1. $n^{10} < 53{,}000{,}000 \times 5^n < 10^n < \sqrt{n!/2} < n!$

Probably the easiest way to fully justify this is to look at the ratio test for each, $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. The term with the largest ρ will have the largest large n behavior.

$$\lim_{n\to\infty} \left|\frac{(n+1)^{10}}{n^{10}}\right| \to \frac{(\infty+1)^{10}}{\infty^{10}} \to \frac{\infty^{10}}{\infty^{10}} = 1$$

$$\lim_{n\to\infty} \left|\frac{53{,}000{,}000 \times 5^{n+1}}{53{,}000{,}000 \times 5^n}\right| = 5$$

$$\lim_{n\to\infty} \left|\frac{10^{n+1}}{10^n}\right| = 10$$

$$\lim_{n\to\infty} \left|\frac{\sqrt{\frac{(n+1)!}{2}}}{\sqrt{\frac{n!}{2}}}\right| = \left|\sqrt{\frac{(n+1)!}{n!}}\right| = \sqrt{n+1}$$

$$\lim_{n\to\infty} \left|\frac{(n+1)!}{n!}\right| = n+1$$

$1 < 5 < 10 < \sqrt{n+1} < n+1$, so

$$\boxed{n^{10} < 53{,}000{,}000 \times 5^n < 10^n < \sqrt{n!/2} < n!}$$

Another solid method is to divide one term by another and take the limit.

2. a. $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ $\lim_{n\to\infty} \ln(n) < \lim_{n\to\infty} n$

therfore $\lim_{n\to\infty} \frac{1}{\ln(n)} > \lim_{n\to\infty} \frac{1}{n}$

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so b/c $\frac{1}{\ln(n)} > \frac{1}{n}$ in the limit of large n, $\boxed{\sum_{n=2}^{\infty} \frac{1}{\ln(n)} \text{ diverges}}$

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b. $\sum_{n=1}^{\infty} \frac{n^n}{10^n}$ Use preliminary test: $\lim_{n\to\infty} \frac{n^n}{10^n}$ is infinite (n^n grows faster than 10^n)

therefore, $\boxed{\sum_{n=1}^{\infty} \frac{n^n}{10^n} \text{ diverges}}$

Note: Special comparison test (a.k.a. Limit comparison test) works well for c and d also.

c. $\sum_{n=1}^{\infty} \frac{1}{n n^{1/n}}$ Compare with $\sum_{n=1}^{\infty} \frac{1}{n \ln(n)}$ (from part e, we know it diverges)

$$\frac{\frac{1}{\cancel{n} n^{1/n}}}{\frac{1}{\cancel{n} \ln(n)}} = \frac{1/n^{1/n}}{1/\ln(n)} = \frac{\ln(n)}{n^{1/n}}$$

$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n^{1/n})} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(n)} = e^0 = 1$

proof

$\lim_{n\to\infty} \frac{\ln(n)}{n^{1/n}} = \infty$ b/c $\ln(n)$ goes to ∞, $n^{1/n}$ goes to 1,

so $\frac{1}{n n^{1/n}} > \frac{1}{n \ln(n)}$ and therefore $\boxed{\sum_{n=1}^{\infty} \frac{1}{n n^{1/n}} \text{ diverges}}$

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d. $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ the convergence can be determined by several different methods. Partial fraction decomposition and then integral test works. I'll show two ways to use comparison, though.

We'll compare with $A\sum_{n=2}^{\infty} \frac{1}{n^2}$ (where A is a constant), which we know converges.

for $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ to converge, $\frac{1/n(n-1)}{A/n^2}$ should be < 1

$$\frac{1/n(n-1)}{A/n^2} < 1$$

$$1/n(n-1) < A/n^2$$

$$\frac{n^{\cancel{2}}}{\cancel{n}(n-1)} < A$$

$$\frac{n}{n-1} < A$$ now look at large n limit: $\lim_{n\to\infty} \frac{n}{n-1} = 1$

alternatively, you can get clever with indexing.
$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ is the same as $\sum_{k=1}^{\infty} \frac{1}{(k+1)k}$ where $k=n-1$

Comparing $\sum_{k=1}^{\infty} \frac{1}{(k+1)k}$ to $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which we know converges,

we see that $\frac{1}{(k+1)k} < \frac{1}{k^2}$, so

$\sum_{k=1}^{\infty} \frac{1}{(k+1)k} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges

At least one person used this successfully

so for $A>1$, $\frac{1}{n(n-1)} < \frac{A}{n^2}$,

which means $\boxed{\sum_{n=2}^{\infty} \frac{1}{n(n-1)} \text{ converges}}$

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e) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ integral test

$$\int^{\infty} \frac{1}{n\ln(n)}\,dn \rightarrow \int^{\ln(\infty)} \frac{1}{u}\,du = \ln u \Big|^{\ln(\infty)} = \ln(\ln(\infty)) \rightarrow \text{diverges}$$

$u = \ln(n)$

$du = 1/n \, dn$

therefore, $\boxed{\sum_{n=2}^{\infty} \frac{1}{n\ln(n)} \text{ diverges}}$

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f) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ compare with $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\frac{1}{n2^n} < \frac{1}{2^n} \quad (n2^n > 2^n)$$

so $\boxed{\sum_{n=1}^{\infty} \frac{1}{n2^n} \text{ converges}}$

3. a. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ alternating series converge if

$|a_{n+1}| < |a_n|$

and

$\lim_{n\to\infty} a_n = 0$

$\frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$ true. $(\ln(n+1) > \ln(n))$

$\lim_{n\to\infty} \frac{1}{\ln(n)} = \frac{1}{\ln(\infty)} = 0$

so $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges conditionally, not absolutely b/c $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges

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b. $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{10^n}$ $\lim \frac{n^n}{10^n} \to \infty$ so it diverges

converges not at all

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c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n n^{1/n}}$ $\frac{1}{(n+1)(n+1)^{1/n+1}} < \frac{1}{n n^{1/n}}$ true

$\lim_{n\to\infty} \frac{1}{n n^{1/n}} \to \frac{1}{n n^0} \to \frac{1}{n} \to 0$

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n n^{1/n}}$ converges <u>conditionally</u> b/c $\sum_{n=1}^{\infty} \frac{1}{n n^{1/n}}$ diverges

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d. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}$ converges absolutely b/c $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges

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e. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln(n)}$ true

$\lim_{n\to\infty} \frac{1}{n\ln n} \to 0$

converges conditionally b/c $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges

f. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n 2^n}$ converges absolutely b/c $\sum_{n=2}^{\infty} \frac{1}{n 2^n}$ converges

4. a) $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$ Ratio test: $\lim_{n\to\infty} \left| \frac{x^{2(n+1)}/2^{(n+1)}}{x^{2n}/2^n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}/2}{x^{2n}} \right| = \lim_{n\to\infty} \left| x^2/2 \right|$

$= |x^2/2|$

for convergence, $|x^2/2| \leq 1$

$\longrightarrow |x| \leq \sqrt{2}$

test endpoints: $\sum_{n=0}^{\infty} \frac{(\sqrt{2})^{2n}}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$

$\longrightarrow$ Diverges

That applies to $\pm\sqrt{2}$, so interval of convergence is $\boxed{|x| < \sqrt{2}}$

b) $\sum_{n=1}^{\infty} \frac{x^{3n}}{n}$ Ratio test: $\lim_{n\to\infty} \left| \frac{x^{3n+3}/n+1}{x^{3n}/n} \right| = \lim_{n\to\infty} \left| \frac{nx^3}{n+1} \right| = |x^3|$

$\rho = |x^3| \leq 1 \rightarrow$ test end points

$x = 1$: $\sum_{n=1}^{\infty} \frac{1^{3n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series, diverges

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^{3n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ We've seen before that this converges, but lets show it.

$|a_{n+1}| < |a_n|$? $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$

decreases monotonically, so yes $|a_{n+1}| < |a_n|$

$\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{|(-1)^n|}{n} = 0$ ✓

so interval of convergence is $\boxed{-1 \leq x < 1}$

4\. c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ Ratio test: $\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right| = \left| \frac{n^2 x}{(n+1)^2} \right| = |x|$

$\rho = |x| \leq 1$

test endpoints

$x = 1$: $\sum_{n=1} \frac{(1)^n}{n^2} = \sum_{n=1} \frac{1}{n^2}$ from prob. 2, we know this converges

$x = -1$: $\sum_{n=1} \frac{(-1)^n}{n^2}$ b/c the absolute value of this series converges, we know this also converges.

interval of convergence = $\boxed{|x| \leq 1}$ or $-1 \leq x \leq 1$

d) $\sum_{n=1}^{\infty} \frac{x^n}{(n!)^2}$ ratio test: $\lim_{n\to\infty} \left| \frac{x^{n+1}/[(n+1)!]^2}{x^n/(n!)^2} \right| = \lim_{n\to\infty} \frac{x(n!)(n!)}{(n+1)!\,(n+1)!}$

$= \frac{x}{(n+1)^2} = 0$

$\rho = 0$ regardless of x, so $-\infty < x < \infty$

or $\boxed{\text{converges for all values of } x}$

e) $\sum_{n=0} \frac{1}{1+x^n}$ ratio test: $\lim_{n\to\infty} \left| \frac{1/(1+x^{n+1})}{1/(1+x^n)} \right| = \lim_{n\to\infty} \left| \frac{1+x^n}{1+x^{n+1}} \right|$

$\rho = \left| \frac{1}{x} \right| \leq 1$

$|x| \geq 1$

test endpoints: $x = 1$: $\sum_{n=0}^{\infty} \frac{1}{1+(1)^n} = \sum_{n=0}^{\infty} \frac{1}{2} \rightarrow$ diverges

$x = -1$: $\sum_{n=0} \frac{1}{1+(-1)^n} \rightarrow$ every other term is undefined, so diverges

Interval of convergence $\boxed{|x| > 1}$

5) $f(x,y) = \sqrt{1+x+y^2}$ expanded around $x=0, y=0$

approximation: $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{n-1} \frac{\frac{\partial^{(i+j)} f}{\partial x^i \partial y^j}(a,b)}{i!\,j!}(x-a)^i(y-b)^j$

Equivalently: $\displaystyle\sum_{i=0}^{n}\frac{\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^i f(x,y)\Big|_{a,b}}{i!}$

for us, $a=0$, $b=0$, $n=3$

$$f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{f_{xx}(0,0)}{2}x^2 + f_{xy}(0,0)xy + \frac{f_{yy}(0,0)}{2}y^2$$
$$+ \frac{f_{xxx}(0,0)}{3!}x^3 + \frac{f_{xxy}(0,0)}{2!\,1!}x^2y + \frac{f_{xyy}(0,0)}{1!\,2!}xy^2 + \frac{f_{yyy}(0,0)}{3!}y^3$$

$f_x = \frac{1}{2}(1+x+y^2)^{-1/2} \rightarrow \boxed{f_x(0,0) = 1/2}$

$f_{xx} = \frac{-1}{4}(1+x+y^2)^{-3/2} \rightarrow \boxed{f_{xx}(0,0) = -1/4}$ $\qquad f_{xy} = \frac{-1}{4}(1+x+y^2)^{-3/2}(2y) \rightarrow \boxed{f_{xy}(0,0) = 0}$

$f_{xxx} = \frac{3}{8}(1+x+y^2)^{-5/2} \rightarrow \boxed{f_{xxx}(0,0) = 3/8}$ $\qquad f_{xyy} = \underbrace{\frac{3}{8}(1+x+y^2)^{-5/2}(2y)^2}_{\text{goes to zero at }(0,0)} - \frac{1}{2}(1+x+y^2)^{-3/2} \rightarrow \boxed{f_{xyy}(0,0) = -1/2}$

f_{xxy} = multiplied by y again, so $\boxed{f_{xxy}(0,0) = 0}$.

$f_y = \frac{1}{2}(1+x+y^2)^{-1/2}(2y) \rightarrow \boxed{f_y(0,0) = 0}$

$f_{yy} = \frac{1}{2}(1+x+y^2)^{-1/2}(2) + \frac{-1}{4}(1+x+y^2)^{-3/2}(2y)^2 \rightarrow \boxed{f_{yy}(0,0) = 1}$

$f_{yyy} = \underbrace{-\tfrac{1}{4}(1+x+y^2)^{-3/2}(2y)}_{\text{goes to zero at }(0,0)} + \underbrace{\tfrac{3}{8}(1+x+y^2)^{-5/2}(2y)^3}_{\text{zero at }(0,0)} - \underbrace{\tfrac{1}{4}(1+x+y^2)^{-3/2}(8y)}_{\text{goes to zero at }(0,0)} \rightarrow \boxed{f_{yyy}(0,0) = 0}$

$f(0,0) = 1$

$$1 + \frac{x}{2} + 0 - \frac{1/4}{2}x^2 + 0 + \frac{1}{2}y^2 + \frac{3/8}{6}x^3 + 0 - \frac{1/2}{2}xy^2$$

$$= \boxed{1 + \frac{x}{2} - \frac{x^2}{8} + \frac{y^2}{2} + \frac{x^3}{16} - \frac{xy^2}{4}}$$