# Homework 5

Ding Zhao
24-677 Special Topics: Linear Control Systems

**Due: Oct 13, 2023, 11:59 pm. Submit within deadline.**

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

**Exercise 1.** *Asymptotic stability and Lyapunov stability.* ***(10 points)***

For each of the systems given below, determine whether it is Lyapunov stable, whether it is asymptotic stable.

(a) **(5 points)**

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}}_{A} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

(b) **(5 points)**

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

(a): $A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} \Rightarrow |A - \lambda I| = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} = (1-\lambda)(-\lambda+0.5) = 0$

$\Rightarrow \lambda_1 = 0.5 \quad \lambda_2 = 1$

so Lyapunov stable but not asymptotically stable.

(b) $A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} \Rightarrow |A - \lambda I| = \left| \begin{bmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{bmatrix} \right| = (1-\lambda)(5-\lambda)(3-\lambda) = 0$

$\lambda_1 = -1, \lambda_2 = -5, \lambda_3 = -3$

so both Lyapunov stable and asymptotically stable.

**Exercise 2.** *Stabilizability (20 points)*

Decompose the state equation

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

to a controllable form. Is the reduced state equation observable, stabilizable, detectable?

$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

$P = [B\ AB\ A^2B] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(P) = 2 < n = 3$

For two independent columns, we take: $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$\therefore \hat{A} = M^{-1}AM = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow A^c = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

$\hat{B} = M^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow B_c = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$\hat{C} = CM = [3\ 0\ 1] \Rightarrow C_c = (3, 0)$

So the controllable form is: $\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} u$

$y = (3, 0) x$

For observability: $Q = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{rank}(Q) = 1 < n = 2$

so not observable.

for $A_c = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda_1 = 0,\ \lambda_2 = -1,\ v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

since it is already in controllable form, we have:

The system is stabilizable, and Lyapunov stable.

so the system is detectable

**Exercise 3.** *Stability* ***(15 points)***

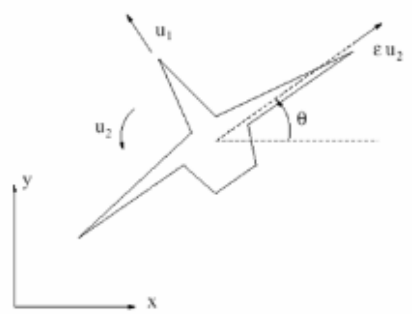



Figure 1: The VTOL. aircraft

The following is a planar model of a Vertical Take-off and Landing (VTOL) aircraft such as Lock-heed's F35 Joint Strike fighter around hover (cf. Figure 1):

$$m\ddot{x} = -u_1 \sin\theta + \epsilon u_2 \cos\theta$$
$$m\ddot{y} = u_1 \cos\theta + \epsilon u_2 \sin\theta - mg$$
$$J\ddot{\theta} = u_2$$

where $x$, $y$ are the position of the center of mass of the aircraft in the vertical plane and $\theta$ is the roll angle of the aircraft. $u1$ and $u2$ are the thrust forces (control inputs). The thrust is generated by a powerful fan and is vectored into two forces $u1$ and $u2$. $J$ is the moment of inertia, and $\epsilon$ is a small coupling constant. Determine the stability of the linearized model around the equilibrium solution

$$\tilde{x}(t), \tilde{y}(t), \tilde{\theta}(t) = 0, \tilde{u}_1(t) = mg; \tilde{u}_2(t) = 0.$$

The linearized model should be time invariant. The state $z = [\theta, \dot{x}, \dot{y}, \dot{\theta}]^T$, $u = [u_1, u_2]^T$

The state of the system is:

$z=[\theta,\dot{x},\dot{y},\dot{\theta}]^T \Rightarrow$ $f[\dot{\theta}\ \ddot{x}\ \ddot{y}\ \ddot{\theta}, u]^T = \begin{bmatrix} \dot{\theta} \\ \frac{-u_1\sin\theta+\epsilon u_2\cos\theta}{m} \\ \frac{u_1\cos\theta+\epsilon u_2\sin\theta-mg}{m} \\ \frac{u_2}{J} \end{bmatrix}$

$\Rightarrow \dot{z}=[\dot{\theta},\ddot{x},\ddot{y},\ddot{\theta}]^T$

For $z$:

At the equilibrium solution: $\tilde{x}(t)=\tilde{y}(t)=\tilde{\theta}(t)=0$, $\tilde{u}_1(t)=mg$, $\tilde{u}_2(t)=0$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 \\ \frac{-u_1\cos\theta-\epsilon u_2\sin\theta}{m} \\ \frac{-u_1\sin\theta+\epsilon u_2\cos\theta}{m} \\ 0 \end{bmatrix} \xrightarrow{\text{@equilibrium}} \begin{bmatrix} 0 \\ \frac{-mg-0}{m} \\ -\frac{0+0}{m} \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -g \\ 0 \\ 0 \end{bmatrix}$$

For u:

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ \frac{-\sin\theta}{m} & \frac{\epsilon\cos\theta}{m} \\ \frac{\cos\theta}{m} & \frac{\epsilon\sin\theta}{m} \\ 0 & 1/J \end{bmatrix} \xrightarrow{@\text{equilibrium}} \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & 1/J \end{bmatrix}$$

The linearized equation can be expressed as:

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

so the jordan form of A is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(eigen values are all 0)

From the J-matrix above, one can easily see that there is a eigen value $\lambda = 1$ with $r = 1$ with $m > 0$, so unstable

**Exercise 4.** *Lyapunov's direct method* ***(10 points)***

An LTI system is described by the equations

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

Use Lyapunov's direct method to determine the range of variable $a$ for which the system is asymptotically stable. Consider the Lyapunov function,

$$V = x_1^2 + x_2^2$$

so we have $V = x_1^2 + x_2^2$ , $\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$$[\dot{x}_1, \dot{x}_2]^T = \begin{bmatrix} ax_1 \\ x_1 - x_2 \end{bmatrix} \Rightarrow \dot{V}(x) = ax_1^2 + 2x_1x_2 - 2x_2^2$$

$$A^TP + PA = \begin{bmatrix} a & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2a & 1 \\ 1 & -2 \end{bmatrix} = -Q$$

$$\Rightarrow Q = \begin{bmatrix} -2a & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = -\sqrt{a^2+2a+2} - a + 1 > 0 \\ \lambda_2 = \sqrt{a^2+2a+2} - a + 1 > 0 \end{cases} \Rightarrow \begin{cases} 4a > -1 \\ 4a > -1 \end{cases} \Rightarrow a > -0.25$$

**Exercise 5.** *Stability of Non-Linear Systems* ***(20 points)***

Consider the following system:

$$\dot{x_1} = x_2 - x_1 x_2^2$$
$$\dot{x_2} = -x_1^3$$

(a) Is the approximated linearized system stable? Based on Lyapunov's Indirect method, is the system stable? [Hint: can you get a firm answer in this case?] **(5 points)**

(b) Based on Lyapunov's Direct method? **(5 points)** Consider the Lyapunov function:

$$V(x_1, x_2) = x_1^4 + 2x_2^2$$

(c) Plot the Phase Portrait plot of the original system and linearized system in a. **(5 points)**. Submit the code to Gradescope.

(d) Generate a 3D plot showing the variation of $\dot{V}$ with respect to $x_1$ and $x_2$. **(5 points)** [Hint: Use Axes3D python library]. Submit the code to Gradescope.

Note: For (c) and (d), include the code along with the plot in the pdf to be submitted. No need to submit .py file.

(a) The linearized system is:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -x_2^2 & -2x_1x_2+1 \\ -3x_1^2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

For equilibrium, $\begin{cases} x_2 - x_1x_2^2 = 0 \\ -x_1^3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \end{cases} \Rightarrow \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \end{matrix}$

For $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 0 \Rightarrow Re = 0$ & $m = 1$, so the system is unstable

(b) $V(x_1, x_2) = x_1^4 + 2x_2^2 \Rightarrow \dot{V} = 4x_1^3\dot{x}_1 + 4x_2\dot{x}_2 = \cancel{4x_1^3}(\cancel{x_2} - x_1x_2^2) + \cancel{4x_2(-x_1^3)}$
$= -4x_1^4x_2^2 < 0$

So based on Lyap's direct method, the system is stable

**Exercise 6.** *BIBO Stability* ***(10 points)***

For each of the systems given below, determine whether it is BIBO stable.

(a) **(5 points)**

$$x(k+1) = \overbrace{\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}}^{A} x(k) + \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}^{B} u(k)$$
$$y(k) = \underbrace{\begin{bmatrix} 5 & 5 \end{bmatrix}}_{C} x(k)$$

(b) **(5 points)**

$$\dot{x} = \overbrace{\begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}}^{B} u$$
$$y = \underbrace{\begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}}_{C} x$$

(a) $G_D(z) = C(zI-A)^{-1}B + D \to 0$

$$= [5\ \ 5]\begin{bmatrix} z-1 & 0 \\ +0.5 & z-0.5 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{5(2z-1)-5+6(z-1)}{(z-1)(2z-1)} = 0$$

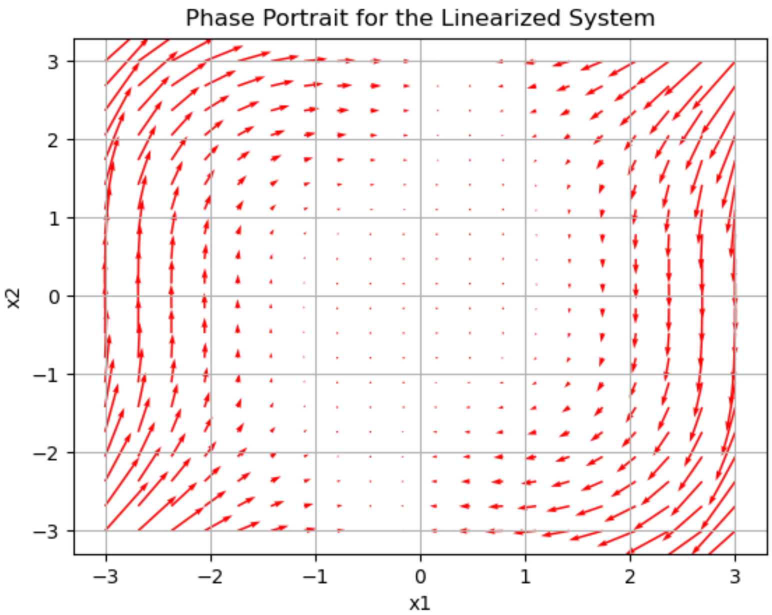
so the system is stable

(b): $G_c(s) = C(sI-A)^{-1}B + D \to 0$

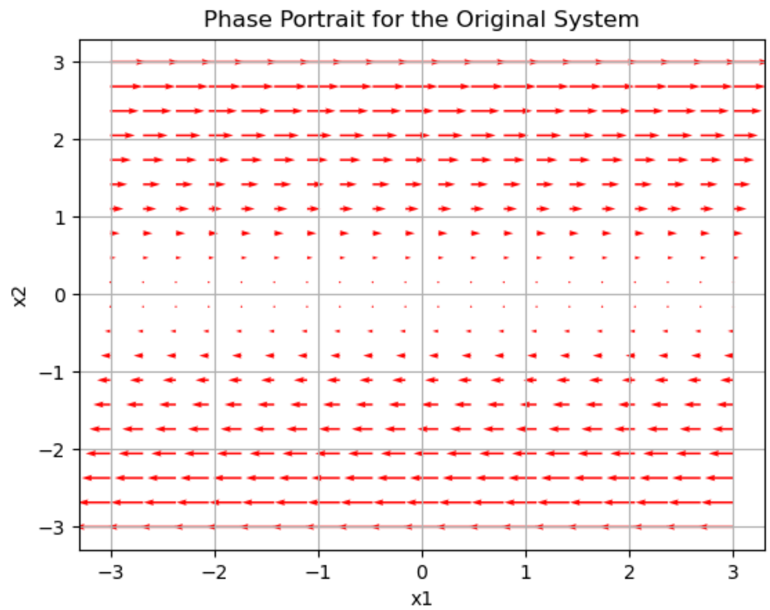
$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} s+7 & +2 & -6 \\ -2 & s+3 & 2 \\ +2 & 2 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow G_c(s) = \begin{pmatrix} -\frac{1}{s+1} & -\frac{1}{s+1} & \frac{2}{s+1} \\ \frac{1}{s+3} & \frac{1}{s+3} & -\frac{1}{s+3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{s+3} & 0 \end{pmatrix}$$

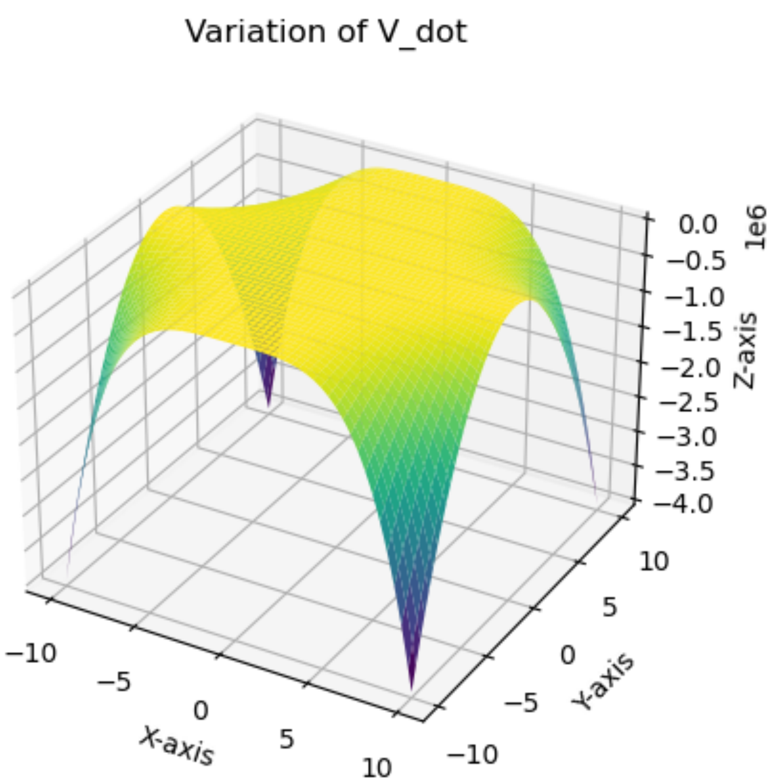
so the pole is $s=-3$ $\longrightarrow$ So it is BIBO Stable



Phase Portrait for the Linearized System
x2
x1

Phase Portrait for the Original System
x2
x1

Variation of V_dot
1e6
Z-axis
X-axis
Y-axis

**Exercise 7.** *BIBO Stability* ***(15 points)***

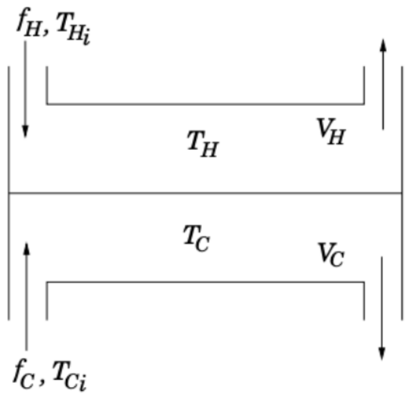



Figure 2: A simple heat exchanger

Consider a simplified model for a heat exchanger shown in Figure 2, in which $f_C$ and $f_H$ are the flows (assumed constant) of cold water and hot water, $T_H$ and $T_C$ represent the temperatures in the hot and cold compartments, respectively, $T_{Hi}$ and $T_{Ci}$ denote the temperature of the hot and cold inflow, respectively, and $V_H$ and $V_C$ are the volumes of hot and cold water. The temperatures in both compartments evolve according to:

$$V_C \frac{dT_C}{dt} = f_C(T_{Ci} - T_C) + \beta(T_H - T_C) \tag{1}$$

$$V_H \frac{dT_H}{dt} = f_H(T_{Hi} - T_H) + \beta(T_C - T_H) \tag{2}$$

Let the inputs to the system be $u_1 = T_{Ci}$, $u_2 = T_{Hi}$, the outputs are $y1 = T_C$ and $y2 = T_H$, and assume that $f_C = f_H = 0.1(m^3/min)$, $\beta = 0.2(m^3/min)$ and $V_H = V_C = 1(m^3)$.

1. Write the state space and output equations for this system. **(5 points)**
2. In the absence of any input, determine $y_1(t)$ and $y_2(t)$. **(5 points)**
3. Is the system BIBO stable? Show why or why not. **(5 points)**

1. $y_1 = T_C$ , $y_2 = T_H$ , $f_c, f_H = 0.1$ , $\beta = 0.2$, $V_H = V_c = 1$

$$\Rightarrow \frac{dy_1}{dt} = 0.1(u_1 - y_1) + 0.2(y_2 - y_1)$$

$$\frac{dy_2}{dt} = 0.1(u_2 - y_2) + 0.2(y_1 - y_2)$$

$$\Rightarrow \dot{x}_1 = 0.1(u_1 - x_1) + 0.2(x_2 - x_1) = -0.3x_1 + 0.2x_2 + 0.1u_1$$

$$\dot{x}_2 = 0.1(u_2 - x_2) + 0.2(x_1 - x_2) = 0.2x_1 - 0.3x_2 + 0.1u_2$$

$$\Rightarrow \dot{x} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = x = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. $A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}$ $\det(\lambda I - A) = 0 \Rightarrow \lambda_1 = -0.5 \quad \lambda_2 = -0.1$

$$\begin{aligned} f(\lambda_1) &= e^{\lambda_1 t} = \beta_1 \lambda_1 + \beta_0 \\ f(\lambda_2) &= e^{\lambda_2 t} = \beta_1 \lambda_2 + \beta_0 \end{aligned} \Rightarrow \begin{aligned} e^{-0.5t} &= -0.5\beta_1 + \beta_0 \\ e^{-0.1t} &= -0.1\beta_1 + \beta_0 \end{aligned} \Rightarrow \begin{aligned} \beta_1 &= -2.5\left(e^{-0.5t} - e^{-0.1t}\right) \\ \beta_0 &= -0.25\left(e^{-0.5t} - 5e^{-0.1t}\right) \end{aligned}$$

$$\therefore e^{At} = \beta_1 A + \beta_0 I = 0.5 \begin{bmatrix} e^{-0.5t} + e^{-0.1t} & e^{-0.5t} - e^{-0.1t} \\ e^{-0.5t} - e^{-0.1t} & e^{-0.5t} + e^{-0.1t} \end{bmatrix}$$

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \cancel{C\int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)d\tau} + \cancel{Du(t)}$$

0 0

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 0.5 \begin{bmatrix} e^{-0.5t} + e^{-0.1t} & e^{-0.5t} - e^{-0.1t} \\ e^{-0.5t} - e^{-0.1t} & e^{-0.5t} + e^{-0.1t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$= 0.5 \begin{bmatrix} x_1(0)\left[e^{-0.5t} + e^{-0.1t}\right] + x_2(0)\left[e^{-0.5t} - e^{-0.1t}\right] \\ x_1(0)\left[e^{-0.5t} - e^{-0.1t}\right] + x_2(0)\left[e^{-0.5t} + e^{-0.1t}\right] \end{bmatrix}$$

3. $G_c(s) = C(sI - A)^{-1}B + D$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s+0.3 & -0.2 \\ -0.2 & s+0.3 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{0.1s + 0.03}{(s+0.1)(s+0.5)} & \frac{0.02}{(s+0.1)(s+0.5)} \\ \frac{0.02}{(s+0.1)(s+0.5)} & \frac{0.1s + 0.03}{(s+0.1)(s+0.5)} \end{pmatrix} \Rightarrow \begin{cases} G_{c11} \quad s_1 = -0.1 \quad s_2 = -0.5 \\ G_{c12} \quad s_1 = -0.1 \quad s_2 = -0.5 \\ G_{c21} \quad s_1 = -0.1 \quad s_2 = -0.5 \\ G_{c22} \quad s_1 = -0.1 \quad s_2 = -0.5 \end{cases}$$

They are all real-negative, so the system is BIBO Stable