Problem Set 4
Physics 104A

Due Wednesday October 20 at 11:59 PM

Late homework may be turned in by October 27 for half credit

Primary topic: vectors, Matrices

1. For $\mathbf{v} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$, $\mathbf{w} = -\hat{\mathbf{x}} + \hat{\mathbf{y}}$ calculate the following:

a) $\mathbf{v} \cdot \mathbf{w}$

b) $\mathbf{v} \times \mathbf{w}$

c) θ, the angle between $\mathbf{v}$ and $\mathbf{w}$

d) $\hat{\mathbf{v}}$, the unit vector in the direction of $\mathbf{v}$

2. When are the scalar products $(a, b, c) \cdot (a, b, c)$ and $(a, b, c) \cdot (b, c, a)$ equal? When one is larger, which one is it? Explain. (No messy calculations necessary!)

3. Consider (non-orthogonal) coordinates $\hat{\mathbf{m}} = \hat{\mathbf{x}}$, $\hat{\mathbf{n}} = (2\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{5}$, where $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ are Cartesian coordinates.

a) Find the Cartesian coordinates for the points $(m, n) = (1, 1) = \hat{\mathbf{m}} + \hat{\mathbf{n}}$ and $(m, n) = (\sqrt{5}, 1/2)$.

b) If we view matrices as coordinate changes, what is the matrix that changes a vector's coordinates in terms of $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ into its coordinates in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$?

c) If we instead view matrices as distorting the plane itself rather than simply renaming directions, what is the matrix that maps physical vectors $\mathbf{V} = \hat{\mathbf{m}}$ and $\mathbf{W} = \hat{\mathbf{n}}$ into $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, respectively? How is this matrix related to the one in part b)?

d) Find the area of the parallelogram defined by $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$, and explain how it relates to the determinant of the matrix of part c).

4. Let $A = \begin{pmatrix} -1 & 7/2 & -1/2 \\ -2 & 9/2 & -1/2 \\ -2 & 7/2 & 1/2 \end{pmatrix}$. The vectors $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are all eigenvectors of A.

a) What are the eigenvalues?

b) What is a matrix T that diagonalizes A; i.e., $T^{-1}AT$ is diagonal? (Since A is not symmetric, T won't be orthogonal.) What is the resulting diagonal matrix?

c) The vector $\mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ is also an eigenvector of A. What is the eigenvalue of $\mathbf{v}$? Find an eigenvector of A which is perpendicular to $\mathbf{v}$.

5. For each matrix, find the eigenvalues and eigenvectors by hand (no computers please!). Then write down a matrix S such that $S^{-1}AS$ (or $S^{-1}BS$) is diagonal, and the resulting diagonal matrix.

a) $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

b) $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$