

# First-Order Logic Syntax

Reading: Chapter 8, ~~9.1-9.2, 9.5.1-9.5.5~~

FOL Syntax and Semantics read: 8.1-8.2

FOL Knowledge Engineering read: 8.3-8.5

~~FOL Inference read: Chapter 9.1-9.2, 9.5.1-9.5.5~~

(Please read lecture topic material before and after each  
lecture on that topic)

## Review: Resolution as Efficient Implication

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(OR A B C D)       $\rightarrow$ Same  $\rightarrow$   
(OR  $\neg$ A E F G)       $\rightarrow$ Same  $\rightarrow$

-----  
(OR B C D E F G)

(NOT (OR B C D))  $\Rightarrow$  A  
A  $\Rightarrow$  (OR E F G)

-----  
(NOT (OR B C D))  $\Rightarrow$  (OR E F G)

-----  
(OR B C D E F G)

# Outline for First-Order Logic (FOL, also called FOPC)

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- Propositional Logic is **Useful** --- but has **Limited Expressive Power**
- First Order Predicate Calculus (FOPC), or First Order Logic (FOL).
  - FOPC has greatly expanded expressive power, though still limited.
- New Ontology
  - The world consists of OBJECTS (for propositional logic, the world was facts).
  - OBJECTS have PROPERTIES and engage in RELATIONS and FUNCTIONS.
- New Syntax
  - Constants, Predicates, Functions, Properties, Quantifiers.
- New Semantics
  - Meaning of new syntax.
- Knowledge engineering in FOL
- ~~Inference in FOL~~

# FOL Syntax: You will be expected to know

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- FOPC syntax
  - Syntax: Sentences, predicate symbols, function symbols, constant symbols, variables, quantifiers
- De Morgan's rules for quantifiers
  - connections between  $\forall$  and  $\exists$
- Nested quantifiers
  - Difference between " $\forall x \exists y P(x, y)$ " and " $\exists x \forall y P(x, y)$ "
  - $\forall x \exists y \text{ Likes}(x, y)$  --- "Everybody likes somebody."
  - $\exists x \forall y \text{ Likes}(x, y)$  --- "Somebody likes everybody."
- Translate simple English sentences to FOPC and back
  - $\forall x \exists y \text{ Likes}(x, y) \Leftrightarrow$  "Everyone has someone that they like."
  - $\exists x \forall y \text{ Likes}(x, y) \Leftrightarrow$  "There is someone who likes every person."

# Pros and cons of propositional logic

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- ☺ Propositional logic is **declarative**
  - Knowledge and inference are separate
- ☺ Propositional logic allows **partial/disjunctive/negated information**
  - unlike most programming languages and databases
- ☺ Propositional logic is **compositional**:
  - meaning of  $B_{1,1} \wedge P_{1,2}$  is derived from meaning of  $B_{1,1}$  and of  $P_{1,2}$
- ☺ Meaning in propositional logic is **context-independent**
  - unlike natural language, where meaning depends on context
- ☹ Propositional logic has **limited expressive power**
  - E.g., cannot say "Pits cause breezes in adjacent squares."
    - except by writing one sentence for each square
  - Needs to refer to objects in the world,
  - Needs to express general rules

# First-Order Logic (FOL), also called First-Order Predicate Calculus (FOPC)

- Propositional logic assumes the world contains **facts**.
- First-order logic (like natural language) assumes the world contains
  - **Objects:** people, houses, numbers, colors, baseball games, wars, ...
  - **Functions:** father of, best friend, one more than, plus, ...
    - Function arguments are objects; function returns an object
  - **Objects generally correspond to English NOUNS**
  - **Predicates/Relations/Properties:** red, round, prime, brother of, bigger than, part of, comes between, ...
    - Predicate arguments are objects; predicate returns a truth value
  - **Predicates generally correspond to English VERBS**
    - **First argument is generally the subject, the second the object**
    - Hit(Bill, Ball) usually means "Bill hit the ball."
    - Likes(Bill, IceCream) usually means "Bill likes IceCream."
    - Verb(Noun1, Noun2) usually means "Noun1 verb noun2."

## Aside: First-Order Logic (FOL) vs. Second-Order Logic

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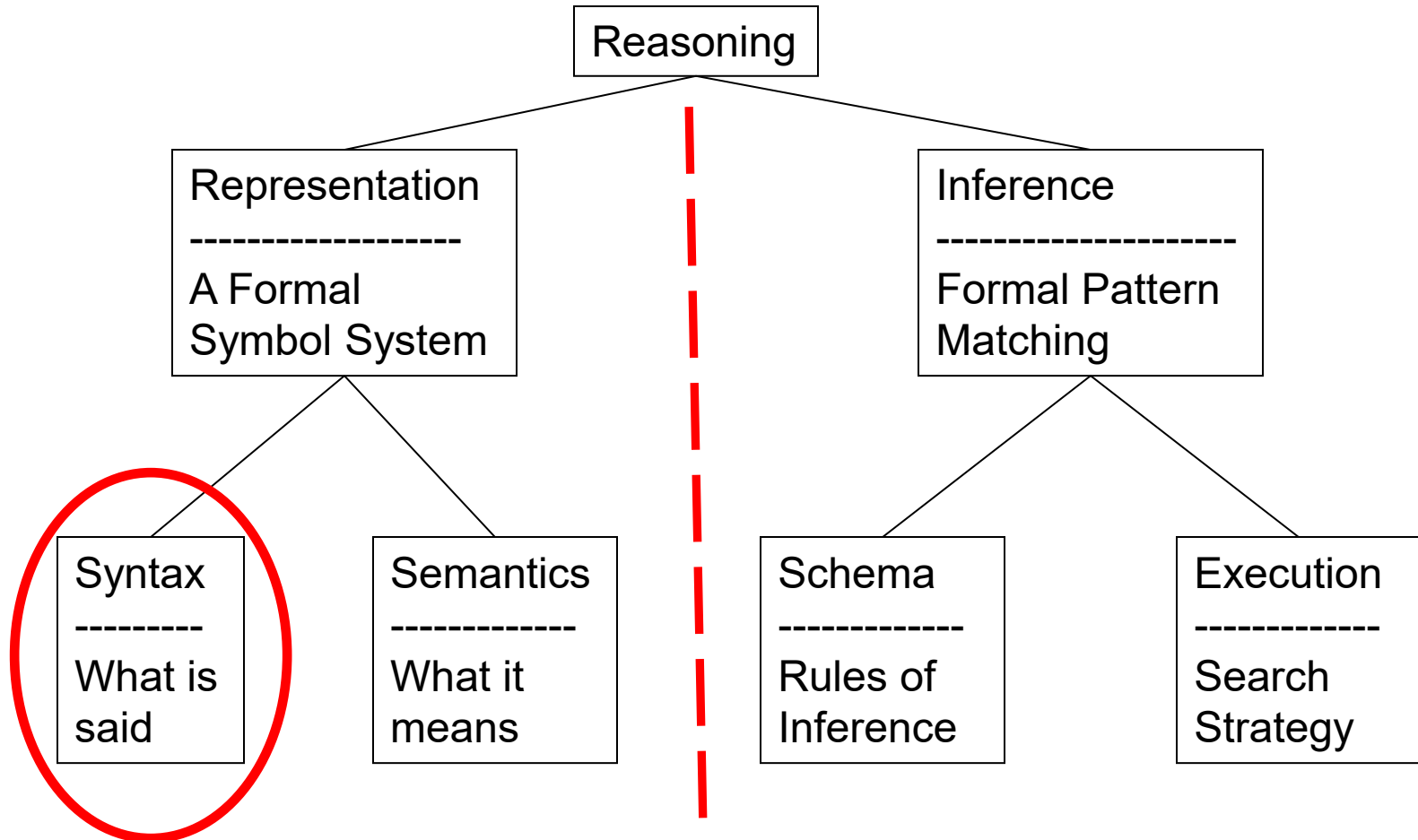
- First Order Logic (FOL) allows variables and general rules
  - “First order” because quantified variables represent objects.
  - “Predicate Calculus” because it quantifies over predicates on objects.
    - E.g., “Integral Calculus” quantifies over functions on numbers.
- Aside: Second Order logic
  - “Second order” because quantified variables can also represent predicates and functions.
    - E.g., can define “Transitive Relation,” which is beyond FOL.
- Aside: In FOL we can state that a relationship is transitive
  - E.g., BrotherOf is a transitive relationship
  - $\forall x, y, z \text{ BrotherOf}(x,y) \wedge \text{BrotherOf}(y,z) \Rightarrow \text{BrotherOf}(x,z)$
- Aside: In Second Order logic we can define “Transitive”
  - $\forall P, x, y, z \text{ Transitive}(P) \Leftrightarrow ( P(x,y) \wedge P(y,z) \Rightarrow P(x,z) )$
  - Then we can state directly, Transitive(BrotherOf)

## FOL (or FOPC) Ontology:

What kind of things exist in the world?

What do we need to describe and reason about?

Objects --- with their relations, functions, predicates, properties, and general rules.





## Syntax of FOL: Basic elements

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- Constants KingJohn, 2, UCI,...
- Predicates Brother, >,...
- Functions Sqrt, LeftLegOf,...
- Variables  $x, y, a, b, \dots$
- Quantifiers  $\forall, \exists$
- Connectives  $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$  (standard)
- Equality  $=$  (but causes difficulties....)

## Syntax of FOL: Basic syntax elements are symbols

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- **Constant** Symbols (correspond to English nouns)
  - Stand for objects in the world.
    - E.g., KingJohn, 2, UCI, ...
- **Predicate** Symbols (correspond to English verbs)
  - Stand for relations (maps a tuple of objects to a **truth-value**)
    - E.g., Brother(Richard, John), greater\_than(3,2), ...
  - $P(x, y)$  is usually read as “x is P of y.”
    - E.g., Mother(Ann, Sue) is usually “Ann is Mother of Sue.”
- **Function** Symbols (correspond to English nouns)
  - Stand for functions (maps a tuple of objects to an **object**)
    - E.g., Sqrt(3), LeftLegOf(John), ...
- **Model** (world) = set of domain objects, relations, functions
- **Interpretation** maps symbols onto the model (world)
  - Very many interpretations are possible for each KB and world!
  - Job of the KB is to rule out models inconsistent with our knowledge.

# Syntax : Relations, Predicates, Properties, Functions

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- Mathematically, all the Relations, Predicates, Properties, and Functions CAN BE represented simply as sets of  $m$ -tuples of objects:
- Let  $W$  be the set of objects in the world.
- Let  $W^m = W \times W \times \dots (m \text{ times}) \dots \times W$ 
  - The set of all possible  $m$ -tuples of objects from the world
- An  **$m$ -ary Relation** is a subset of  $W^m$ .
  - Example: Let  $W = \{John, Sue, Bill\}$
  - Then  $W^2 = \{<John, John>, <John, Sue>, \dots, <Sue, Sue>\}$
  - E.g.,  $MarriedTo = \{<John, Sue>, <Sue, John>\}$
  - E.g.,  $FatherOf = \{<John, Bill>\}$
- Analogous to a constraint in CSPs
  - The constraint lists the  $m$ -tuples that satisfy it.
  - The relation lists the  $m$ -tuples that participate in it.

# Syntax : Relations, Predicates, Properties, Functions

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- A **Predicate** is a list of  $m$ -tuples making the predicate true.
  - E.g., PrimeFactorOf =  $\{ \langle 2, 4 \rangle, \langle 2, 6 \rangle, \langle 3, 6 \rangle, \langle 2, 8 \rangle, \langle 3, 9 \rangle, \dots \}$
  - This is the same as an  $m$ -ary Relation.
  - Predicates (and properties) generally correspond to English verbs.
- A **Property** lists the  $m$ -tuples that have the property.
  - Formally, it is a predicate that is true of tuples having that property.
  - E.g., IsRed =  $\{ \langle \text{Ball-5} \rangle, \langle \text{Toy-7} \rangle, \langle \text{Car-11} \rangle, \dots \}$
  - This is the same as an  $m$ -ary Relation.
- A **Function** CAN BE represented as an  $m$ -ary relation
  - the first  $(m-1)$  objects are the arguments and the  $m^{\text{th}}$  is the value.
  - E.g., Square =  $\{ \langle 1, 1 \rangle, \langle 2, 4 \rangle, \langle 3, 9 \rangle, \langle 4, 16 \rangle, \dots \}$
- An **Object** CAN BE represented as a function of zero arguments that returns the object.
  - This is just a 1-ary relationship.

## Syntax of FOL: Terms

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- **Term** = logical expression that **refers to an object**
- **There are two kinds of terms:**
  - **Constant Symbols** stand for (or name) objects:
    - E.g., KingJohn, 2, UCI, Wumpus, ...
  - **Function Symbols** map tuples of objects to an object:
    - E.g., LeftLeg(KingJohn), Mother(Mary), Sqrt(x)
    - This is nothing but a complicated kind of name
      - No “subroutine” call, no “return value”

## Syntax of FOL: Atomic Sentences

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- **Atomic Sentences** state facts (logical truth values).
  - An **atomic sentence** is a Predicate symbol, optionally followed by a parenthesized list of any argument terms
  - E.g., *Married( Father(Richard), Mother(John) )*
  - An **atomic sentence** asserts that some relationship (some predicate) holds among the objects that are its arguments.
- An **Atomic Sentence is true** in a given model if the relation referred to by the predicate symbol holds among the objects (terms) referred to by the arguments.

# Syntax of FOL: Atomic Sentences

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- Atomic sentences in logic state facts that are true or false.
- Properties and  $m$ -ary relations do just that:
  - LargerThan(2, 3) is false.
  - BrotherOf(Mary, Pete) is false.
  - Married(Father(Richard), Mother(John)) could be true or false.Properties and  $m$ -ary relations are Predicates that are true or false.
- Note: Functions refer to objects, do not state facts, and form no sentence:
  - Brother(Pete) refers to John (his brother) and is neither true nor false.
  - Plus(2, 3) refers to the number 5 and is neither true nor false.
- BrotherOf( Pete, Brother(Pete) ) is True.

↑  
Binary relation  
is a truth value.

↑  
Function refers to John, an object in the  
world, i.e., John is Pete's brother.  
(Works well iff John is Pete's only brother.)

# Syntax of FOL: Connectives & Complex Sentences

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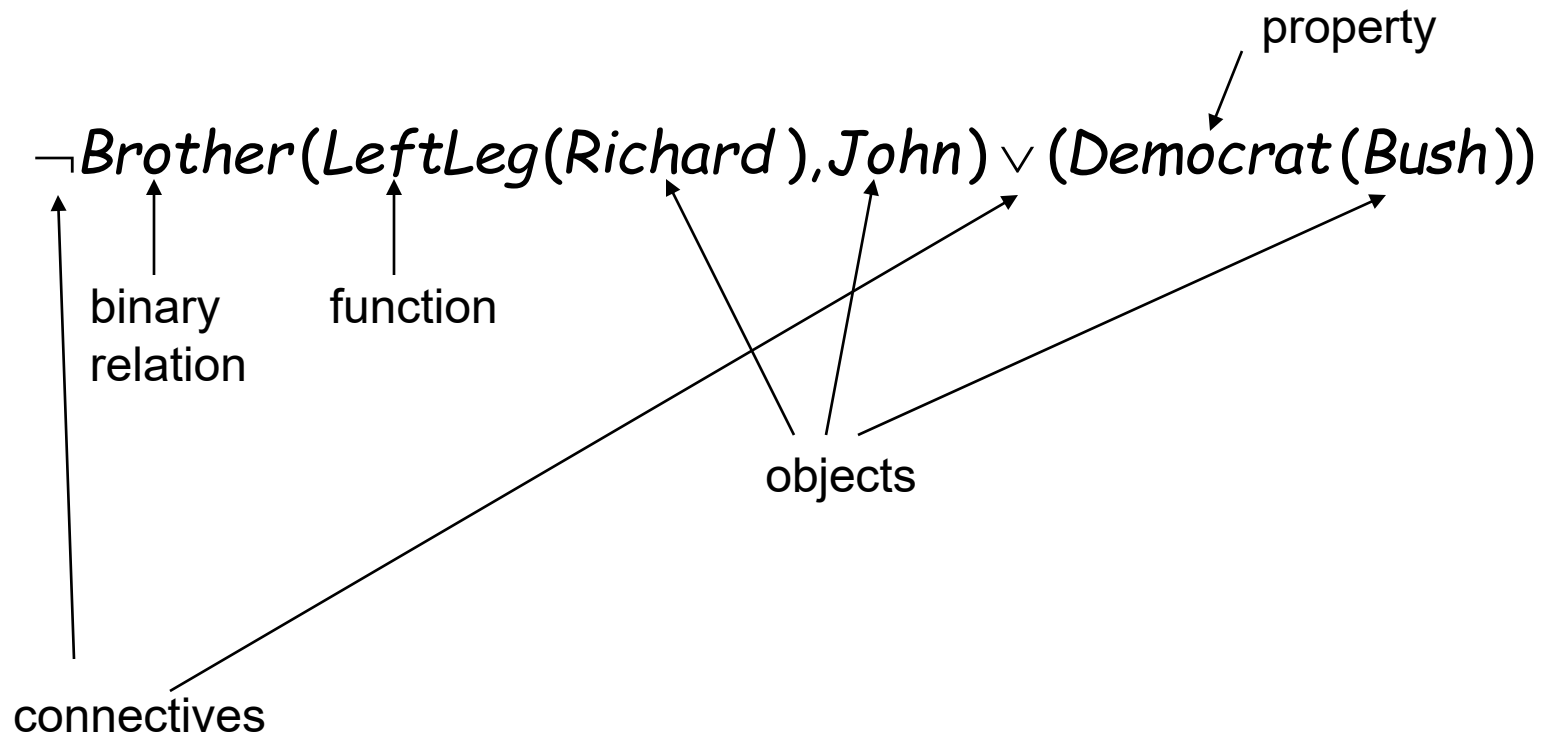
- **Complex Sentences** are formed in the same way, and are formed using the same logical connectives, as we already know from propositional logic
- The **Logical Connectives**:
  - $\Leftrightarrow$  biconditional
  - $\Rightarrow$  implication
  - $\wedge$  and
  - $\vee$  or
  - $\neg$  negation
- **Semantics** for these logical connectives are the same as we already know from propositional logic.



# Complex Sentences

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- We make complex sentences with connectives (just like in propositional logic).



## Examples

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- $\text{Brother}(\text{Richard}, \text{John}) \wedge \text{Brother}(\text{John}, \text{Richard})$
- $\text{King}(\text{Richard}) \vee \text{King}(\text{John})$
- $\text{King}(\text{John}) \Rightarrow \neg \text{King}(\text{Richard})$
- $\text{LessThan}(\text{Plus}(1,2), 4) \wedge \text{GreaterThan}(1,2)$

(Semantics of complex sentences are the same as in propositional logic)

## Syntax of FOL: Variables

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- **Variables** range over objects in the world.
- A **variable** is like a **term** because it represents an object.
- A **variable** may be used wherever a **term** may be used.
  - **Variables** may be arguments to functions and predicates.
- (A **term with NO variables** is called a **ground term**.)
- (A **variable not bound by a quantifier** is called **free**.)

# Syntax of FOL: Logical Quantifiers

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- There are two **Logical Quantifiers**:
  - **Universal:**  $\forall x P(x)$  means "For all x, P(x)."
    - The "upside-down A" reminds you of "ALL."
  - **Existential:**  $\exists x P(x)$  means "There exists x such that, P(x)."
    - The "upside-down E" reminds you of "EXISTS."
- Syntactic "sugar" --- we really only need one quantifier.
  - $\forall x P(x) \equiv \neg \exists x \neg P(x)$
  - $\exists x P(x) \equiv \neg \forall x \neg P(x)$
  - You can ALWAYS convert one quantifier to the other.
- **RULES:**  $\forall \equiv \neg \exists \neg$  and  $\exists \equiv \neg \forall \neg$
- **RULE:** To move negation "in" across a quantifier, change the quantifier to "the other quantifier" and negate the predicate on "the other side."
  - $\neg \forall x P(x) \equiv \exists x \neg P(x)$
  - $\neg \exists x P(x) \equiv \forall x \neg P(x)$

# Universal Quantification $\forall$

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- $\forall$  means “for all”
- Allows us to make statements about all objects that have certain properties
- Can now state general rules:

$\forall x \text{ King}(x) \Rightarrow \text{Person}(x)$  “All kings are persons.”

$\forall x \text{ Person}(x) \Rightarrow \text{HasHead}(x)$  “Every person has a head.”

$\forall i \text{ Integer}(i) \Rightarrow \text{Integer}(\text{plus}(i,1))$  “If  $i$  is an integer then  $i+1$  is an integer.”

Note that

$\forall x \text{ King}(x) \wedge \text{Person}(x)$  is not correct!

This would imply that all objects  $x$  are Kings and are People

$\forall x \text{ King}(x) \Rightarrow \text{Person}(x)$  is the correct way to say this

Note that  $\Rightarrow$  is the natural connective to use with  $\forall$ .

## Universal Quantification $\forall$

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- Universal quantification is equivalent to:
  - Conjunction of all sentences obtained by substitution of an object for the quantified variable.
- All Cats are Mammals.
  - $\forall x \text{ Cat}(x) \Rightarrow \text{Mammal}(x)$
- Conjunction of all sentences obtained by substitution of an object for the quantified variable:  
*Cat(Spot)  $\Rightarrow$  Mammal(Spot)  $\wedge$*   
*Cat(Rick)  $\Rightarrow$  Mammal(Rick)  $\wedge$*   
*Cat(LAX)  $\Rightarrow$  Mammal(LAX)  $\wedge$*   
*Cat(Shayama)  $\Rightarrow$  Mammal(Shayama)  $\wedge$*   
*Cat(France)  $\Rightarrow$  Mammal(France)  $\wedge$*   
*Cat(Felix)  $\Rightarrow$  Mammal(Felix)  $\wedge$*   
*...*

# Existential Quantification $\exists$

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- $\exists x$  means “there exists an  $x$  such that....” (at least one object  $x$ )
- Allows us to make statements about some object without naming it
- Examples:

$\exists x \text{ King}(x)$  “Some object is a king.”

$\exists x \text{ Lives\_in}(\text{John}, \text{Castle}(x))$  “John lives in somebody’s castle.”

$\exists i \text{ Integer}(i) \wedge \text{GreaterThan}(i, 0)$  “Some integer is greater than zero.”

Note that  $\wedge$  is the natural connective to use with  $\exists$

(And note that  $\Rightarrow$  is the natural connective to use with  $\forall$  )

## Existential Quantification $\exists$

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- Existential quantification is equivalent to:
  - Disjunction of all sentences obtained by substitution of an object for the quantified variable.
- Spot has a sister who is a cat.
  - $\exists x \text{ Sister}(x, \text{Spot}) \wedge \text{Cat}(x)$
- Disjunction of all sentences obtained by substitution of an object for the quantified variable:
  - $\text{Sister}(\text{Spot}, \text{Spot}) \wedge \text{Cat}(\text{Spot}) \vee$
  - $\text{Sister}(\text{Rick}, \text{Spot}) \wedge \text{Cat}(\text{Rick}) \vee$
  - $\text{Sister}(\text{LAX}, \text{Spot}) \wedge \text{Cat}(\text{LAX}) \vee$
  - $\text{Sister}(\text{Shayama}, \text{Spot}) \wedge \text{Cat}(\text{Shayama}) \vee$
  - $\text{Sister}(\text{France}, \text{Spot}) \wedge \text{Cat}(\text{France}) \vee$
  - $\text{Sister}(\text{Felix}, \text{Spot}) \wedge \text{Cat}(\text{Felix}) \vee$
  - ...



## Combining Quantifiers --- Order (Scope)

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The order of “unlike” quantifiers is important.

**Like nested variable scopes in a programming language**

**Like nested ANDs and ORs in a logical sentence**

$\forall x \exists y \text{ Loves}(x,y)$

- For everyone (“all x”) there is someone (“exists y”) whom they love.
- There might be a different y for each x (y is inside the scope of x)

$\exists y \forall x \text{ Loves}(x,y)$

- There is someone (“exists y”) whom everyone loves (“all x”).
- Every x loves the same y (x is inside the scope of y)

Clearer with parentheses:  $\exists y ( \forall x \text{ Loves}(x,y) )$

The order of “like” quantifiers does not matter.

**Like nested ANDs and ANDs in a logical sentence**

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

## Fun with sentences

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$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x).$$

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One's mother is one's female parent

$$\forall x, y \text{ Mother}(x, y) \Leftrightarrow (\text{Female}(x) \wedge \text{Parent}(x, y)).$$

A first cousin is a child of a parent's sibling

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$$\forall x, y \text{ Mother}(x, y) \Leftrightarrow (\text{Female}(x) \wedge \text{Parent}(x, y)).$$

A first cousin is a child of a parent's sibling

$$\forall x, y \text{ FirstCousin}(x, y) \Leftrightarrow \exists p, ps \text{ Parent}(p, x) \wedge \text{Sibling}(ps, p) \wedge \text{Parent}(ps, y)$$

## Connections between Quantifiers

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- Asserting that all  $x$  have property  $P$  is the same as asserting that does not exist any  $x$  that does not have the property  $P$

$$\forall x \text{ Likes}(x, \text{CS-171 class}) \Leftrightarrow \neg \exists x \neg \text{Likes}(x, \text{CS-171 class})$$

- Asserting that there exists an  $x$  with property  $P$  is the same as asserting that not all  $x$  do not have the property  $P$

$$\exists x \text{ Likes}(x, \text{IceCream}) \Leftrightarrow \neg \forall x \neg \text{Likes}(x, \text{IceCream})$$

In effect:

- $\forall$  is a conjunction over the universe of objects
- $\exists$  is a disjunction over the universe of objects

Thus, DeMorgan's rules can be applied

## De Morgan's Law for Quantifiers

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De Morgan's Rule

$$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$$

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

Generalized De Morgan's Rule

$$\forall x P \equiv \neg \exists x (\neg P)$$

$$\exists x P \equiv \neg \forall x (\neg P)$$

$$\neg \forall x P \equiv \exists x (\neg P)$$

$$\neg \exists x P \equiv \forall x (\neg P)$$

Rule is simple: if you bring a negation inside a disjunction or a conjunction, always switch between them (or  $\rightarrow$  and, and  $\rightarrow$  or).



## Aside: More syntactic sugar --- uniqueness

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- $\exists!$  x is “syntactic sugar” for “There exists a unique x”
  - “There exists one and only one x”
  - “There exists exactly one x”
  - Sometimes  $\exists!$  is written as  $\exists^1$
- For example,  $\exists! x \text{ PresidentOfTheUSA}(x)$ 
  - “There is exactly one PresidentOfTheUSA.”
- This is just syntactic sugar:
  - $\exists! x P(x)$  is the same as  $\exists x P(x) \wedge (\forall y P(y) \Rightarrow (x = y))$

# Equality

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- $term_1 = term_2$  is true under a given interpretation if and only if  $term_1$  and  $term_2$  refer to the same object
- E.g., definition of *Sibling* in terms of *Parent*:

$$\begin{aligned} \forall x, y \text{ Sibling}(x, y) \Leftrightarrow \\ [\neg(x = y) \wedge \\ \exists m, f \neg (m = f) \wedge \text{Parent}(m, x) \wedge \text{Parent}(f, x) \\ \wedge \text{Parent}(m, y) \wedge \text{Parent}(f, y)] \end{aligned}$$

**Equality can make reasoning much more difficult!**

**(See R&N, section 9.5.5, page 353)**

You may not know when two objects are equal.

E.g., Ancients did not know (MorningStar = EveningStar = Venus)

You may have to prove  $x = y$  before proceeding

E.g., a resolution prover may not know  $2+1$  is the same as  $1+2$

# Syntactic Ambiguity

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- FOPC provides many ways to represent the same thing.
- E.g., “Ball-5 is red.”
  - HasColor(Ball-5, Red)
    - Ball-5 and Red are objects related by HasColor.
  - Red(Ball-5)
    - Red is a unary predicate applied to the Ball-5 object.
  - HasProperty(Ball-5, Color, Red)
    - Ball-5, Color, and Red are objects related by HasProperty.
  - ColorOf(Ball-5) = Red
    - Ball-5 and Red are objects, and ColorOf() is a function.
  - HasColor(Ball-5(), Red())
    - Ball-5() and Red() are functions of zero arguments that both return an object, which objects are related by HasColor.
  - ...
- This can GREATLY confuse a pattern-matching reasoner.
  - Especially if multiple people collaborate to build the KB, and they all have different representational conventions.

# Syntactic Ambiguity --- Partial Solution

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- FOL can be TOO expressive, can offer TOO MANY choices
- Likely confusion, especially for **teams** of Knowledge Engineers
- Different team members can make different representation choices
  - E.g., represent “Ball43 is Red.” as:
    - a predicate (= verb)? E.g., “Red(Ball43)” ?
    - an object (= noun)? E.g., “Red = Color(Ball43)” ?
    - a property (= adjective)? E.g., “HasProperty(Ball43, Red)” ?
- PARTIAL SOLUTION:
  - An upon-agreed **ontology** that settles these questions
  - Ontology = what exists in the world & how it is represented
  - The Knowledge Engineering teams agrees upon an ontology BEFORE they begin encoding knowledge

# Summary

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- First-order logic:
  - Much more expressive than propositional logic
  - Allows objects and relations as semantic primitives
  - Universal and existential quantifiers
- Syntax: constants, functions, predicates, equality, quantifiers
- Nested quantifiers
  - Order of unlike quantifiers matters (the outer scopes the inner)
    - Like nested ANDs and ORs
  - Order of like quantifiers does not matter
    - like nested ANDs and ANDs
- Translate simple English sentences to FOPC and back