

Basic Rules of Algebra, Algebraic Fractions, Laws of Exponents, and Roots/Radicals

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If you find yourself having difficulty with basic algebraic operations, exponents, and/or roots and radicals, this document provides a handy review along with some practice exercises. Without these skills and a working knowledge of trigonometry, success in calculus is not possible. The earlier you correct these weaknesses the better.

1. Basic rules of algebra

A. Letters and numbers ... What's the difference? Letters or other symbols should be considered placeholders for numbers. We typically refer to them as variables or parameters. When they are in an equation (or inequality), or in a set of equations (or inequalities), their values may be limited by the requirement(s) imposed by the equation(s) (or inequality(ies)). All that means is that the equation(s) or inequality(ies) may be true only for certain values of the variables or parameters.

B. Commutative and associative properties of multiplication and addition Basically these properties say that, for a list of items that you want to add or multiply, it doesn't matter how you order the list (commutative property) or which additions or multiplications you perform first (associative property). Therefore you should feel comfortable making the following re-arrangements (**note how parentheses dictate the order of computation**):

$$a + b + c = (a + b) + c = a + (b + c) = (c + a) + b = (b + c) + a \quad \text{etc.}$$

$$abc = (ab)c = a(bc) = (ac)b = (cb)a = c(ba) = c(ab) = (ca)b \quad \text{etc.}$$

C. Distributive property: $a(b + c) = ab + ac$ This property is one of the most important reasons for using parentheses when you write mathematics. If you want to write "a" times "the sum of b + c" you must write it as $a(b + c)$. If you just write $ab + c$ you are merely adding "c" to the product of "a" and "b". These are NOT the same thing! Of course you could write it as $(c + b)a$ using the commutative properties.

D. Do unto the LEFT side of an equality as you would do unto the RIGHT side. I'm not sure who wrote this, but it should be engraved in the cortex of all students who graduate high school. Unfortunately it is not. The principle is obvious IF you think about it. If you have two numbers OR two algebraic expressions that **equal** each other ... i.e., they are **truly identical in their values** (their form or appearance are typically different) ... then, if you do the same thing to both, the resulting values must be the same. Here are some examples starting with the identity $7 = 7$:

$$7 = 7 \text{ so } 7 + 5 = 7 + 5 \text{ and } 7 - \sqrt{\pi} = 7 - \sqrt{\pi} \text{ and } 4 * 7 = 4 * 7 \text{ and } \frac{7}{9} = \frac{7}{9} \text{ and } \sqrt{7} = \sqrt{7} \text{ and } 7^{\frac{3}{4}} = 7^{\frac{3}{4}}$$

$$\text{Thus, if } f(x) = g(x) \text{ then } f(x) \pm a = g(x) \pm a \text{ and } af(x) = ag(x) \text{ and } \frac{f(x)}{a} = \frac{g(x)}{a} \text{ and } f^p(x) = g^p(x)$$

There is only one caution: DO NOT DIVIDE BY ZERO!

Note that when items are not equal the situation requires additional care. Consider the true statement: $-5 < -3$. If you were to multiply both sides by -7 you would get: $35 < 21$... a false statement. Lesson: Whenever you multiply or divide an inequality by a negative number, you must reverse the inequality. An additional example that occurs frequently in mathematics involves absolute values. E.g., $|x - 5| < 3$. To eliminate the absolute value operation, you should read this as "The distance between x and 5 is less than 3 ." What values of x are within three units of 5 ? The answer is $x \in (2, 8)$ or, equivalently, $2 < x < 8$. You did correctly translate $x \in (2, 8)$?

E. You can do some things to ONLY one side of an equation or to any single element within an equation but ONLY if you do NOT change its value!

Three common examples are: 1. Replacing an item with something of equal value, 2. Adding ZERO, or 3. Multiplying by ONE. For example, if $f(x) = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$, you can use the ability to multiply by one to eliminate the factor of Δx from the denominator. Note how the "1" was then replaced.

$$f(x) = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \right) * 1 = \left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \right) * \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right) = \frac{x+\Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \frac{1}{(\sqrt{x+\Delta x} + \sqrt{x})}$$

Note: This algebraic transformation is frequently used in Calculus I. You will be expected to carry out transformations such as this without error. Did you note the use of the identity: $(a+b)(a-b) = a^2 - b^2$?

F. Using parentheses One of the quickest ways to fail a mathematics course is to fail to understand and use parentheses. Many points are lost on every calculus exam because of missing parentheses.

Parentheses ensure that a coefficient that is intended to multiply the sum of several terms does not just end up multiplying the first term in the sum. For example, if you intent to subtract twice the sum of $x + y$ from z you should write: $z - 2(x + y)$. If you write: $z - 2x + y$, you have just failed that problem. Overusing parentheses rarely causes a problem. If you fail to use them when required, you will not to get the correct answer.

G. Problems

1. Simplify the following: $(a-b)(a^2 + ab + b^2)$, $(a-b)(a+b-c) - (a^2 - b^2)$, $\left(\frac{x}{y} - x\right)\left(\frac{y}{x} - y\right)(x+y)$

2. Solve for x : $\frac{2x-5}{7x+3} = 2$, $2xy - 3x + 4 = y$, $x - y^2 + 2xzy - 3xy = yz - xz$, $\frac{3}{x} = x + 2$, $x + 3 = \frac{7}{x-3}$

2. Operations with algebraic fractions

A. Manipulating fractions with variables or parameters Fractions that include variables (e.g., x, y, z, t) or parameters (e.g., a, b, c, k, n, m, p) can undergo the same operations as illustrated previously. For example:

$$\frac{ax}{b} = \left(\frac{a}{b}\right)x \text{ and } \frac{x}{a} \pm \frac{y}{b} = \frac{bx \pm ay}{ab} \text{ and } \frac{x}{a} \frac{y}{b} = \frac{xy}{ab} \text{ and } \frac{\left(\frac{z}{7}\right)}{n} = \frac{z}{7n} \text{ and } \frac{t}{\left(\frac{x}{\sqrt{2}}\right)} = \frac{\sqrt{2}t}{x} \text{ and } \frac{y/p}{k/x} = \frac{xy}{kp}$$

Note that the slanted division symbol in the final example is NO DIFFERENT than the horizontal division symbol. You may make fewer mistakes if you always use the horizontal symbol. Also note the fourth example (with “n” starting in the denominator). This form frequently confuses students. Many students end up re-writing this and placing the “n” in the numerator. The correct solution is evident if you replace the n in the denominator with $\frac{n}{1}$ and then “inverting and multiplying.” Now n ends up in the denominator of the resulting expression where it belongs.

B. Problems

Simplify the following:

$$\frac{x}{x^2-1} + \frac{3}{x+1}, \quad \frac{1}{2x^2+4x} - \frac{1}{4x}, \quad \frac{3}{xy} - y\left(\frac{2}{x} - \frac{4}{y}\right), \quad \frac{\frac{1}{xy} - \frac{2}{y}}{\frac{1}{zx} - \frac{3}{yz}}, \quad \frac{a}{bc} - \frac{c}{ba} - \frac{b}{ac},$$

$$\frac{\frac{2}{ab} - \frac{b}{c}}{\frac{1}{cb} + \frac{3}{ac}} - \frac{a+b-c}{abc}, \quad \frac{\sqrt{x}-\sqrt{y}}{x+y} * \frac{\sqrt{x}+\sqrt{y}}{x-y}, \quad \frac{4x-3y}{2xy} \div \frac{\frac{x}{y} - \frac{y}{x}}{y-x}, \quad \frac{a-3}{b+3} + \frac{a+3}{b-3}, \quad \frac{(a+b)(a+b)(a+b) - a^3 - b^3}{ab}$$

3. Exponents and related laws

A. Positive integer exponents If you can count and add, you can handle positive integer exponents. The laws of exponents are some of the simplest rules in mathematics, yet a very high percentage of students find them to be very confusing. The key is to remember what x^n means. It means n x 's multiplied together. It is simple! Here are some examples:

$$x^3 = xxx \quad x^5 = xxxxx \quad x^8 = xxxxxxxx \quad x^{13} = xxxxxxxxxxxxxx \quad \text{Just count and check!}$$

Putting the “n” in the exponent position is just a notational convention, nothing more. Mathematicians could have made it a subscript ... they just didn't. Maybe they voted. Who knows? Who cares? Just go with it.

OK, now lets multiply x^3 by x^5 . If you replace these by what the equal (see above) you get:

$$x^3 x^5 = (xxx)(xxxxx) = xxxxxxxx = x^8 \quad \text{which, not surprisingly, is } x^{3+5}.$$

We've just discovered a law of exponents, namely: $x^n x^m = x^{n+m}$. It is really **just a law of this notation** since the equality of $(xxx)(xxxxx)$ and $xxxxxxxx$ is obvious from the Associative Law of Multiplication (you didn't forget it, did you?). It is a simple matter of counting!

B. Allowing zero and negative integer exponents One day, a mathematician wrote the equation $x^7 x^m = x^3$ and suddenly became very confused! The quantity on the left MUST equal x^{7+m} based on the law we just established and, therefore, $7+m$ must equal 3. That means m must equal -4. That is very odd! x^m , in this case, must equal x^{-4} ... a negative exponent!

Clearly, the effect of $x^m = x^{-4}$ in the original equation is to reduce the number of x 's multiplied together by 4 (i.e., from 7 to 3). The only way to do that is to divide x^7 by 4 x 's multiplied together (i.e., by x^4). Now a brilliant light when on above his head! Multiplying by x^{-4} MUST be the same as dividing by x^4 . Therefore,

$x^{-4} = \frac{1}{xxxx} = \frac{1}{x^4}$ or, in general, $x^{-m} = \frac{1}{x^m}$ and therefore our first law, $x^n x^m = x^{n+m}$, can be made to work with negative integers as long as we interpret negative exponents as division instead of multiplication. Here are some examples: $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, $\frac{x^2}{x^5} = x^{-3}$, $\frac{y^{-4}}{y^{-6}} = y^{-4-(-6)} = y^2$, $\frac{x^{11}z^{-3}y^7}{x^{17}z^{-5}y^5} = x^{-6}y^2z^2 = \frac{y^2z^2}{x^6}$

But our rule yields one very weird result. What do we do with this: $x^7 x^{-7} = x^{7-7} = x^0$? Ouch!!

Do we multiply or divide? And if we picked one ... how can we perform the desired operation with zero elements??

Well, let's think about what $x^7 x^{-7}$ means. Using our insight that negative exponents indicate division, we should have:

$$x^7 x^{-7} = \frac{x^7}{x^7} = \frac{xxxxxxx}{xxxxxxx} = 1 \text{ but since } x^7 x^{-7} = x^{7-7} = x^0, \quad x^0 \text{ MUST equal 1.}$$

C. What about $(a^n)^m$? The rule for $(a^n)^m$ can be determined directly from the meaning of exponents. Consider cubing x^4 or $(x^4)^3$ which, by definition, means $(xxxx)(xxxx)(xxxx) = x^{4*3} = 4^{12}$ (just count the x 's).

This result is general and also works for negative exponents. Try $(x^{-4})^3 = (\frac{1}{xxxx})(\frac{1}{xxxx})(\frac{1}{xxxx}) = x^{-4*3} = x^{-12}$.

The general rule, in this case, for exponent notation is $(x^n)^m = x^{nm}$ where x is any real number and n and m are integers.

D. Summary of exponent laws for integer exponents For integer exponents (whether positive, negative or zero) our Laws of Exponents are as follows (See the table. We've used the properties of multiplication and fractions to extend the laws to include products, xy , and fractions, x/y , as the base):

$$x^n x^m = x^{n+m}$$

$$x^n x^{-m} = \frac{x^n}{x^m} = x^{n-m}$$

$$(x^n)^m = x^{nm}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} = \left(\frac{y}{x}\right)^{-n}$$

$$(x)^0 = 1$$

where x is any quantity or algebraic expression

Remember, these laws are really just a consistent set of rules for this notation that uses x^n to signify (n) x 's multiplied together ... **nothing more**. We use this notation because it is much easier to write $(x + \sqrt{2})^3$ than $(x + \sqrt{2})(x + \sqrt{2})(x + \sqrt{2})$. Can you imagine if we had to write out $(x + \sqrt{2})^{5000}$? By the way, were you surprised when I used $(x + \sqrt{2})$ instead of just x ? Anything can be raised to the n^{th} power. It just means that n of those things, whatever it is, are multiplied together (or divided, if n is negative).

Finally, note how the fraction, $\left(\frac{y}{x}\right)$, in the next to last rule is inverted if the sign of the exponent is changed.

E. Problems Write the following as fractions where all terms have positive exponents. Then, write them just using multiplication but permitting negative exponents.

$$\frac{xy^3z^7}{x^2yz^2}, \quad \frac{xy/z^3}{zx^{-3}/y^4}, \quad \frac{x^5y^{-2}z}{x^6y^{-1}z^{-2}}, \quad \frac{x^{-7}y^5z^2}{x^8y^0z}, \quad \frac{(xy^{-3}z^3)^2}{(x^2y^{-1}z^{-3})^{-1}}, \quad \frac{y^3z^5}{xyz} \div \frac{x^{-2}y}{z^4x^{-4}}, \quad (x^{-2} + y^{-1})^{-1}, \quad \left(\frac{x^{-1}y^5}{z^2x^3}\right)^{-3}$$

4. Roots and radicals

A. Fractional exponents If exponents can be positive or negative integers or zero, why can't they be fractions!! One response might be, "Because you can't multiply something by itself $\frac{1}{2}$ a time!." Well don't be too sure!

Consider the equation $a^p = \sqrt{a}$. We know that $\sqrt{a}\sqrt{a} = a$ so it must be true that $a^p a^p = a$. But these are equivalent to $a^{p+p} = a^{2p} = a^1$ and, therefore, $2p = 1$ or $p = \frac{1}{2}$ and it must be true that $a^{\frac{1}{2}} = \sqrt{a}$.

This same process can be repeated for $a^p = \sqrt[m]{a}$ yielding $p = \frac{1}{m}$ or, for any quantity a , $a^{1/m} = \sqrt[m]{a}$.

B. Beware of even roots of negative numbers What does $(-4)^{1/2}$ mean? It means that whatever this is, if you multiply it by itself, $(-4)^{1/2}(-4)^{1/2}$, you will get $(-4)^1 = -4$. But there is no real number that squared will yield a negative result. In fact $(-4)^{1/m} = \sqrt[m]{-4}$ makes no sense for any even number m , since multiplying any real number by itself an even number of times cannot yield a negative result. Even roots of negative numbers makes no sense for real numbers.

In a future course you will be exposed to the imaginary number "i" that, when multiplied by itself, yields -1. In the world of "imaginary numbers" $(-4)^{1/2} = \pm 2i$ because $(\pm 2i)^2 = (2*2)i^2 = 4(-1) = -4$. For now, forget that you saw this ... consider it TOP SECRET. For now we are living just in the world of Real numbers and we **EXCLUDE even roots of negative numbers.**

What about odd roots of negative numbers? No problem! Consider $(-8)^{1/3} = \sqrt[3]{-8}$. We need a number that when three of them are multiplied together we get -8. Of course there is such a number, namely -2, since $(-2)(-2)(-2) = -(2^3) = -8$.

So whenever you see $(something)^{1/m}$, beware when the "something" is less than zero and m is even because, in that case, there is no real solution.

C. More general fractional exponents You now know what $a^{1/m}$ means and when it is meaningful, but $\frac{1}{m}$ is not the most general fraction. What does $a^{p/q}$ mean? Well, we know that $\frac{p}{q} = (\frac{1}{q})p = p(\frac{1}{q})$. Using the rule that $(a^n)^m = a^{nm}$ we can write $a^{p/q} = a^{\frac{1}{q}p} = (a^{1/q})^p = (\sqrt[q]{a})^p$ and also $a^{p/q} = a^{p\frac{1}{q}} = (a^p)^{1/q} = \sqrt[q]{a^p}$ both of which make sense based on our earlier discussion.

One caution: Consider $(-8)^{2/6}$. Is this meaningful when working with real numbers? You might consider changing the $\frac{2}{6}$ to $\frac{1}{3}$ and conclude that $(-8)^{2/6} = (-8)^{1/3} = \sqrt[3]{-8} = -2$ **OR** you might consider writing the original expression as $\sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$ **OR** you might consider writing the original expression as $(\sqrt[6]{-8})^2$ and declare that the even root of -8 doesn't exist and thus there is no real number representation. Which is correct? The answer is to declare there is no real number representation unless you can demonstrate that the underlying process represented by the $\frac{2}{6}$

power is inherently a odd root (e.g., in this case, $\frac{1}{3}$ power). If this is the case, you should be able to re-do your analysis and avoid the $\frac{2}{6}$ power altogether.

D. Irrational exponents OK, the clever student will note that we only defined a^q when q is an integer or a rational number (i.e., $\frac{n}{m}$). What if you want to know what 7^π equals or if it even is meaningful. The answer is that you take the first 100 digits of the irrational exponent, q , express the resulting number as a fraction, and compute an estimated value of the desired quantity. Of course that estimate is not perfect, so you use 101 digits. Better, but still not perfect. Then 102, then 103, etc. For a finite number of digits the result is never perfect, BUT, in the limit, as the number of digits goes to infinity, the number you get is the desired value. You didn't really have to know this, but doesn't it make you feel better?

E. Final summary of the laws of exponents The previous summary (page 4) only needs the addition of the rule for general fractional exponents plus an understanding that fractional exponents can be used, in the limit, to determine the value when the exponent is irrational (e.g., 7^π). Here is the final summary:

F. Problems Simplify the following expressions (answers should only have positive exponents):

$$\frac{x^{2/3}y^{-1/7}z^{9/4}}{xyz^{1/2}}, \quad \frac{\sqrt{x}y^2z^{-2/3}}{\sqrt[3]{x}\sqrt[4]{yz}^{-5/6}}, \quad \left(\frac{a^{1/5}b^2c^{-3/4}}{ab^{3/2}c^2}\right)^{-3}, \quad \sqrt[3]{\frac{\sqrt{xy}}{\sqrt[3]{xy^2}}}$$

$$(\sqrt[5]{x^2} - \sqrt[3]{y^5})(\sqrt[5]{x^2} + \sqrt[3]{y^5}), \quad (\sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x})^2, \quad \left(\frac{x^2\sqrt[3]{xy^2z}}{\sqrt{xyz}}\right)$$

$$(\sqrt{xy} - \sqrt[3]{xy})(\sqrt[4]{xz} + \sqrt{yz})$$

$$x^n x^m = x^{n+m}$$

$$x^n x^{-m} = \frac{x^n}{x^m} = x^{n-m}$$

$$(x^n)^m = x^{nm}$$

$$x^{n/m} = (\sqrt[m]{x})^n = \sqrt[m]{x^n}$$

$$\left(\frac{xy}{z}\right)^{n/m} = \frac{(\sqrt[m]{x})^n (\sqrt[m]{y})^n}{(\sqrt[m]{z})^n}$$

$$(x)^0 = 1$$

where x , y and z are any quantities or algebraic expressions with the following restrictions: even roots of negative quantities and division by zero are prohibited. Note that the application of the laws to products and ratios are indicated in the next-to-last law.

Note that solutions to all problems in this handout are available via a separate link on the Calculus 1A / 1B website.