1. a) (Not Graded)

Hermitian means $H = H^\dagger$

Start with $H|v\rangle = \lambda|v\rangle$

Conjugate transpose: $(H|v\rangle)^\dagger = (\lambda|v\rangle)^\dagger$

$\langle v|H^\dagger = \lambda^* \langle v|$

right multiply by $|v\rangle$: $\langle v|H^\dagger|v\rangle = \lambda^* \langle v|v\rangle$

b/c $H = H^\dagger$: $\langle v|H|v\rangle = \lambda^* \langle v|v\rangle$

$\lambda \cancel{\langle v|v\rangle} = \lambda^* \cancel{\langle v|v\rangle}$

$\boxed{\lambda = \lambda^* \text{ so } \lambda \text{ must be real}}$

b) Start w/ $H|v\rangle = \lambda_v|v\rangle$ where $|v\rangle$ is an eigenvector with associated eigenvalue λ_v

left multiply by $\langle u|$, where $|u\rangle$ is an eigenvector with eigenvalue λ_u: $\langle u|H|v\rangle = \lambda_v \langle u|v\rangle$

b/c $\langle u|H = (H^\dagger|u\rangle)^\dagger = (H|u\rangle)^\dagger = (\lambda_u|u\rangle)^\dagger = \lambda_u^* \langle u| = \lambda_u \langle u|$

$\rightarrow \lambda_u \langle u|v\rangle = \lambda_v \langle u|v\rangle$

if $\lambda_u \neq \lambda_v$, then $\langle u|v\rangle = 0$ and $|u\rangle$ and $|v\rangle$ are orthogonal

---

Notation notes for those unfamiliar with it:

$V \equiv |v\rangle$

$V^\dagger \equiv \langle v|$

1. a) $U|v\rangle = \lambda|v\rangle$

unitary: $UU^\dagger = I$

(that is, $U^{-1} = U^\dagger$)

$U^\dagger U|v\rangle = \lambda U^\dagger|v\rangle$

$|v\rangle = \lambda U^\dagger|v\rangle$

$\rightarrow U^\dagger|v\rangle = \frac{1}{\lambda}|v\rangle$

so $U^\dagger$ has the same eigenvectors as U with eigenvalues $1/\lambda$, where λ represents the eigenvalues of U

b) $\langle Uv|v\rangle \equiv (Uv)^\dagger v$

so $\langle Uv|v\rangle = (|Uv\rangle)^\dagger|v\rangle = (U|v\rangle)^\dagger|v\rangle$

$(U|v\rangle)^\dagger|v\rangle = (\lambda|v\rangle)^\dagger|v\rangle = \lambda^*\langle v|v\rangle$

$(U|v\rangle)^\dagger|v\rangle = \langle v|U^\dagger|v\rangle = \langle v|(\frac{1}{\lambda}|v\rangle) = \frac{1}{\lambda}\langle v|v\rangle$

so $\frac{1}{\lambda}\cancel{\langle v|v\rangle} = \lambda^*\cancel{\langle v|v\rangle}$

$1 = \lambda^*\lambda \rightarrow 1 = \lambda^2 \rightarrow \boxed{|\lambda| = 1}$

2. a) $g(x) = x$ on $[-1, 1]$

Need periodicity over $[-1, 1]$, so use $e^{in\pi x}$ (see page 361)

$$g(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2}\int_{-1}^{1} g(x) e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} x e^{-in\pi x}\, dx$$

integrate by Parts! $u = x$ $dv = e^{-in\pi x}dx$

$du = dx$ $v = \frac{-e^{-in\pi x}}{in\pi}$

$$uv\Big|_a^b - \int_a^b v\,du = \frac{1}{2}\left[\frac{-xe^{-in\pi x}}{in\pi}\Big|_{-1}^{1} + \frac{1}{in\pi}\int_{-1}^{1} e^{-in\pi x}\,dx\right]$$

$$= \frac{1}{2in\pi}\left[-e^{-in\pi} - e^{in\pi} - \frac{1}{in\pi}\left(e^{-in\pi x}\Big|_{-1}^{1}\right)\right]$$

$$= \frac{1}{2in\pi}\left[-2\cos(n\pi) - \frac{1}{in\pi}\left(e^{-in\pi} - e^{in\pi}\right)\right]$$

$$= \frac{1}{2in\pi}\left[-2\cos(n\pi) - \frac{2i\sin(n\pi)}{in\pi}\right]$$

$\sin(n\pi) = 0$ for all integers n

$\cos(n\pi) = (-1)^n$ for all integers n

$$= \frac{1}{\cancel{2}in\pi}\left(-\cancel{2}(-1)^n\right) = \frac{-(-1)^n}{in\pi}\cdot\frac{i}{i} = \frac{i(-1)^n}{n\pi}$$

Notice: for $n=0$, c_n is undefined, so treat it separately

$$c_0 = \frac{1}{2}\int_{-1}^{1} x\,dx \quad \text{(can see from $\ast$ that it's zero by symmetry)}$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\Big|_{-1}^{1}\right] = \frac{1}{4}[1-1] = 0$$

So $\boxed{g(x) = x = \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n\pi} e^{in\pi x}}$ (okay to have $\cos(n\pi)$ instead of $(-1)^n$)

2. b) $h(x) = |x|$ on $[-1, 1]$ use $e^{in\pi x}$ again

$C_n = \frac{1}{2}\int_{-1}^{1} |x| e^{-in\pi x}\, dx$, split into two intervals, $[-1, 0)$ and $[0, -1]$

on $[-1, 0)$, $h(x) = -x$; on $[0, 1]$, $h(x) = x$

$= \frac{1}{2}\int_{-1}^{0} -x e^{-in\pi x}\, dx + \frac{1}{2}\int_{0}^{1} x e^{-in\pi x}\, dx$

integrate by parts again, same for each

$u = x \qquad dv = e^{-in\pi x}dx$

$du = dx \qquad v = -\frac{e^{-in\pi x}}{in\pi}$

$= \frac{1}{2}\left[\frac{x e^{-in\pi x}}{in\pi}\Big|_{-1}^{0} - \frac{1}{in\pi}\int_{-1}^{0} e^{-in\pi x}\, dx - \frac{x e^{-in\pi x}}{in\pi}\Big|_{0}^{1} + \frac{1}{in\pi}\int_{0}^{1} e^{-in\pi x}\, dx\right]$

$= \frac{1}{2in\pi}\left[e^{in\pi} + \frac{e^{-in\pi x}}{in\pi}\Big|_{-1}^{0} - e^{-in\pi} - \frac{e^{-in\pi x}}{in\pi}\Big|_{0}^{1}\right]$

$= \frac{1}{2in\pi}\left(2i\sin(n\pi) + \frac{1}{in\pi}\left(1 - e^{in\pi} - e^{-in\pi} + 1\right)\right)$

$= \frac{1}{2in\pi}\left[2i\sin(n\pi) + \frac{2 - 2\cos(n\pi)}{in\pi}\right]$ $\quad \sin(n\pi) = 0$

$= \frac{-1}{(n\pi)^2}\left(1 - (-1)^n\right) = \frac{(-1)^n - 1}{n^2\pi^2}$ $\quad \cos(n\pi) = (-1)^n$

for n even: $C_n = 0$

for n odd: $C_n = \frac{-2}{n^2\pi^2}$

but notice, for $n = 0$, C_n is undefined, so treat it separately

$n = 0$: $C_0 = \frac{1}{2}\int_{-1}^{1} |x|\, dx$ (b/c $e^{-in\pi x} = 1$ for $n = 0$)

$= \frac{1}{2}\left[\int_{-1}^{0} -x\, dx + \int_{0}^{1} x\, dx\right] = \frac{1}{2}\left[-\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1}\right]$

$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}\right] = 1/2$

$$\boxed{\text{so } h(x) = |x| = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n \neq 0}}^{\infty} \frac{-2}{n^2\pi^2} e^{in\pi x}}$$

equivalently: $h(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} e^{in\pi x}$ or $\frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{-2}{(2m+1)^2\pi^2} e^{i(2m+1)\pi x}$

c) $f(x) = \begin{cases} 0 & : x < 0 \\ x & : x \geq 0 \end{cases}$ on $[-1, 1]$ use $e^{in\pi x}$

We can construct this from $g(x) = x$ and $h(x) = |x|$. You can see it by inspection, but I'll show a methodical approach.

 

on $[-1, 0]$: $f(x) = 0 = ag(x) + bh(x)$

$g(x) = x$, $h(x) = -x$ $\rightarrow$ $0 = ax - bx$ $\rightarrow$ $a = b$

on $[0, 1]$: $f(x) = x = ag(x) + bh(x)$

$g(x) = x$, $h(x) = x$ $\rightarrow$ $\cancel{x} = a\cancel{x} + b\cancel{x}$ $\rightarrow$ $1 = a + b$ $\rightarrow$ $1 = a + a$ $\rightarrow$ $a = \frac{1}{2}$, $b = \frac{1}{2}$

so $f(x) = \frac{1}{2}(g(x) + h(x))$

$$= \boxed{\frac{1}{2}\left[\sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{i(-1)^n}{n\pi} e^{in\pi x} + \frac{1}{2} + \sum_{\substack{-\infty \\ n \text{ odd} \\ n \neq 0}}^{\infty} \frac{-2}{n^2\pi^2} e^{in\pi x}\right]}$$

by integration: $c_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-in\pi x}$

$$= \frac{1}{2}\left[\cancel{\int_{-1}^{0} 0 e^{-in\pi x}} + \int_0^1 x e^{-in\pi x}\right]$$

from earlier integration by parts

$$= \frac{1}{2}\left[\frac{-e^{-in\pi}}{in\pi} + \frac{e^{-in\pi x}}{n^2\pi^2}\Big|_0^1\right] = \frac{1}{2n\pi}\left[ie^{-in\pi} + \frac{e^{-in\pi} - 1}{n\pi}\right]$$

Simplify: $\frac{1}{2n\pi}\left[i(\cos(n\pi) - \underbrace{i\sin(n\pi)}_{0}) + \frac{\cos(n\pi) - \overbrace{i\sin(n\pi)}^{0} - 1}{n\pi}\right]$

$$= \frac{1}{2n\pi}\left[i(-1)^n + \frac{(-1)^n - 1}{n\pi}\right]$$

treat $n = 0$ separately: $\frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\Big|_0^1\right] = \frac{1}{4}$

all together: $f(x) = \frac{1}{4} - \frac{1}{2}\sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{1}{n\pi}\left[i(-1)^n + \frac{(-1)^n - 1}{n\pi}\right] e^{in\pi x}$

$$= \boxed{\frac{1}{2}\left[\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i(-1)^n}{n\pi} e^{in\pi x} + \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0 \\ n \text{ odd}}}^{\infty} \frac{-2}{n^2\pi^2} e^{in\pi x}\right]}$$ Same as above

2. d) $q(x) = \begin{cases} -x/2 & : x<0 \\ 3x & : x \geq 0 \end{cases}$ for $[-1,1]$

$$q(x) = a g(x) + b h(x)$$

$[-1,0)$: $\quad -x/2 = a g(x) + b h(x) = ax - bx$

$$-1/2 = a - b \rightarrow a = b - 1/2$$

$[0,1]$:

$$3x = a g(x) + b h(x) = ax + bx$$

$$3 = a + b \Rightarrow 2b - 1/2$$

$$b = 7/4 \qquad a = 5/4$$

$$q(x) = 5/4\, g(x) + 7/4\, h(x)$$

$$= \boxed{\frac{1}{4}\left[5\sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n\pi} e^{in\pi x} + 7/2 - 14\sum_{\substack{-\infty \\ n\neq 0 \\ n \text{ odd}}}^{\infty} \frac{e^{in\pi x}}{n^2\pi^2}\right]}$$

3. a) $g(x) = \cos(x) + \sqrt{7}\sin(5x)$ on $[0, 2\pi]$
so use e^{inx}

$n=0$: $\frac{1}{2\pi}\int_0^{2\pi}(\cos(x) + \sqrt{7}\sin(5x))dx$
$= 0$ b/c $\cos(x)$ & $\sin(5x)$ are periodic over $[0,2\pi]$
(avg value of each on $[0,2\pi]$ is 0)

$$c_n = \frac{1}{2\pi}\int_0^{2\pi}(\cos(x) + \sqrt{7}\sin(5x))e^{-inx}dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(e^{ix} + e^{-ix} + \frac{\sqrt{7}}{i}e^{5ix} - \frac{\sqrt{7}}{i}e^{-5ix}\right)e^{-inx}dx$$
$$= \frac{1}{4\pi}\left[\int_0^{2\pi} e^{i(1-n)x}dx + \int_0^{2\pi} e^{-i(1+n)x}dx + \frac{\sqrt{7}}{i}\int_0^{2\pi} e^{i(5-n)x}dx - \frac{\sqrt{7}}{i}\int_0^{2\pi} e^{-i(5+n)x}dx\right]$$

from eq. 5.4 (pg. 351), we know e^{ikx} ingrated over a period is zero unless $k=0$

So $c_n = 0$ except for $n = \pm 1, \pm 5$

$n=1$: $\frac{1}{4\pi}\int_0^{2\pi} e^0 dx = \frac{1}{4\pi}\left(x\Big|_0^{2\pi}\right) = \frac{1}{2}$ $\quad n=5$: $\frac{\sqrt{7}}{i4\pi}\int_0^{2\pi}dx = \frac{\sqrt{7}}{2i}$

$n=-1$: $\frac{1}{4\pi}\int_0^{2\pi}dx = \frac{1}{2}$ $\quad n=-5$: $-\frac{\sqrt{7}}{i4\pi}\int_0^{2\pi}dx = -\frac{\sqrt{7}}{2i}$

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \boxed{\frac{1}{2}\left[e^{inx} + e^{-inx} + \frac{\sqrt{7}e^{5ix}}{i} - \frac{\sqrt{7}e^{-5ix}}{i}\right]}$$

notice $\frac{e^{inx} + e^{-inx}}{2} = \cos x$ and $\frac{\sqrt{7}}{2i}\left(e^{5ix} - e^{-5ix}\right) = \frac{\sqrt{7}}{2i}\sin(5x)$

so we get $g(x)$ back

b) $f(x) = e^{ix/3}$ on $[0, 2\pi]$ (notice $[0,2\pi]$ does not correspond to an integer number of periods for $e^{ix/3}$)

use e^{inx}

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{ix/3}e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(1/3-n)x}dx = \frac{1}{2\pi}\left(\frac{1}{i(1/3-n)}e^{i(1/3-n)x}\Big|_0^{2\pi}\right)$$
$$= \frac{1}{2\pi}\left[\frac{i}{(n-1/3)}\left(e^{2\pi i(1/3-n)} - 1\right)\right] = \frac{i}{2\pi(n-1/3)}\left(e^{2\pi i/3}\underbrace{e^{-2\pi i n}}_{\cos(2\pi n) = 1} - 1\right)$$
$$= \frac{i(e^{2\pi i/3} - 1)}{2\pi(n-1/3)}$$ (also valid for $n=0$)

$$f(x) = \boxed{\sum_{n=-\infty}^{\infty}\frac{i(e^{2\pi i/3} - 1)}{2\pi(n-1/3)}e^{inx}} = \frac{i(e^{2\pi i/3} - 1)}{2\pi}\sum_{n=-\infty}^{\infty}\frac{e^{inx}}{n-1/3}$$

$$= \frac{i(\cos(2\pi/3) + i\sin(2\pi/3) - 1)}{2\pi}\sum_{n=-\infty}^{\infty}\frac{e^{inx}}{n-1/3} = \frac{i(-1/2 + i\sqrt{3}/2 - 1)}{2\pi}\sum_{n=-\infty}^{\infty}\frac{e^{inx}}{n-1/3}$$
$$= \sum_{n=-\infty}^{\infty}\frac{\sqrt{3} + 3i}{4\pi(1/3-n)}e^{inx}$$ (equivalently)

3. c) $e^{ix/3} = \cos(x/3) + i\sin(x/3)$

so $\sin(x/3) = \mathrm{Im}[e^{ix/3}]$

$$= \mathrm{Im}\left[\sum_{-\infty}^{\infty} \frac{\sqrt{3}+3i}{4\pi(1/3-n)} e^{inx}\right] = \mathrm{Im}\left[\sum_{-\infty}^{\infty} \frac{(\sqrt{3}+3i)}{4\pi(1/3-n)} (\cos(nx) + i\sin(nx))\right]$$

$$= \boxed{\sum_{-\infty}^{\infty} \frac{3\cos(nx) + \sqrt{3}\sin(nx)}{4\pi(1/3-n)}}$$

but cosine & sine series should go from $0 \to \infty$

$$= \sum_{0}^{\infty} \frac{3\cos(nx) + \sqrt{3}\sin(nx)}{4\pi(1/3-n)} + \sum_{-\infty}^{-1} \frac{3\cos(nx) + \sqrt{3}\sin(nx)}{4\pi(1/3-n)}$$

$$= \frac{3}{4\pi(1/3)} + \sum_{1}^{\infty} \frac{3\cos(nx) + \sqrt{3}\sin(nx)}{4\pi(1/3-n)} + \sum_{1}^{\infty} \frac{3\cos(nx) - \sqrt{3}\sin(nx)}{4\pi(1/3+n)}$$

pull out $n=0$ term

substitute $-n$ for n to get series from $1 \to \infty$

$$= 9/4\pi + \sum_{1}^{\infty} \frac{1}{4\pi(1/9-n^2)} \left(3n\cos(nx) + \cos(nx) + \tfrac{\sqrt{3}}{3}\sin(nx) + \sqrt{3}n\sin(nx) + \cos(nx) - 3n\cos(nx) - \tfrac{\sqrt{3}}{3}\sin(nx) + \sqrt{3}n\sin(nx)\right)$$

$$= 9/4\pi + \sum_{1}^{\infty} \frac{(2\cos(nx) + 2\sqrt{3}\sin(nx))}{4\pi(1/9-n^2)}$$

$$= \boxed{9/4\pi + \sum_{n=1}^{\infty} \frac{\cos(nx) + 2\sqrt{3}\sin(nx)}{2\pi(1/9-n^2)}}$$

can also use $\sin(x/3) = \dfrac{e^{ix/3} - e^{-ix/3}}{2i}$, getting the series for $e^{-ix/3}$ by taking complex conjugate of $f(x)$ from part b. It doesn't work to say $e^{-ix/3} = f(-x)$, it needs to be $e^{-ix/3} = (f(x))^*$.

if $e^{ix/3} = f(x) = \sum_{n=-\infty}^{\infty} \frac{i(e^{2\pi i/3}-1)}{2\pi(n-1/3)} e^{inx} = \sum_{n=-\infty}^{\infty} \frac{\sqrt{3}+3i}{4\pi(1/3-n)} e^{inx}$

then $f(-x) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{3}+3i}{4\pi(1/3-n)} e^{-inx}$ but $(f(x))^* = \sum_{n=-\infty}^{\infty} \frac{\sqrt{3}-3i}{4\pi(1/3-n)} e^{-inx}$ which is clearly not equivalent. ~~Using f(-x)~~

4. a) $\delta(x)$ on $[-\pi, \pi]$, so use e^{inx}

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x) e^{-inx}\,dx = \frac{1}{2\pi} e^{-in(0)} = \frac{1}{2\pi}$$ (valid for $n=0$ as well)

$$\boxed{\delta(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx}}$$

b) $$\delta(\phi_1 - \phi_2) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi_1 - \phi_2)}$$

if true, then $$\int_{-\pi}^{\pi} f(\phi_1) \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi_1 - \phi_2)}\,d\phi_1 = f(\phi_2) \quad (1)$$

$$f(\phi_1) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi_1} \text{ on } [-\pi, \pi]$$

(1) becomes → $$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{in(\phi_1)} \left(\frac{1}{2\pi}\right) \sum_{m=-\infty}^{\infty} e^{im(\phi_1 - \phi_2)}\,d\phi_1$$

note: both $e^{in\phi_1}$ and $e^{im(\phi_1 - \phi_2)}$ are periodic on $[-\pi, \pi]$

$$= \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n \sum_{m=-\infty}^{\infty} e^{in\phi_1 + im\phi_1 - im\phi_2}\,d\phi_1$$

pull sums & constants out:

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n \sum_{m=-\infty}^{\infty} e^{-im\phi_2} \underbrace{\int_{-\pi}^{\pi} e^{i\phi_1(n+m)}\,d\phi_1}_{\text{this is 0 unless } m=-n \text{ b/c it's periodic on } [-\pi,\pi]}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n e^{in\phi_2} \int_{-\pi}^{\pi} d\phi_1 = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n e^{in\phi_2}(2\pi) = \boxed{\sum_{n=-\infty}^{\infty} c_n e^{in\phi_2}}$$

$\boxed{\text{which is } f(\phi_2)}$

5. 11.5, 11.6, 11.7, 11.8

11.5 $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

from 9.6: $f(x) = \begin{cases} -1, & -l < x < 0 \\ 1, & 0 < x < l \end{cases} = \frac{4}{\pi}\left(\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\left(\frac{5\pi x}{l}\right) \dots\right)$

$\rightarrow b_n = \frac{4}{\pi(2n-1)}$

$a_n = 0 \quad \frac{1}{2}a_0 = 0$

Parseval's theorem: average of $[f(x)]^2$ over a period $= \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1 a_n^2 + \frac{1}{2}\sum_1^{\infty} b_n^2$

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx = \frac{1}{2l}\left[\int_{-l}^{0}(-1)^2 dx + \int_0^l (1)^2 dx\right] = \frac{1}{2l}\left[\int_{-l}^{l} dx\right]$$

$$= \frac{1}{2l}\left[x \Big|_{-l}^{l}\right] = \frac{1}{2l}(l-(-l)) = 1$$

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^{\infty} a_n^2 + \frac{1}{2}\sum_1^{\infty} b_n^2$$

$$1 = 0 + 0 + \frac{1}{2}\left(\frac{4}{\pi}\sum_1^{\infty}\frac{1}{(2n-1)}\right)^2$$

$$1 = \frac{8}{\pi^2}\sum_1^{\infty}\frac{1}{(2n-1)^2}$$

$$\sum_1^{\infty}\frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \boxed{\frac{\pi^2}{8}}$$

11.6 $\sum_{n=1}^{\infty} \frac{1}{n^4}$ from 9.9: $f(x) = x^2$ on $-\frac{1}{2} < x < \frac{1}{2}$

$$f(x) = \frac{1}{12} - \frac{1}{\pi^2}\left(\cos(2\pi x) - \frac{1}{2^2}\cos(4\pi x) + \frac{1}{3^2}\cos(6\pi x) + \dots\right)$$

$$\frac{1}{2}a_0 = \frac{1}{12} \qquad a_n = \frac{(-1)^n}{\pi^2 n^2} \qquad b_n = 0$$

parsevals: $\frac{1}{\text{period}}\int_{\text{period}} [f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$

$$\frac{1}{1}\int_{-1/2}^{1/2} (x^2)^2 dx = \frac{1}{144} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\pi^4 n^4} + 0$$

$$\left.\frac{x^5}{5}\right|_{-1/2}^{1/2} = \frac{1}{144} + \frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{5}\left(\frac{1}{32} + \frac{1}{32}\right) - \frac{1}{144} = \frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{80} - \frac{1}{144} = \frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

$\frac{1}{180}$

11.7 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ from 5.8: $f(x) = 1 + x = 1 + 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x)\right) - \ldots$ (on $(-\pi, \pi)$)

$\frac{1}{2}a_0 = 1 \quad a_n = 0 \quad b_n = \frac{2}{n}(-1)^n$

Parseval's: $\frac{1}{\text{period}}\int_{\text{period}} [f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+x)^2 dx = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+2x+x^2)dx = 1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{2\pi}\left(x\Big|_{-\pi}^{\pi} + x^2\Big|_{-\pi}^{\pi} + \frac{x^3}{3}\Big|_{-\pi}^{\pi}\right) - 1 = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{4\pi}\left(2\pi + 0 + \frac{2\pi^3}{3}\right) - \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{n^2} \longrightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

11.8 $\sum_{n \text{ odd}} \frac{1}{n^4}$ from 9.10: $f(x) = |x|$ for $-\pi/2 < x < \pi/2$

$f(x)$ is even, so $a_n = \frac{1}{(\pi/2)}\int_{-\pi/2}^{\pi/2} |x|\cos(2nx)\,dx$

$b_n = 0$

$a_0 = \frac{1}{(\pi/2)}\int_{-\pi/2}^{\pi/2}|x|\,dx$

$$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}|x|\,dx = \frac{4}{\pi}\int_0^{\pi/2} x\,dx = \frac{2}{\pi}x^2\Big|_0^{\pi/2} = \pi/2$$

(b/c $|x|$ is even)

$u = x \quad du = dx \qquad dv = \cos(2nx)dx \quad v = \frac{1}{2n}\sin(2nx)$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}|x|\cos(2nx)dx = \frac{4}{\pi}\int_0^{\pi/2} x\cos(2nx)dx = \frac{4}{\pi}\left(\frac{x}{2n}\sin(2nx)\Big|_0^{\pi/2} - \frac{1}{2n}\int_0^{\pi/2}\sin(2nx)dx\right)$$

(b/c $|x|$ and $\cos(2nx)$ are odd)

$$= \frac{4}{\pi}\left(0 + \frac{1}{(2n)^2}\cos(2nx)\Big|_0^{\pi/2}\right) = \frac{1}{\pi n^2}\left(\cos(n\pi) - 1\right) = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

n even: $\longrightarrow 0$

n odd: $\longrightarrow \frac{-2}{\pi n^2}$

Parseval's: $\frac{1}{\text{period}}\int_{\text{period}}[f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1} a_n^2 + \frac{1}{2}\sum_{n=1} b_n^2$

$\longrightarrow$

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (|x|)^2 dx = \left(\frac{\pi}{4}\right)^2 + \frac{1}{2}\sum_{n\,odd} \frac{4}{\pi^2 n^4} + 0$$

$$(|x|)^2 = x^2 \rightarrow \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x^2 dx = \pi^2/16 + \frac{2}{\pi^2}\sum_{n\,odd} \frac{1}{n^4}$$

$$\frac{1}{\pi}\left(x^3/3 \Big|_{-\pi/2}^{\pi/2}\right) = \frac{\pi^2}{12} = \pi^2/16 + \frac{2}{\pi^2}\sum_{n\,odd} \frac{1}{n^4}$$

$$\sum_{n\,odd} \frac{1}{n^4} = \left(\frac{4\pi^2}{48} - \frac{3\pi^2}{48}\right)\frac{\pi^2}{2} = \boxed{\frac{\pi^4}{96}}$$