3. a) $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ $\quad \begin{vmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) - 4 = 0$

$$18 - 9\lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - 9 + 14 = 0$$
$$(\lambda - 7)(\lambda - 2) = 0$$

$$\lambda = 2, 7$$

$D = C^{-1}AC$ where $D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ and C is composed of $\vec{v}_1$ and $\vec{v}_2$

so $\boxed{D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}}$

b) $D = C^{-1}AC \rightarrow CDC^{-1} = A$

$e^A = Ce^DC^{-1}$

not-super-rigorous proof: (not required for full credit)

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$
$$= CC^{-1} + CDC^{-1} + \frac{CDC^{-1}CDC^{-1}}{2!} + \frac{CDC^{-1}CDC^{-1}CDC^{-1}}{3!} + \dots$$
$$= CC^{-1} + CDC^{-1} + \frac{CD^2C^{-1}}{2!} + \frac{CD^3C^{-1}}{3!} + \dots$$
$$= C \sum_{n=0}^{\infty} \frac{D^n}{n!} C^{-1} = \boxed{Ce^DC^{-1}}$$

Need to find C:

$\lambda = 2$: $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow 3a + 2b = 2a \rightarrow a = -2b$

$\cancel{2a}_{-4b} + 6b = 2b \rightarrow b = b$, choose $b = 1$

for $\lambda_1 = 2$, $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda = 7$: $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 7\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow 3a + 2b = 7a \rightarrow a = b/2$

$\cancel{2a}_{b} + 6b = 7b \rightarrow b = b$, choose $b = 2$

for $\lambda_2 = 7$, $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$C = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$ (needs to correspond to $D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$, so $\vec{v}_1$ associated with $\lambda_1 = 2$ needs to be first)

note: solution is the same for any valid eigenvectors, e.g. $v_1 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

continued →

$C = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$, for 2×2 matrix, $C^{-1} = \frac{1}{\det(C)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = -4-1 = -5$$

$$C^{-1} = -\frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

Quick check

$$\frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

So $e^A = Ce^DC^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e^7 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\frac{1}{5}$

$$= \frac{1}{5}\begin{pmatrix} -2e^2 & e^7 \\ e^2 & 2e^7 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} = \boxed{\frac{1}{5}\begin{pmatrix} 4e^2+e^7 & -2e^2+2e^7 \\ -2e^2+2e^7 & e^2+4e^7 \end{pmatrix}}$$

for diagonal matrices, you can exponentiate the non-zero terms b/c ~~###~~ for a diagonal matrix:

$$e^D = \sum_{n=0}^{\infty}\frac{D^n}{n!} = I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \dots$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix} + \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \lambda_1^3 & & 0 \\ & \lambda_2^3 & \\ 0 & & \ddots \end{pmatrix} + \dots$$

$$= \begin{pmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots \end{pmatrix}$$