1a. Let $f(x) = x^3 + 2x^2 - 1$

Expand in Legendre series: $f(x) = \sum_\ell c_\ell P_\ell(x)$

given 3rd order polynomial we need $\ell \leq 3$

$$\to f(x) = c_0 + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x)$$

$$= c_0 + c_1 (x) + c_2 \left(\tfrac{1}{2}(3x^2 - 1)\right) + c_3 \left(\tfrac{1}{2}(5x^3 - 3x)\right)$$

$$= c_0 - \frac{c_2}{2} + x\left(c_1 - \frac{3}{2} c_3\right) + x^2 \cdot \frac{3}{2} c_2 + x^3 \cdot \frac{5}{2} c_3$$

We get the following eqn's:

$$\frac{5}{2} c_3 = 1 \to c_3 = \frac{2}{5}, \quad \frac{3}{2} c_2 = 2 \to c_2 = 4/3$$

$$c_1 - \frac{3}{2}\left(\frac{2}{5}\right) = 0 \to c_1 = 3/5, \quad c_0 - \frac{1}{2}\left(4/3\right) = -1 \to c_0 = -1/3$$

Or determine c_ℓ directly from: $c_\ell = \dfrac{2\ell+1}{2} \displaystyle\int_{-1}^{1} f(x) P_\ell(x) dx$

This amounts to doing integrals of type

$\displaystyle\int_{-1}^{1} a x^n dx$ which is straight forward so I spare the details.

b) let $f(x) = x^5$, note $f(x)$ is odd so we need consider odd ℓ, and only up to order 5.

$$c_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} x^5 P_\ell(x) dx$$

$$c_1 = \frac{3}{2} \int_{-1}^{1} x^6 dx = \frac{3}{2} \cdot \frac{1}{7} \cdot 2 = \frac{3}{7}$$

$$c_3 = \frac{7}{2} \cdot \frac{1}{2} \int_{-1}^{1} x^5 (5x^3 - 3x) dx = \frac{7}{4} \left[\frac{5}{9}(2) - \frac{3}{7}(2) \right] = \frac{4}{\ldots}$$

$$c_3 = \frac{7}{2}\cdot\frac{1}{2}\int_{-1}^{1} x^5(5x^3-3x)\,dx = \frac{7}{4}\left[\frac{5}{9}(2) - \frac{3}{7}(2)\right] = \frac{4}{9}$$

$$c_5 = \frac{11}{2}\cdot\frac{1}{8}\int_{-1}^{1} x^5(63x^5-70x^3+3) = \frac{11}{16}\left[\frac{63}{11}\cdot 2 - \frac{70}{9}\cdot 2 + \frac{3}{6}\cdot 2\right]$$

$$= \frac{8}{63}$$

2a) Rodrigues' Formula: $P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

Consider $\ell = 2$

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2$$

$$\frac{1}{4\cdot 2}\frac{d}{dx}\left(4x(x^2-1)\right)$$

$$= \frac{1}{4\cdot 2}\left[4(x^2-1) + 4x\cdot 2x\right] = \frac{1}{4\cdot 2}\left[12x^2 - 4\right]$$

$$= \frac{1}{2}(3x^2-1)$$

Now we explicitly check orthogonality:

$$I = \int_{-1}^{1} P_2(x)P_4(x)\,dx = \frac{1}{8}\cdot\frac{1}{2}\int_{-1}^{1}(3x^2-1)(35x^4-30x^2+3)\,dx$$

$$= \frac{1}{16}\left[\int_{-1}^{1}\left[3\cdot 35x^6 - 90x^4 + 9x^2 - 35x^4 + 30x^2 - 3\right]dx\right.$$

$$= \frac{2}{16}\left[\frac{3\cdot 35}{7} - \frac{125}{5} + \frac{39}{3} - 3\right] = 0$$

Now check normalization by checking:

$$\int_{-1}^{1} [P_\ell(x)]^2 dx = \frac{2}{2\ell+1}$$

3a) Evaluate $\int_{-1}^{1} P_\ell(x)\, P'_{\ell-1}(x)\, dx = I_0$

Using recursion relation $P'_{\ell-1} = x P'_\ell - \ell P_\ell$

To get our integral, we multiply by P_ℓ and integrate:

$$\rightarrow \underbrace{\int_{-1}^{1} P_\ell(x)\, P'_{\ell-1}(x)\, dx}_{I_0} = \underbrace{\int_{-1}^{1} x P_\ell(x)\, P'_\ell(x)\, dx}_{I_1} - \underbrace{\ell \int_{-1}^{1} [P_\ell(x)]^2 dx}_{I_2}$$

I_1 : we can write $\frac{d}{dx}[P_\ell(x)]^2 = 2 P_\ell(x) \cdot P'_\ell(x)$

$$I_1 = \frac{1}{2}\int_{-1}^{1} x \frac{d}{dx}[P_\ell(x)]^2 dx \quad \text{, now IBP} \quad dv = \frac{d}{dx}[P_\ell(x)]^2 dx$$

$$v = P_\ell(x)^2$$

$$= \frac{1}{2}\left[x P_\ell(x)^2 \Big|_{-1}^{1} - \int_{-1}^{1} [P_\ell(x)]^2 dx \right] \qquad u = x \quad du = dx$$

$$\rightarrow P_\ell(1)^2 = P_\ell(-1)^2 = 1$$

$$= \frac{1}{2}\left[2 - \frac{2}{2\ell+1} \right] = 1 - \frac{1}{2\ell+1} = \frac{2\ell+1}{2\ell+1} - \frac{1}{2\ell+1} = \frac{2\ell}{2\ell+1}$$

$$I_2 = -\ell \int_{-1}^{1} [P_\ell(x)]^2 dx = -\frac{2\ell}{2\ell+1}$$

$$I_0 = I_1 + I_2 = \frac{2\ell}{2\ell+1} - \frac{2\ell}{2\ell+1} = 0$$

b) Consider $\int_{-1}^{1} P_3(x)\,[P_4(x)]^2\,dx$, we know odd order L. poly's are odd functions. Even are Even.

Here we have an odd function multiplied by an even one, so the product is an odd function.
An odd function integrated over symmetric interval is zero.

4. Let $f(x) = \delta(x)$, $c_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} \delta(x)\,P_\ell(x)\,dx = \frac{2\ell+1}{2}\,P_\ell(0)$

$$\Rightarrow f(x) = \sum \frac{2\ell+1}{2} P_\ell(0)\,P_\ell(x) = \delta(x)$$

We can use: $(\ell+1)P_{\ell+1}(x) = (2\ell+1)\,x\,P_\ell(x) - P_{\ell-1}(x)$

to work out $(\ell+1)\,P_{\ell+1}(0) = -P_{\ell-1}(0)$, which skips an ℓ.

$\Rightarrow$ For odd ℓ, this tells us that all will be zero since

$$P_1(0) = 0 \propto P_3(0) \propto P_5(0) \ldots$$

We can try to work out $P_{2\ell}(0)$ from Rodrigues' Formula

$$P_{2\ell}(x) = \frac{1}{2^{(2\ell)}\,2\ell!}\cdot\frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^{2\ell}$$

Use Binomial Thm. $(x^2-1)^{2\ell} = \sum_{k=0}^{2\ell}\binom{2\ell}{k}(x^2)^{2\ell-k}(-1)^k$

So we have derivative $D^{2\ell}$ applied to this evaluated at $x=0$, so all will be zero except $k=\ell$. Here we have

$$(-1)^\ell\binom{2\ell}{\ell}D^{2\ell}x^{4\ell-2\ell} = (-1)^\ell\binom{2\ell}{\ell}\,2\ell!$$

$$\Rightarrow P_{2\ell}(0) = \frac{1}{2^{(2\ell)}\,2\ell!}\cdot(-1)^\ell\binom{2\ell}{\ell}\,2\ell! = \frac{(-1)^\ell}{2^{(2\ell)}}\binom{2\ell}{\ell}$$

Finally

$$\rightarrow \quad \delta(x) = \sum_{\ell} \frac{4\ell+1}{2} \left[\frac{(-1)^{\ell}}{2^{(2\ell)}} \binom{2\ell}{\ell}\right] P_{2\ell}(x)$$

5. Consider $f(x) = \sin 2\pi x$, this is an odd function so we know all even ℓ coeffs are zero.

since we consider coeffs up to order 3, this leaves c_1, c_3

$$c_1 = \frac{3}{2}\int_{-1}^{1} P_1(x)\sin 2\pi x\, dx = \frac{3}{2}\int_{-1}^{1} x \sin 2\pi x\, dx$$

IBP: $dv = \sin 2\pi x\, dx \qquad du = dx$

$V = -\frac{\cos 2\pi x}{2\pi} \qquad u = x$

$$= \frac{3}{2}\left[-x\frac{\cos 2\pi x}{2\pi}\Big|_{-1}^{1} + \frac{1}{2\pi}\int_{-1}^{1}\cos 2\pi x\, dx\right]$$

$$= \frac{3}{2}\left[-\frac{(1(1) - (-1)(1))}{2\pi} + 0\right] = -\frac{3}{2\pi}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1}(5x^3 - 3x)\sin 2\pi x\, dx = \frac{7}{2}\left[5\underbrace{\int_{-1}^{1} x^3 \sin 2\pi x\, dx}_{I_1} - 3\underbrace{\int_{-1}^{1} x \sin 2\pi x\, dx}_{I_2 = -\frac{1}{}}\right]$$

$\underbrace{\qquad\qquad}_{I_1}$ $\qquad$ $\underbrace{\qquad\qquad}_{I_2 = -\frac{1}{\pi}}$

We can determine I_2 from above.

I_1 can be done with repeated I.B.P. each time dropping the order of x by 1.

I_1:
$$dv = \sin 2\pi x\, dx \qquad du = 3x^2$$
$$v = -\frac{\cos 2\pi x}{2\pi} \qquad u = x^3$$

$$I_1 = \underbrace{-x^3 \frac{\cos 2\pi x}{2\pi}\Bigg|_{-1}^{1}}_{-\frac{1}{\pi}} + \frac{3}{2\pi}\int_{-1}^{1} x^2 \cos 2\pi x\, dx$$

IBP again.
$$dv = \cos 2\pi x\, dx \qquad du = 2x$$
$$v = \frac{\sin 2\pi x}{2\pi} \qquad u = x^2$$

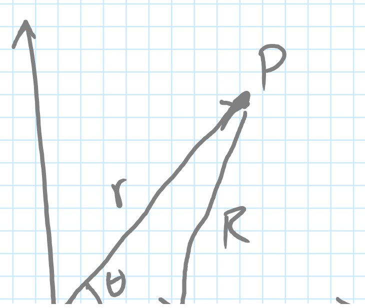
$$= -\frac{1}{\pi} + \frac{3}{2\pi}\left[\underbrace{x^2 \frac{\sin 2\pi x}{2\pi}\Bigg|_{-1}^{1}}_{0} - \frac{1}{\pi}\underbrace{\int_{-1}^{1} x \sin 2\pi x\, dx}_{I_2} \right]$$

$$= -\frac{1}{\pi} + \frac{3}{2\pi}\left[-\frac{1}{\pi}\cdot\left(-\frac{1}{\pi}\right)\right] = -\frac{1}{\pi} + \frac{3}{2\pi^3}$$

$$\to C_3 = \frac{7}{2}\left[5\left[-\frac{1}{\pi} + \frac{3}{2\pi^3}\right] + \frac{3}{\pi}\right] = \frac{7}{2}\left[-\frac{2}{\pi} + \frac{15}{2\pi^3}\right]$$

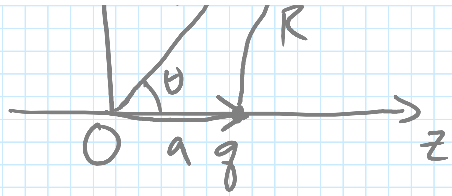
6. Find distribution of charges along an axis such that for large distance R away from potential we get: $V(R) \sim \frac{1}{R^4}$

First consider Potential from some point charge



$$V = k\frac{q}{R}$$

$$R = (r^2 + a^2 - 2ar\cos\theta)^{1/2}$$ either from vector analysis or



$R = (r^2 + a^2 - 2ar\cos\theta)^{1/2}$ from vector analysis or law of cosine

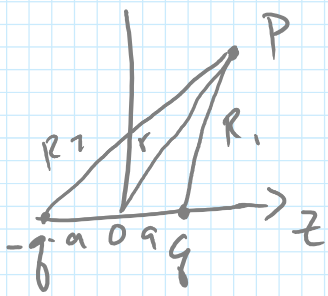
R is nearly the generating function for Legendre polynomials.

→ In particular:

$$\frac{1}{R} = \frac{1}{r}\left(1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right)^{-1/2} = \frac{1}{r}\sum_{l=0} P_l(\cos\theta)\left(\frac{a}{r}\right)^l \quad \text{for } r > a$$

Note for $l = 0$, $\frac{1}{R} = \frac{1}{r}$, which is exactly the form of single charge far away or for $r >> a$ such that $r \sim R$

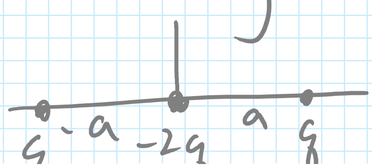
This is the idea behind the multipole expansion. $l = 1$ is the dipole or 2 charges in the far away limit.



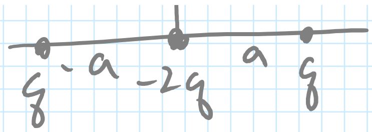
Here we get a term $V \sim \frac{1}{r^2}$

To get $V \sim \frac{1}{r^4}$ we need the $l = 3$ term which is the octupole

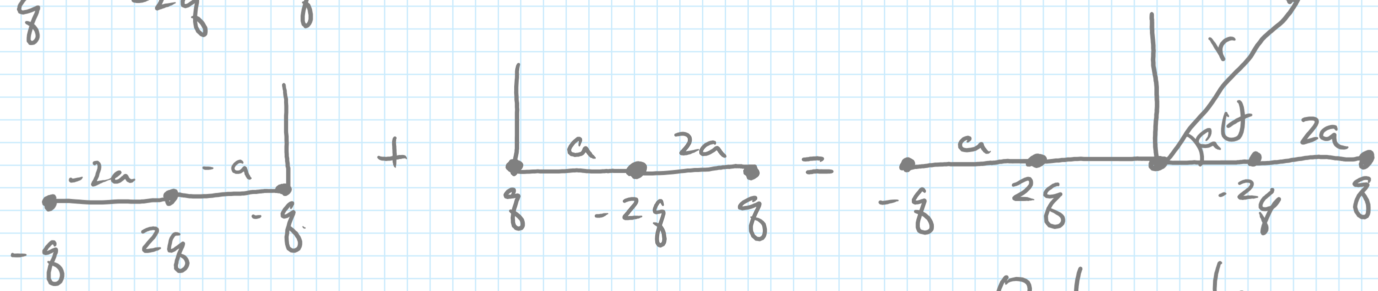
We constructed the dipole by taking the monopole and alinging them symmetrically opposite each other. The quadrapole can be obtained by doing the same with the dipole.



Quadrapole



Quadrapole



Octupole

$V(r) = K q \cdot P_3(\cos\theta) \cdot \frac{a^3}{r^4}$, where $P_3(\cos\theta)$ has the angular dependence.