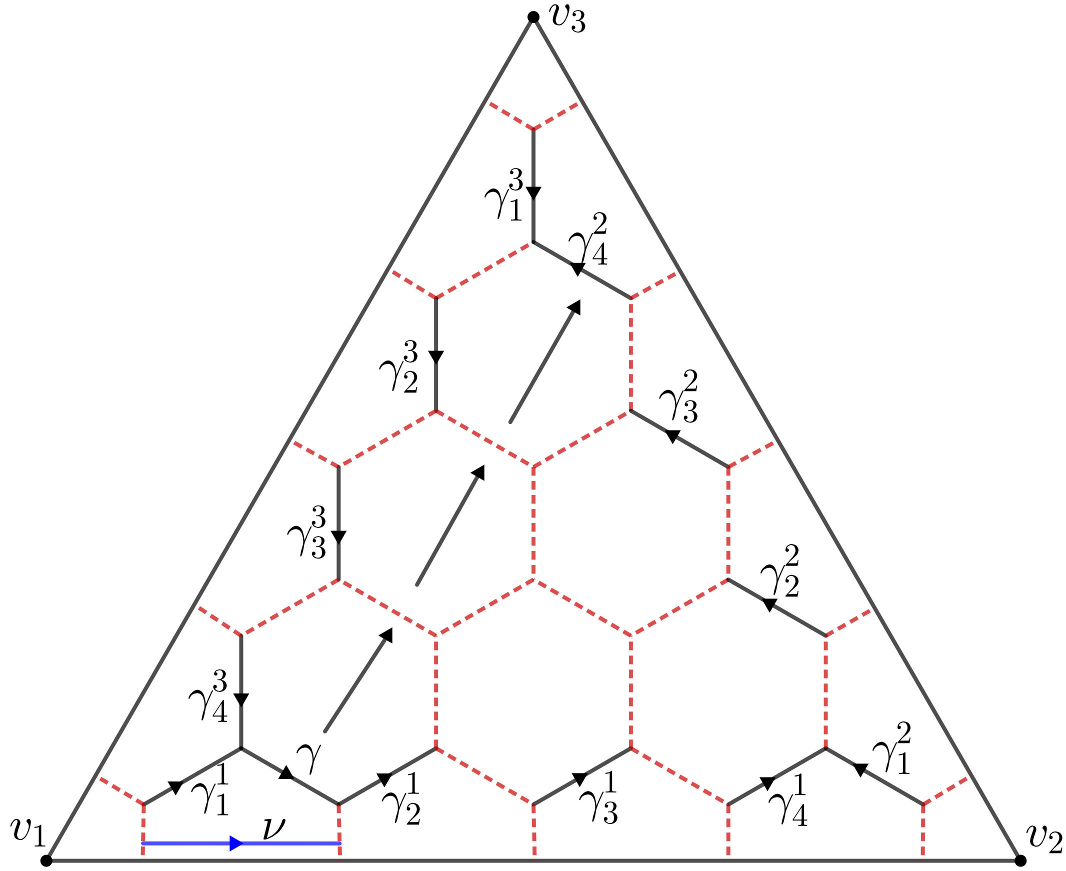

FIGURE 2. A generating set for $H_1^{tf}(\pi^{-1}(\mathcal{U}_{\sigma^2}))$.

decomposition into six 4×4 submatrices:

$$\delta^L_{\sigma^2,\sigma^1} := \begin{array}{c|ccc} & \widetilde{\gamma}^1_j & \widetilde{\gamma}^2_j & \widetilde{\gamma}^3_j \\ \hline \nu^1_j & \star & \star & \star \\ \nu^2_j & \star & \star & \star \end{array}.$$

Here rows correspond to the eight generators $[\nu^i_j]$ of $H_1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1}))$, and the columns to the twelve generators $[\widetilde{\gamma}^i_j]$ of $H_1^{tf}(\pi^{-1}(\mathcal{U}_{\sigma^2}))$. We first determine, up to sign, an expression for $[\nu^i_j]$ in $H_1^{tf}(U_\sigma)$ in terms of the classes $[\widetilde{\gamma}^i_j]$. To do this, we consider a homotopy of the circle ν^i_j onto $\pi^{-1}(\Delta)$. For example, if σ^1 is the edge with vertices v_1 and v_2 (see Figure 2) we can consider a homotopy from the segment ν to the union of the segments γ^1_1 and γ^1_2. This lifts to a homotopy of each of the circles ν^i_1 to the circles $\widetilde{\gamma}^1_1$ and $\widetilde{\gamma}$ (where $\widetilde{\gamma}$ is as defined in Example 1). In particular, the circle ν^1_1 breaks up into the wedge union of circles homologous to $\widetilde{\gamma}^1_1$ and $\widetilde{\gamma}^2_4$ respectively. Generalising this example, and letting I_4 denote the

4×4 identity matrix and letting

$$J_4 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

we have that

$$\delta^L_{\sigma^2,\sigma^1} := \begin{array}{c|ccc} & \widetilde{\gamma}^1_j & \widetilde{\gamma}^2_j & \widetilde{\gamma}^3_j \\ \hline \nu^1_j & \pm I_4 & \pm J_4 & 0 \\ \nu^2_j & \pm I_4 & \pm J_4 & 0 \end{array}$$

in this case. Note that the signs of the six 4×4 submatrices of $\delta^L_{\sigma^2,\sigma^1}$ may vary. Similarly, if σ^1 is the edge with vertices v_2 and v_3, each of the circles in $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1})$ are homotopic to wedge unions of pairs of circles homologous to $\widetilde{\gamma}^2_i$ and $\widetilde{\gamma}^3_{5-i}$ respectively, for some $i \in \{1, \ldots, 4\}$. Thus, in this case, we have that:

$$\delta^L_{\sigma^2,\sigma^1} := \begin{array}{c|ccc} & \widetilde{\gamma}^1_j & \widetilde{\gamma}^2_j & \widetilde{\gamma}^3_j \\ \hline \nu^1_j & 0 & \pm I_4 & \pm J_4 \\ \nu^2_j & 0 & \pm I_4 & \pm J_4. \end{array}$$

Finally, if σ^1 is the edge of σ^2 with vertices v_1 and v_3, we have that

$$\delta^L_{\sigma^2,\sigma^1} := \begin{array}{c|ccc} & \widetilde{\gamma}^1_j & \widetilde{\gamma}^2_j & \widetilde{\gamma}^3_j \\ \hline \nu^1_j & \pm I_4 & 0 & \pm J_4 \\ \nu^2_j & \pm I_4 & 0 & \pm J_4. \end{array}$$

Note the order of I_4 and J_4 in the final matrix, circles ν^i_j are indexed from v_1 to v_3 as j varies, while the circles γ^3_j are indexed from v_3 to v_1, see Figure 2.

To keep track of which vertices are connected by edges of σ^2, we use the variable `place`. The edge σ^1 of σ^2 has *place* $i \in \{1, 2, 3\}$ if, ordering the vertices v_i of σ^2 for $i \in \{1, 2, 3\}$ in increasing order of their index, σ^1 does not contain the vertex v_i.

Listing 2 describes the block construction of this submatrix of δ. In particular, we note that $\delta^L_{\sigma^2,\sigma^1}$ consists of three 8×4 submatrices whose order depends on the value of the variable `place`. These 8×4 blocks are stored in the list `LH_blocks`, while the list `gluing_table[i]` describes the order these should be arranged in when `place = i`. In this way we obtain an 8×12 matrix (up to sign). To summarise, each row of the matrix $\delta^L_{\sigma^2,\sigma^1}$ corresponds to a circle ν^i_j, and each column to a circle $\widetilde{\gamma}^a_b$ and this matrix contains a value of ± 1 if the circle $\widetilde{\gamma}^a_b$ appears in the expression for ν^i_j.

Listing 2. A further block decomposition for submatrices of δ

```
LH_I := VerticalJoin(IdentityMatrix(Integers(),4),IdentityMatrix(Integers(),4));
LH_J :=
   Matrix(Z,8,4,[<1,4,1>,<2,3,1>,<3,2,1>,<4,1,1>,<5,4,1>,<6,3,1>,<7,2,1>,<8,1,1>]);
```

```
LH_Z := Matrix(Z,8,4,[]);
LH_blocks := [LH_I,LH_J,LH_Z];

gluing_table := [[3,1,2],[1,3,2],[1,2,3]];

...

// We obtain the value 'place' of the given edge.
deleted_vertex := [v : v in vertices_of_fs[f] | v notin
   vertices_of_es[m[2]]][1];
place := Index(vertices_of_fs[f],deleted_vertex);

// We use the function find_face_signs to obtain any necessary sign changes for
   the blocks
// LH_I, and LH_J.
signs := find_face_signs(f,m[2],place);
LH_sign_I :=
   DiagonalJoin(ScalarMatrix(4,signs[1][1]),ScalarMatrix(4,signs[2][1]))*LH_I;
LH_sign_J :=
   DiagonalJoin(ScalarMatrix(4,signs[1][2]),ScalarMatrix(4,signs[2][2]))*LH_J;

// We form the 8x12 block LH_total by joining these three blocks, as described
// by the triples stored in 'gluing_table'.
LH_sign_blocks := [LH_sign_I,LH_sign_J,LH_Z];
LH_total := HorizontalJoin(
HorizontalJoin(LH_sign_blocks[gluing_table[place][1]],
LH_sign_blocks[gluing_table[place][2]]),
LH_sign_blocks[gluing_table[place][3]]);
```

The signs are determined by the function `find_face_signs`. This function takes in the face, edge and place associated with a circle ν_j^i. The function determines the index of tetrahedron `ts_of_fs[f][1]` containing σ^2, which we use to fix a bijection between a fibre of $\breve{\pi}$ contained in this tetrahedron and $\{1,\ldots,7\}$. This determines the indices of the sheets used in the arcs of the circles ν_j^i, and these indices are stored in `sheets_used`. Expressing ν_j^i as a sum $\pm\gamma_b^a \pm \gamma_d^c$, as above, the pair of sheets labelling each arc of γ_b^a and γ_d^c are stored in `combining_sheets_1`, and `combining_sheets_2`.

Recalling that ν_j^i is homotopic to a wedge union of circles in the pre-image of Δ, we need to compare the orientation of ν_j^i with the given orientation of each of this pair of circles. Listing 3 implements the required assignment of signs, based on the position of the lowest index sheet `sheets_used[i][1]` defining ν_j^i. Note that if σ^1 has `place = 2` we also need to reverse all signs, since σ^1 is oriented from v_1 to v_3, but the vertices (v_1, v_2, v_3) of σ^2 are cyclically ordered.

LISTING 3. Determining relative cycle orientations.

```
for i in [1,2] do
   if sheets_used[i][1] in combining_sheets_1[2] then
      signs[i][1] := -1;
   end if;

   if sheets_used[i][1] in combining_sheets_2[1] then
      signs[i][2] := -1;
   end if;
end for;
```

We now consider the right hand half of the matrix $[\delta]$. Columns of this matrix are divided into ten blocks of 12, corresponding to edges σ^1 of $\Delta_{\mathbb{P}^4}$. To fix a basis for $H_1(\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})) \cong \mathbb{Z}^{12}$ we fix a basepoint v, equal to `vertices_of_es[e][1]`, where $e \in \{1, \ldots, 10\}$ is the index of the edge σ^1. Labelling the fibre $\breve{\pi}^{-1}(v)$ with elements of $\{1, \ldots, 7\}$, circles in $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})$ consist of a pair of arcs, each labelled with an element of $\{1, \ldots, 7\}$ as above. We orient these circles by orienting the arc of lower index in each circle in the direction from `vertices_of_es[e][1]` to `vertices_of_es[e][2]`, or – in the notation used in Figure 3 – from v_1 to v_2, and endowing the remaining arcs with the reverse orientation.

Let $\delta^R_{\sigma^2,\sigma^1}$ denote the submatrix of δ whose columns correspond to the edge $\sigma^1 := \sigma^1_e$, and whose rows correspond to the pair (σ^2, σ^1), where $\sigma^2 := \sigma^2_f$ is a two-dimensional face containing σ^1. The submatrix $\delta^R_{\sigma^2,\sigma^1}$ itself decomposes into six 4×4 submatrices. Blocks of rows of this decomposition are indexed by the two connected components of $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1})$ with non-trivial first homology groups, while blocks of columns are indexed by the three connected components of $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})$ with non-trivial first homology groups. We let μ^i_j denote the 12 circles in $\breve{\pi}^{-1}(U_{\sigma^1})$, where $i \in \{1, 2, 3\}$ records the connected component containing μ^i_j, while $j \in \{1, \ldots, 4\}$ records the position of μ^i_j along σ^1, see Figure 3.

$$\delta^R_{\sigma^2,\sigma^1} = \begin{array}{c|ccc} & \mu^1_j & \mu^2_j & \mu^3_j \\ \hline \nu^1_j & \star & \star & \star \\ \nu^1_j & \star & \star & \star \end{array}$$

We order the three connected components of $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})$ lexicographically by indices of the labels associated with arcs of μ^i_j. For example, if these connected components are labelled with the pairs of indices $\{1, 7\}, \{3, 6\}$, and $\{4, 5\}$ respectively, these are ordered as listed. The three components of $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})$ and two components of $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1})$ with non-trivial first homology are illustrated in Figure 3.

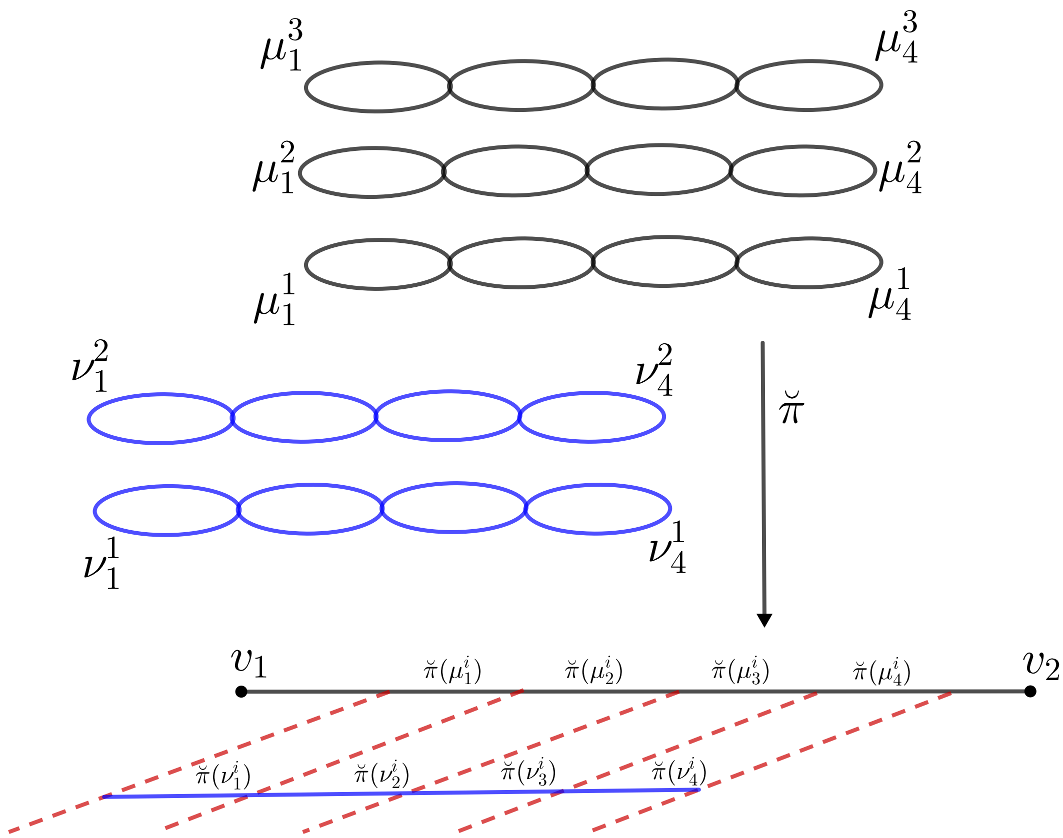

FIGURE 3. The block decomposition of the map $H_1(\pi^{-1}(\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1})) \hookrightarrow H_1(\pi^{-1}(\mathcal{U}_{\sigma^1}))$

All but two of the six blocks in $\delta^R_{\sigma^2,\sigma^1}$ are equal to zero, and up to reordering the 3 block columns and changing signs, $\delta^R_{\sigma^2,\sigma^1}$ contains an 8×8 identity submatrix, and all other entries vanish. The signs and ordering of the block columns are determined by the function `find_edge_matrix` which returns a 2×3 matrix with two non-zero entries, equal to ± 1. The assembly of the block $\delta^R_{\sigma^2,\sigma^1}$ is shown in Listing 4.

LISTING 4. Assembling blocks in $\delta^R_{\sigma^2,\sigma^1}$.

```
// The function find_edge_matrix returns a 2x3 matrix with entires -1, 0, or +1.
edge_mtx := find_edge_matrix(m[1],e);

// We then form a block matrix by replacing each entry in edge_mtx by a 4x4
   scalar matrix.
block := VerticalJoin(
   HorizontalJoin(
      HorizontalJoin(ScalarMatrix(4,edge_mtx[1,1]),ScalarMatrix(4,edge_mtx[1,2])),
         ScalarMatrix(4,edge_mtx[1,3])),
      HorizontalJoin(
```

```
        HorizontalJoin(ScalarMatrix(4,edge_mtx[2,1]),ScalarMatrix(4,edge_mtx[2,2])),
            ScalarMatrix(4,edge_mtx[2,3])));
curr_col := VerticalJoin(curr_col,block);
```

The function find_edge_matrix first records the labels in $\{1, \dots, 7\}$ associated to the cycles ν_j^i appearing over $\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1}$, relative to a given trivialisation of $\breve{\pi}$ over a point in the interior of the tetrahedron

```
nearby := Explode([t :  t in [1..5] | f in faces_of_ts[t]]);
```

containing σ^2, stored in sheets_used. The function then determines the sheets appearing in each of the 12 circles contained in $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})$ (see Figure 3), using a trivialisation of $\breve{\pi}$ over the vertex

```
first_vertex := vertices_of_es[e][1];
```

of σ^1. These indices are stored in sheets_used_p. We then use the permutation

```
Paste[[nearby,first_vertex]]
```

to ensure that the same trivialisation is used to label sheets of both $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^2} \cap \mathcal{U}_{\sigma^1})$ and $\breve{\pi}^{-1}(\mathcal{U}_{\sigma^1})$. The subsequent change in order and orientation of the cycles ν_j^i is stored in edge_mtx. This matrix determines the matrix $\delta^R_{\sigma^2,\sigma^1}$ and, iterating over faces σ^1 and σ^2, this completes the construction of the matrix δ.