(1) $v = 2\hat{x} + 3\hat{y} \quad w = -\hat{x} + \hat{y}$

a) $v \cdot w = -2 + 3 = 1$

b) $v \times w = (2+3)\hat{z} = 5\hat{z}$

c) angle b/w v, w is: $\cos\theta = \frac{v \cdot w}{|v||w|} = \frac{1}{\sqrt{13}\sqrt{2}}$

d) $\hat{v} = \frac{2}{\sqrt{13}}\hat{x} + \frac{3\hat{y}}{\sqrt{13}}$

(2) $(a,b,c)\cdot(a,b,c) = a^2+b^2+c^2$

$(a,b,c)\cdot(b,c,a) = ab+bc+ca$

let $\alpha = (a,b,c)\cdot(a,b,c) - (a,b,c)\cdot(b,c,a)$

$$= a^2+b^2+c^2-ab-bc-ca$$

$$= \frac{1}{2}a^2 - ab + \frac{b^2}{2} + \frac{1}{2}a^2 - ac + \frac{1}{2}c^2 + \frac{1}{2}b^2 - bc + \frac{1}{2}c^2$$

$$= \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 \geq 0$$

so $\alpha \geq 0 \implies (a,b,c)\cdot(a,b,c) \geq (a,b,c)\cdot(b,c,a)$

always as long as $a,b,c \in \mathbb{R}$

& they are equal when $a=b=c$

③ $\hat{m} = \hat{x}$ $\quad \hat{n} = \frac{2\hat{x}+\hat{y}}{\sqrt{5}}$

Let's define $(a,b)_{m,n}$ to be the coordinate of the vectors in terms of $\hat{m}, \hat{n}$ vectors.

$$(a,b)_{m,n} = a\hat{m} + b\hat{n}$$
$$= a\hat{x} + \frac{2b}{\sqrt{5}}\hat{x} + \frac{b}{\sqrt{5}}\hat{y} = \left(a + \frac{2b}{\sqrt{5}}\right)\hat{x} + \frac{b}{\sqrt{5}}\hat{y}$$

a) $= \left(a + \frac{2b}{\sqrt{5}}, \frac{b}{\sqrt{5}}\right)_{x,y}$ in terms of $\hat{x}, \hat{y}$ vectors!

$(1,1)_{m,n}$ & $(\sqrt{5}, 1/2)_{m,n}$ replace a,b with the appropriate values.

b) let's take a vector $(a,b)_{m,n}$, which is $\left(a + \frac{2b}{\sqrt{5}}, \frac{b}{\sqrt{5}}\right)_{x,y}$.

So let O be the matrix that does this transformation?

$$O\ (a,b)^T_{m,n} = \left(a + \frac{2b}{\sqrt{5}}, \frac{b}{\sqrt{5}}\right)^T_{x,y}$$

$$\Rightarrow \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 2b/\sqrt{5} \\ b/\sqrt{5} \end{pmatrix}$$

So $O = \begin{pmatrix} 1 & 2/\sqrt{5} \\ 0 & 1/\sqrt{5} \end{pmatrix}$

c) Now we want a matrix M such that

$M V = \hat{x} \quad M W = \hat{y}$

$\Rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\alpha = 1$ — ① $\quad \gamma = 0$ — ②

$\frac{2\alpha}{\sqrt{5}} + \frac{\beta}{\sqrt{5}} = 0$ — ③ $\quad \frac{2\gamma}{\sqrt{5}} + \frac{\delta}{\sqrt{5}} = 1$ — ④

$\rightarrow \beta = -2 \qquad \delta = \sqrt{5}$

So $M = \begin{pmatrix} 1 & -2 \\ 0 & \sqrt{5} \end{pmatrix}$

So let's calculate MO:

$$\begin{pmatrix} 1 & 2/\sqrt{5} \\ 0 & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & \sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & -2+2 \\ 0 & 1 \end{pmatrix} = I$$

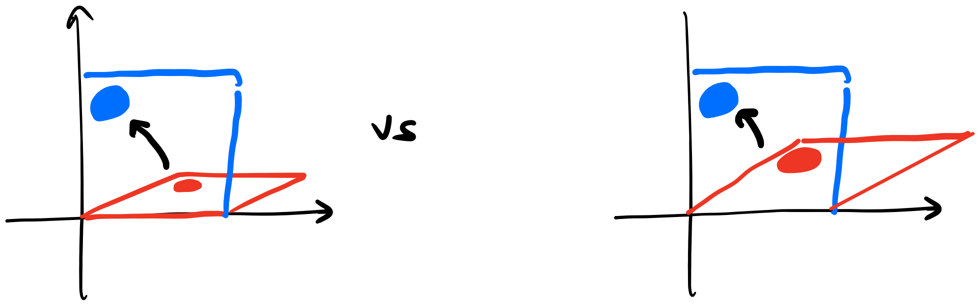
So O is the inverse of M

d) $\det(M) = \sqrt{5}$

& area of "llgm" is $\hat{m} \times \hat{n} = \left|\hat{x} \times \left(\frac{2\hat{x}}{\sqrt{5}} + \frac{\hat{y}}{\sqrt{5}}\right)\right|$

$= 1/\sqrt{5}$

This makes sense, because, the matrix M maps a space into a distorted space,

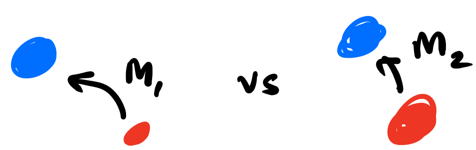


facts → In order to map the entire space, I can just map the ||'gm to the □. Since I can consider other ||'gms displaced, mapped to the displaced □ etc.

Now if I map the red blob on the ||'gm to the blue blob on the □, The larger the ||'gm, the larger the blob to be mapped.

Basically, the more the area of ||'gm, the smaller the mapping has to be! And what is the map? In this case it's M :)

In graphics:



M_1 needs to "stretch" the blob a lot more to map it to the blue blob since the red blob is small!

This "stretching" is measured in terms of det(M).

$$A = \begin{pmatrix} -1 & 7/2 & -1/2 \\ -2 & 9/2 & -1/2 \\ -2 & 7/2 & 1/2 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are the 3 eigenvectors

a) eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = 2$

b) The matrix that diagonalizes A is the matrix of eigenvectors:

$$T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$
$$\begin{array}{ccc} \hookrightarrow v_1 & \hookrightarrow v_2 & \hookrightarrow v_2 \end{array}$$

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

c) $v = v_1 + v_2$, so $\quad Av = \lambda v$

$$\Rightarrow \lambda = \lambda_1 + \lambda_2 = -1$$

we want to find an eigenvector of A that is $\perp$ to v:

let $\quad A\omega = x\,\omega$

$\omega = \alpha v_1 + \beta v_2 + \gamma v_3$ since that is the most general eigenvector you can construct out of the 3 eigenvectors.

so, $A\omega = x\omega \Rightarrow x = \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3$

& since $\omega \perp v$, then

$\omega \cdot v = 0$

so $\alpha|v_1|^2 + \beta|v_2|^2 + \alpha\, v_1\cdot v_2 + \beta\, v_2\cdot v_1 + \gamma\, v_3\cdot v_1 + \gamma\, v_3\cdot v_2 = 0$

$\Longrightarrow \alpha(11) + \beta(6) + \alpha(0) + \beta(0) + \gamma(5) + \gamma(-2) = 0$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Longrightarrow 11\alpha + 6\beta + 3\gamma = 0$ —— 1

We can choose a non zero solution from this solution set. For example:

if $\alpha = 0,\ \beta = a$ then $\gamma = -2a$

let's verify: $\omega = a v_2 - 2a v_3$

$$\begin{aligned}
\omega\cdot v &= (a v_2 - 2a v_3)\cdot(v_1 + v_2) \\
&= a\, v_2\cdot v_1 + a|v_2|^2 - 2a\, v_3\cdot v_1 - 2a\, v_3\cdot v_2 \\
&= a(0) + a(6) - 2a(5) - 2a(-2) \\
&= 0 \times a = 0
\end{aligned}$$

So $\omega \perp v$. And since ω was a linear combination of eigenvectors, it is itself an eigenvector.

5 a) $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}$ The eigenvalue eq^n is $AX = \lambda X$

To solve it, take RHS to LHS & we get:

$(A-\lambda I)X=0 \implies \det(A-\lambda I)=0$ since $X\neq 0$

$$\implies \left|\begin{pmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{pmatrix}\right| = 0$$

$$\implies (2-\lambda)\left((2-\lambda)(-1-\lambda)\right) + 2(-(2-\lambda)2) = 0$$

$$\implies -(2-\lambda)^2(1+\lambda) - 4(2-\lambda) = 0$$

$$\implies (2-\lambda)\left[(2-\lambda)(1+\lambda) + 4\right] = 0$$

so $\lambda = 2$ or $(2-\lambda)(1+\lambda)+4=0$

$$\implies 2+2\lambda-\lambda-\lambda^2+4=0$$

$$\implies \lambda^2-\lambda-6=0$$

$$\implies (\lambda+2)(\lambda-3)=0$$

$$\implies \lambda=-2,3$$

so the eigenvalues are $\lambda = -2, 2, 3$

The eigenvectors are:

$$AX_1 = -2X_1 \implies \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a \\ -2b \\ -2c \end{pmatrix}$$

$$\implies \begin{aligned} 2a+2c &= -2a \\ 2b &= -2b \\ 2a-c &= -2c \end{aligned} \implies \begin{aligned} &b=0; \\ &4a+2c=0; \\ &2a+c=0; \end{aligned}$$

so $X_1 = a(1,0,-2)^T$ is a solution of these eq^{n}s & is indeed an eigenvector.

Similarly, $AX_2 = 2X_2$; $AX_3 = 3X_3$ will be the other two eigenequations.

Now we can find a matrix S s.t. $S^{-1}AS$ is diagonal by writing the eigenvector matrix: $S = \begin{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} & \begin{pmatrix} \\ \\ \end{pmatrix} & \begin{pmatrix} \\ \\ \end{pmatrix} \end{pmatrix}$
$\qquad x_1 \quad x_2 \quad x_3$

The same procedure will work for B as well.