1. a. $\frac{\partial^2 f}{\partial \theta^2} + r\frac{\partial f}{\partial r} = 0$ say $f(r,\theta) = f_1(r)f_2(\theta)$

$$\frac{\partial^2}{\partial\theta^2}(f_1(r)f_2(\theta)) + r\frac{\partial}{\partial r}(f_1(r)f_2(\theta)) = 0$$

$$f_1(r)\frac{\partial^2 f(\theta)}{\partial\theta^2} + rf_2(\theta)\frac{\partial f_1(r)}{\partial r} = 0$$

Multiply each side by $\frac{1}{f_1(r)f_2(\theta)}$

$$\frac{\partial^2 f_2(\theta)}{\partial\theta^2}\frac{1}{f_2(\theta)} + \frac{\partial f_1(r)}{\partial r}\frac{r}{f_1(r)} = 0$$

only depends on θ — only depends on r

so for that to cancel to zero, each of those terms must equal a constant

So $\frac{\partial^2 f_2(\theta)}{\partial\theta^2}\frac{1}{f_2(\theta)} = -\frac{\partial f_1(r)}{\partial r}\frac{r}{f_1(r)}$

$$\frac{\partial^2 f_2(\theta)}{\partial\theta^2}\frac{1}{f_2(\theta)} = \alpha^2 \rightarrow \boxed{\frac{\partial^2 f_2(\theta)}{\partial\theta^2} = \alpha^2 f_2(\theta)}$$

(the choice of α^2 is arbitrary, could choose α)

So $-\frac{\partial f_1(r)}{\partial r}\frac{r}{f_1(r)} = \alpha^2 \rightarrow \boxed{r\frac{\partial f_1(r)}{\partial r} = -\alpha^2 f_1(r)}$

b. $\frac{\partial^2 f}{\partial\theta^2} + \theta\frac{\partial f}{\partial r} = 0$ say $f(r,\theta) = f_1(r)f_2(\theta)$

$$\rightarrow f_1(r)\frac{\partial^2 f_2(\theta)}{\partial\theta^2} + \theta f_2(\theta)\frac{\partial f_1(r)}{\partial r} = 0$$

Multiply both sides by $\frac{1}{f_1(r)f_2(\theta)}$:

$$\rightarrow \frac{\partial^2 f_2(\theta)}{\partial\theta^2}\frac{1}{f_2(\theta)} + \theta\frac{\partial f_1(r)}{\partial r}\frac{1}{f_1(r)} = 0$$

$$\rightarrow \frac{\partial^2 f_2(\theta)}{\partial\theta^2}\frac{1}{\theta f_2(\theta)} + \frac{\partial f_1(r)}{\partial r}\frac{1}{f_1(r)} = 0$$

$$\frac{\partial^2 f_2(\theta)}{\partial\theta^2}\frac{1}{\theta f_2(\theta)} = \alpha^2 \rightarrow \boxed{\frac{\partial^2 f_2(\theta)}{\partial\theta^2} = \alpha^2\theta f_2(\theta)}$$

$$\frac{\partial f_1(r)}{\partial r}\frac{1}{f_1(r)} = -\alpha^2 \rightarrow \boxed{\frac{\partial f_1(r)}{\partial r} = -\alpha^2 f_1(r)}$$

1c. $x^2 \frac{\partial^2 f}{\partial x^2} + (x+y)\frac{\partial f}{\partial y} = 0 \qquad f(x,y) = f_1(x) f_2(y)$

$\rightarrow x^2 f_2(y) \frac{\partial^2 f_1(x)}{\partial x^2} + (x+y) f_1(x) \frac{\partial f_2(y)}{\partial y} = 0$

$\rightarrow \frac{\partial^2 f_1(x)}{\partial x^2} \frac{x^2}{f_1(x)} + x \frac{\partial f_2(y)}{\partial y} \frac{1}{f_2(y)} + y \frac{\partial f_2(y)}{\partial y} \frac{1}{f_2(y)} = 0$

cannot separate into terms that only depend on x and terms that only depend on y

d. $\frac{\partial^2 f}{\partial x \partial y} + e^{xy} \partial f/\partial y = 0 \qquad f(x,y) = f_1(x) f_2(y)$

$\rightarrow \frac{\partial f_1(x)}{\partial x} \frac{\partial f_2(y)}{\partial y} + e^{xy} f_1(x) \frac{\partial f_2(y)}{\partial y} = 0 \rightarrow \frac{\partial f_2(y)}{\partial y}$ in each term

$\rightarrow \frac{\partial f_1(x)}{\partial x} + e^{xy} f_1(x) = 0 \rightarrow e^{xy} = -\frac{\partial f_1(x)}{\partial x} \frac{1}{f_1(x)}$

$\rightarrow xy = \ln\left(\frac{-f_1'(x)}{f_1(x)}\right) \rightarrow y = \frac{1}{x} \ln\left(\frac{-f_1'(x)}{f_1(x)}\right)$

$\boxed{y = \text{constant} = \alpha}$ $\longrightarrow$ But this is a trivial case

$\frac{1}{x} \ln\left(\frac{-f_1'(x)}{f_1(x)}\right) = \alpha \rightarrow -\frac{f_1'(x)}{f_1(x)} = e^{\alpha x}$

$\rightarrow \boxed{\frac{\partial f_1(x)}{\partial x} = -f_1(x) e^{\alpha x}}$

leads to trivial solutions

So not separable for non-trivial solution

1\. e. $\frac{\partial^2 f}{\partial xy} + xy\frac{\partial^2 f}{\partial y^2} = 0 \qquad f(x,y) = f_1(x) f_2(y)$

$$\rightarrow \frac{\partial f_1(x)}{\partial x}\frac{\partial f_2(y)}{\partial y} + xy f_1(x)\frac{\partial^2 f_2(y)}{\partial y^2} = 0$$

$$\rightarrow \frac{\partial f_1(x)}{\partial x}\frac{1}{x f_1(x)} + y\frac{\partial^2 f_2(y)}{\partial y^2}\frac{1}{\partial f_2(y)/\partial y} = 0$$

$$\frac{\partial f_1(x)}{\partial x}\frac{1}{x f_1(x)} = \alpha \rightarrow \boxed{\frac{\partial f_1(x)}{\partial x} = \alpha x f_1(x)}$$

$$y\frac{\partial^2 f_2(y)}{\partial y^2}\frac{1}{\partial f_2(y)/\partial y} = -\alpha \rightarrow \boxed{\frac{\partial^2 f_2(y)}{\partial y^2} = \frac{-\alpha}{y}\frac{\partial f_2(y)}{\partial y}}$$

---

f. $\frac{\partial^2 f}{\partial r \partial \theta} + r\theta\frac{\partial f}{\partial \theta} + r^2 f = 0 \qquad f(r,\theta) = f_1(r) f_2(\theta)$

$$\rightarrow \frac{\partial f_1(r)}{\partial r}\frac{\partial f_2(\theta)}{\partial \theta} + r\theta f_1(r)\frac{\partial f_2(\theta)}{\partial \theta} + r^2 f_1(r) f_2(\theta) = 0$$

$$\rightarrow \frac{\partial f_1(r)}{\partial r}\frac{1}{r f_1(r)} + \theta + r f_2(\theta)\frac{1}{\partial f_2(\theta)/\partial\theta} = 0$$

cannot separate

Alternate solution (not required for full credit, just an fyi)

1. c) $x^2\frac{\partial^2 f}{\partial x^2} + (x+y)\frac{\partial f}{\partial y} = 0 \qquad f(x,y) = f_1(x)f_2(y)$

$\rightarrow x^2 f_2(y) f_1''(x) + (x+y) f_1(x) f_2'(y) = 0$

$\rightarrow \frac{x^2 f_1''(x)}{f_1(x)} + x\frac{f_2'(y)}{f_2(y)} + y\frac{f_2'(y)}{f_2(y)} = 0$

differentiate both sides w.r.t. x; $\frac{\partial}{\partial x}(\ldots)$

$\frac{\partial}{\partial x}\left(\frac{x^2 f_1''(x)}{f_1(x)}\right) + \frac{f_2'(y)}{f_2(y)} + 0 = 0$

$\frac{\partial}{\partial x}\left(\frac{x^2 f_1''(x)}{f_1(x)}\right) = \frac{-f_2'(y)}{f_2(y)} = \alpha$

$\rightarrow \boxed{f_2'(y) = -\alpha f_2(y)}$

$\rightarrow \frac{\partial}{\partial x}\left(\frac{x^2 f_1''(x)}{f_1(x)}\right) = \alpha$ integrate both sides

$\rightarrow \frac{x^2 f_1''(x)}{f_1(x)} = \alpha x + c$

$x^2 f_1''(x) = (\alpha x + c) f_1(x)$

2. a) $\nabla^2 f + Cf = 0 \qquad f(x,y,z) = f_1(x) f_2(y) f_3(z)$

$\rightarrow \nabla^2 (f_1(x) f_2(y) f_3(z)) + C f_1(x) f_2(y) f_3(z) = 0$

$\rightarrow f_2(y) f_3(z) f_1''(x) + f_1(x) f_3(z) f_2''(y) + f_1(x) f_2(y) f_3''(z) + C f_1(x) f_2(y) f_3(z) = 0$

divide both sides by $f_1 f_2 f_3$ $\rightarrow \dfrac{f_1''(x)}{f_1(x)} + \dfrac{f_2''(y)}{f_2(y)} + \dfrac{f_3''(z)}{f_3(z)} + C = 0$

$$\frac{f_1''(x)}{f_1(x)} = \text{constant} \rightarrow \boxed{f_1''(x) = K f_1(x)}$$

$K > 0$: let $K = \alpha^2$

$f_1''(x) = \alpha^2 f_1(x) \rightarrow f_1(x) = Ae^{\alpha x} + Be^{-\alpha x}$

$f_1'(x) = A\alpha e^{\alpha x} - B\alpha e^{-\alpha x}$

$f_1''(x) = \alpha^2 (Ae^{\alpha x} + Be^{-\alpha x}) = \alpha^2 f_1(x)$ ✓

$$\boxed{f_1(x) = Ae^{\alpha x} + Be^{-\alpha x}}$$

$K < 0$: let $K = -\alpha^2$

$f_1''(x) = -\alpha^2 f_1(x) \rightarrow f_1(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$f_1'(x) = -A\alpha\sin(\alpha x) + \alpha B\cos(\alpha x)$

$f_1''(x) = -\alpha^2 A\cos(\alpha x) - \alpha^2 B\sin(\alpha x) = -\alpha^2 f_1(x)$ ✓

$$\boxed{f_1(x) = A\cos(\alpha x) + B\sin(\alpha x)}$$

$K = 0$: $f_1''(x) = 0$

$f_1'(x) = \text{constant} = A$ (call it A)

$\boxed{f_1(x) = Ax + B}$ (B is another integration constant)

2 b) $\left.\frac{\partial f_1}{\partial x}\right|_{x=0} = \left.\frac{\partial f_1}{\partial x}\right|_{x=1} = 0$

try $f_1(x) = Ae^{\alpha x} + Be^{-\alpha x}$

$\left.\frac{\partial f_1}{\partial x}\right|_{x=0} = \alpha Ae^{\alpha x} - B\alpha e^{-\alpha x}\Big|_{x=0} = \alpha A - \alpha B = 0$

$A = B$ so

$f_1(x) = A(e^{\alpha x} + e^{-\alpha x})$

$\left.\frac{\partial f_1}{\partial x}\right|_{x=1} = \alpha Ae^{\alpha} - \alpha Ae^{-\alpha} = 0$

$= \alpha A(e^{\alpha} - e^{-\alpha}) = 2\alpha A \sinh(\alpha) = 0$

$\alpha = 0$

that would mean $f_1(x)$ would be constant, which is a pretty trivial solution.

$f_1(x) = Ax + B$ will give us the same thing

try $f_1(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$\left.\frac{\partial f_1}{\partial x}\right|_{x=0} = -\alpha A\sin(\alpha x) + \alpha B\cos(\alpha x)\Big|_{x=0} = \alpha B = 0 \rightarrow B = 0$

so $f_1(x) = A\cos(\alpha x)$

$\left.\frac{\partial f_1}{\partial x}\right|_{x=1} = -\alpha A\sin(\alpha) = 0 \rightarrow \alpha = n\pi x$

where n is any integer

so $\boxed{f_1(x) = \sum_n A_n \cos(n\pi x)}$

2. c) $\frac{\partial f}{\partial x}(0,y,z) = \frac{\partial f}{\partial x}(1,y,z) = \frac{\partial f}{\partial y}(x,0,z) = \frac{\partial f}{\partial y}(x,1,z) = \frac{\partial f}{\partial z}(x,y,0)$
$= \frac{\partial f}{\partial z}(x,y,1) = 0$

from part b, $f_1(x) = \sum_n A_n \cos(n\pi x)$, $f_2(y) = \sum_m B_m \cos(m\pi x)$,

$f_3(z) = \sum_p C_p \cos(p\pi x)$ (each needs its own index)

from part a,

$$\frac{f_1''(x)}{f_1(x)} + \frac{f_2''(y)}{f_2(y)} + \frac{f_3''(z)}{f_3(z)} + C = 0$$

b/c each turned out to be a sin/cos solution, each term is a negative separation constant, say

$$(-\alpha^2) + (-\beta^2) + (-\gamma^2) + C = 0$$

where $\alpha = n\pi$ $\beta = m\pi$ $\gamma = p\pi$

So our equation becomes $-n^2\pi^2 - m^2\pi^2 - p^2\pi^2 + C = 0$

try: $n = m = p = 1$: $C = 3\pi^2 = 29.6 \rightarrow f(x,y,z) = A\cos(\pi x)\cos(\pi y)\cos(\pi z)$

$n = m = 1$, $p = 2$ (and $n = p = 1$, $m = 2$; $m = p = 1$, $n = 2$) : $C = 6\pi^2 = 59.2 \rightarrow f(x,y,z) = A\cos(\pi x)\cos(\pi y)\cos(2\pi z)$
or $A\cos(\pi x)\cos(2\pi y)\cos(\pi z)$
or $A\cos(2\pi x)\cos(\pi y)\cos(\pi z)$

$n = 1$, $m = p = 2$ (and $m = 1$, $n = p = 2$; $p = 1$, $n = m = 2$) : $C = 9\pi^2 = 88.8 \rightarrow f(x,y,z) = A\cos(\pi x)\cos(2\pi y)\cos(2\pi z)$
or $A\cos(2\pi x)\cos(2\pi y)\cos(\pi z)$
or $A\cos(2\pi x)\cos(\pi y)\cos(2\pi z)$

$n = m = p = 2$: $C = 118.4$ $C > 100$

$n = m = p = 0$: $C = 0 \rightarrow f(x,y,z) = A$

$n = m = 0$, $p = 1$ (and $n = p = 0$ $m = 1$; $m = p = 0$ $n = 1$) : $C = \pi^2 = 9.87 \rightarrow f(x,y,z) = A\cos(\pi x)$ or $A\cos(\pi y)$ or $A\cos(\pi z)$

$n = m = 1$ $p = 0$ (and $n = p = 1$ $m = 0$; $m = p = 1$ $n = 0$) : $C = 2\pi^2 = 19.7 \rightarrow f(x,y,z) = A\cos(\pi x)\cos(\pi y)$ or $A\cos(\pi x)\cos(\pi z)$ or $A\cos(\pi y)\cos(\pi z)$

more on next page

2c) cont'd

$n=2,\ m=p=0$
or $n=m=0\ p=2$
or $n=p=0\ m=2$

$C=4\pi^2=39.5$ $\quad f(x,y,z)=A\cos(2\pi x)$ or $A\cos(2\pi y)$ or $A\cos(2\pi z)$

$n=m=2\ p=0$
or $n=p=2\ m=0$
or $m=p=2\ n=0$

$C=8\pi^2=79.0$ $\quad f(x,y,z)=A\cos(2\pi x)\cos(2\pi y)$ or $A\cos(2\pi x)\cos(2\pi z)$ or $A\cos(2\pi y)\cos(2\pi z)$

$n=1,\ m=0,\ p=3$
or $n=1\ m=3\ p=0$
or $n=3\ m=0\ p=1$
or $n=3\ m=1\ p=0$
or $n=0\ m=3\ p=1$
or $n=0\ m=1\ p=3$

$C=10\pi^2=98.7$ $\quad f(x,y,z)=A\cos(3\pi x)\cos(\pi y)$
or $A\cos(\pi x)\cos(3\pi y)$
or $A\cos(3\pi x)\cos(\pi z)$
or $A\cos(\pi x)\cos(3\pi z)$
or $A\cos(3\pi y)\cos(\pi z)$
or $A\cos(\pi y)\cos(3\pi z)$

$n=m=0\ p=3$
or $n=p=0\ m=3$
or $m=p=0\ n=3$

$C=9\pi^2=88.8$ $\quad f(x,y,z)=A\cos(3\pi x)$ or $A\cos(3\pi y)$ or $A\cos(3\pi z)$

$n=0\ m=1\ p=2$
or perturbations

$C=5\pi^2=49.3$ $\quad f(x,y,z)=A\cos(\pi x)\cos(2\pi y)$
or $A\cos(\pi x)\cos(2\pi z)$
or $A\cos(2\pi x)\cos(\pi y)$
or $A\cos(2\pi x)\cos(\pi y)$
or $A\cos(\pi y)\cos(2\pi z)$
or $A\cos(2\pi y)\cos(\pi z)$

3. $\frac{\partial T}{\partial t} = D\nabla^2 T \xrightarrow{\text{Steady-State}} \nabla^2 T = 0$

Assume $T = T_1(x)T_2(y) \rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

$\rightarrow T_2(y)T_1''(x) + T_1(x)T_2''(y) = 0$

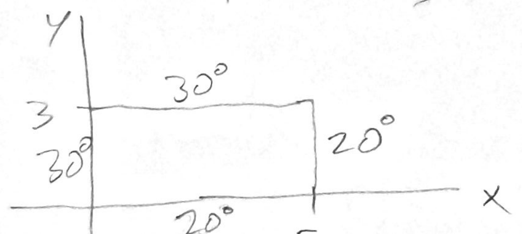
multiply both sides by $\frac{1}{T_1(x)T_2(y)}$: $\frac{T_1''(x)}{T_1(x)} + \frac{T_2''(y)}{T_2(y)} = 0$

only depends on x — only depends on y — ∴ both must be constant

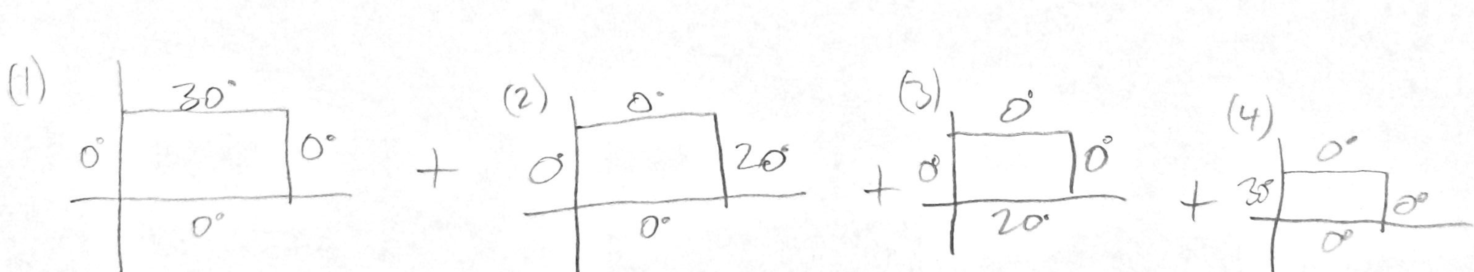
$\frac{T_1''(x)}{T_1(x)} = \pm K^2 \rightarrow T_1''(x) = \pm K^2 T_1(x)$

$\frac{T_2''(y)}{T_2(y)} = \mp K^2 \rightarrow T_2''(y) = \mp K^2 T_2(y)$ (needs to be opposite sign from eqn for x)

$+K^2$ will give exponentials, $-K^2$ will give sine/cosine, we'll look at boundary conditions to help us decide.



is the same as



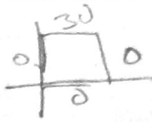
Start w/ (1): $T(0,y) = T(5,y) = T(x,0) = 0$ and $T(x,3) = 0$

notice the x boundaries are both zero, that tips us off that a periodic solution might fit for $T_1(x)$

So $T_1''(x) = -K^2 T_1(x) \rightarrow T_1(x) = A\cos(Kx) + B\sin(Kx)$

meaning $T_2''(y) = K^2 T_2(y) \rightarrow T_2(y) = Ce^{Ky} + De^{-Ky}$

So in general for (1), $T(x,y) = (A\cos(Kx) + B\sin(Kx))(Ce^{Ky} + De^{-Ky})$

3. cont'd   solution for (1)  starting from $T(x,y) = (A\cos(kx) + B\sin(kx))(Ce^{ky} + De^{-ky})$

$T(0,y) = 0 = (A+0)(Ce^{ky}+De^{-ky}) \to A = 0 \to T(x,y) = B\sin(kx)(Ce^{ky}+De^{-ky})$

can get absorbed into C and D

$T(5,y) = 0 = \sin(5k)(Ce^{ky}+De^{-ky}) \to \sin(5k) = 0 \to 5k = n\pi \to k = \frac{n\pi}{5}$

$\to T(x,y) = \sin\left(\frac{n\pi x}{5}\right)\left(Ce^{n\pi y/5} + De^{-n\pi y/5}\right)$

$T(x,0) = 0 = \sin\left(\frac{n\pi x}{5}\right)(Ce^{0} + De^{0}) \to C + D = 0 \to C = -D$

$\to T(x,y) = \sin\left(\frac{n\pi x}{5}\right)\left(Ce^{n\pi y/5} - Ce^{-n\pi y/5}\right) = \sin\left(\frac{n\pi x}{5}\right)\left(2C\sinh\left(\frac{n\pi y}{5}\right)\right)$

→ Arbitrary constant, so absorb the 2

$T(x,y) = C\sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{n\pi y}{5}\right)$

$T(x,3) = 30 = C\sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{3n\pi}{5}\right)$ → we can't solve this w/ only one integer n, we therefore need a sum of solutions over the integer n

$\to 30 = \sum_n C_n \sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{3n\pi}{5}\right)$

Now we can use orthogonality of the set of functions $\sin\left(\frac{n\pi x}{5}\right)$ over the interval $[0,5]$

$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \frac{l}{2}\delta_{m,n}$ (for us $l=5$, so $= \frac{5}{2}\delta_{m,n}$)

So we multiply both sides by $\sin\left(\frac{m\pi x}{5}\right)$ and integrate from $0 \to 5$

$$\int_0^5 30\sin\left(\frac{m\pi x}{5}\right)dx = \sum_n C_n\sinh\left(\frac{3n\pi}{5}\right)\underbrace{\int_0^5 \sin\left(\frac{n\pi x}{5}\right)\sin\left(\frac{m\pi x}{5}\right)dx}_{=\frac{5}{2}\delta_{m,n} \text{ which collapses our sum}}$$

$$30\int_0^5 \sin\left(\frac{m\pi x}{5}\right)dx = C_m\sinh\left(\frac{3m\pi}{5}\right)(5/2)$$

$$C_m = \left(\frac{2}{5}\right)\left(\frac{30}{\sinh\left(\frac{3m\pi}{5}\right)}\right)\left(\frac{5}{m\pi}\right)\left[-\cos\left(\frac{m\pi x}{5}\right)\Big|_0^5\right] = \frac{60}{m\pi\sinh\left(\frac{3m\pi}{5}\right)}\underbrace{\left(-\cos(m\pi) + 1\right)}_{\substack{=0 \text{ for } m \text{ even} \\ =2 \text{ for } m \text{ odd}}}$$

$$= \frac{120}{m\pi\sinh\left(\frac{3m\pi}{5}\right)}$$

so for (1), $T(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}} \frac{120}{n\pi\sinh\left(\frac{3n\pi}{5}\right)}\sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{n\pi y}{5}\right)$

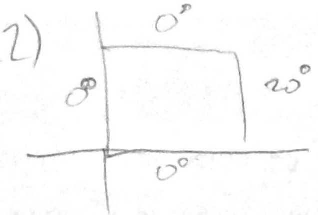
note: what we did above is the same as saying $30 = \sum_n \underbrace{C_n\sinh\left(\frac{3n\pi}{5}\right)}_{b_n \text{ (fourier coefficient}}\sin\left(\frac{n\pi x}{5}\right)$

so $C_n = \frac{b_n}{\sinh\left(\frac{3n\pi}{5}\right)}$ $b_n = \frac{2}{5}\int_0^5 30\sin\left(\frac{n\pi x}{5}\right)dx$

gives same answer

Next case: (2)

(3. cont'd)



Notice this time, the two sides w/ zero are on the y boundaries, so let's assume

$$T_2''(y) = -k^2 T_2(y) \rightarrow A\cos(ky) + B\sin(ky)$$

$$T_1''(x) = k^2 T_1(x) \rightarrow Ce^{kx} + De^{-kx}$$

The boundary conditions are essentially the same as for (1)

$$T(x,0) = 0 \rightarrow A = 0 \rightarrow T(x,y) = B\sin(ky)\left(Ce^{kx} + De^{-kx}\right)$$

$$T(x,3) = 0 = B\sin(3k)\left(Ce^{kx} + De^{-kx}\right) \rightarrow \sin(3k) = 0 \rightarrow 3k = n\pi \rightarrow k = n\pi/3$$

$$\rightarrow T(x,y) = B\sin(n\pi y/3)\left(Ce^{n\pi x/3} + De^{-n\pi x/3}\right)$$

$$T(0,y) = 0 = B\sin\left(\frac{n\pi y}{3}\right)(C+D) \rightarrow C = -D$$

$$\rightarrow T(x,y) = B\sin\left(\frac{n\pi y}{3}\right)(C)\underbrace{\left(e^{n\pi x/3} - e^{-n\pi x/3}\right)}_{2\sinh(n\pi x/3)}$$

$$= B\sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{n\pi x}{3}\right)$$

Absorb 2C into B (b/c they're arbitrary anyway at this point)

$$T(5,y) = 20° = B\sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{5n\pi}{3}\right)$$

Same as last time, we'll need a series to solve it

$$\rightarrow 20° = \sum_n B_n \sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{5n\pi}{3}\right)$$

multiply both sides by $\sin\left(\frac{m\pi y}{3}\right)$ integrate from $0 \rightarrow 3$

$$\int_0^3 20\sin\left(\frac{m\pi y}{3}\right)dy = \sum_n B_n \sinh\left(\frac{5n\pi}{3}\right)\underbrace{\int_0^3 \sin\left(\frac{n\pi y}{3}\right)\sin\left(\frac{m\pi y}{3}\right)dy}_{=\frac{3}{2}\delta_{m,n}}$$

$$-20\left(\frac{3}{m\pi}\right)\cos\left(\frac{m\pi y}{3}\right)\Big|_0^3 = \frac{3}{2}B_m \sinh\left(\frac{5m\pi}{3}\right)$$

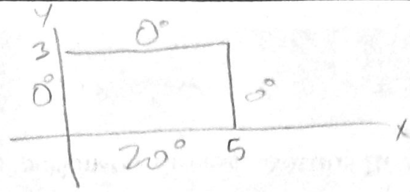
$$\frac{1}{\sinh\left(\frac{5m\pi}{3}\right)}\left(\frac{40}{m\pi}\right)\underbrace{\left(-\cos(m\pi) + 1\right)}_{\substack{=0 \text{ for } m \text{ even} \\ =2 \text{ for } m \text{ odd}}} = B_m \longrightarrow B_m = \frac{80}{m\pi\sinh\left(\frac{5m\pi}{3}\right)} \quad (m \text{ odd})$$

So for (2)

$$T(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{80\sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{n\pi x}{3}\right)}{n\pi\sinh\left(\frac{5n\pi}{3}\right)}$$

Case (3)

3. cont'd Case (3)



the boundaries for x are the same as for case (1), so we already know $T_1(x) = B\sin\left(\frac{n\pi x}{5}\right)$

so $T(x,y) = B\sin\left(\frac{n\pi x}{5}\right)\left(Ce^{n\pi y/5} + De^{-n\pi y/5}\right)$

let's apply $T(x,3) = 0 = B\sin\left(\frac{n\pi x}{5}\right)\left(Ce^{n\pi 3/5} + De^{-n\pi 3/5}\right)$

$\rightarrow Ce^{3n\pi/5} = -De^{-3n\pi/5}$ we could use this, but we can use a trick to make it look nicer.

we can define $T_2(y) = Ce^{n\pi y/5} + De^{-n\pi y/5}$ as:

$Ce^{n\pi(y-3)/5} + De^{-n\pi(y-3)/5}$ w/o any loss of generality b/c

$= \underbrace{Ce^{3n\pi/5}}_{\text{arbitrary constant}} e^{n\pi y/5} + \underbrace{De^{3n\pi/5}}_{\text{arbitrary constant}} e^{-n\pi y/5}$

using this, reapply $T(x,3) = 0 = B\sin(n\pi x/5)\left(Ce^{n\pi(3-3)/5} + De^{-n\pi(3-3)/5}\right)$

$= B\sin(n\pi x/5)(C+D)$

$\rightarrow$ now $C = -D$

so $T_2(y) = C\left(e^{n\pi(y-3)/5} - e^{-n\pi(y-3)/5}\right)$

$= 2C\sinh\left(\frac{n\pi(y-3)}{5}\right)$

so $T(x,y) = B\sin(n\pi x/5)\sinh\left(\frac{n\pi(y-3)}{5}\right)$

now $T(x,0) = 20° = B\sin(n\pi x/5)\sinh\left(-\frac{3n\pi}{5}\right)$ use infinite series

also $\sinh\left(\frac{-3n\pi}{5}\right) = -\sinh\left(\frac{3n\pi}{5}\right)$

$20° = -\sum_n B_n \sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{3n\pi}{5}\right)$

as before: $\int_0^5 20\sin\left(\frac{m\pi x}{5}\right)dx = -\sum_n B_n\sinh\left(\frac{3n\pi}{5}\right)\underbrace{\int_0^5 \sin\left(\frac{n\pi x}{5}\right)\sin\left(\frac{m\pi x}{5}\right)dx}_{= \frac{5}{2}\delta_{mn}}$

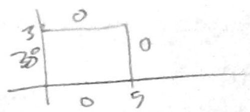
$20(2)\left(\frac{5}{m\pi}\right) = -\frac{5}{2}B_m\sinh\left(\frac{3m\pi}{5}\right)$

for odd n, 0 for even (from before) $\rightarrow B_n = \frac{-80}{n\pi\sinh\left(\frac{3n\pi}{5}\right)}$ for n odd

So for (3): $T(x,y) = -\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{80}{n\pi\sinh\left(\frac{3n\pi}{5}\right)}\sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{n\pi(y-3)}{5}\right)$

$= \boxed{\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{80}{n\pi\sinh\left(\frac{3n\pi}{5}\right)}\sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{n\pi(3-y)}{5}\right)}$ for (3)

3 cont'd, case (4):



the y solution is the same as case (2): $T_2(y) = B\sin\left(\frac{n\pi y}{3}\right)$

for x, we'll use a trick similar to case (3): let $T_1(x) = Ce^{\frac{n\pi(5-x)}{3}} + De^{-\frac{n\pi(5-x)}{3}}$

[using 5-x this time b/c of our final solution from case (3)]

$$T(y,5) = 0 = B\sin\left(\frac{n\pi y}{3}\right)\left(Ce^0 + De^0\right)$$

$$\rightarrow C = -D \rightarrow T_1(x) = 2C\sinh\left(\frac{n\pi(5-x)}{3}\right)$$

$$\rightarrow T(x,y) = \sum_n B_n \sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{n\pi(5-x)}{3}\right)$$

now $T(y,0) = 30° = \sum_n B_n \sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{5n\pi}{3}\right)$

$$\rightarrow \int_0^3 30\sin\left(\frac{m\pi y}{3}\right)dy = \sum_n B_n \sinh\left(\frac{5n\pi}{3}\right)\int_0^3 \sin\left(\frac{n\pi y}{3}\right)\sin\left(\frac{m\pi y}{3}\right)dy$$

$$= 30\left(\frac{3}{m\pi}\right)(2) = \frac{3}{2}B_m \sinh\left(\frac{5m\pi}{3}\right)$$

↳ for odd m, as before
= 0 for even m

$$\rightarrow B_m = \frac{120}{m\pi\sinh\left(\frac{5m\pi}{3}\right)}$$

for (4), $T(x,y) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{120}{n\pi\sinh\left(\frac{5n\pi}{3}\right)}\sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{n\pi(5-x)}{3}\right)$

Full solution is (1) + (2) + (3) + (4):

$$\boxed{T(x,y) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty}\left[\frac{120}{n\pi}\left(\frac{\sin\left(\frac{n\pi x}{5}\right)\sinh\left(\frac{n\pi y}{5}\right)}{\sinh\left(\frac{3n\pi}{5}\right)} + \frac{\sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{n\pi(5-x)}{3}\right)}{\sinh\left(\frac{5n\pi}{3}\right)}\right) + \frac{80}{n\pi}\left(\frac{\sin\left(\frac{n\pi y}{3}\right)\sinh\left(\frac{n\pi x}{3}\right)}{\sinh\left(\frac{5n\pi}{3}\right)} + \frac{\sin\left(\frac{n\pi x}{5}\right)\sinh\left(n\pi(3-y)\right)}{\sinh\left(\frac{3n\pi}{5}\right)}\right)\right]}$$

4.  $f(0,t) = f(L,t) = 0$

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$ assume $f(x,t) = f_1(x) f_2(t)$

$$\rightarrow f_2(t) f_1''(x) = \frac{1}{v^2} f_1(x) f_2''(t)$$

multiply both sides by $\frac{1}{f_1(x) f_2(t)} \rightarrow \frac{f_1''(x)}{f_1(x)} = \frac{1}{v^2}\frac{f_2''(t)}{f_2(t)} = -k^2$

$$\rightarrow f_1''(x) = -k^2 f_1(x) \longrightarrow f_1(x) = C\cos(kx) + D\sin(kx)$$

$$f_2''(t) = -v^2k^2 f_2(t) \rightarrow f_2(t) = A\cos(vkt) + B\sin(vkt)$$

Before any B.C.'s: $f(x,t) = (C\cos(kx) + D\sin(kx))(A\cos(vkt) + B\sin(vkt))$

$f(0,t) = 0 = (C)(f_2(t)) \longrightarrow C = 0$

$f(L,t) = 0 = D\sin(kL)(f_2(t)) \rightarrow kL = n\pi \rightarrow k = \frac{n\pi}{L}$

So for all parts, $f(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A\cos\left(\frac{n\pi v t}{L}\right) + B\sin\left(\frac{n\pi v t}{L}\right)\right]$

But likely need series solutions: $f(x,t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi v t}{L}\right) + B_n\sin\left(\frac{n\pi v t}{L}\right)\right]$

The rest is just applying initial conditions

a) at $t=0$, $f(x,0) = 0$, $\frac{\partial f}{\partial t}(x,0) = L\delta(x-a)$

$f(x,0) = 0 = \sum_n \sin\left(\frac{n\pi x}{L}\right)[A_n + 0] \rightarrow A_n = 0$

so now $f(x,t) = \sum_n B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi v t}{L}\right)$

$$\left.\frac{\partial f}{\partial t}\right|_{t=0} = \sum_n \frac{n\pi v}{L} B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right)\Big|_{t=0} = \sum_n \frac{n\pi v}{L} B_n \sin\left(\frac{n\pi x}{L}\right) = L\delta(x-a)$$

$\rightarrow L\delta(x-a) = \sum_n \frac{n\pi v}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$ use orthogonality: $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{m,n}$

So multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from 0 to L

$$\int_0^L L\delta(x-a)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_n \frac{n\pi v}{L} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \rightarrow L\sin\left(\frac{m\pi a}{L}\right) = \frac{m\pi v}{L} B_m \left(\frac{L}{2}\right)$$

$$\rightarrow B_n = \frac{2L\sin\left(\frac{n\pi a}{L}\right)}{n\pi v}$$

$$\rightarrow \boxed{f(x,t) = \sum_{n=1}^{\infty} \frac{2L\sin(n\pi a/L)}{n\pi v}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi v t}{L}\right)}$$

b) $t<0$: $f(x,t) = \begin{cases} \sin\left(\frac{4\pi x}{L}\right)\sin\left(\frac{4\pi v t}{L}\right) & \text{for } x< L/2 \\ 0 & \text{for } x > L/2) \end{cases}$

So $f(x,0) = 0$ everywhere

$$\left.\frac{\partial f}{\partial t}\right|_{t=0} = \begin{cases} \left.\frac{4\pi v}{L}\sin\left(\frac{4\pi x}{L}\right)\cos\left(\frac{4\pi v t}{L}\right)\right|_{t=0} = \frac{4\pi v}{L}\sin\left(\frac{4\pi x}{L}\right) & \text{for } x<L/2 \\ 0 & \text{for } x>L/2 \end{cases}$$

Start w/ $f(x,t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi v t}{L}\right) + B_n\sin\left(\frac{n\pi v t}{L}\right)\right]$

$$f(x,0) = 0 = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left[A_n + 0\right] \rightarrow A_n = 0$$

$$f(x,t) = \sum_n B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi v t}{L}\right)$$

$$\frac{\partial f(x,t)}{\partial t} = \sum_n \frac{n\pi v}{L} B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right)$$

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0} = \text{piecewise above} = \sum_n \frac{n\pi v}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Same as in a), multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$, integrate from $0 \to L$

$$\underbrace{\int_0^L \left(\left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{\text{split up into two integrals, one from } 0\to L/2 \text{, other from } L/2 \to L} = \sum_n \frac{n\pi v}{L} B_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{=\frac{L}{2}\delta_{n,m}}$$

$$\int_0^{L/2} \frac{4\pi v}{L}\sin\left(\frac{4\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx + \underbrace{\int_{L/2}^{L}(0)\sin\left(\frac{m\pi x}{L}\right)dx}_{=0} = \frac{m\pi v}{L}B_m\left(\frac{L}{2}\right)$$

$$\rightarrow B_n = \frac{8}{nL}\text{ (integral)}$$

$$\boxed{\text{So } f(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi v t}{L}\right) \quad \text{where } B_n = \frac{8}{nL}\int_0^{L/2}\sin\left(\frac{4\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx}$$

thankfully, we're told we don't have to evaluate the integral

c) $$f(x,0) = \begin{cases} 0 & x < L/4 \\ 15 & L/4 \le x < 3L/4 \\ 0 & x > L/4 \end{cases}$$

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0} = 0 \text{ everywhere}$$

Starting from $f(x,t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v t}{L}\right)\right)$

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0} = 0 = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi v}{L}\right)\left.\left[-A_n \sin\left(\frac{n\pi v t}{L}\right) + B_n \cos\left(\frac{n\pi v t}{L}\right)\right]\right|_{t=0}$$

$$\rightarrow 0 = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi v}{L}\right)(B_n) \Rightarrow B_n = 0$$

So $f(x,t) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right)$

$f(x,0) = \text{piecewise above} = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$

$$\int_0^L f(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_n A_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{m,n}}$$

$$\int_0^{L/4}(0)\sin\left(\frac{m\pi x}{L}\right)dx + \int_{L/4}^{3L/4} 15\sin\left(\frac{m\pi x}{L}\right)dx + \int_{3L/4}^{L}(0)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}A_m$$

$$\boxed{\begin{array}{l} \text{So } A_n = \frac{30}{L}\int_{L/4}^{3L/4}\sin\left(\frac{n\pi x}{L}\right)dx \\ \text{and } f(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right) \end{array}}$$

(which is $\frac{2}{L}\int_0^L f(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$, the sine fourier coefficient for $f(x,0)$)