1. a) $$\int_0^{\infty} f(x)\,\delta(2x-3)\,dx = \int_0^{\infty} f(x)\,\frac{1}{2}\,\delta(x-3/2)$$
$$= \frac{1}{2} f(3/2)$$

b) $$\int_0^{2\pi} \frac{\sin z}{\cos^2(2z)+\tan^2(z/2)}\,\delta(\pi - z) = \frac{0}{1+0} = 0$$

c) $$I = \int_0^{3} f(x)\,\delta(2x^2-x-1)\,dx \quad , g(x) = 2x^2-x-1$$
$$= (2x+1)(x-1)$$
$$I = \int_0^{3} f(x)\,\delta((2x+1)(x-1))\,dx \quad g'(x) = 4x-1$$
$$= \left[\frac{f(1)}{|g'(1)|}\right] = \frac{f(1)}{3}$$

d) $$I = \int_{-\infty}^{\infty} \cos(x)\,\delta(x^3-\pi^2 x)\,dx \quad , g(x) = x^3-\pi^2 x = x(x^2-\pi^2)$$
$$= x(x+\pi)(x-\pi)$$
$$g'(x) = 3x^2-\pi^2$$
$$g'(0) = -\pi^2, \; g'(\pi) = 2\pi^2$$
$$g'(-\pi) = 2\pi^2$$
$$= \left[\frac{\cos(0)}{\pi^2} + \frac{2\cos(\pi)}{2\pi^2}\right]$$
$$= 0$$

e) $$I = \int_{-\infty}^{\infty} e^{x}\,\delta(x^3)\,dx \quad \text{take } u = x^3, \; du = 3x^2\,dx$$

$$= \int_{-\infty}^{\infty} e^{u^{1/3}} \delta(u) \frac{du}{3x^2}$$ $\delta(u)=0$ for $u=x=0$, but integral diverges for $x \to 0$

f) $$I = \int_0^{2\pi} \sin\theta \, \delta(e^\theta \cos\theta) \, d\theta$$, $g(\theta) = e^\theta \cos\theta$

$g'(\theta) = e^\theta \cos\theta - e^\theta \sin\theta$

$$= \left[\frac{\sin(\pi/2)}{e^{\pi/2}} + \frac{\sin(3\pi/2)}{e^{3\pi/2}} \right]$$

$g'(\pi/2) = -e^{\pi/2}$

$g'(3\pi/2) = e^{3\pi/2}$

$$= \frac{1}{e^{\pi/2}} - \frac{1}{e^{3\pi/2}}$$

g) $$I = \int_{-\infty}^{\infty} f(x) \delta(2x^2+1)$$ $g(x) = 2x^2+1 = 2(x+i/2)(x-i/2)$

$= 0$ has no real roots

h) $$I = \int^{r} f(x) \delta(\sqrt{r^2-x^2}-2) \, dx$$, $r > 2$

$g(x) = \sqrt{r^2-x^2}-2 \Rightarrow g(x) = 0$ for $x = \pm\sqrt{r^2-4}$

$g'(x) = \frac{1}{2}(r^2-x^2)^{-1/2} \cdot (-2x) = -x(r^2-x^2)^{-1/2}$

$g'(\pm\sqrt{r^2-4}) = \pm\sqrt{r^2-4}\,(4)^{-1/2} = \pm\frac{1}{2}\sqrt{r^2-4}$

$$I = 2\int_0^{r} f(x) \left[\frac{\delta(x-\sqrt{r^2-4}) + \delta(x+\sqrt{r^2-4})}{\sqrt{r^2-4}} \right] dx$$

for $r>2$, the roots are real, however because $r>2$ $x = -\sqrt{r^2-4}$ will be less than zero and will not contribute

$$I = \frac{2 f\left(\sqrt{r^2-4}\right)}{\sqrt{r^2-4}}$$

for $r < 2$, all roots are not simple real roots so $I = 0$

$$2\text{a)}\quad I = \int_{-\infty}^{\infty} f(x)\, \frac{d\delta(x-1)}{dx}\, dx$$

$$dv = \frac{d\delta(x-1)}{dx}\, dx$$

$$v = \delta(x-1)$$

$$u = f(x),\ du = f'(x)\, dx$$

$$= -\int_{-\infty}^{\infty} \delta(x-1) f'(x)\, dx$$

$$= -f'(1)$$

$$\text{b)}\quad I = \int_{-5\pi/2}^{5\pi/2} e^{-2x}\, \delta'(\sin x)\, dx = -\int_{-5\pi/2}^{5\pi/2} \delta(\sin x)\, \frac{d}{dx} e^{-2x}\, dx$$

$$= 2\int_{-5\pi/2}^{5\pi/2} \delta(\sin x)\, e^{-2x}\, dx \qquad g(x) = \sin x,\ g = 0 \text{ for } x = 0, \pm\pi, \pm 2\pi$$

$$= 2\left[1 + e^{2\pi} + e^{-2\pi} + e^{\pi} + e^{-\pi}\right] = 4\left[2 + \cosh(\pi) + \cosh(2\pi)\right]$$

$$\infty \qquad \tfrac{1}{2}(x+2) \qquad \sqrt{6}\left(z + \tfrac{1}{\sqrt{6}}\right)$$

c) $$I = \int_{-\infty}^{\infty} dx\,dy\,dz\, f(x,y,z)\, \delta\Big(\overset{\nearrow \frac{1}{2}(x+2)}{\tfrac{x}{2}+1}\Big)\, \delta(y-3)\, \delta\Big(\overset{\nearrow \sqrt{6}\left(z+\frac{1}{\sqrt{6}}\right)}{\sqrt{6}\,z+1}\Big)$$

$$= \int_{-\infty}^{\infty} dy\,dz\; 2 f(-2,y,z)\, \delta(y-3)\, \delta\big(\sqrt{6}\,(z+1/\sqrt{6})\big)$$

$$= 2\int_{-\infty}^{\infty} dz\, f(-2,3,z)\, \delta\big(\sqrt{6}\,(z+1/6)\big)$$

$$= \frac{2}{\sqrt{6}}\, f(-2,3,-1/\sqrt{6})$$

d) $$\int_{-\infty}^{\infty} dx\,dy\,(x^2+xy+1)\,\delta(3x-5)\,\delta(y)$$

$$= \int_{-\infty}^{\infty} dx\,(x^2+1)\,\delta(3(x-5/3)) = \frac{1}{3}\left((5/3)^2+1\right)$$

e) $$\int_0^{\infty} dx \int_0^{\infty} dy\, \frac{\delta(x-y)}{(4x^2+5y^2+1)^{3/2}} = \int_0^{\infty} dy\,(9y^2+1)^{-3/2}$$

$$= \frac{y}{\sqrt{9y^2+1}}\Big|_0^{\infty} = 1/3$$

$$I_2 = \int_{-\infty}^{\infty} dx \int_0^{\infty} dy\, \frac{\delta(x-y)}{(4x^2+5y^2+1)^{3/2}} = \int_0^{\infty} dy\, \frac{1}{(9y^2+1)^{3/2}} = 1/3$$

$$I_3 = \int^{\infty} dx \int^{\infty} dy\, \frac{\delta(x-y)}{(4x^2+5y^2+1)} = \int^{\infty} dy\, \frac{1}{(9y^2+1)^{3/2}} = \frac{y}{\ldots}\Big|^{\infty} = \frac{1}{\ldots} - (-1/\ldots)$$

$$I_3 = \int_{-\infty}^{}dx\int_{-\infty}^{}dy\ \frac{\delta(x-y)}{(4x^2+5y^2+1)} = \int_{-\infty}^{} dy \frac{1}{(9y^2+1)^{3/2}} = \frac{y}{(9y^2+1)^{1/2}}\Bigg|_{-\infty}^{\infty} = \frac{1}{3} - \left(-\frac{1}{3}\right)$$

$$= 2/3$$

f) $$I = \int_0^{\pi/4} dx \int_0^{\pi/4} dy\ \delta(x-2y)\cos(\sqrt{2xy})$$

for $x = 2y$ $\quad x \to 0,\ y \to 0$

$\quad x \to \pi/4 \quad y \to \pi/8$

$$I = \int_0^{\pi/8} dy \cos(2y) = \frac{1}{2\sqrt{2}}$$

Other way

$$I = \int_0^{\pi/4} dx \int_0^{\pi/4} dy\ \delta\left(-2\left(y - \frac{x}{2}\right)\right) \cos(\sqrt{2xy})$$

$y \to 0,\ x \to 0$

$y \to \pi/4,\ x \to \pi/2$, but only going up to $\pi/4$

$$I = \frac{1}{2}\int_0^{\pi/4} dx \cos(x) = \frac{1}{2\sqrt{2}}$$

3. a) $$I = \int_{-\pi/2}^{\pi/2} \cos\theta\ \delta((\theta - \pi/4)\sin\theta)\, d\theta$$

$$g(\theta) = (\theta - \pi/4)\sin\theta \quad , g = 0 \text{ for } \theta = \pi/4, 0$$

$$g'(\theta) = \sin\theta + (\theta - \pi/4)\cos\theta$$

$$|g'(0)| = \pi/4 \quad , |g'(\pi/4)| = \sin\pi/4 = \frac{1}{\sqrt{2}}$$

$$I = \int_{-\pi/2}^{\pi/2} \cos\theta \left[\frac{\delta(\theta - \pi/4)}{\sin\pi/4} + \frac{\delta(\theta)}{\pi/4} \right]$$

$$= \frac{\cos\pi/4}{\sin\pi/4} + \frac{1}{\pi/4} = 1 + \frac{4}{\pi}$$

b) $$I = \int_0^\infty dy \int_0^5 dx\, \delta(x-y)$$

$$= \int_0^5 dy = 5$$