$$\nabla^2 \Phi = 0$$

$$\Rightarrow \frac{1}{\cancel{r^2}} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\cancel{r^2} \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

$$+ \frac{1}{\cancel{r^2} \sin^2\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right) = 0$$

let's assume that there is Φ-symmetry!

That is, $\frac{\partial V}{\partial \Phi} = 0$ [This is bc the boundary conditions are agnostic to Φ]

& $V = V(r)V(\theta)$

$$\Rightarrow \text{let } \frac{1}{V(r)} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V(r)}{\partial r}\right) = \ell(\ell+1) \quad \text{(1)}$$

$$\Rightarrow \frac{1}{V(\theta)} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = -\ell(\ell+1) \quad \text{(2)}$$

$$\text{(1)} \Rightarrow \frac{1}{V_r} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V_r}{\partial r}\right) = \ell(\ell+1)$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_r}{\partial r}\right) = V_r \, \ell(\ell+1)$$

$$\Rightarrow 2r \frac{\partial V_r}{\partial r} + r^2 \frac{\partial^2 V_r}{\partial r^2} = V_r \, \ell(\ell+1)$$

let $V_r = \sum_{n=0}^{\infty} a_n r^n$

We choose $\ell(\ell+1)$ bc it will lead to a familiar differential equation

$$2r \sum_{n=1}^{\infty} n a_n r^{n-1} + r^2 \sum_{n=2}^{\infty} n(n-1) a_n r^{n-2} = \ell(\ell+1) \sum_{n=0}^{\infty} a_n r^n$$

so $n(n-1) + 2n - \ell(\ell+1) = 0$

$\Longrightarrow n(n+1) = \ell(\ell+1)$

so $n = \ell$ or $-\ell-1$

so $V_r = a_\ell r^\ell + b_\ell r^{-\ell-1}$

similarly, we derive V_θ, turns out

$V_\theta = P_\ell(\cos\theta)$ which is a legendre poly.

Hence, $V_r V_\theta = \sum_\ell (a_\ell r^\ell + b_\ell r^{-\ell-1}) P_\ell(\cos\theta)$

but initial value condition is :

We want V to be finite inside the sphere ($r=0$), so for (a) $b_\ell = 0$, and for (b) we want V to be finite at $r \to \infty$, so $a_\ell = 0$

$V(r=1) = 35\cos^2\theta$

$\Longrightarrow$ So for (a), we have

$\sum_\ell (a_\ell) P_\ell(\cos\theta) = 35\cos^2\theta$

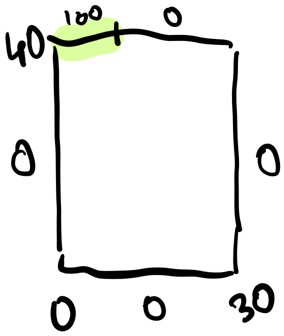
$$\Longrightarrow 35\cos^2\theta = \frac{35\times 2}{3}\left[\frac{1}{2}(3\cos^2\theta - 1)\right] + \frac{35}{3}$$

$$= \frac{35\times 2}{3} P_2(\cos\theta) + \frac{35}{3} P_0(\cos\theta)$$

so, $a_0 = 35/3 \quad a_2 = 35\times 2/3 \quad C_n = 0 \quad n \neq 0, 2$

$V(r,\theta) = 35/3\, P_0(\cos\theta) + 70/3\, r^2 P_2(\cos\theta)$

Similarly, for (b), $V(r,\theta) = 35/3\, 1/r\, P_0(\cos\theta) + 70/3\, 1/r^3\, P_2(\cos\theta)$

Q3) 

Since the plate is rectangular, we choose the cartesian coordinates to solve Laplace's eq^n.

Our differential eq^n is:

$$\underbrace{\frac{\partial^2}{\partial x^2} T}_{= k^2} + \underbrace{\frac{\partial^2}{\partial y^2} T}_{= -k^2} = 0$$

let $T = T_x T_y$ then we have 2 separable eq^{n}s —

$$\frac{\partial^2}{\partial x^2} T_x = k^2 T_x \qquad \frac{\partial^2}{\partial y^2} T_y = -k^2 T_y$$

if $k^2 > 0$, then:

$$T_x = a_1 e^{kx} + a_2 e^{-kx} \qquad T_y = b_1 e^{iky} + b_2 e^{-iky}$$

if $k^2 < 0$, then:

$$T_x = c_1 e^{ikx} + c_2 e^{-ikx} \qquad T_y = d_1 e^{-ky} + d_2 e^{ky}$$

so if $k^2 \geq 0$,

We know that for $(x, y=0)$, $T = 0$

$$b_2 = -b_1$$

$(x=0, y)$, $T=0$ $\qquad$ $(x=30, y)$, $T=0$

$a_1 = -a_2$

$$\underbrace{a_1 e^{30k} + a_2 e^{-30k} = 0}_{\text{this is never true if } a_1 = -a_2!}$$

unless $k = 0$

but that yeilds a trivial sol^n

So, k^2 MUST be $k^2<0$!

Now we do it again:

$(x, y=0), T=0 \implies d_1 + d_2 = 0$

$(x=0, y), T=0 \implies c_1 + c_2 = 0$

$(x=30, y), T=0 \implies c_1 e^{ik \times 30} + c_2 e^{-ik \times 30} = 0$

$\implies e^{ik \times 30} - e^{-ik30} = 0$

$\implies 2i \sin(30k) = 0 \implies$

$k = n\pi/30 \quad n \in \mathbb{Z}$

So $$T_x T_y = \sum_{n=-\infty}^{\infty} c_n \sin\left(\frac{n\pi x}{30}\right) \sinh\left(\frac{n\pi y}{30}\right)$$

finally, we have the last condition

$$\left.\begin{aligned} y = 40 \quad & x \in [0, 10] \quad T = 100 \\ & x \in (10, 30] \quad T = 0 \end{aligned}\right\} \#4$$

Now, before we proceed further, let's see how to compute the coeff. c_n!

We begin with considering:

$$\int_0^{30} \sin \frac{m\pi x}{30} \sin \frac{n\pi x}{30} \, dx = \delta_{mn} \times \frac{30}{2}$$

$$\left[\int_0^{30} \sin^2 \frac{m\pi x}{30} \, dx = \int_0^{30} \frac{1 - \cos \frac{2m\pi x}{30}}{2} \, dx = \frac{30}{2} \right]$$

So, we get —

$$T = \sum_n C_n \sin\left(\frac{n\pi x}{30}\right) \sinh\left(\frac{n\pi y}{30}\right)$$

$$\int_0^{30} T \sin\left(\frac{m\pi x}{30}\right) dx = \sum_{n=-\infty}^{\infty} C_n \left(\int_0^{30} \sin\frac{n\pi x}{30} \sin\frac{m\pi x}{30}\, dx\right) \sinh\left(\frac{n\pi y}{30}\right)$$

$$= C_m \times \frac{30}{2} \times \sinh\left(\frac{m\pi y}{30}\right)$$

$$\text{so } C_m = \frac{\frac{2}{30}\int_0^{30} T \sin\left(\frac{m\pi x}{30}\right) dx}{\sinh \frac{m\pi y}{30}}$$

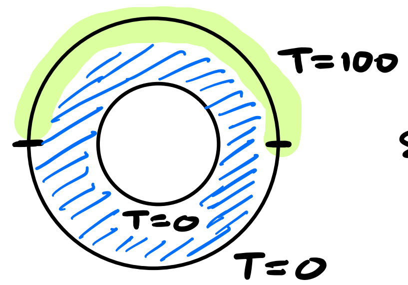
Now let's bring back the boundary condition #4

$$C_m = \frac{\frac{2}{30}\left[\int_0^{10} \underbrace{100}_{T=100 \text{ for } x\in[0,10]} \sin\frac{m\pi x}{30}\, dx + \int_{10}^{30} \underbrace{0}_{T=0 \text{ for } x\in(10,30]} \sin\frac{m\pi x}{30}\, dx\right]}{\sinh\left(\frac{m\times\pi\times \underbrace{40}_{y=40}}{30}\right)}$$

$$= \frac{\frac{2}{30} \times 100 \times \frac{30}{m\pi}\left(-\cos\left(\frac{m\pi x}{30}\right)\right)\Big|_0^{10}}{\sinh\left(\frac{4\pi m}{3}\right)}$$

$$= \frac{200\left(1 - \cos\left(\frac{m\pi}{3}\right)\right)}{m\pi \sinh\left(\frac{4\pi m}{3}\right)}$$

$$\text{So } T = \sum_{n=-\infty}^{\infty}\left(\frac{200\left(1-\cos\left(n\pi/3\right)\right)}{m\pi \sinh\left(\frac{4\pi n}{3}\right)}\right) \sin\left(\frac{n\pi x}{30}\right) \sinh\left(\frac{n\pi y}{30}\right)$$

Q4) 

Now, as we have the same laplace eq^n for Q4 as for Q2, we have the same generic sol^n.

$$T = \sum_{\ell=0}^{\infty} \underbrace{\left(a_\ell r^\ell + b_\ell r^{-\ell-1}\right)}_{\text{radial part}} \underbrace{P_\ell(\cos\theta)}_{\text{angular part}}$$

[We can only use this form if no ϕ dependence for T!]

Now we know, if $r=1$, $T=0$

so $$0 = \sum_{\ell=0}^{\infty} (a_\ell + b_\ell) P_\ell(\cos\theta)$$ call this x

which is true if $a_\ell = -b_\ell \ \forall \ \ell$

if $r=2$, $T=100$ for the upper half of the sphere, $T=0$ for the lower half.

$$\text{So } \sum_{\ell=0}^{\infty} \left(a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}}\right) P_\ell(x) = \underbrace{\begin{cases} 100 & x \in [0,1] \\ 0 & x \in [-1,0) \end{cases}}$$

converted the θ units to $\cos\theta$

This has been worked out in the text book Ch. 12 section 9

use the integral
$$\int P_\ell(x) P_m(x)\, dx = \frac{2\delta_{m\ell}}{2m+1}$$

$$\sum_{\ell=0}^{\infty}\left(a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}}\right)P_\ell(x) = 100\left[\frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x)\right]$$

...

so $\left(a_0 + \frac{b_0}{2}\right) = \frac{100}{2}$

$$\left(2a_1 + \frac{b_1}{4}\right) = \frac{300}{4}$$

$$\left(4a_2 + \frac{b_2}{8}\right) = -\frac{700}{16} \quad \text{etc.}$$

now from our previous condition, $a_\ell = -b_\ell$

so $a_0 = 100 \qquad b_0 = -100$

$a_1 = \frac{600}{7} \qquad b_1 = -\frac{600}{7}$

$a_2 = -\frac{700}{62} \qquad b_2 = \frac{700}{62} \quad \ldots \quad$ etc

So
$$T = \left(100 - \frac{100}{r}\right)P_0(\cos\theta) + \left(\frac{600}{7}r - \frac{600}{7r^2}\right)P_1(\cos\theta) - \left(\frac{700}{62}r^2 - \frac{700}{62r^3}\right)P_2(\cos\theta) + \cdots$$