Exerciții - limite

1) Det. $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și prec. dacă există $\lim\limits_{n\to+\infty} x_n$:

a) $x_n = \dfrac{(1-(-1)^n)\cdot 2^n + 1}{2^n+3}$, $\forall n \in \mathbb{N}$

Sol:

$$x_{2k} = \frac{(1-(-1)^{2k})\cdot 2^{2k}+1}{2^{2k}+3} = \frac{(1-1)\cdot 2^{2k}+1}{2^{2k}+3} = \frac{0\cdot 2^{2k}+1}{2^{2k}+3}$$

$$= \frac{1}{2^{2k}+3} \xrightarrow[k\to+\infty]{} \frac{1}{2^{\infty}+3} = \frac{1}{\infty} = \boxed{0}$$

$$x_{2k+1} = \frac{(1-(-1)^{2k+1})\cdot 2^{2k+1}+1}{2^{2k+1}+3} = \frac{(1+1)\cdot 2^{2k+1}+1}{2^{2k+1}+3}$$

$$= \frac{2^{2k+2}+1}{2^{2k+1}+3} = \frac{\cancel{2^{2k}}\left(2+\underbrace{\frac{1}{2^{2k+2}}}_{\to 0}\right)}{\cancel{2^{2k}}\left(1+\underbrace{\frac{1}{2^{2k+1}}}_{\to 0}\right)} \xrightarrow[k\to+\infty]{} \frac{2}{1} = \boxed{2}$$

$\Rightarrow$ $\mathbb{N} = 2\mathbb{N} \cup (2\mathbb{N}+1)$

$\mathcal{L}((x_n)_n) = \{0, 2\} \Rightarrow \underline{\lim}\, x_n = 0$, $\overline{\lim}\, x_n = 2$ $\Rightarrow$ $\underline{\lim}\, x_n \neq \overline{\lim}\, x_n$ $\Downarrow$ $(\nexists) \lim\limits_{n\to\infty} x_n$ $\square$

b) $x_n = \dfrac{2 + (-1)^n}{1 + n^{(-1)^n}} + \sin \dfrac{n\pi}{2}, \forall n \in \mathbb{N}$

Sol:

$$x_{4k} = \frac{2 + (-1)^{4k}}{1 + (4k)^{(-1)^{4k}}} + \sin\left(\frac{4k\pi}{2}\right)$$

$$= \frac{2+1}{1+4k} + \sin(2k\pi) = \underbrace{\frac{3}{1+4k}}_{\to 0} + \underbrace{\sin(2k\pi)}_{\to 0} \xrightarrow[k\to\infty]{} \boxed{0}$$

$$x_{4k+1} = \frac{2 + (-1)^{4k+1}}{1 + (4k+1)^{(-1)^{4k+1}}} + \sin\left(\frac{(4k+1)\pi}{2}\right)$$

$$= \frac{2-1}{1 + \underbrace{\frac{1}{4k+1}}_{\to 0}} + \sin\left(\frac{4k\pi}{2} + \frac{\pi}{2}\right) \xrightarrow[k\to\infty]{} \frac{1}{1} + \sin\frac{\pi}{2} = 1 + 1 = \boxed{2}$$

$$x_{4k+2} = \frac{2 + (-1)^{4k+2}}{1 + (4k+2)^{(-1)^{4k+2}}} + \sin\left(\frac{(4k+2)\pi}{2}\right)$$

$$= \underbrace{\frac{2}{1+4k+2}}_{\to 0} + \sin\pi \xrightarrow[k\to\infty]{} \sin\pi = \boxed{0}$$

$$x_{4k+3} = \frac{2 + (-1)^{4k+3}}{1 + (4k+3)^{(-1)^{4k+3}}} + \sin\left(\frac{(4k+3)\pi}{2}\right)$$

$$= \frac{2-1}{1 + \underbrace{\frac{1}{4k+3}}_{\to 0}} + \sin\left(\frac{3\pi}{2}\right) \xrightarrow[k\to\infty]{} 1 + (-1) = \boxed{0}$$

$\mathbb{N} = 4\mathbb{N} \cup (4\mathbb{N}+1) \cup (4\mathbb{N}+2) \cup (4\mathbb{N}+3)$

$\mathcal{L}((x_n)_n) = \{0, 2\}$

$$\left.\begin{aligned} \Rightarrow \overline{\lim}\, x_n &= 2 \\ \underline{\lim}\, x_n &= 0 \end{aligned}\right\} \Rightarrow \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow (\nexists) \lim_{n\to\infty} x_n$$

$\square$

c) $x_n = \frac{n^2+1}{2n^2+3n+1}\sin\left(\frac{(-1)^n\pi}{2}\right) + \frac{n^3+2}{3n^3+3n+4}\cos\left(\frac{n\pi}{3}\right)$

• $\boxed{x_{6k}} = \underbrace{\frac{(6k)^2+1}{2(6k)^2+3(6k)+1}}_{\to\frac{1}{2}}\sin\left(\frac{(-1)^{6k}\pi}{2}\right) + \underbrace{\frac{(6k)^3+2}{3(6k)^3+3(6k)+4}}_{\to\frac{1}{3}}\underbrace{\cos\left(\frac{6k\pi}{3}\right)}_{\to 1}$

$\xrightarrow{k\to\infty} \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\ \cancel{\cos(2}\cdot 1 = \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$

• $\boxed{x_{6k+1}} = \underbrace{\frac{(6k+1)^2+1}{2(6k+1)^2+3(6k+1)+1}}_{\to\frac{1}{2}}\sin\left(\frac{(-1)^{6k+1}\pi}{2}\right) + \underbrace{\frac{(6k+1)^3+2}{3(6k+1)^3+3(6k+1)+4}}_{\to\frac{1}{3}}\cos\left(\frac{(6k+1)\pi}{3}\right)$

$\xrightarrow{k\to\infty} \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{6k\pi}{3} + \frac{\pi}{3}\right)$

$= -\frac{1}{2} + \frac{1}{3}\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = -\frac{1}{2} + \frac{1}{6} = -\frac{2}{6} = \boxed{-\frac{1}{3}}$ $\Big\}$ $\cos(x + 2k\pi) = \cos(x)$

(mai direct de acum)

• $\boxed{x_{6k+2}} = \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{2\pi}{3}\right) = \frac{1}{2} + \frac{1}{3}\cdot\left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$

• $\boxed{x_{6k+3}} = \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) + \frac{1}{3}\cos(\pi) = -\frac{1}{2} + \frac{1}{3}(-1) = -\frac{1}{2} - \frac{1}{3} = \boxed{-\frac{5}{6}}$

• $\boxed{x_{6k+4}} = \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{4\pi}{3}\right) = \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot\left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{6} = \boxed{\frac{1}{3}}$

• $\boxed{x_{6k+5}} = \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}\cdot(-1) + \frac{1}{3}\cdot\frac{1}{2} = -\frac{1}{2} + \frac{1}{6} = -$

$\mathbb{N} = 6\mathbb{N} \cup (6\mathbb{N}+1) \cup (6\mathbb{N}+2) \cup (6\mathbb{N}+3) \cup (6\mathbb{N}+4) \cup (6\mathbb{N}+5)$

$\mathcal{L}((x_n)_n) = \left\{-\frac{5}{6}, -\frac{1}{3}, \frac{1}{3}, \frac{5}{6}\right\}$

$\Rightarrow \left.\begin{array}{l} \overline{\lim}\, x_n = \frac{5}{6} \\ \underline{\lim}\, x_n = -\frac{5}{6} \end{array}\right\} \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow (\nexists) \lim_{n\to\infty} x_n$ □

E T 1

Exerciții -> serii

2) Studiați convergența seriilor

(Cursula final)

IV/a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,(n+3)!}{(2n+1)!\cdot x^n}$, $x \in (0,\infty)$.

Sol: Folosim criteriul raportului

Fie $x_n = \dfrac{n!\,(n+3)!}{(2n+1)!\cdot x^n}$, $\forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\overset{n+1}{\cancel{(n+1)!}}\;\overset{n+1}{\cancel{(n+4)!}}}{\underset{(2n+2)(2n+3)}{\cancel{(2n+3)!}}\cdot \underset{x}{\cancel{x^{n+1}}}} \cdot \frac{\cancel{(2n+1)!}\cdot\cancel{x^n}}{\cancel{n!}\;\cancel{(n+3)!}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{(n+1)(n+4)}{(2n+2)(2n+3)}}_{\frac{1}{4}} \cdot \frac{1}{x} = \frac{1}{4x}.$$

Conform Criteriului raportului avem:

1) Dacă $\dfrac{1}{4x} < 1$ (i.e. $x \in \left(\frac{1}{4}, \infty\right)$), seria este conv.

2) Dacă $\dfrac{1}{4x} > 1$ (i.e. $x \in \left(0, \frac{1}{4}\right)$), seria este div.

3) Dacă $\dfrac{1}{4x} = 1$ (i.e. $x = \frac{1}{4}$) crit. nu decide

(fol. crit Raabe-D.)

Fie $x = \frac{1}{4} \Rightarrow x_n = \dfrac{n!\,(n+3)!}{(2n+1)!\left(\frac{1}{4}\right)^n} = \dfrac{n!\,(n+3)!\cdot 4^n}{(2n+1)!}$ $\forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n!\,(n+3)!\cdot 4^n}{(2n+1)!}\cdot\frac{(2n+3)!}{(n+1)!\,(n+4)!\cdot 4^{n+1}}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)}{(n+1)(n+4)\cdot 4} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+20n+16} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{4n^2+10n+6-4n^2-20n-16}{4n^2+20n+16}\right) = \lim_{n\to\infty} \frac{-10n^2-10n}{4n^2+20n+16}$$

$= -\dfrac{10}{4} = -\dfrac{5}{2} < 1 \Rightarrow$ Conform Crit Raabe-Duhamel seria e divergentă.

Am obținut:

$$\sum_{n=1}^{\infty} \frac{n!(n+3)!}{(2n+1)!\, x^n} \begin{cases} \text{convergentă, dacă } x \in \left(\frac{1}{4}, \infty\right) \\ \text{divergentă, dacă } x \in \left(0, \frac{1}{4}\right] \end{cases} \quad \square$$

I b) $\sum_{n=1}^{\infty} e^{-n^2}$

Sol: Fol. crit de comp cu ineg (se poate și cu lim)

Fie $x_n = e^{-n^2}$

și $y_n = e^{-n}$

! $e^{n^2} \geq e^n$, dar $\frac{1}{e^{n^2}} \leq \frac{1}{e^n}$.

~~Conform Crit de Comp cu ineg.~~ , $\sum$

Deci $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n}_{\text{conv}}$ $\Big\}$ $\frac{1}{e} < 0$ serie geometrică

=> Conform Crit de Comp cu ineg, $\sum_{n=1}^{\infty} x_n$ e convergentă

II c) $\sum_{n=1}^{\infty} a^n \left(1+\frac{1}{n}\right)^n$, $a > 0$

<u>Sol</u>: Fol. crit ~~raportului raportului~~ radicalului

Fie $x_n = a^n \left(1+\frac{1}{n}\right)^n \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{a^n \left(1+\frac{1}{n}\right)^n}$

$= \lim_{n\to\infty} \sqrt[n]{a^n} \cdot \sqrt[n]{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} a \cdot \left(1+\frac{1}{n}\right) = a \cdot 1 = a$

Conform Criteriului ~~Raportului~~ Radicalului:

1) Dacă $l < 1$ (i.e. $a < 1$), at $\sum_n x_n$ este conv

2) Dacă $l > 1$ (i.e. $a > 1$), at $\sum_n x_n$ este div.

3) Dacă $l = 1$ crit nu decide

ETI

Fie $a=1 \Rightarrow \sum_{n}^{\infty} \left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e \Rightarrow$ divergentă

$\Rightarrow$ Am aflat

$$\sum_{n}^{\infty} a^n \left(1+\frac{1}{n}\right)^n \begin{cases} \text{conv, pt } a \in (0,1) \\ \text{div, pt } a \in [1,\infty) \end{cases}$$

III d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^r}, \; r>0$

Sol: Fol. crit. condensării

Fie $x_n = \frac{1}{n(\ln n)^r}$

$$\sum_{n}^{\infty} 2^n \cdot x_{2^n} = \sum_{n}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n}(\ln 2^n)^r} = \sum_{n}^{\infty} \frac{1}{(n \ln 2)^r}$$

$$= \sum_{n}^{\infty} \underbrace{\frac{1}{(\ln 2)^r}}_{=C} \cdot \frac{1}{n^r} = C \cdot \sum_{n}^{\infty} \frac{1}{n^r} \begin{cases} \text{conv pt } r>1 \\ \text{div pt } r \leq 1 \end{cases}$$

const >0
nu se ia în calcul

Cum $\sum_{n}^{\infty} x_n \sim \sum_{n}^{\infty} 2^n x_{2^n} \Rightarrow \sum_{n} x_n \begin{cases} \text{conv pt } r>1 \\ \text{div pt } r \leq 1 \end{cases}$