1) $A = \left(\begin{array}{cc|cc} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 3 \\ \hline 4 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{array}\right)$

let $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \alpha \quad \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} = \beta \Rightarrow A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$

to find eigenvalues, we find the sol^n of eq^n

$AX = \lambda X$ where X are 1×4 column vector.

we can write $X = \begin{pmatrix} \Omega \\ \Gamma \end{pmatrix}$ where Ω is a 1×2 vector
Γ is a 1×2 vector

So: $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \Omega \\ \Gamma \end{pmatrix} = \lambda \begin{pmatrix} \Omega \\ \Gamma \end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} \alpha\Omega + \beta\Gamma &= \lambda\Omega \\ \beta\Omega + \alpha\Gamma &= \lambda\Gamma \end{aligned}\right\} \begin{aligned} (\alpha - \lambda I_{2\times2})\Omega + \beta\Gamma &= 0 \quad ☺ \\ \beta\Omega + (\alpha - \lambda I_{2\times2})\Gamma &= 0 \quad ☹ \end{aligned}$$

Now lets explore α, β! They are diagonal matrices, & so will be very convenient for us to compute with.

$\Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ & $\Gamma = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$ where ω_i, τ_i are constants

So ☺ becomes: $\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = 0$

$$\Rightarrow (1-\lambda)\omega_1 + 4\tau_1 = 0 \quad (1)$$

$$\& \ (2-\lambda)\omega_2 + 3\tau_2 = 0 \quad (2)$$

& ☹ becomes: $\begin{pmatrix}4&0\\0&3\end{pmatrix}\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix} + \begin{pmatrix}1-\lambda&0\\0&2-\lambda\end{pmatrix}\begin{pmatrix}\tau_1\\\tau_2\end{pmatrix}=0$

$\Rightarrow 4\omega_1 + (1-\lambda)\tau_1 = 0$ — ③

$3\omega_2 + (2-\lambda)\tau_2 = 0$ — ④

use ①, ③ $\Rightarrow (1-\lambda)\omega_1 - 4\left(\dfrac{4\omega_1}{1-\lambda}\right)=0$

$\Rightarrow \omega_1\left((1-\lambda) - \dfrac{16}{(1-\lambda)}\right) = 0 \Rightarrow (1-\lambda)^2 - 16 = 0$

if $\omega_1 \neq 0$

$\Rightarrow 1-\lambda = \pm 4$

$\lambda = -3, 5$

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use ②, ④ $\Rightarrow \dfrac{(2-\lambda)(-2+\lambda)}{3}\tau_2 + 3\tau_2 = 0$ if $\tau_2 \neq 0$

$\Rightarrow -\dfrac{(2-\lambda)^2}{3} + 3 = 0 \Rightarrow (2-\lambda)^2 = 9$

$\Rightarrow \lambda = 5, -1$

So the 4 eigenvalues are $\lambda = -3, -1, 5, 5$

Now, to find the eigenvectors:

$AX_1 = -X_1 \Rightarrow$ use eq^n ①, ②, ③, ④ w/ $\lambda = -1$

$2\omega_1 + 4\tau_1 = 0 \qquad 4\omega_1 + 2\tau_1 = 0$

$3\omega_2 + 3\tau_2 = 0 \qquad 3\omega_2 + 3\tau_2 = 0$

$\Rightarrow \omega_1 = 0,\ \tau_1 = 0\ ;\ \omega_2 = -\tau_2$

then $X_1 = (0\ 1\ 0\ -1)$ (say)

similarly to find X_2, replace $\lambda = 5$

then: $4\omega_1 + (1-\lambda)\tau_1 = 0 \qquad (1-\lambda)\omega_1 + 4\tau_1 = 0$

$3\omega_2 + (2-\lambda)\tau_2 = 0 \qquad (2-\lambda)\omega_2 + 3\tau_2 = 0$

$\omega_1 = \tau_1 \quad \omega_2 = \tau_2 \Rightarrow x_2 = (1,0,1,0) \quad x_3 = (0,1,0,1)$

$x_{3,4} \rightarrow \lambda = -3$

$4\omega_1 + 4\tau_1 = 0 \qquad 4\omega_1 + 4\tau_1 = 0$

$3\omega_2 + 5\tau_2 = 0 \qquad 5\omega_2 + 3\tau_2 = 0$

$\left.\right\} \begin{array}{l} \omega_1 = -\tau_1 \\ \omega_2 = \tau_2 = 0 \end{array}$

$x_4 = (1,0,-1,0)$

as a check: let's solve the eigen equation w/ x_2 & see if we get $\lambda = 5$ as the eigenvalue:

$$\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 3 \\ 4 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \lambda = 5 \quad \text{so verified}$$

* Notes: ① We could've brute forced it to find the $\det(A-\lambda I)$ but this solution is certainly more elegant & easier to keep track of.

② Whenever we have an eigenvector solution that has many possible solution, we shouldn't worry about it too much. All solutions will be an eigenvector so we can choose one of them without loss of generality as the possible eigenvectors already span the entire space of possible eigenvectors (i.e. we will recover the other sol^n after taking a linear combo of $x_1, x_2 \cdots$)

② a) $[A, B] = I$ so,

$AB = I + BA$

$[A^2, B] = A^2B - BA^2$

(that's the definition)

$= A(AB) - (BA)A$

$= A(I + BA) - (AB - I)A$

$= A + ABA - ABA + A = 2A$

[$[\cdot,\cdot]$ is known as a commutator if you haven't seen it before. $[A,B]$ is defined as $AB - BA$.

It behaves just like any other operator. You can think of operators as matrices if you're more comfortable w/ that.

b) $A = \frac{\partial}{\partial x}$ $\quad B = x_{op}$

Firstly, operators act on something. In this case operators are actually acting on functions. So, A only makes sense when it acts on the function: $A f(x)$ or $\frac{\partial}{\partial x} f(x)$!

Coming back to the question:

$[\partial^2/\partial x^2, x_{op}]$ is also an operator, so to evaluate it, we act it on $f(x)$:

$$[\partial^2/\partial x^2, x_{op}] f(x) = \partial^2/\partial x^2 (x f(x)) - x \frac{\partial^2}{\partial x^2} f(x)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x f(x))\right) - x \frac{\partial^2}{\partial x^2} f(x)$$

$$= \frac{\partial}{\partial x}\left(f(x) + x \frac{\partial}{\partial x} f(x)\right) - x \frac{\partial^2}{\partial x^2} f(x)$$

$$= \frac{\partial f(x)}{\partial x} + \frac{\partial}{\partial x}(f(x)) + \cancel{x \frac{\partial^2}{\partial x^2} f(x)} - \cancel{x \frac{\partial^2}{\partial x^2} f(x)}$$

$$= 2 \frac{\partial}{\partial x} f(x)$$

c) $[A,B]f(x) = [\partial/\partial x, x]f(x) = f(x)$ (verify yourself)

so $[A,B] = I$ since it's a "do nothing" operator!

Hence, $[A^2, B] = 2A$ as verified in (a), & that's what you get in part (b)

So, basically, operator algebra is consistent & hopefully makes sense.

3 a) $\frac{df}{dx} = \lambda f$ let $f = e^{kx}$ $k \in \mathbb{C}$

then $\frac{df}{dx} = \lambda f \Rightarrow \lambda = k$ eigenvalue

$f = e^{kx}$ eigenfunction

b) $x\frac{df}{dx} = \lambda f$

$$\Rightarrow \frac{df}{f} = \lambda\frac{dx}{x} \Rightarrow \int\frac{df}{f} = \lambda\int\frac{dx}{x}$$

$$\Rightarrow \ln f = \lambda \ln x + c$$

$$\Rightarrow f = c' x^{\lambda} \quad \text{eigenfunction}$$

λ eigenvalue

4 a) GS method —

$v_1 = (1,1,0)$ $v_2 = (1,1,-1)$ $v_3 = (3,0,4)$

e_1, e_2, e_3 are the orthonormal bases we want.

let $e_1 = \frac{v_1}{|v_1|} = \left(\frac{1}{\sqrt{2}}, 1/\sqrt{2}, 0\right)$

then $u_2 = v_2 - \langle e_1, v_2 \rangle e_1 = (1,1,-1) - \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$= (0,0,-1)$

$e_2 = \frac{u_2}{|u_2|} = (0,0,-1)$

$u_3 = v_3 - \langle e_1, v_3 \rangle e_1 - \langle e_2, v_3 \rangle e_2$

$= (3,0,4) - \left(\frac{3}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - (-4)(0,0,-1)$

$= \left(3/2, -3/2, 0\right)$

$e_3 = \frac{u_3}{|u_3|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

b) We can do the GS method on functions too, since they are technically a vector space too (just not the typical ones we're used to! Verify)

$F_1(x) = 1 \quad F_2(x) = e^{-x}$

now, we have to define $\langle f_1, f_2 \rangle$ on this space for this to work! It is known as an inner product.

$\langle f_1, f_2 \rangle = \int_0^1 f_1 f_2 \, dx$ (it's basically a generalizat^n of $v_1 \cdot v_2 = \sum_i v_1^i v_2^i$ for continuous functions. It's not hard to imagine, think of vectors with infinite components)

then $\|f\|^2 = \int_0^1 f^2 dx$

$\cos\theta = \frac{\langle v_1, v_2 \rangle}{|v_1||v_2|}$ for usual vectors

$= \frac{\langle f_1, f_2 \rangle}{\|f_1\| \, \|f_2\|}$ for functions!

$$\text{so } \cos\theta = \frac{\int_0^1 1 \times e^{-x}\,dx}{\sqrt{\int_0^1 1\,dx}\sqrt{\int_0^1 e^{-2x}\,dx}} = \frac{-(e^{-1}-1)}{1 \times (-e^{-2}+1)^{1/2} \cdot \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}(1-e^{-1})}{(1-e^{-2})^{1/2}}$$

$$g_1(x) = \frac{F_1(x)}{\|F_1(x)\|} = \frac{F_1(x)}{\int_0^1 F_1\,dx} = 1$$

$$\ell(x) = F_2(x) - \langle F_2(x), g_1(x)\rangle g_1(x)$$

$$= e^{-x} - \left(\int_0^1 e^{-x}\,dx\right) 1 = e^{-x} + (e^{-1}-1)$$

$$g_2(x) = \frac{\ell(x)}{\|\ell(x)\|} = \frac{e^{-x}+(e^{-1}-1)}{\sqrt{\int_0^1 (e^{-x}+\underbrace{(e^{-1}-1)}_{\alpha})^2\,dx}}$$

$$\int_0^1 (e^{-2x} - 2\alpha e^{-x} + \alpha^2)\,dx$$

$$= \frac{1}{-2}(e^{-2}-1) + \frac{2\alpha}{-1}(e^{-1}-1) + \alpha^2$$

$$\text{so } g_2(x) = \frac{e^{-x}+\alpha}{(-\frac{1}{2}(e^{-2}-1) - \alpha^2)^{1/2}} \qquad \alpha = \left(\frac{1}{e}-1\right)$$

$$\leadsto \beta = (-\tfrac{3}{2}e^{-2} + 2e^{-1} - \tfrac{1}{2})^{1/2}$$

c) $h(x) = 3e^{-x} - 2$

$$\left[g_2(x) = \frac{e^{-x}}{\beta} + \frac{\alpha}{\beta} \right]$$

$$= 3(g_2(x)\beta - \alpha) - 2$$

$$= 3\beta g_2(x) - (3\alpha+2)$$

$$\Rightarrow g_2(x)\beta + \alpha = e^{-x}$$

$$= 3\beta g_2(x) - (3\alpha+2)\,g_1(x)$$

That's all folks!

The end

:)