Q1] We want to solve $\frac{d^2 y(t)}{dt^2} + \omega^2 y(t) = \sin \alpha t$

But we already know the sol^n to $\frac{d^2 G(t,t')}{dt^2} + \omega^2 G(t,t') = \delta(t-t')$

so we can multiply $\sin \alpha t$ on both sides,

& integrate to find the solution:

$$\int_0^\infty \left(\frac{d^2 G(t,t')}{dt^2} + \omega^2 G(t,t')\right) \sin\alpha t' \, dt' = \int_0^\infty \delta(t-t') \sin \alpha t' \, dt'$$

$$= \left(\frac{d^2}{dt^2} + \omega^2\right) \underbrace{\int_0^\infty G(t,t') \sin \alpha t' \, dt'}_{y(t)} = \sin \alpha t$$

We want the explicit form of $y(t)$, since that will solve the diff eq $\left(\frac{d^2}{dt^2} + \omega^2\right) y(t) = \sin \alpha t$

$$y(t) = \int_0^\infty \sin \alpha t' \frac{1}{\omega} \sin \omega (t-t') \, dt'$$

$$G(t,t') = 0 \quad t < t'$$

$$G(t,t') = \frac{1}{\omega} \sin \omega (t-t') \quad t > t'$$

$$\text{so } y(t) = \int_0^t \sin \alpha t' \frac{1}{\omega} \sin \omega (t-t') \, dt'$$

use the identity
$\sin A \sin B$
$= \cos(A+B) - \cos(A-B)$

$$= \frac{1}{\omega}\left[\frac{\alpha \sin(\omega t) - \omega \sin(\alpha t)}{\alpha^2 - \omega^2}\right]$$

$$\lim_{\alpha \to \omega} = \frac{1}{\omega}\left[\frac{(\omega + a)\sin(\omega t) - \omega \sin((\omega + a)t)}{(\omega + a)^2 - \omega^2}\right]$$

$$= \frac{1}{\omega}\left[\frac{(\omega+a)\sin(\omega t) - \omega(\sin\omega t + \cos\omega t(at))}{2\omega a}\right]$$

$$= \frac{1}{2\omega^2 \cancel{a}}\left[\cancel{a}\sin(\omega t) - \omega\cos\omega(\cancel{a}t)\right]$$

$$= \frac{\sin(\omega t) - \omega\cos(\omega t)t}{2\omega^2}$$

as $t \to \infty$, $y \to \infty$ hence proved resonance exists

Q2) Now we do the same thing for e^{-t}!

$$y(t) = \int_0^\infty G(t-t')e^{-t'}dt'$$

$$= \int_0^t \frac{1}{\omega}\sin\omega(t-t')e^{-t'}dt'$$

$$= \int_0^t \frac{1}{\omega}\left(\frac{e^{i\omega(t-t')} - e^{-i\omega(t-t')}}{2i}\right)e^{-t'}dt'$$

$$= \frac{1}{2\omega i}e^{i\omega t}\left[\frac{e^{(-i\omega-1)t'}}{-i\omega-1}\right]_0^t - \frac{e^{-i\omega t}}{2\omega}\left[\frac{e^{(i\omega-1)t'}}{i\omega-1}\right]_0^t$$

$$= \frac{e^{i\omega t}}{2\omega i}\left[\frac{e^{(-i\omega-1)t} - 1}{-i\omega-1}\right] - \frac{e^{-i\omega t}}{2\omega i}\left[\frac{e^{(i\omega-1)t} - 1}{i\omega-1}\right]$$

$$= \frac{(e^{-t} - e^{i\omega t})(1-i\omega)}{-2\omega i(1+\omega^2)} - \frac{(e^{-t} - e^{-i\omega t})(1+i\omega)}{-2\omega i(1+\omega^2)}$$

$$= \frac{1}{2\omega(1+\omega^2)}\left[(e^{-t} - e^{-i\omega t})(1+i\omega) - (e^{-t} - e^{i\omega t})(1-i\omega)\right]$$

$$= \left[\cancel{e^{-t}} + i\omega e^{-t} - e^{-i\omega t} - i\omega e^{-i\omega t} - \cancel{e^{-t}} + i\omega e^{-t} + e^{i\omega t} - i\omega e^{i\omega t} \right] \Big/ 2\omega i(1+\omega^2)$$

$$= \left[\frac{\omega e^{-t} + \sin\omega t - \omega\cos\omega t)}{\omega(1+\omega^2)} \right]$$

Q3) a) $xy' = y$

let $y = \sum_{n=0}^{\infty} a_n x^n$

then $xy' - y = 0 \Rightarrow \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$

now n is a dummy variable, let's relabel the indices $\sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$\Rightarrow -a_0 + \sum_{n=1}^{\infty} (n a_n - a_n) x^n = 0$$

for LHS to be 0, every coeff of x^n $\forall n$ should be 0, since x^n is arbitrary values.

$\Rightarrow a_0 = 0 \quad \underbrace{a_1 = a_1}_{\text{this is an identity}}, \quad a_n = n a_n \Rightarrow a_n = 0 \;\forall n > 1$

so

$y = a_1 x$ is the answer.

b) $y' = 3x^2 y$ let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 3 a_n x^{n+2} = 0$$

let's rewrite $\sum_{n=1}^{\infty} n a_n x^{n-1}$ as

$$a_1 + a_2 x + \sum_{n=0}^{\infty} (n+3) a_{n+3} x^{n+2}$$

then: $\Rightarrow a_1 + a_2 x + \sum_{n=0}^{\infty} (n+3) a_{n+3} x^{n+2} - 3 a_n x^{n+2} = 0$

$\Rightarrow a_1 = 0 \quad a_2 = 0 \quad \& \ (n+3) a_{n+3} = 3 a_n$

$\Rightarrow a_{n+3} = \frac{3}{n+3} a_n$

but $a_1 = a_2 = 0 \Rightarrow a_{3n+1} = a_{3n+2} = 0$

$$a_{3n} = \frac{3}{3n} a_{3n-3} = \frac{1}{n} a_{3n-3} = \frac{1}{n} \times \frac{3}{(3n-3)} a_{n-6} \ldots$$

$$= \left[\frac{1}{n} \times \frac{1}{(n-1)} \times \frac{1}{(n-2)} \cdots \right] a_0$$

$$= \frac{a_0}{n!}$$

so $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=3k;\, k=0}^{\infty} \frac{a_0 (x^3)^k}{k!} = a_0 e^{x^3}$

Q4) $y'' - 4xy' + (4x^2 - 2)y = 0$

let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 4n a_n x^n + \underbrace{\sum_{n=0}^{\infty} 4a_n x^{n+2}} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

the lowest power of x in this term is x^2, so we use this as a basis to rewrite every other series.

$$\Rightarrow 2a_2 + 6a_3 x - 4a_1 x - 2a_0 - 2a_1 x$$

$$+ \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}x^n - 4na_n x^n + 4a_{n-2}x^n - 2a_n x^n = 0$$

so: $2a_2 - 2a_0 = 0 \quad 6a_3 - 6a_1 = 0$

$$a_{n+2} = \frac{(4n+2)a_n - 4a_{n-2}}{(n+2)(n+1)}$$

it's not obvious what the series is! We can

(a) Explicitly calculate the first few terms

(b) Prove by induction (solving the ODE another way). I'm sure you all plugged the eq^n in wolfram alpha, which would give you a solution that is a sum of exponentials $y(x) = a_0 e^{x^2} + a_1 x e^{x^2}$

Motivated by this,

let $\boxed{a_{2k} = \frac{a_0}{k!}}$ $\quad a_{2k+2} = \dfrac{(8k+2)\frac{a_0}{k!} - \frac{4a_0}{(k-1)!}}{(2k+2)(2k+1)}$

$$= \dfrac{\frac{(8k+4)a_0}{k!} - \frac{2a_0}{k!} - \frac{4a_0}{(k-1)!}}{(2k+2)(2k+1)} = \frac{2a_0}{(k+1)!} - \dfrac{\frac{(\cancel{2}+4k)a_0}{(k)!}}{(\cancel{2}k+\cancel{2})\cancel{(2k+1)}}$$

$$= \frac{2a_0}{(k+1)!} - \frac{a_0}{(k+1)!} = \frac{a_0}{(k+1)!}.$$

hence proved! same proof of a_{2k+1} as well.

So $y(x) = \sum_{k=0}^{\infty} a_0 \frac{x^{2k}}{k!} + \sum_{k=0}^{\infty} a_1 \frac{x^{2k+1}}{k!}$

$$\Longrightarrow \boxed{y(x) = a_0 e^{x^2} + a_1 x e^{x^2}}$$

Q4) $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

There's a variety of ways to get these — Rodrigues' formula, generating function, integral identity etc.

The easiest way to define $P_n(x)$ is :

$$\int_{-1}^{1} P_m(x) P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}$$ (orthogonal)

Now, let $P_0(x) = 1$, then the easiest way to make the integral is to choose an odd function $\because \int_{-x}^{x}$ odd function $= 0$

So $P_{2n+1}(x)$ should be odd!

And if $P_{2n+1}(x)$ is odd, then odd × even function is an odd function. And $\int_{-x}^{x}$ odd function $= 0$

so we should choose $P_{2m}(x)$ as even functions.

Thus, if $P_{2n+1}(x)$ consists of odd powers of x, and $P_{2m}(x)$ contains even powers of x, then $\int_{-1}^{1} P_{2n+1}(x) P_{2m}(x)\,dx$ will always be zero. So they satisfy orthogonality by design.