Solution to Problem 2.

a) Consider the series

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n^{0.5}}$$

to find the region of absolute convergence we apply ratio test

$$\frac{|a_{n+1}|}{|a_n|} = \frac{+2^{n+1}|x|^{n+1}}{(n+1)^{0.5}} \cdot \frac{n^{0.5}}{2^n |x|^n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2|x|$$

Absolute convergence for $|x| < 1/2$

must diverge for $|x| > 1/2$

Now, need to check end points more carefully

at $x = +1/2$, series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$$

Alternating series converges conditionally as $|a_n| \to 0$ but it does not converge absolutely as $\frac{1}{n^{0.5}} > \frac{1}{n}$ and latter diverges.

At $x = -1/2$ $\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$ diverges for some reason

Series converges for $-1/2 < x \leq 1/2$, where at $x = 1/2$ it converges conditionally.

(b) $E(v) = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$ [expression for energy in relativity]

It is best to use binomial expansion

$$E(v) = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$= mc^2\left[1 + \left(-\tfrac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \frac{(-1/2)(-3/2)}{2!}\left(-\frac{v^2}{c^2}\right)^4 + \cdots\right]$$

$$= mc^2 + \frac{m\cancel{c^2}}{2}\frac{v^2}{\cancel{c^2}} + \frac{3}{8}mc^2\cdot\frac{v^4}{c^4}$$

$$= \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{non-relativistic Kinetic energy}} + \underbrace{\frac{3}{8}m\frac{v^4}{c^2}}_{\text{leading relativistic correction to Kinetic energy}}$$

Alternatively

If you want to use Taylor expansion it is best to regard expansion variable as $x = \frac{v^2}{c^2}$. It would make algebra easier.

$$E(x) = E(0) + xE'(0) + \frac{x^2}{2!}E''(0)$$

$$E(0) = mc^2$$

$$E'(x) = -\tfrac{1}{2}mc^2(1-x)^{-3/2}(-1) \Rightarrow E'(0) = \frac{mc^2}{2}$$

$$E''(x) = -\tfrac{1}{2}mc^2\left(-\tfrac{3}{2}\right)(1-x)^{-5/2}(+1) \Rightarrow E''(0) = \frac{3mc^2}{4}$$

$$E(v) = \underbrace{E(0)}_{mc^2} + \frac{v^2}{c^2}\cdot\frac{mc^2}{2} + \frac{1}{2}\frac{v^4}{c^4}\cdot\frac{3mc^2}{4}$$

$$= mc^2 + \tfrac{1}{2}mv^2 + \tfrac{3}{8}m\frac{v^4}{c^2}$$

Rubrik

2a

(A) Correct +10

(α) No useful work shown −10

(β) Major error −8

(β') Major error after a few steps −6

(γ) Minor error / missed end points −4

(δ) Missed one end point / conditional convergence −2

2b

(A) Correct +10

(α) No useful work shown −10

(β) Major error in getting to v^2 −8

(γ) Error in v^2 term −4

(δ) Error in v^4 term −2

(only) Numerical error in v^4 term −1