Homework 3

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24-677 Special Topics: Linear Control Systems

Due: Sept 29, 2023, 11:59 pm. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Controllability and Observability* ***(10 points)***

Is the state equation

$$\dot{x} = \overset{A}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}} x + \overset{B}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} u$$
$$y = \underbrace{\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}}_{C} x$$

Controllable?(**5 points**) Observable?(**5 points**) Provide your derivation.

For controllability:

$$P = [B \quad AB \quad A^2B] = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \text{rank}(P) = 3$$

So $\text{rank}(P) = 3 = n$, the system is controllable

For observability:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} \quad C = [1 \ 2 \ 1] \quad CA = [1 \ 2 \ 1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = [-1 \ -2 \ -1]$$

$$CA^2 = [1 \ 2 \ 1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}^2 = [1 \ 2 \ 1]$$

$$\Rightarrow Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow \text{rank}(Q) = 1 < n,$$

The system is non-observable

Exercise 2. *Jordan form test* ***(15 points)***

Is the Jordan-form state equation controllable **(7.5 points)** and observable? **(7.5 points)**

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$$

For controllability

$$\hat{B}^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \text{rank}(\hat{B}^2) = 3$$

$$\hat{B}^1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \text{rank}(\hat{B}^1) = 2$$

So the system is controllable

For observability

$$\hat{C}^2 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(\hat{C}^2) = 2 < 3$$

So the system is non-observable.

Exercise 3. *Controllability* ***(10 points)***

Recall the Exercise 2 of Homework 2 from last week. Is that system controllable? **(5 points)** Why?
Now lets move the inlet pipe from tank 1 to tank 2, as shown in the figure. Is this system controllable now? **(5 points)** Why?

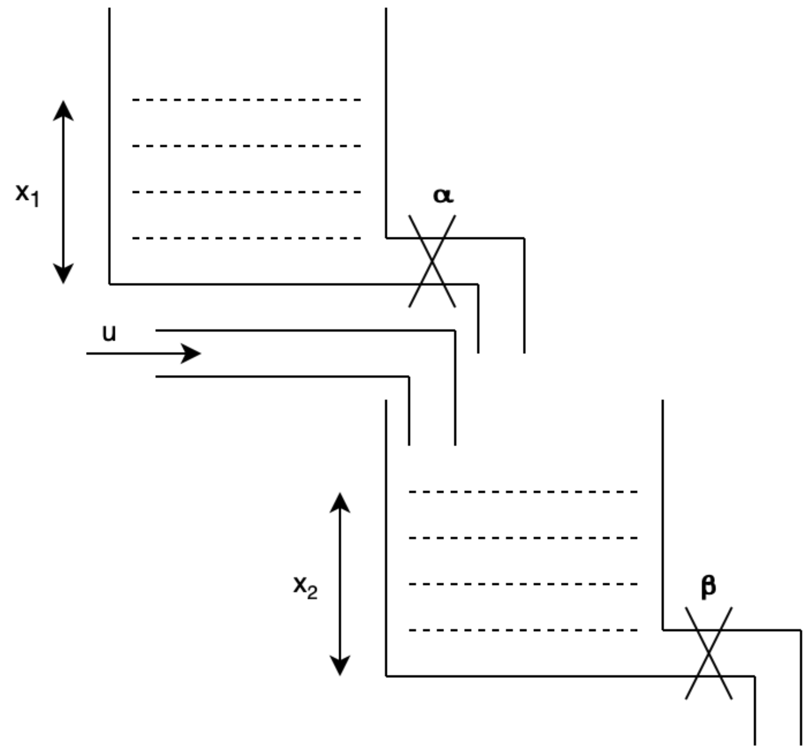


Figure 1: Revised Tank Problem

The system dynamics are

$$\frac{dx_1}{dt} = -\alpha x_1$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 + u$$

From homework 2, $A = \begin{bmatrix} -\alpha & 0 \\ \alpha & \beta \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $[B \vdots AB] = \begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix}$

$\text{rank}\left(\begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix}\right) = 2 = n$ so the system is controllable

If inlet pipe moved from pipe 1 to 2, then the matrix A&B:

$A = \begin{bmatrix} -\alpha & 0 \\ \alpha & \beta \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $[B \vdots AB] = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}$ $\text{rank}\left(\begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}\right) = 1 < n$

so now the system is non-controllable

Exercise 4. *Gauss Elimination and LU Decomposition* ***(20 points)***

1. Solve the following system of linear equations using Gauss Elimination Method

$$\begin{aligned} a)\, x + \;\; y + \;\; z &= 3 \\ x + 2y + 3z &= 0 \\ x + 3y + 2z &= 3 \end{aligned}$$

$$\begin{aligned} b)\;\; x + 2y - \;\; z &= 1 \\ 2x + 5y - \;\; z &= 3 \\ x + 3y + 2z &= 6 \end{aligned}$$

$$\begin{aligned} c)\;\; x_1 + \;\; x_2 - \;\; x_3 + x_4 &= 1 \\ 2x_1 + 3x_2 + \;\; x_3 \qquad &= 4 \\ 3x_1 + 5x_2 + 3x_3 - x_4 &= 5 \end{aligned}$$

2. Solve the following system of linear equations using LU Decomposition Method

$$\begin{aligned} x_1 + 2x_2 + \;\; 4x_3 &= 3 \\ 3x_1 + 8x_2 + 14x_3 &= 13 \\ 2x_1 + 6x_2 + 13x_3 &= 4 \end{aligned}$$

Provide your derivation.

1. a) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & -3 \\ -1 & 0 & 1 & -6 \\ 1 & 1 & 1 & 3 \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & -6 \end{array}\right] \Rightarrow$ $x = 4$, $y = 1$, $z = -2$

b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \\ 0 & 0 & 2 & 4 \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 0 & 5 \\ 2 & 6 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow$ $x = 5$, $y = -1$, $z = 2$

c) $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & 3 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$

$\hookrightarrow$ No such solution

2. $\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 3 & 8 & 14 & 13 \\ 2 & 6 & 13 & 4 \end{array}\right]$ where $\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} U_{11} = 1 & U_{12} = 2 & U_{13} = 4 \\ L_{21} = 3 & \overset{3}{\cancel{L_{21}}} \cdot 2 + U_{22} & \overset{3}{\cancel{L_{21}}} \cdot 4 + U_{23} \\ L_{31} = 2 & \underset{2}{\cancel{L_{31}}} \cdot 2 + L_{32} \cdot U_{22} & \underset{2}{\cancel{L_{31}}} \cdot 4 + L_{32} \cdot U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 + U_{22} = 8 & 12 + U_{23} = 14 \\ 4 + L_{32} \underset{2}{\cancel{U_{22}}} = 6 & 8 + L_{32} \underset{2}{\cancel{U_{23}}} + U_{33} = 13 \end{bmatrix} = \begin{bmatrix} U_{22} = 2 & U_{23} = 2 \\ L_{32} = 1 & U_{33} = 3 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

For $Lz = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = A \Rightarrow \begin{bmatrix} z_1 \\ 3z_1 + z_2 \\ 2z_1 + z_2 + z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} z_1 = 3 \\ z_2 = 4 \\ z_3 = -6 \end{bmatrix}$$

For $Ux = z$:

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ 2x_2 + 2x_3 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 = 3 \\ x_2 = 4 \\ x_3 = -2 \end{bmatrix}$$

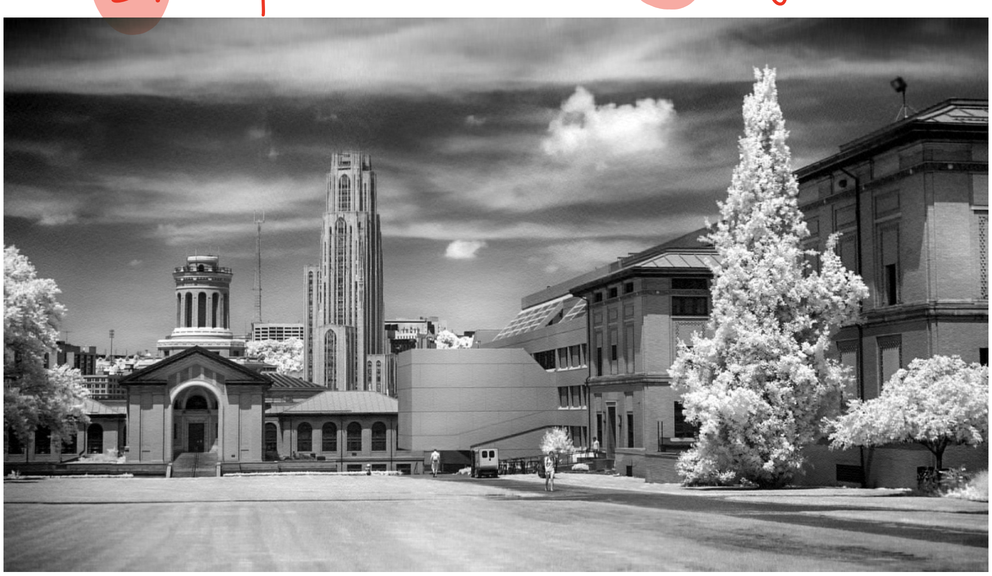
Exercise 5. *SVD (15 points)*

Use SVD to compress the following image to 50%, 10%, and 5% of the original storage required. You will find the image in the Canvas homework folder. For this problem you need to upload code and attached the corresponding compressed images as well as the number of singular value you used for each level of compression. **Note**: you don't need to care about the actual file size after compression since PNG itself is a more complex data structure that contains the grayscale data and other metadata of the image. You only need to care about the storage required in theory, i.e., how many values you need to store the pixel information. For example, for a greyscale image of size $m \times n$, you need $m \times n$ values to store the information if without compression.

You are expected to **attach the images at each compression level in the writeup and also mention the no. of singular values** you used to obtain the compression at each level.

Figure 2: CMU_Grayscale.png

50% compression ratio with 216 singular values



10% compression ratio with 43 singular values

5% compression ratio with 22 singular values

Exercise 6. *Design for Controllability and Observability* ***(20 points)***

Given the following Linear Time Invariant (LTI) system with a tunable parameter γ,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. What values of γ makes the system controllable but not observable? **(10 points)**
2. What values of γ makes the system observable but not controllable? **(10 points)**

For the system:

Controllability: $P = [B \quad AB] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} -3 \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix}$

observability: $Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{matrix} [1 \quad 1] \\ \vdots \\ [-3+\gamma \quad -1] \end{matrix} = \begin{bmatrix} 1 & 1 \\ \gamma-3 & -1 \end{bmatrix}$

If controllable & not observable:

Rank(P) = n & Rank(Q) < m

$\Rightarrow$ $\gamma = 2$

If not controllable & observable:

Rank(P) < n   Rank(Q) = m

$\Rightarrow$ $\gamma = 0$

Exercise 7. *State Space Representation, Controllability* **(10 points)**

We have an LED strip with 5 red LEDs whose brightnesses we want to set. These LEDs are addressed as a queue: at each time step, we can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

1. Model the system as a discrete system with input $u(t)$ as the brightness command to the left-most LED. The state to be the brightness of the five LEDs. Output equals to the state. Write out the state equations in matrix form. **(5 points)**

2. Check the system's controllability. Explain intuitively what the controllability means in this system. **(5 points)**

Note: you do NOT need to consider the 0-255 constraints on the input.

The system can be summarized

$$x(k+1) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

$$\therefore P = [B, A^1B, A^2B, A^3B, A^4B]$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \text{rank}(P) = 5 = n$, the system is controllable

The "controllable" means we can control the system to any state between initial and final state, which is we can control the brightness state of the LEDs, even though there will be some delay.