An Analysis of Angles and their Units

Motivation

Many people just calculate angles in radians. Other people use degrees. Many people know how to convert between these two. But many don't know that some facts (e.g. the derivative of $\sin(x)$) change depending on what units you use for angles.
In the following, I will illustrate the properties of angles and their units, categorize well-known angle units according to them and introduce a lesser-known unit as an alternative to degrees and radians.

1 Angles and linear Units

1.1 Angles

What is an angle?
The reader might jump to a definition they've heard somewhere: "Angles are ratios of lengths!"
However, this does not tell us much. Why do we even need angles? What do they express?
Upon pondering on this for a while, the answer which underlies the following analysis is: Angles are measures of rotation. We need them to say how much something rotates.
This also implies that angles should not be treated as dimensionless.

In this light, we define:
The zero angle α_0 is the angle that depicts no rotation at all.
The full angle $\hat{\alpha}$ is the angle that depicts the second smallest rotation such that every point ends up where it started (a full turn).

1.2 Angle Units

We will looks at Angle Units which set $\alpha_0 = 0$ and assign $\hat{\alpha}$ some other value.

Any angle α can be expressed as

$$\alpha = f \cdot \hat{\alpha} \tag{1}$$

for some real number f.
The right angle will be denoted ϱ and is equal to a quarter turn, so $\varrho = \frac{\hat{\alpha}}{4}$

We will only be working with angles in the interval $[0; \hat{\alpha}]$.

1.3 The Circular Sector

We will now look at the following: A line segment $\overline{AB}$ of length r is rotated around the point A (see figure 1), where the angle α measures the rotation.
The path described by point B during this rotation is called a *circular arc*.
Connecting the circular arc with the original line segment and the rotated line segment, we obtain a *circular sector*.
The length of the circular arc is also called the arc length of the circular sector.

Now we will examine how to express the arc length $L_\circ$ and the area A of the circular sector in terms of α:

1. Doubling α yields two copies of the circular sector. Thus A and $L_\circ$ are doubled.

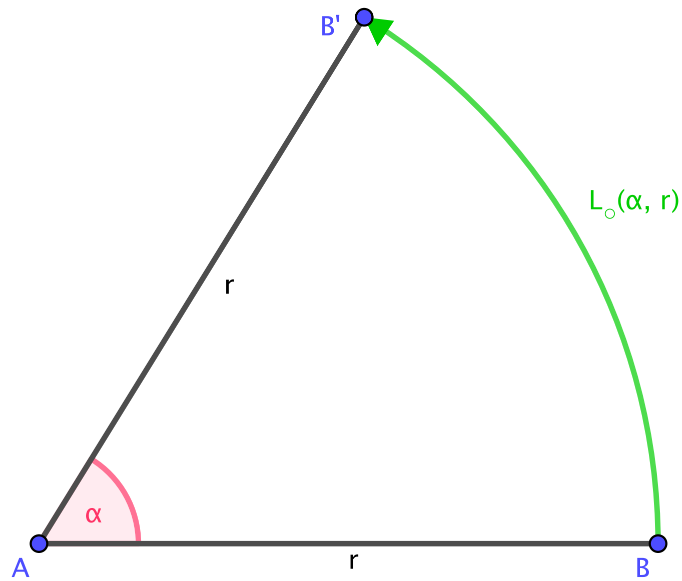

Figure 1: A circular arc is created by rotating the line segment $\overline{AB}$ around the point A

Similar arguments show that for any c, the transformation $\alpha \to c \cdot \alpha$ yields $A \to c \cdot A$ and $L_\circ \to c \cdot L_\circ$.
Therefore A and $L_\circ$ are proportional to α, i.e. $A = k_1 \cdot \alpha$ and $L_\circ = k_2 \cdot \alpha$ for two constants k_1 and k_2.

2. If $\alpha = \hat{\alpha}$, the circular sector is just a circle. Then the usual formulas for the area and circumference of a circle apply:
$A = k_1 \cdot \hat{\alpha} = \pi r^2$ and $L_\circ = k_2 \cdot \hat{\alpha} = 2\pi r$

3. From this we can determine the constants: $k_1 = \frac{\pi r^2}{\hat{\alpha}}$ and $k_2 = \frac{2\pi r}{\hat{\alpha}}$.

In summary:

$$A(\alpha, r) = \pi r^2 \frac{\alpha}{\hat{\alpha}} = \pi r^2 f \tag{2}$$

$$L_\circ(\alpha, r) = 2\pi r \frac{\alpha}{\hat{\alpha}} = 2\pi r f \tag{3}$$

1.3.1 An important value

The value of

$$\lim_{\alpha \to 0} \frac{\sin(\alpha)}{\alpha} \tag{4}$$

appears often when dealing with angles. It will be denoted $\mathfrak{L}$.

Existence of $\mathfrak{L}$ We have to show that $\mathfrak{L}^+ = \lim_{\alpha\to 0^+} \frac{\sin(\alpha)}{\alpha}$ and $\mathfrak{L}^- = \lim_{\alpha\to 0^-} \frac{\sin(\alpha)}{\alpha}$ both exist and have the same value. Should this be the case, then $\mathfrak{L}^+ = \mathfrak{L}^- = \mathfrak{L}$.

We first show that $\mathfrak{L}^- = \mathfrak{L}^+$, if $\mathfrak{L}^+$ exists:

$$\begin{aligned}\mathfrak{L}^- &= \lim_{\alpha\to 0^-} \frac{\sin(\alpha)}{\alpha} = \lim_{\alpha\to 0^+} \frac{\sin(-\alpha)}{-\alpha} \\ &= \lim_{\alpha\to 0^+} \frac{-\sin(\alpha)}{-\alpha} = \lim_{\alpha\to 0^+} \frac{\sin(\alpha)}{\alpha} \\ &= \mathfrak{L}^+\end{aligned}$$

Now we determine $\mathfrak{L}^+$.

Consider a circular sector with radius r and angle α.
Its area is given by

$$A = \pi r^2 f \tag{5}$$

Now we construct two triangles (see figure 2). First consider the triangle ABB': its height h, which is orthogonal to $\overline{AB}$, is

$$h = |\overline{AB}| \cdot \sin(\alpha) = r \cdot \sin(\alpha)$$

. Therefore the ABB''s area is

$$A_{ABB'} = \frac{1}{2} \cdot |\overline{AC}| \cdot h = \frac{1}{2} \cdot r^2 \cdot \sin(\alpha) = \frac{1}{2} r^2 \sin(f\hat{\alpha}) \tag{6}$$

.

Now consider the triangle ABC. Using basic trigonometry, one can easily show that the following holds.

$$|\overline{BC}| = |\overline{AB}| \cdot \tan(\alpha) = r \cdot \tan(\alpha)$$

Hence the area of ABC is

$$A_{\Delta ABC} = \frac{1}{2} \cdot |\overline{OA}| \cdot |\overline{AC}| = \frac{1}{2} \cdot r^2 \cdot \tan(\alpha) = \frac{1}{2} r^2 \tan(f\hat{\alpha}) \tag{7}$$

To calculate $\mathfrak{L}^+$, we rewrite its definition to $\mathfrak{L}^+ = \lim_{f\to 0^+} \frac{\sin(f\hat{\alpha})}{f\hat{\alpha}}$.
Since this is a right sided limit, f is strictly positive.
By looking at figure 2, you can easily confirm that $A_{\Delta ABB'} < A(\alpha, r) < A_{\Delta ABC}$ always holds for any r, α.

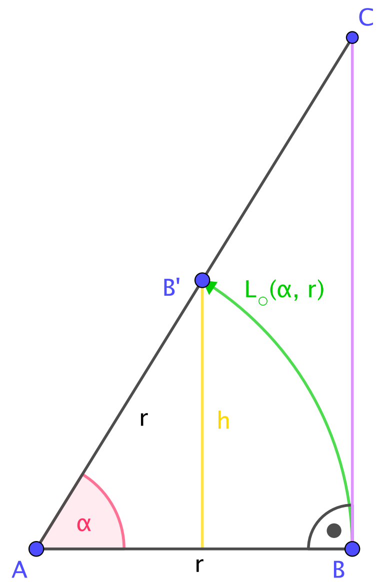

Figure 2: A circular sector and two triangles

Starting from this, we derive

$$\begin{aligned}
A_{\Delta ABB'} &< A(\alpha, r) && < A_{\Delta ABC} \\
\iff \frac{1}{2}r^2 \sin(f\hat{\alpha}) &< \pi r^2 f && < \frac{1}{2}r^2 \tan(f\hat{\alpha}) \\
\iff \sin(f\hat{\alpha}) &< 2\pi f && < \tan(f\hat{\alpha}) \\
\iff 1 &< 2\pi \frac{f}{\sin(f\hat{\alpha})} && < \frac{\tan(f\hat{\alpha})}{\sin(f\hat{\alpha})} \\
\iff 1 &> \frac{1}{2\pi}\frac{\sin(f\hat{\alpha})}{f} > \frac{\sin(f\hat{\alpha})}{\tan(f\hat{\alpha})} && = \cos(f\hat{\alpha}) \\
\iff 1 &> \frac{\hat{\alpha}}{2\pi}\frac{\sin(f\hat{\alpha})}{f\hat{\alpha}} > \frac{\sin(f\hat{\alpha})}{\tan(f\hat{\alpha})} && = \cos(f\hat{\alpha})
\end{aligned}$$

It is trivial to confirm that $\lim_{\alpha\to 0^+}\cos(\alpha) = 1$ and $\lim_{\alpha\to 0^+} 1 = 1$. Using this knowledge and the squeeze theorem to obtain

$$\begin{aligned}
\lim_{f\to 0^+}\frac{\hat{\alpha}}{2\pi}\frac{\sin(f\hat{\alpha})}{f\hat{\alpha}} &= 1 \\
\iff \frac{\hat{\alpha}}{2\pi}\lim_{f\to 0^+}\frac{\sin(f\hat{\alpha})}{f\hat{\alpha}} &= 1 \\
\iff \frac{\hat{\alpha}}{2\pi}\cdot \mathfrak{L}^+ &= 1
\end{aligned}$$

. Therefore

$$\mathfrak{L} = \mathfrak{L}^{+} = \frac{2\pi}{\hat{\alpha}} \tag{8}$$

.

Area and arc length can now be expressed in terms of $\mathfrak{L}$:

$$A(\alpha, r) = \frac{\mathfrak{L}}{2}\alpha r^2 \tag{9}$$

$$L_{\circ}(\alpha, r) = \mathfrak{L}\alpha r \tag{10}$$

1.4 Derivatives of Sine and Cosine

The classic proof for the derivatives of sine and cosine will be done here, not in radians, but using arbitrary units.
Necessary prior knowledge for this proof are the identities:

$$\begin{aligned} &\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta), \\ &1 - \cos^2(\alpha) = \sin^2(\alpha) \end{aligned}$$

as well as the following.

1.4.1 A useful limit

$$\begin{aligned} &\lim_{\alpha \to 0} \frac{\cos(\alpha) - 1}{\alpha} \\ &= \lim_{\alpha \to 0} \frac{(\cos(\alpha) - 1)(\cos(\alpha) + 1)}{\alpha \cdot (\cos(\alpha) + 1)} \\ &= \lim_{\alpha \to 0} \frac{\cos^2(\alpha) - 1}{\alpha \cdot (\cos(\alpha) + 1)} \\ &= \lim_{\alpha \to 0} \frac{-\sin^2(\alpha)}{\alpha \cdot (\cos(\alpha) + 1)} \\ &= -\lim_{\alpha \to 0} \frac{\sin(\alpha)}{\alpha} \cdot \frac{\sin(\alpha)}{\cos(\alpha) + 1} \\ &= -(\lim_{\alpha \to 0} \frac{\sin(\alpha)}{\alpha}) \cdot (\lim_{\alpha \to 0} \frac{\sin(\alpha)}{\cos(\alpha) + 1}) \\ &= -\mathfrak{L} \cdot \frac{\sin(0)}{\cos(0) + 1} \\ &= -\mathfrak{L} \cdot \frac{0}{2} \\ &= 0 \end{aligned}$$

1.4.2 Sine

Now for the proof.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\alpha}\ \sin(\alpha) = \lim_{\varepsilon\to 0} \frac{\sin(\alpha+\varepsilon)-\sin(\alpha)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\sin(\alpha)\cdot\cos(\varepsilon)+\cos(\alpha)\cdot\sin(\varepsilon)-\sin(\alpha)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}(\sin(\alpha)\frac{\cos(\varepsilon)-1}{\varepsilon}+\cos(\alpha)\frac{\sin(\varepsilon)}{\varepsilon}) \\
&= \sin(\alpha)\cdot(\lim_{\varepsilon\to 0}\frac{\cos(\varepsilon)-1}{\varepsilon})+\cos(\alpha)\cdot(\lim_{\varepsilon\to 0}\frac{\sin(\alpha)}{\alpha}) \\
&= \sin(\alpha)\cdot 0+\cos(\alpha)\cdot\mathfrak{L} \\
&= \mathfrak{L}\ \cos(\alpha)
\end{aligned}
$$

An unexpected result!

1.4.3 Cosine

This one is quite short since $\cos(\alpha) = \sin(\varrho - \alpha)$.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\alpha}\ \cos(\alpha) \\
&= \frac{\mathrm{d}}{\mathrm{d}\alpha}\ \sin(\varrho-\alpha) \\
&= \mathfrak{L}\cdot\cos(\varrho-\alpha)\cdot(-1) \\
&= \mathfrak{L}\cdot(-\sin(\alpha))
\end{aligned}
$$

1.5 Euler's formula

Every complex number z can be expressed in polar coordinates. Therefore it holds for certain r and α that

$$z = r(\cos(\alpha) + i\sin(\alpha))$$

And since e^{iz} is also a complex number:

$$e^{iz} = r(\cos(\alpha) + i\sin(\alpha))$$

By differentiating (not assuming anything about $\frac{\mathrm{d}\alpha}{\mathrm{d}z}$ and $\frac{\mathrm{d}r}{\mathrm{d}z}$), we obtain

$$
\begin{aligned}
ie^{iz} &= r(-\mathfrak{L}\sin(\alpha)+i\mathfrak{L}\cos(\alpha))\frac{\mathrm{d}\alpha}{\mathrm{d}z}+\frac{\mathrm{d}r}{\mathrm{d}z}(\cos(\alpha)+i\sin(\alpha)) \\
\iff i(r(\cos(\alpha)+i\sin(\alpha))) &= r(-\mathfrak{L}\sin(\alpha)+i\mathfrak{L}\cos(\alpha))\frac{\mathrm{d}\alpha}{\mathrm{d}z}+\frac{\mathrm{d}r}{\mathrm{d}z}(\cos(\alpha)+i\sin(\alpha)) \\
\iff -r\sin(\alpha)+ir\cos(\alpha) &= (-r\mathfrak{L}\sin(\alpha)\frac{\mathrm{d}\alpha}{\mathrm{d}z}+\frac{\mathrm{d}r}{\mathrm{d}z}\cos(\alpha))+i(r\mathfrak{L}\cos(\alpha)\frac{\mathrm{d}\alpha}{\mathrm{d}z}+\frac{\mathrm{d}r}{\mathrm{d}z}\sin(\alpha))
\end{aligned}
$$

This equation can be split into real and imaginary parts. Firstly:

$$
\begin{aligned}
-r\sin(\alpha) &= -r\mathfrak{L}\sin(\alpha)\frac{\mathrm{d}\alpha}{\mathrm{d}z} + \frac{\mathrm{d}r}{\mathrm{d}z}\cos(\alpha) \\
\iff r\sin(\alpha)\cdot(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1) &= \frac{\mathrm{d}r}{\mathrm{d}z}\cos(\alpha) \\
\iff \frac{\mathrm{d}r}{\mathrm{d}z} = \frac{r\sin(\alpha)\cdot(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1)}{\cos(\alpha)} &= r\tan(\alpha)\cdot(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1)
\end{aligned}
$$

Secondly:

$$
\begin{aligned}
r\cos(\alpha) &= r\mathfrak{L}\cos(\alpha)\frac{\mathrm{d}\alpha}{\mathrm{d}z} + \frac{\mathrm{d}r}{\mathrm{d}z}\sin(\alpha) \\
\iff r\cos(\alpha)\cdot(1 - \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z}) &= \frac{\mathrm{d}r}{\mathrm{d}z}\sin(\alpha) \\
\iff \frac{\mathrm{d}r}{\mathrm{d}z} = \frac{r\cos(\alpha)\cdot(1 - \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z})}{\sin(\alpha)} &= r\cot(\alpha)\cdot(1 - \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z}) \\
&= -r\cot(\alpha)\cdot(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1)
\end{aligned}
$$

From this, it follows that:

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}z} = r\tan(\alpha)\cdot(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1) &= -r\cot(\alpha)\cdot(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1) \\
\iff r(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1)\cdot(\tan(\alpha) + \cot(\alpha)) &= 0 \\
\iff (\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1)\cdot(\tan(\alpha) + \cot(\alpha)) &= 0 \\
\iff \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1 = 0 \vee \tan(\alpha) + \cot(\alpha) &= 0 \\
\iff \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} = 1 \vee \frac{\sin(\alpha)}{\cos(\alpha)} + \frac{\cos(\alpha)}{\sin(\alpha)} &= 0 \\
\iff \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} = 1 \vee \frac{\sin^2(\alpha) + \cos^2(\alpha)}{\sin(\alpha)\cos(\alpha)} &= 0 \\
\iff \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} = 1 \vee \frac{1}{\sin(\alpha)\cos(\alpha)} &= 0 \\
\iff \mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} &= 1 \\
\iff \frac{\mathrm{d}\alpha}{\mathrm{d}z} &= \frac{1}{\mathfrak{L}}
\end{aligned}
$$

By substitution, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}z} &= r\tan(\alpha)\cdot(\mathfrak{L}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - 1) \\
&= r\tan(\alpha)\cdot(\frac{\mathfrak{L}}{\mathfrak{L}} - 1) \\
&= 0
\end{aligned}
$$

.

It now holds that:

$$\frac{\mathrm{d}r}{\mathrm{d}z} = 0 \Rightarrow r = c_r$$
$$\frac{\mathrm{d}\alpha}{\mathrm{d}z} = \frac{1}{\mathfrak{L}} \Rightarrow \alpha = \frac{1}{\mathfrak{L}}z + c_\alpha$$

Because of $e^{i\cdot 0} = e^0 = 1$, $c_r = 1$ and $c_\alpha = 0$ hold.
Therefore the following is true:

$$e^{iz} = r(\cos(\alpha) + i\sin(\alpha)) = 1 \cdot (\cos(\frac{z}{\mathfrak{L}}) + i\sin(\frac{z}{\mathfrak{L}}))$$

Or equivalently:

$$e^{i\mathfrak{L}z} = \cos(z) + i\sin(z) \tag{11}$$

An finally:

$$e^{i\pi} = \exp(i\frac{2\pi}{\hat{\alpha}}\frac{\hat{\alpha}}{2}) = \exp(i\mathfrak{L}\frac{\hat{\alpha}}{2}) = \cos(\frac{\hat{\alpha}}{2}) + i\sin(\frac{\hat{\alpha}}{2}) = -1 \tag{12}$$

1.6 Integrating in polar coordinates

The surface element $\mathrm{d}S$ is an annulus sector (a circular sector with a smaller circular sector of the same angle cut out of it). It has the inner arc length $L_\circ = \mathfrak{L} \cdot r \cdot \mathrm{d}\alpha$ and the thickness $W = \mathrm{d}r$. Therefore,

$$\mathrm{d}S = \mathfrak{L}r\ \mathrm{d}s\ \mathrm{d}\alpha \tag{13}$$

.

1.6.1 Gaussian Integral

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-x^2} \,\mathrm{d}x \\
I^2 &= \int_{-\infty}^{\infty} e^{-x^2} \,\mathrm{d}x \cdot \int_{-\infty}^{\infty} e^{-y^2} \,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} \,\mathrm{d}x \,\mathrm{d}y \\
&= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \,\mathrm{d}S \\
&= \int_0^{\hat{\alpha}} \int_0^{\infty} e^{-r^2} \mathfrak{L} r \,\mathrm{d}r \,\mathrm{d}\alpha \\
&= \int_0^{\hat{\alpha}} -\frac{\mathfrak{L}}{2} \cdot [e^{-r^2}]_0^{\infty} \,\mathrm{d}\alpha \\
&= \int_0^{\hat{\alpha}} -\frac{\mathfrak{L}}{2} \cdot (-1) \,\mathrm{d}\alpha \\
&= \int_0^{\hat{\alpha}} \frac{\mathfrak{L}}{2} \,\mathrm{d}\alpha \\
&= \hat{\alpha} \cdot \frac{\mathfrak{L}}{2} \\
&= \hat{\alpha} \cdot \frac{2\pi}{2\hat{\alpha}} \\
&= \pi \\
&\Rightarrow I = \sqrt{\pi}
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{14}
$$

2 Degrees (deg, $^\circ$)

Degrees are defined by $\hat{\alpha} = 360°$.
In degrees, you can work well with many angles since 360 has many divisors.
The other values in degrees are:

$$
\begin{aligned}
&\mathfrak{L} = \frac{\pi}{180^\circ}\\
&\varrho = 90^\circ\\
&L_\circ(r,\alpha) = \frac{\pi}{180^\circ} r \cdot \alpha\\
&\frac{\mathrm{d}}{\mathrm{d}\alpha}\ \sin(\alpha) = \frac{\pi}{180^\circ}\cos(\alpha)\\
&\exp(\frac{i\pi\alpha}{180^\circ}) = \cos(\alpha) + i\sin(\alpha)\\
&\mathrm{d}S = \frac{\pi}{180^\circ} r\ \mathrm{d}r\ \mathrm{d}\alpha
\end{aligned}
$$

3 Radians (rad, ^{r}, τ)

Radians are defined by $\hat{\alpha} = 2\pi^r$.
Radians make many formulas quite nice since $\mathfrak{L} = 1\ \mathrm{rad}^{-1}$.
The other values in radians are:

$$
\begin{aligned}
&\varrho = \frac{\pi}{2}\ \mathrm{rad}\\
&L_\circ(r,\alpha) = r \cdot \alpha\ \mathrm{rad}^{-1}\\
&\frac{\mathrm{d}}{\mathrm{d}\alpha}\ \sin(\alpha) = \cos(\alpha)\ \mathrm{rad}^{-1}\\
&e^{i\alpha\ \mathrm{rad}^{-1}} = \cos(\alpha) + i\sin(\alpha)\\
&\mathrm{d}S = r\ \mathrm{rad}^{-1}\ \mathrm{d}r\ \mathrm{d}\alpha
\end{aligned}
$$

You can ignore the rad^{-1} when actually working with angles. It is just there for dimensional analysis.

4 Revolutions (rev)

Revolutions are defined by $\hat{\alpha} = 1$.
Revolutions are easy to grasp, easy to work with and formulas are not much more difficult than in radians since most only differ by a factor of 2π.
The other values in revolutions are:

$$
\begin{aligned}
&\mathfrak{L} = 2\pi \text{ rev}^{-1} \\
&\varrho = \frac{1}{4} \text{ rev} \\
&L_\circ(r, \alpha) = 2\pi r\alpha \text{ rev}^{-1} \\
&\frac{d}{d\alpha} \sin(\alpha) = 2\pi \cos(\alpha) \text{ rev}^{-1} \\
&e^{2i\pi\alpha \text{ rev}^{-1}} = \cos(\alpha) + i \sin(\alpha) \\
&\mathrm{d}S = 2\pi r \text{ rev}^{-1} \ \mathrm{d}r \ \mathrm{d}\alpha
\end{aligned}
$$

You can ignore the rev^{-1} when actually working with angles. It is just there for dimensional analysis.

Revolutions are not taught in school and not really known by most people, even though they are the, in my opinion, truest description of angles.

5 Nonlinear Units

There are also systems for quantifying angles which do not adhere to $a = f\hat{\alpha}$.
One example uses the inclination of a triangle's hypotenuse and thus characterizes α by $\tan(\alpha)$.
These systems have other scopes and thus are not further handled here.

6 Reflection

Looking back, we have encountered angles in a new light.
Whenever you use angles from now on, the additional insight you gained might become useful (e.g. the derivative of $\sin(x)$ in degrees).
Even if it won't you have now seen the, in my opinion, true face of angles.