**24 - 677**
**Fall 2023**
**Mid-term Exam**
**10/24/23**
**Time: 24 Hours**

**Name:** Letian Leng

**Andrew id:** lleng

*Print your initials on each page that has your answers*

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This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use your equation sheet and calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- You are allowed to use course slides, homework solution sheets as references. You are allowed to search for the knowledge needed on the internet.
- You must conduct the exam independently. Discussion or seeking help from others, online or in-person, is prohibited.
- All answers need to be derived by hand to get points. You are allowed to use a calculator for basic calculation of scalars. You can use calculate/-computer programs to verify your answers but the effort does not account as credits.
- You can ask questions on campuswire but only to the TAs and instructors.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 20 | |
| Total: | 100 | |

1. Please state whether each of the following statement is **True** or **False**. Explanation is not required.
   (a) (3 points) The system $y(t) = \sin(t)u(3t)$ is linear.
   (b) (3 points) Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system. Assuming $A^{-1}$ exists, the system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable.
   (c) (3 points) The following continuous time system is BIBO stable.
   $$\dot{x} = u, \quad y = x$$
   (d) (3 points) The following DT system is controllable if $a \neq 0$.
   $$x[k+1] = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$$
   (e) (3 points) The system given in (d) is stabilizable when $a = 0$.

(a) $\alpha y_1(t) = \alpha \sin(t) u_1(3t)$ $H(\alpha u_1 + \beta u_2) = \sin(t)(\alpha u_1 + \beta u_2)(3t)$
$\beta y_2(t) = \beta \sin(t) u_2(3t)$ $= \alpha u_1 \sin(t)(3t) + \beta u_2 \sin(t)(3t)$

so true

(b) All $\lambda$ is negative real part. $A^{-1}$: same

so true

(c) $A=0$ $B=1$ $C=1$ $D=0$

$G_c(s) = 1(s-0)^{-1} \cdot 1 = \frac{1}{s}$

False

(d) True

(e) True

2. Consider a model of fisheries management. State $x_1$ is the population level of a prey species, $x_2$ is the population level of a predator species, and $x_3$ is the effort expended by humans in fishing the predator species. The model is

$$\begin{aligned}\dot{x}_1 &= (r_1 - x_2)x_1\\ \dot{x}_2 &= (r_2 - x_3)x_2\\ \dot{x}_3 &= u\\ y &= x_2\end{aligned}$$

where $u$ is the input, $y$ is the measurement of the predator species, and $r_1 = 10$ and $r_2 = 25$

(a) (5 points) Find the equilibrium point if the prey species population is known to be $\bar{x}_1 = 20$.

(b) (5 points) Linearize the model using the equilibrium point from (a)

(c) (5 points) Find the transfer function of the linearized state model from (b)

(a) @ equilibrium point $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$

$$\begin{cases} \dot{x}_1 = 0 \Rightarrow (10 - x_2)x_1 = 0 & \Rightarrow x_2 = 10 \\ \dot{x}_2 = 0 \Rightarrow (25 - x_3)x_2 = 0 & \Rightarrow x_3 = 25 \\ \dot{x}_3 = 0 = u & x_1 = 20 \end{cases}$$

So $\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 20 & 10 & 25 \end{bmatrix} \end{array}$

(b) $\frac{\partial f}{\partial x_1} = 10 - x_2 \quad \frac{\partial f}{\partial x_2} = 25 - x_3 \quad \frac{\partial f}{\partial x_3} = 0$

$\frac{\partial f}{\partial u} = 1 \quad \Rightarrow \begin{bmatrix} 10-x_2 & -x_1 & 0 \\ 0 & 25-x_3 & -x_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$

$\dot{x} \Rightarrow \begin{bmatrix} 0 & -20 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(c) $G(s) = C(sI - A)^{-1}B + D$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} s & 20 & 0 \\ 0 & s & 10 \\ 0 & 0 & s \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\left[\begin{array}{ccc|ccc} s & 20 & 0 & 1 & 0 & 0 \\ 0 & s & 10 & 0 & 1 & 0 \\ 0 & 0 & s & 0 & 0 & 1 \end{array}\right]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{s} & -\frac{20}{s^2} & \frac{200}{s^3} \\ 0 & \frac{1}{s} & -\frac{10}{s^2} \\ 0 & 0 & 1/s \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{10}{s^2}$$

3. (15 points) For the following dynamical system

$$\dot{x}(t) = \overbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{B} u(t)$$

compute $x(0)$ when $u(t) = 0$ and $x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1$$

$$\left.\begin{aligned} f(\lambda) &= e^{\lambda t} = \beta_1 \lambda + \beta_0 \\ \frac{df(\lambda)}{d\lambda} &= te^{\lambda t} = \beta_1 \end{aligned}\right\} \xrightarrow{\text{for } \lambda = 1} \begin{cases} e^t = \beta_1 + \beta_0 \\ te^t = \beta_1 \end{cases} \Rightarrow \begin{cases} \beta_1 = te^t \\ \beta_0 = (1-t)e^t \end{cases}$$

$$e^{At} = \beta_0 I + \beta_1 A = \begin{bmatrix} e^t & 0 \\ 3te^t & e^t \end{bmatrix}$$

For $t = 2$, $x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $t_0 = 0$

$$x(2) = \begin{bmatrix} e^2 & 0 \\ 6e^2 & e^2 \end{bmatrix} x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x(0) \Rightarrow \begin{bmatrix} \frac{1}{e^2} \\ -\frac{6}{e^2} \end{bmatrix}$$

4. Given an LTI system with state space representation

$$\dot{x}(t) = \overbrace{\begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 1 \\ \alpha \end{bmatrix}}^{B} u(t)$$
$$y(t) = \begin{bmatrix} 1 & \alpha \end{bmatrix} x(t) + u(t)$$

where $\alpha \in \mathbb{R}$.

(a) (5 points) Find the range of $\alpha$ for which the system is exponentially stable.

(b) (10 points) For the supremum (least upper bound) of the range of $\alpha$ determined in ($\boxed{a}$), check whether the given system is BIBO stable.

(a) $A = \begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix} \Rightarrow \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -1 \quad \lambda_2 = 1-\alpha$

If we want the system asymptotically stable,

$$\begin{cases} \lambda_1 < 0 & \checkmark \\ \lambda_2 < 0 & 1-\alpha < 0 \Rightarrow \alpha > 1 \end{cases}$$

(b) So $\alpha = 1$

$$G_c(s) = C(sI - A)^{-1}B + D$$
$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1$$
$$= \frac{1}{s+1} + \frac{s}{s(s+1)} + 1 = \frac{\cancel{s} + \cancel{s} + \cancel{s}(s+1)}{\cancel{s}(s+1)} = \frac{s+3}{s+1}$$

Pole $= -1 \Leftarrow$ negative real part, So BIBO stable

5. Consider the following nonlinear system

$$\dot{x}_1 = -\frac{x_2}{1+x_1^2} - 2x_1$$
$$\dot{x}_2 = \frac{x_1}{1+x_1^2}$$

(a) (5 points) Using the candidate Lyapunov function $V(x) = x_1^2 + x_2^2$ and Lyapunov Direct method, first find the equilibrium point and then find the stability of the system at the equilibrium point

(b) (5 points) Linearize the system about the equilibrium point and find the stability of the linearized system using Lyapunov indirect method

(a) For a equilibrium point: $\dot{x}_1 = 0$ $\dot{x}_2 = 0$

$$\begin{cases} -\frac{x_2}{1+x_1^2} - 2x_1 = 0 \\ \frac{x_1}{1+x_1^2} = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

so $V(x) = x_1^2 + x_2^2 = 0$

stable @ eq point

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2\dot{x}_2 x_2$$
$$= 2x_1\left(-\frac{x_2}{1+x_1^2} - 2x_1\right) + 2x_2\left(\frac{x_1}{1+x_1^2}\right)$$
$$= -\cancel{\frac{2x_1x_2}{1+x_1^2}} - 4x_1^2 + \cancel{\frac{2x_2x_1}{1+x_1^2}}$$
$$= -4x_1^2$$

⇐ always real negative part, so asymptotically stable

(b)

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{2x_1x_2}{(x_1^2+1)^2} - 2 & -\frac{1}{x_1^2+1} \\ \frac{1-x_1^2}{(1+x_1^2)^2} & 0 \end{bmatrix}$$

For $x_1 = x_2 = 0$

$$\dot{x} = \underbrace{\begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}}_{A}$$

$\det(A - \lambda I) = 0$

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = -1$

real negative parts

Asymptotically stable

6. (10 points) Find the minimal realization for

$$G(s) = \begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}.$$

(6) $G(s) = \begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}$ $G(\infty) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow G(s) - G(\infty) = \begin{bmatrix} \frac{-1}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix} = G_{sp}$

$\Rightarrow \Delta(s) = s(s+1) = s^2 + s \Rightarrow \alpha_1 = 1$

$\Rightarrow G_{sp} = \frac{1}{s(s+1)} \begin{bmatrix} -s \\ 1 \end{bmatrix} \Rightarrow N_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $N_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$N_1 = N_2 \in R_{2\times 1} \Rightarrow P = 1$

$\Rightarrow A = [-1]$ $B = [1]$

$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So the state-space of this system is:

$$\begin{cases} \dot{x} = [-1]x + [1]u \\ y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \end{cases}$$

For whether the mirimal relization

P-matrix $= [1] = n$ $\rightarrow$ controllable

Q-matrix $= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = n$ $\rightarrow$ observable

So $\begin{cases} \dot{x} = [-1]x + [1]u \\ y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \end{cases}$ is a minimal relization

7. Suppose you are invited as a control engineering consultant to investigate a critical safety issue for an airplane company. You are provided with an approximate linear model of the lateral dynamics of the aircraft which has the state and control vectors

$$x = \begin{bmatrix} p & r & \beta & \phi \end{bmatrix}^T \quad \text{and} \quad u = \begin{bmatrix} \delta_a & \delta_r \end{bmatrix}^T$$

where $p$ and $r$ are incremental roll and yaw rates, $\beta$ is an incremental sideslip angle, and $\phi$ is an incremental roll angle. The control inputs are the incremental changes in the aileron angle $\delta_a$ and in the rudder angle $\delta_r$, respectively. In a consistent set of units, the linearized model is given as $\dot{x} = Ax + Bu$ with

$$\mathbf{A} = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Answer the following questions with derivations. A simple yes or no without explanation will not get 0 credit.

(a) (5 points) Is the linearized aircraft model asymptotically stable? Is the linearized aircraft model stable i.s.L.?

(b) (5 points) Is the aircraft controllable with just $\delta_r$? Is the aircraft controllable with both $\delta_r$ and $\delta_a$?

(c) (5 points) Suppose a malfunction prevents manipulation of the rudder angle $\delta_r$, is it possible to control the aircraft using only the aileron angle $\delta_a$?

(d) (5 points) If you only have budget to measure one state, which one to measure (choose one from $\{p, r, \beta, \phi\}$) so that the whole system is observable?

7. (a) We have $A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \lambda(\lambda+10)(\lambda^2+\lambda+1) = 0$

$\therefore \lambda_1 = 0, \ \lambda_2 = 10, \ \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \ \lambda_4 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Since the eigenvalues have positive and imagine part, it is not asymptotically stable, however, there is $\lambda_1 = 0$ such that $Re = 0$ & $Im = 0$, so the system is stable i.s.L

(b) For $\delta r$, the B-matrix becomes $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

The P-matrix is $P = [B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 0 & 0 & -1 & 11 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 11 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow rank(P) = 4 = n$

So the aircraft is controllable

For $\delta r \ \delta a$, the B-matrix becomes $\begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

The P-matrix is $P = [B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 10 & 0 & -100 & 0 & 1000 & -1 & -10000 & 11 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 10 & 0 & -100 & 0 & 1000 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{1}{10} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -10 & 0 & 100 & -\frac{1}{10} \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \Rightarrow rank(P) = 4 = n$

So the aircraft is controllable

(c): For $\delta a$, $B = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $P = [B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 10 & -100 & 1000 & -10000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & -100 & 1000 \end{bmatrix}$

$\Rightarrow rank(P) = 2 < n = 4$ So impossible to control

(d). $C_1 = [1 \ 0 \ 0 \ 0]$ $C_2 = [0 \ 1 \ 0 \ 0]$ $C_3 = [0 \ 0 \ 1 \ 0]$ $C_4 = [0 \ 0 \ 0 \ 1]$

$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$

$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \\ -1000 & -11 & -99 & 0 \end{bmatrix}$ $Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $Q_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$Q_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix}$ only $Rank(Q_4) = 4 = n$, with $C = [0 \ 0 \ 0 \ 1]$

So choose $\phi$