Homework 4

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24-677 Special Topics: Linear Control Systems

Due: Oct 6, 2023, 11:59 pm. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Canonical forms* ***(10 points)***

Consider the system given by:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Find the controllable canonical form state space representation.

So the transfer function is:

$$G(s) = \frac{s+3}{s^2+3s+2}$$

From $G(s)$ $n=2$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

Exercise 2. *Realization matrix form of realizable MIMO system* ***(15 points)***

Find a state-space realization for

$$\hat{G}_1(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix}$$

$$A = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & -\alpha_3 I_p \\ I_p & 0 & 0 \\ 0 & I_p & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I_p \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Also, C & D matrix:

$$C = \begin{bmatrix} 1 & 2 & 4 & 0 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

So the state-space is:

$$\dot{x} = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 2 & 4 & 0 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Exercise 3. *Minimum Realizations (20 points)*

Are the two state equations

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 2 \end{bmatrix} x$$

and

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 0 \end{bmatrix} x$$

equivalent, i.e. do they have the same transfer function? Are they minimal realizations?

So for the first equation:

$$G(s) = C(sI-A)^{-1}B + \overset{\to 0}{D}$$

$$= \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2\cancel{(s-1)}}{(s-2)\cancel{(s-1)}} = \frac{2}{s-2}$$

This is not a minimal realization, the minimal realization is $\frac{s}{s-2}$, Also from

$$P = (B \cdot AB) = \boxed{\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}} \quad \& \quad Q = \begin{pmatrix} C \\ CA \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}}$$

↳ not controllable ↳ not observable

$$G(s_2) = C(sI-A)^{-1}B + \overset{\to 0}{D}$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \frac{1}{(s-2)(s+1)} \begin{bmatrix} s+1 & 0 \\ -1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2\cancel{(s+1)}}{(s-2)\cancel{(s+1)}}$$

This is not a minimal realization, the minimal realization is $\frac{s}{s-2}$, Also from

$$P = (B \cdot AB) = \boxed{\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}} \quad \& \quad Q = \begin{pmatrix} C \\ CA \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}}$$

↳ is controllable ↳ not observable

Exercise 4. *Realization* ***(15 points)***

Consider the following transfer function

$$g(s) = \frac{2s-4}{s^3-7s+6}$$

(a) Determine the canonical controllable realization. **(5 points)**

(b) Determine the canonical observable realization. **(5 points)**

(c) Determine a minimal realization. **(5 points)**

(a)

From $g(s) = \frac{2s-4}{s^3-7s+6}$ we can have $\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 0 \end{bmatrix}$

so the canonical controllable from is,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = [-4, 2, 0]\, x$$

(b) the canonical observable realization is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 1 \\ 6 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} u \qquad y = [1, 0, 0]\, x$$

(c) for the transfer function:

$$g(s) = \frac{2s-4}{s^3-7s+6} = \frac{2(s-2)}{(s-2)(s-1)(s+3)} = \frac{2}{(s-1)(s+3)}$$

the controllable canonical form is $\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

check for observability: $Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad y = [2, 0]\, x$

controllable

Exercise 5. *Controllable decomposition* ***(10 points)***

Reduce the state equation

$$\dot{x} = \overbrace{\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{B} u$$
$$y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} x$$

to a controllable form. Is the reduced state equation observable?

For controllability matrix

$P = [B \; AB] = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ $\text{rank}(P) = 1 < 2 = n$ & $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\hat{A} = M^{-1}AM = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix} \Rightarrow A_c = 3$

$\hat{B} = M^{-1}B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $B_c = 1$

$\hat{C} = CM = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$ $C_c = 2$

So the state space representation is

$\dot{x} = 3x + u \qquad y = 2x$

$Q = [2]$ $\text{rank}(Q) = 1 \Rightarrow$ observable

Exercise 6. *kalman decomposition* ***(10 points)***

Decompose the state equation

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix} x$$

to a form that is both controllable and observable.
Hint: If you use the correct approach, you don't need to compute any complicated algebra and matrix calculations.

so for the transfer function:

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

For controllability: $P = [B \ AB \ A^2B \ A^3B \ A^4B]$

$$\Rightarrow P = \begin{bmatrix} 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{rank}(P) = 3 < n = 5$$

The state equation can be aparted to

$$A = \left[\begin{array}{ccc|cc} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] \quad B = \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ \hline 0 \\ 0 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|cc} 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

and can be reduced to:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \qquad y = [0 \ 1 \ 1] x$$

For observability: $Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & \lambda_1^2 & \lambda_2^2 \end{bmatrix} \Rightarrow \text{rank}(Q) = 2$

(If want to be observable) needs to be

so the original state space equation can be expressed by:

$$\dot{x} = \left[\begin{array}{cc|c} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_2 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \\ \hline 1 \end{array}\right] u$$

reduced $\Rightarrow$

$$\boxed{\begin{aligned} \dot{x} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x \end{aligned}}$$

Exercise 7. *Controllable Canonical Form* ***(20 points)***

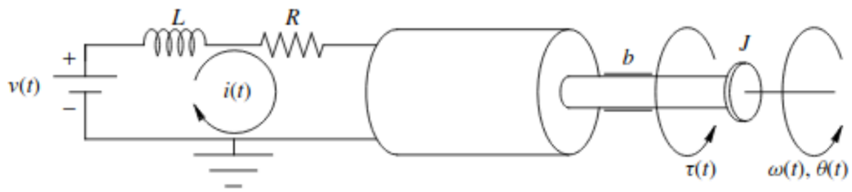

Figure 1: An electromechanical system

The dynamic model of this system can be derived in three segments: a circuit model, electromechanical coupling, and a rotational mechanical model. For the circuit model, Kirchoff's voltage law yields a first order differential equation relating the armature current to the armature voltage; that is,

$$L\frac{di(t)}{dt} + Ri(t) = v(t) \tag{1}$$

Motor torque is modeled as being proportional to the armature current, so the electromechanical coupling equation is

$$\tau(t) = k_T i(t) \tag{2}$$

where k_T is the motor torque constant. For the rotational mechanical model, Euler's rotational law results in the following second-order differential equation relating the motor shaft angle $\theta(t)$ to the input torque $\tau(t)$.

$$J\ddot{\theta}(t) + b\dot{\theta}(t) = \tau(t) \tag{3}$$

Converting the ODEs into transfer functions and multiplying them together, we eliminate the intermediate variables to get the overall transfer function:

$$\frac{\Theta(s)}{V(s)} = \frac{k_T}{(Ls+R)(Js^2+bs)} \tag{4}$$

Write the controllable canonical form of this system.

The transfer function is $\frac{\Theta(s)}{V(s)} = \frac{k_T}{(Ls+R)(Js^2+bs)} = \frac{k_T}{LJs^3+(RJs^2+Lbs^2)+Rbs}$

$$\Rightarrow \frac{k_T/JL}{s^3+[(RJ+Lb)LJ]s^2+Rbs/LJ}$$

$$\Rightarrow \Delta(s) = s^3+[(RJ+Lb)LJ]s^2+\left[\frac{Rb}{LJ}\right]s$$

$$\Rightarrow \quad b_0 = \frac{k_T}{LJ} \quad b_1 = 0 = b_2 = b_3$$

$$\Rightarrow \quad a_0 = 0 \quad a_1 = \frac{bR}{JL} \quad a_2 = \frac{bL + JR}{JL}$$

So the controllable form is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -Rb/LJ & \frac{RJ+Lb}{LJ} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[\frac{k_T}{JL}, 0, 0 \right] x$$