$x_n = a(-1)^{n+1}\ln\left(1+\frac{1}{2n}\right) + \cos\frac{n\pi}{2}$

$\overline{\lim}\, x_n$ și $\underline{\lim}\, x_n$

$(-1)^n = \begin{cases} -1, & n = 2k \\ 1, & n = 2k+1 \end{cases}$

$\ln\left(1+\frac{1}{2n}\right) = 0$

$\cos\frac{n\pi}{2} = \begin{cases} 1, & n = 4k \\ 0, & n = 2k+1 \\ -1, & n = 4k+2 \end{cases}$

Subșiruri:

$\lim\limits_{k\to\infty} x_{4k} = a \cdot 1 \cdot 0 + 1 = 1$

$\lim\limits_{k\to\infty} x_{2k+1} = a \cdot (-1) \cdot 0 + 0 = 0$

$\lim\limits_{k\to\infty} x_{4k+2} = a \cdot 1 \cdot 0 + (-1) = -1$

$L = \{1, 0, -1\} \Rightarrow \begin{array}{l} \overline{\lim}\, x_n = 1 \\ \underline{\lim}\, x_n = -1 \end{array} \Bigg| \Rightarrow \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow \nexists \lim x_n$

$\overline{\lim}\, x_n$, $\underline{\lim}\, x_n$ și $\lim x_n$

$x_n = \left(1+\frac{1}{n}\right)^{n(-1)^n} + \sin\frac{n\pi}{2}$

$(-1)^n = \begin{cases} -1, & n = 2k+1 \\ 1, & n = 2k \end{cases}$

$\sin\frac{n\pi}{2} = \begin{cases} 0 = 2k \\ 1 = 4k+1 \\ -1 = 4k+3 \end{cases}$

Subșiruri:

$\lim\limits_{k\to\infty} x_{2k} \overset{\lim\limits_{k\to\infty}}{=} \left(1+\frac{1}{2k}\right)^{2k\cdot 1} + 0 = \lim\limits_{k\to\infty} \left(1+\frac{1}{2k}\right)^{2k} = e$

$\lim\limits_{k\to\infty} x_{4k+1} = \lim\limits_{k\to\infty} \left(1+\frac{1}{4k+1}\right)^{(4k+1)\cdot(-1)} + 1 = e^{-1} + 1 = \frac{1}{e} + 1$

$\lim\limits_{k\to\infty} x_{4k+3} = \lim\limits_{k\to\infty} \left(1+\frac{1}{4k+3}\right)^{(4k+3)(-1)} - 1 = e^{-1} - 1 = \frac{1}{e} - 1$

$L = \left\{\frac{1}{e} - 1, \frac{1}{e} + 1, e\right\} \Rightarrow \begin{array}{l} \overline{\lim}\, x_n = \frac{1}{e} - 1 \\ \underline{\lim}\, x_n = e \end{array} \Bigg| \Rightarrow \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow \nexists \lim x_n$

$\overline{\lim}\, x_n$, $\underline{\lim}\, x_n$ și $\lim x_n$

$$x_n = \frac{\left(1+\frac{(-1)^n}{n}\right)^{n+1}}{2-\cos\frac{n\pi}{2}} = \frac{\left(1+\frac{(-1)^n}{n}\right)^{n}\cdot\left(1+\frac{(-1)^n}{n}\right)}{2-\cos\frac{n\pi}{2}} = \frac{\left(1+\frac{(-1)^n}{n}\right)^{\frac{n}{(-1)^n}\cdot(-1)^n}\left(1+\frac{(-1)^n}{n}\right)}{2-\cos\frac{n\pi}{2}}$$

$$(-1)^n = \begin{cases} -1, & n=2k+1 \\ 1, & n=2k \end{cases}$$

$$\cos\frac{n\pi}{2} = \begin{cases} 1, & n=4k \\ 0, & n=2k+1 \\ -1, & n=4k+2 \end{cases}$$

Subșiruri:

$$\lim_{k\to\infty} x_{4k} = \lim_{k\to\infty} \frac{\left(1+\frac{1}{4k}\right)^{4k}\cdot\left(1+\frac{1}{4k}\right)}{2-1} = e$$

$$\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} \frac{\left(1+\frac{1}{2k+1}\right)^{(2k+1)\cdot(-1)}\cdot\left(1+\frac{1}{2k+1}\right)}{2-0} = \frac{e^{-1}}{2} = \frac{1}{2e}$$

$$\lim_{k\to\infty} x_{4k+2} = \lim_{k\to\infty} \frac{\left(1+\frac{1}{4k+2}\right)^{4k+2}\cdot\left(1+\frac{1}{4k+2}\right)}{2+1} = \frac{e}{3}$$

$$L = \left(\frac{1}{2e}, \frac{e}{3}, e\right) \Rightarrow \begin{array}{l} \overline{\lim}\, x_n = e \\ \underline{\lim}\, x_n = \frac{1}{2e} \end{array} \Bigg| \Rightarrow \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow \nexists \lim x_n$$

$$\sum_{n=0}^{\infty} \frac{1^2 \cdot 4^2 \cdot 7^2 \cdot \ldots \cdot (3n+1)^2}{(n!)^2} \cdot a^n$$

$$x_n = \frac{1^2 \cdot 4^2 \cdot \ldots \cdot (3n+1)^2}{(n!)^2} a^n$$

$$s_n = \cancel{a^x} + \frac{4^2}{(2!)^2} a^2 + \ldots + \frac{1^2 4^2 \cdot \ldots \cdot (3n+1)^2}{(n!)^2} a^n$$

$$\lim_{n \to \infty} s_n = ?$$

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \frac{\dfrac{1^2 \cdot 4^2 \cdot 7^2 \cdot \ldots \cdot (3n+1)^2 \cdot (3n+4)^2}{((n+1)!)^2} a^{n+1}}{\dfrac{1^2 \cdot 4^2 \cdot \ldots \cdot (3n+1)^2}{(n!)^2} a^n} =$$

$$= \lim_{n \to \infty} \frac{\cancel{(n!)^2} \cdot \cancel{1^2 \cdot 4^2 \cdot \ldots \cdot (3n+1)^2} \cdot (3n+4)^2 \cdot a^{\cancel{n+1}}}{\cancel{(n!)^2} \cdot (n+1)^2 \cdot \cancel{1^2 \cdot 4^2 \cdot \ldots \cdot (3n+1)^2} \cdot \cancel{a^n}} =$$

$$= \lim_{n \to \infty} \frac{(3n+4)^2 \cdot a}{(n+1)^2} = \lim_{n \to \infty} \frac{\cancel{n^2}\left(3 + \frac{4}{n}\right)^2}{\cancel{n^2}\left(1 + \frac{1}{n}\right)^2} a =$$

$$= \cancel{\lim_{n \to \infty}} \ \cancel{9a} \ \frac{(3+0)^2}{(1+0)^2} a = 9a \Rightarrow l = 9a$$

Discuție:

dacă $l > 1 \iff 9a > 1 \iff a > \frac{1}{9}$, atunci $\sum_{n=0}^{\infty} x_n$ conv.

dacă $l < 1 \iff 9a < 1 \iff a < \frac{1}{9}$, atunci $\sum_{n=0}^{\infty} x_n$ div.

dacă $l = 1 \iff 9a = 1 \iff a = \frac{1}{9}$, crit. raportului nu ne ajută

(2

$$a = \frac{1}{9} \Rightarrow x_n = \frac{1^2 \cdot 4^2 \cdot \ldots \cdot (3n+1)^2}{(n!)^2 \cdot 9^n}$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(n+1)^2}{(3n+4)^2 \cdot \frac{1}{9}} - 1\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{9(n+1)^2}{(3n+4)^2} - 1\right) = \lim_{n\to\infty} n \cdot \frac{9n^2 + 18n + 9 - (3n+4)^2}{9n^2 + 24n + 16} =$$

$$= \lim_{n\to\infty} n \frac{\cancel{9n^2} + 18n + 9 - \cancel{9n^2} - 24n - 16}{9n^2 + 24n + 16} = \lim_{n\to\infty} \frac{-6n^2 - 7n}{9n^2 + 24n + 16} =$$

$$= \lim_{n\to\infty} \frac{n^2\left(-6 - \frac{7}{n}\right)}{n^2\left(9 + \frac{24}{n} + \frac{16}{n^2}\right)} = \frac{-6}{9} = -\frac{2}{3}$$

$l = -\frac{2}{3} < 1 \Rightarrow \sum_{n=0}^{\infty} x_n$ divergentă

Deci: $\sum_{n=0}^{\infty} x_n$ convergentă pt. $a \in \left(\frac{1}{9}, +\infty\right)$

$\sum_{n=0}^{\infty} x_n$ divergentă pt. $a \in \left(-\infty, \frac{1}{9}\right]$

$$\sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot \ldots \cdot (3n-2)}{3 \cdot 7 \cdot 11 \cdot \ldots \cdot (4n-1)} a^n$$

$\lim_{n\to\infty} s_n = ?$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{\cancel{1 \cdot 4 \cdot \ldots \cdot (3n-2)} \cdot (3n+1)}{\cancel{3 \cdot 7 \cdot 11 \cdot \ldots \cdot (4n-1)} \cdot (4n+3)} a^{n+1}}{\frac{\cancel{1 \cdot 4 \cdot \ldots \cdot (3n-2)}}{\cancel{3 \cdot 7 \cdot 11 \cdot \ldots \cdot (4n-1)}} \cancel{a^n}} =$$

$$= \lim_{n\to\infty} \frac{3n+1}{4n+3} a = \frac{3}{4} a = l$$

Discuție:

$l > 1 \iff \frac{3}{4} a > 1 \iff a > \frac{4}{3} \Rightarrow \sum_{n=0}^{\infty} x_n$ convergent.

$l < 1 \iff \frac{3}{4} a < 1 \iff a < \frac{4}{3} \Rightarrow \sum_{n=0}^{\infty} x_n$ divergent.

$l = 1 \iff \frac{3}{4} a = 1 \iff a = \frac{4}{3} \Rightarrow$ ~~$\sum_{n=0}^{\infty}$~~ nu putem studia natura seriei

$$a = \frac{4}{3} \Rightarrow x_n = \frac{1 \cdot 4 \cdot \ldots \cdot (3n-2)}{3 \cdot 7 \cdot \ldots \cdot (4n-1)} \frac{4}{3}$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(4n+3) \cdot 3}{(3n+1) \cdot 4} - 1\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{12n+9}{12n+4} - 1\right) = \lim_{n\to\infty} n \frac{\cancel{12n} + 9 - \cancel{12n} - 4}{12n+4} =$$

$$= \lim_{n\to\infty} n \frac{5}{12n+4} = \lim_{n\to\infty} \frac{5n}{12n+4} = \frac{5}{12} < 1 \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} x_n \text{ divergentă pt. } a = \frac{4}{3}$$

Deci: $\sum_{n=0}^{\infty} x_n$ convergentă pt. $a \in \left(\frac{4}{3}, +\infty\right)$

$\sum_{n=0}^{\infty} x_n$ divergentă pt. $a \in \left(-\infty, \frac{4}{3}\right]$

$$\sum_{n=0}^{\infty} \left(\frac{n+1}{an}\right)^{n^2}$$

$\lim_{n\to\infty} s_n = ?$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{an}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n+1}{an}\right)^n =$$

$$= \lim_{n\to\infty} \left(\frac{n}{an} + \frac{1}{an}\right)^n = \lim_{n\to\infty} \left(\frac{1}{a} + \frac{1}{an}\right)^n = \lim_{n\to\infty} \underbrace{\frac{1}{a^n}}_{0} \underbrace{\left(1+\frac{1}{n}\right)^n}_{e} =$$

$= 0 < 1 \Rightarrow$ cf. criteriului radicalului că $\sum_{n=0}^{\infty} x_n$ este divergentă

$$\sum_{n=0}^{\infty} \frac{n!(n+3)!}{(2n+1)!\, a^n}$$

$\lim_{n\to\infty} s_n = ?$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!(n+4)!}{(2n+3)!\, a^{n+1}}}{\frac{n!(n+3)!}{(2n+1)!\, a^n}} = \lim_{n\to\infty} \frac{(n+1)(n+4)}{(2n+1)(2n+2)a} =$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{4}{n}\right)}{4n^2\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)a} = \frac{1}{4a} = l$$

Discuție:

$l > 1 \Leftrightarrow \frac{1}{4a} > 1 \Leftrightarrow a < \frac{1}{4} \Rightarrow \sum_{n=0}^{\infty} x_n$ convergentă

$l < 1 \Leftrightarrow \frac{1}{4a} < 1 \Leftrightarrow a > \frac{1}{4} \Rightarrow \sum_{n=0}^{\infty} x_n$ divergentă

$l = 1 \Leftrightarrow \frac{1}{4a} = 1 \Leftrightarrow a = \frac{1}{4} \Rightarrow$ nu poate fi determinat

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(2n+1)(2n+2)}{4(n+1)(n+4)} - 1\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{4n^2+10n+2}{4n^2+16n+16} - 1\right) = \lim_{n\to\infty} n \left\{\frac{4n^2+10n+2-4n^2-16n-16}{4n^2+16n+16}\right\} =$$

$$= \lim_{n\to\infty} n \frac{-6n-14}{4n^2+16n+16} = \lim_{n\to\infty} \frac{-6n^2-14n}{4n^2+16n+16} = \frac{-6}{4} = \frac{-3}{2} < 1 \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} x_n \text{ divergentă pt. } a = \frac{1}{4}$$

$$\sum_{n=0}^{\infty} \frac{(n+7)! \cdot x^n}{a(a+1)\cdots(a+n)}$$

! Obs: x – necunoscută

x_n = șir din $\mathbb{R}$

$\lim\limits_{x\to\infty} s_n = ?$

$$\lim_{x\to\infty} \frac{x_{n+1}}{x_n} = \lim_{x\to\infty} \frac{\dfrac{(n+8)! \cdot x^{n+1}}{a(a+1)\cdots(a+n)(a+n+1)}}{\dfrac{(n+7)! \cdot x^n}{a(a+1)\cdots(a+n)}} = \lim_{n\to\infty} \frac{(n+8)x}{a+n+1} =$$

$$= \lim_{n\to\infty} \frac{n(1+\frac{8}{n})x}{n(1+\frac{a+1}{n})} = x = l_1$$

Discuție:

$l_1 > 1 \iff x > 1 \Rightarrow \sum_{n=0}^{\infty} x_n$ convergentă

$l_1 < 1 \iff x < 1 \Rightarrow \sum_{n=0}^{\infty} x_n$ divergentă

$l_1 = 1 \iff x = 1 \Rightarrow$ nu putem decide

$$x_n = \frac{(n+7)!}{a(a+1)\cdots(a+n)}$$

$$b_n = n\left(\frac{x_n}{x_{n+1}} - 1\right) = n\left(\frac{a+n+1}{n+8} - 1\right) = n\,\frac{a+n+1-n-8}{n+8} =$$

$$= n\,\frac{a-7}{n+8}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n(a-7)}{n+8} = \lim_{n\to\infty} \frac{n(a-7)}{n(1+\frac{8}{n})} = a-7 = l_2$$

Discuție:

$l_2 > 1 \iff a-7 > 1 \iff a > 8 \Rightarrow \sum_{n=0}^{\infty} x_n$ convergentă

$l_2 < 1 \iff a-7 < 1 \iff a < 8 \Rightarrow \sum_{n=0}^{\infty}$ divergentă

$l_2 = 1 \iff a-7 = 1 \iff a = 8 \Rightarrow$ nu putem decide

$$x_n = \frac{(n+8)!}{8(8+1)\cdots(8+n)} = \frac{(n+8)!}{\dfrac{(n+8)!}{7!}} = \frac{1}{7!} \Rightarrow \lim_{n\to\infty} x_n = \frac{1}{7!} \Rightarrow$$

$\Rightarrow \sum_{n=0}^{\infty} x_n$ divergentă

Deci:

dacă $x \in (1, +\infty)$ atunci $\sum_{n=0}^{\infty} x_n$ este convergentă

dacă $x \in (-\infty, 1)$ atunci $\sum_{n=0}^{\infty} x_n$ este divergentă

dacă $x = 1$ atunci:

- dacă $a \in (-\infty, 8]$, $\sum_{n=0}^{\infty} x_n$ este divergentă
- dacă $a \in (8, +\infty)$, $\sum_{n=0}^{\infty} x_n$ este convergentă