DM 3 - convex optimisation

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Question 1

We introduce the auxiliary variable

$$z = Xw - y \qquad \iff \qquad z + y - Xw = 0,$$

so the problem becomes

$$\min_{w,z} \frac{1}{2}\|z\|_2^2 + \lambda\|w\|_1 \quad \text{s.t.} \quad z + y - Xw = 0.$$

The Lagrangian :

$$L(w,z,\mu) = \frac{1}{2}\|z\|_2^2 + \lambda\|w\|_1 + \mu^\top(z + y - Xw) = \mu^\top y + \left(\frac{1}{2}\|z\|_2^2 + \mu^\top z\right) + \left(\lambda\|w\|_1 - \mu^\top Xw\right).$$

Thus the dual function is

$$g(\mu) = \inf_{w,z} L(w,z,\mu) = \mu^\top y + \inf_z \left(\frac{1}{2}\|z\|_2^2 + \mu^\top z\right) + \inf_w \left(\lambda\|w\|_1 - \mu^\top Xw\right).$$

For the z–term, note that

$$\frac{1}{2}\|z\|_2^2 + \mu^\top z = \sum_{i=1}^n \left(\frac{1}{2}z_i^2 + \mu_i z_i\right)$$

so the infimum is attained at $z = -\mu$ and equals $-\frac{1}{2}\|\mu\|_2^2$.

For the w–term, write $a_j = \mu^\top x_j$ where x_j is the j-th column of X, so

$$\lambda\|w\|_1 - \mu^\top Xw = \lambda\sum_{j=1}^d |w_j| - \sum_{j=1}^d a_j w_j.$$

If $w_j = t$, with $t > 0$ and $w_i = 0$ if $i \neq j$ one has:

$$\lambda\|w\|_1 - \mu^\top Xw = \lambda t - a_j t = (\lambda - a_j)t.$$

If $\lambda - a_j < 0$ then $(\lambda - a_j)t \to -\infty$ as $t \to +\infty$, so the function is unbounded below in this direction.

If $w_j = -t$, with $t > 0$ and $w_i = 0$ if $i \neq j$, one has:

$$\lambda\|w\|_1 - \mu^\top Xw = \lambda t - a_j(-t) = (\lambda + a_j)t.$$

If $\lambda + a_j < 0$ then $(\lambda + a_j)t \to -\infty$ as $t \to +\infty$ and the function is unbounded below in this direction.

Therefore, for the objective to be bounded below for all such directions, we must have

$$\lambda - a_j \geq 0 \quad \text{and} \quad \lambda + a_j \geq 0,$$

i.e.

$$-\lambda \le a_j \le \lambda \qquad \forall j,$$

which is exactly

$$\|X^\top \mu\|_\infty = \max_j |a_j| \le \lambda.$$

Now assume that this condition holds, i.e. $\|X^\top \mu\|_\infty \le \lambda$. Since $\|X^\top \mu\|_\infty \le \lambda$, we have $|a_j| \le \lambda$ for all j. Write

$$\lambda |w_j| - a_j w_j = \lambda\Big(|w_j| - \tfrac{a_j}{\lambda} w_j\Big).$$

Since $|\frac{a_j}{\lambda}| \le 1$, we deduce that for all w, the function is positive, with it attaining a minimum of 0 for $w = 0$.

Hence the dual problem is

$$\max_{\mu \in \mathbb{R}^n} \; \mu^\top y - \frac{1}{2}\|\mu\|_2^2 \quad \text{s.t.} \quad \|X^\top \mu\|_\infty \le \lambda.$$

Rewriting as a minimization yields

$$\min_{\mu \in \mathbb{R}^n} \; \frac{1}{2}\|\mu\|_2^2 - y^\top \mu \quad \text{s.t.} \quad X^\top \mu \le \lambda \mathbf{1}_d, \qquad -X^\top \mu \le \lambda \mathbf{1}_d.$$

This fits the quadratic program $\min v^\top Q v + p^\top v$ with

$$Q = \frac{1}{2} I_n, \qquad p = -y,$$

and constraint matrix

$$A = \begin{pmatrix} X^\top \\ -X^\top \end{pmatrix}, \qquad b = \lambda \begin{pmatrix} \mathbf{1}_d \\ \mathbf{1}_d \end{pmatrix}.$$

$$\boxed{\min_{v \in \mathbb{R}^n} \; v^\top \Big(\tfrac{1}{2} I_n\Big) v - y^\top v \quad \text{s.t.} \quad \begin{pmatrix} X^\top \\ -X^\top \end{pmatrix} v \le \lambda \begin{pmatrix} \mathbf{1}_d \\ \mathbf{1}_d \end{pmatrix}.}$$

Question 2

Given the quadratic objective

$$f_0(x) = x^\top Q x + p^\top x,$$

and the inequality constraints $Ax \le b$, the barrier–augmented function for a parameter $\tau > 0$ is

$$F_\tau(x) = \tau\big(x^\top Q x + p^\top x\big) \; - \; \sum_{i=1}^{m} \log\big(b_i - a_i^\top x\big),$$

where $a_i^\top$ is the i-th row of A.

The gradient splits as

$$\nabla F_\tau(x) = \tau \, \nabla(x^\top Q x + p^\top x) \; + \; \sum_{i=1}^{m} \nabla\Big[-\log(b_i - a_i^\top x)\Big].$$

For the quadratic part,

$$\nabla(x^\top Q x) = 2Qx, \qquad \nabla(p^\top x) = p,$$

so

$$\tau \, \nabla(x^\top Q x + p^\top x) = \tau(2Qx + p).$$

For the barrier term,

$$\nabla\Big[-\log(b_i - a_i^\top x)\Big] = \frac{a_i}{b_i - a_i^\top x}.$$

Thus the full gradient is

$$\nabla F_\tau(x) = \tau(2Qx + p) \; + \sum_{i=1}^{m} \frac{a_i}{b_i - a_i^\top x}.$$

For the Hessian, differentiate again:

$$\nabla\left(\frac{a_i}{b_i - a_i^\top x}\right) = \frac{a_i a_i^\top}{(b_i - a_i^\top x)^2}.$$

Hence the full Hessian is

$$\nabla^2 F_\tau(x) = 2\tau Q \; + \sum_{i=1}^{m} \frac{a_i a_i^\top}{(b_i - a_i^\top x)^2}.$$

Question 3

In Figure 1 and Figure 2, the effect of different values of μ is illustrated. Figure 1 corresponds to 10 datapoints in dimension 100, while Figure 2 corresponds to 100 datapoints in dimension 1000. In both cases, the results a similar.

The left side of the figures show how convergence changes for different values of μ. We see that higher mu values lead to significantly faster convergence, which is no surprise. The L2 distance matrix to the right side of the plot compares the final estimation of the w values. They are very similar (considering that these distance values are for high dimension), notably showing that increasing μ does not hinder performance.

As such, a high value of μ like 50 or 100 to get both low compute time and good results is the best configuration.

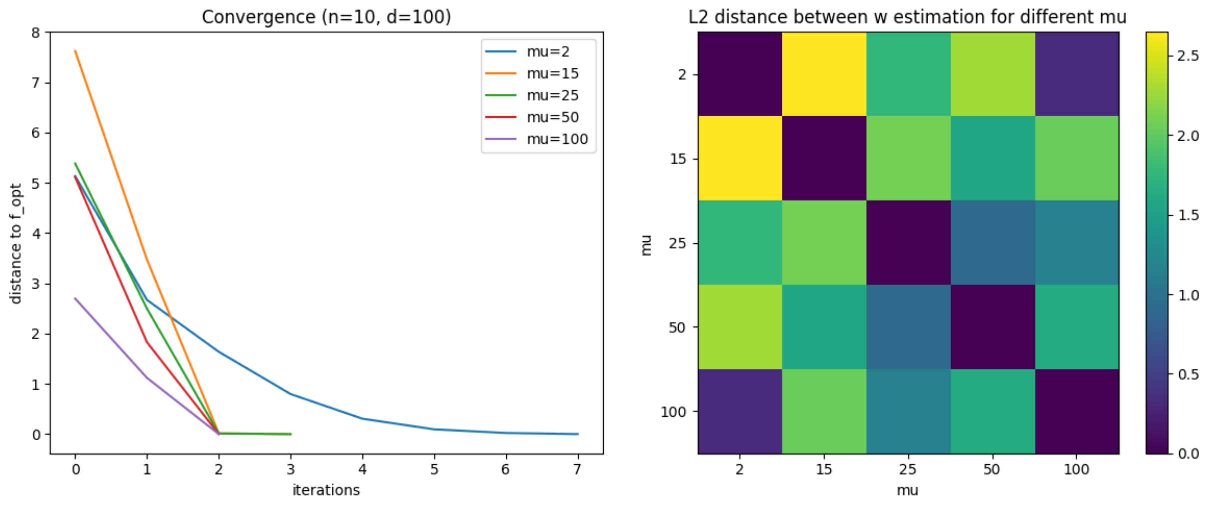


Figure 1: Convergence of barr_method and distance matrix of the final w for different values of μ for $n = 10$ and $d = 100$

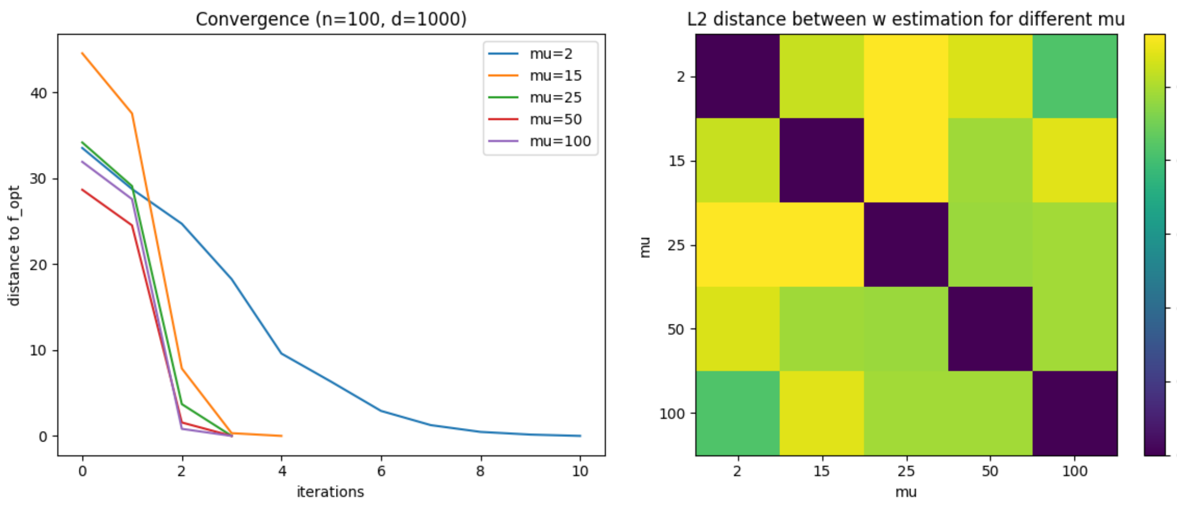


Figure 2: Convergence of barr_method and distance matrix of the final w for different values of μ for $n = 100$ and $d = 100$

Importing packages

In [1]:
```python
import numpy as np
import matplotlib.pyplot as plt
```

I. Centering step

Functions for gradient and hessian computation

In [2]:
```python
def compute_gradient_ft(t, x, Q, p, A ,b):
    # t the scalar that charachterises the functionn we are trying to minimise
    # x of dimension (n), the point in which we are evaluating the function
    # Q of dimension (n,n)
    # p of dimenson (n)
    # A of dimension (m,n)
    # b of dimension (m)

    m = b.size
    n = x.size
    left_side = t*(2*Q @ x + p)
    right_side = np.zeros(n)
    for i in range (m):
#       print((b[i] - A[i] @ x).shape)
#       print(f"A shape: {A.shape}")
#       print(f"x shape: {x.shape}")
#       print(f"b shape: {b.shape}")
        right_side += A[i]  / (b[i] - A[i] @ x)



    return left_side + right_side


def compute_hessian_ft(t, x, Q, A, b):
    # t the scalar that charachterises the functionn we are trying to minimise
    # x of dimension (n), the point in which we are evaluating the function
    # Q of dimension (n,n)
    # A of dimension (m,n)
    # b of dimension (m)

    m = b.size
    n = x.size
    left_side = 2 * t * Q
    right_side = np.zeros((n,n))
    for i in range(m):
        right_side += (np.outer(A[i], A[i]) ) / (b[i] - A[i]  @ x)**2

    return left_side + right_side
```

Additonal utility functions

In [3]:
```python
# returns bool to check if the stop condition of the newton algorithm is met
def newton_dercrement(grad, delta_x, eps):
    return - grad @ delta_x/ 2 <= eps

def compute_delta_x(grad, hessian):
    return np.linalg.solve(hessian, -grad)

def backtracking(ft, x, delta_x, grad, alpha, beta): # returns the optimal x con
    # x of shape (n)
    # delta_x of shape (n)
    # grad of shape (n)
    step = 1
    while ft(x + step * delta_x) >= ft(x) + alpha * step * grad @ delta_x:
        step = beta * step
    return x + step * delta_x
```

Centring step function

In [4]:
```python
def centering_step(Q, p, A, b, t, v0, eps):
    # initialisation
    v_running = v0
    list_v_points = []
    count = 0  # to ensure there is at least one iteration
    alpha = 0.3
    beta = 0.5

    while True:
        # do the matrix computations
        grad = compute_gradient_ft(t, v_running, Q, p, A ,b)
        hessian = compute_hessian_ft(t, v_running, Q, A, b)
        delta_v = compute_delta_x(grad, hessian)

        # check for the stop condition
        if newton_dercrement(grad, delta_v, eps) and count > 0:
            break

        # define the f_t function so it can be used in backtracking
        def ft(x):
            s = b - A @ x
            if np.any(s <= 0):  # value at +infinity if out of bounds
                return np.inf

            right_side = t * (x @ Q @ x + p @ x)
            left_side  = -np.sum(np.log(s))
            return right_side + left_side

        # backtracking to upgrade v with the optimal step
        v_running = backtracking(ft, v_running, delta_v, grad, alpha, beta)
        list_v_points.append(v_running)
        count += 1
    return list_v_points

# dimensions
n = 10
d = 2

#LASSO hyperprameter
lambda_val = 10
```

```
# primal matrix sampling
X = np.random.randn(n, d)
y = np.random.randn(n)

# dual variables reformating
Q  = np.eye(n) / 2
p = -y
A = np.concatenate([X.T, - X.T], axis=0)
b = lambda_val * np.ones(2 * d)

# interior points and Newton hyperparametres
v0 = np.zeros(n)
eps_newton = 0.01

t = 1
return_array = centering_step(Q, p, A, b, t, v0, eps_newton)
```

Log-barrier method (interior points)

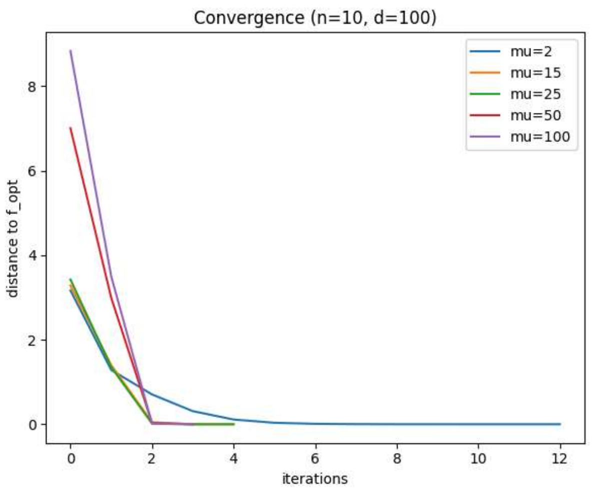
In [5]:
```
def barr_method(Q, p, A, b, v0, eps_barr=0.1, eps_newton=0.1, mu = 1.2):
    # set the parameters needed
    m = b.size
    t = 1

    # run Newton once
    list_v_barr = [v0]
    v_running = centering_step(Q, p, A, b, t, v0, eps_newton)[-1]
    list_v_barr.append(v_running)

    # iterate through the algorithm as long as m/t < eps is not verivied
    while eps_barr <= m / t:
        t = mu * t
        v_running = centering_step(Q, p, A, b, t, v0, eps_newton)[-1]
        list_v_barr.append(v_running)

    return list_v_barr

# dimensions
n = 100
d = 30

#LASSO hyperprameter
lambda_val = 10

def generate_variables(n, d, lambda_val):
    # primal matrix sampling
    X = np.random.randn(n, d)
    y = np.random.randn(n)

    # dual variables reformating
    Q  = np.eye(n) / 2
    p = -y
    A = np.concatenate([X.T, - X.T], axis=0)
    b = lambda_val * np.ones(2 * d)

    v0 = np.zeros(n)
    return Q, p, A, b, v0


Q, p, A, b, v0= generate_variables(n, d, lambda_val)

# interior points and Newton hyperparametres

eps_barr = 0.1
mu = 1.2

list_v_barr = barr_method(Q, p, A, b, v0, eps_barr=eps_barr, mu=mu)
```

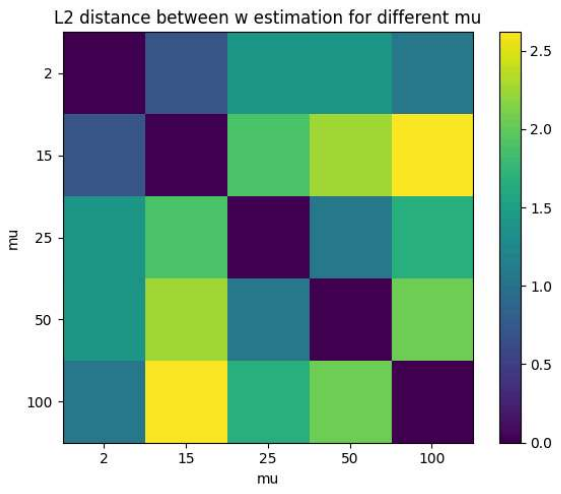
Running and plotting results

In [12]:

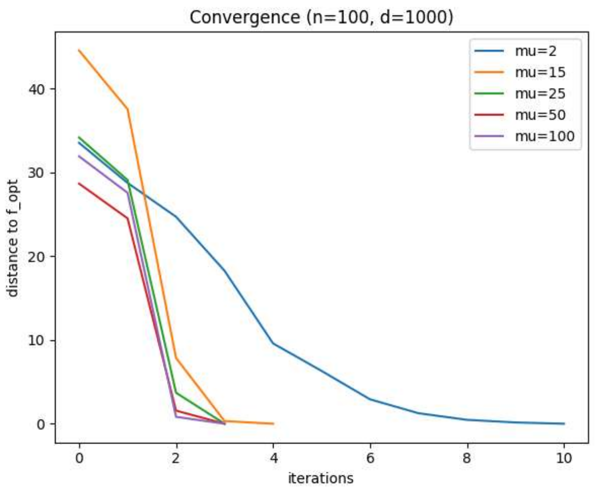
```
def f(x, Q, p):
    return x @ Q @ x + p @ x


def plot_all_in_one(n, d, mus, lambda_val,eps_barr=0.1, eps_newton=0.1):
    fig, (ax_traj, ax_mat) = plt.subplots(1, 2, figsize=(12, 5))

    final_vectors = []  # store list_v_barr[-1] for each mu

    # left: convergence curves
    for mu in mus:
        Q, p, A, b, v0 = generate_variables(n, d, lambda_val)
        list_v_barr = barr_method(Q, p, A, b, v0,
                                  eps_barr=eps_barr,
                                  eps_newton=eps_newton,
                                  mu=mu)

        x_final = list_v_barr[-1]
        final_vectors.append(x_final)

        f_opt = f(x_final, Q, p)
        f_diff = [f(v, Q, p) - f_opt for v in list_v_barr]
        ax_traj.plot(f_diff, label=f"mu={mu}")

    ax_traj.set_title(f"Convergence (n={n}, d={d})")
    ax_traj.set_xlabel("iterations")
    ax_traj.set_ylabel("distance to f_opt")
    ax_traj.legend()

    # right: MSE matrix between final vectors
    X = np.vstack(final_vectors)                # shape (k, d)
    diffs = X[:, None, :] - X[None, :, :]       # shape (k, k, d)
    mse_mat = np.mean(diffs**2, axis=2)         # shape (k, k)

    im = ax_mat.imshow(mse_mat, interpolation='nearest')
    plt.colorbar(im, ax=ax_mat)
    ax_mat.set_title("L2 distance between w estimation for different mu")
    ax_mat.set_xticks(range(len(mus)))
    ax_mat.set_yticks(range(len(mus)))
    ax_mat.set_xticklabels(mus)
    ax_mat.set_yticklabels(mus)
    ax_mat.set_xlabel("mu")
    ax_mat.set_ylabel("mu")
    plt.tight_layout()
    plt.show()
```

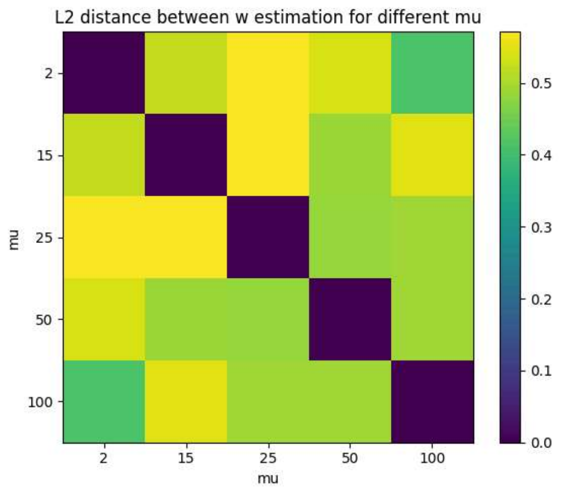
In [13]:
```
lambda_val = 10
n = 10
d = int(1e2)
mu_list = [2, 15, 25, 50, 100]
plot_all_in_one(n, d, mu_list, lambda_val)
```





In [9]:
```
n = 100
d = int(1e3)
mu_list = [2, 15, 25, 50, 100]
plot_all_in_one(n, d, mu_list)
```





observation: increasing μ reduces the number of iterations and, considering the high-dimension w is in, the mse between (w_μ) is small. So a high value of μ like 50 or 100 should be selected so as to have fewer steps and faster computation.