Exercise 3 Solutions
Fuid Dynamics II SS 2022

4.5.2022

1 Exercise: Averages

The Burgers vortex is a special solution of the Navier-Stokes equation for an axisymmetric velocity field $\mathbf{u}(\mathbf{x},t) = \mathbf{u}_{ext}(\mathbf{x},t) + [w_x(x,y,t), w_y(x,y,t), 0]$ with an external strain field $\mathbf{u}_{ext}(\mathbf{x},t) = \left[-\frac{a}{2}x, -\frac{a}{2}y, az\right]$. The velocity field in cylindrical coordinates can be derived as

$$u_r(r,z,t) = -\frac{a}{2}r\ , \quad u_\varphi(r,z,t) = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{r_B^2}}\right)\ , \quad u_z(r,z,t) = az\ . \tag{1}$$

Here, $r_B^2 = \frac{4\nu}{a}$, and the strain parameter a is assumed to be constant.

1. The field lines for different parameters are depicted in Fig. 1.
2. The divergence in polar coordinates is given according to

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial\left(-\frac{a}{2}r^2\right)}{\partial r} + \frac{\partial az}{\partial z} = -a + a = 0. \tag{2}$$

3. In cylindrical coordinates the Lagrangian path is given as

$$\mathbf{r}(r_0,\varphi_0,z_0,t) = r(r_0,\varphi_0,z_0,t)\mathbf{e}_r(r_0,\varphi_0,z_0,t) + z(r_0,\varphi_0,z_0,t)\mathbf{e}_z \tag{3}$$

where $\mathbf{e}_r(r_0,\varphi_0,z_0,t) = [\cos\varphi(r_0,\varphi_0,z_0,t), \sin\varphi(r_0,\varphi_0,z_0,t), 0]$. Deriving Eq. (3) with respect to time yields

$$\begin{aligned}\dot{\mathbf{r}}(r_0,\varphi_0,z_0,t) =& \dot{r}(r_0,\varphi_0,z_0,t)\mathbf{e}_r(r_0,\varphi_0,z_0,t) + r(r_0,\varphi_0,z_0,t)\dot{\mathbf{e}}_r(r_0,\varphi_0,z_0,t) + \dot{z}(r_0,\varphi_0,z_0,t)\mathbf{e}_z \\ =& \dot{r}(r_0,\varphi_0,z_0,t)\mathbf{e}_r(r_0,\varphi_0,z_0,t) + r(r_0,\varphi_0,z_0,t)\dot{\varphi}(r_0,\varphi_0,z_0,t)\mathbf{e}_\varphi(r_0,\varphi_0,z_0,t) \\ &+ \dot{z}(r_0,\varphi_0,z_0,t)\mathbf{e}_z\end{aligned} \tag{4}$$

where $\mathbf{e}_\varphi(r_0,\varphi_0,z_0,t) = [-\sin\varphi(r_0,\varphi_0,z_0,t), \cos\varphi(r_0,\varphi_0,z_0,t), 0]$.
Moreover, in order to evaluate $\dot{\mathbf{r}}(r_0,\varphi_0,z_0,t) = \mathbf{u}(\mathbf{r}(r_0,\varphi_0,z_0,t),t)$ we have to insert the position of the tracer into Eq. (1) and obtain

$$\dot{r}(r_0,t) = -\frac{a}{2}r(r_0,t)\ , \tag{5}$$

$$r(r_0,t)\dot{\varphi}(r_0,t) = \frac{\Gamma}{2\pi r(r_0,t)}\left(1 - e^{-\frac{r^2(r_0,t)}{r_B^2}}\right)\ , \tag{6}$$

$$\dot{z}(z_0,t) = az(z_0,t)\ . \tag{7}$$

Here, it can be observed that the motion in the $r-\varphi$-plane is decoupled from the z-direction. We can solve the system of equations as

$$
\begin{aligned}
r(r_0,t) =& r_0 e^{-\frac{a}{2}t} && (8)\\
\varphi(r_0,t) =& \frac{\Gamma}{2\pi r_0^2}\int_0^t \mathrm{d}t' e^{\frac{a}{2}t'}\left(1-e^{-\frac{r_0^2}{r_B^2}e^{-at'}}\right)+\varphi_0 && (9)\\
z(z_0,t) =& z_0 e^{at} && (10)
\end{aligned}
$$

4. The correlation functions in the radial and vertical direction can now be calculated as

$$
\begin{aligned}
\langle r(r_0,t)r(r_0,t+\tau)\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\mathrm{d}t' r(r_0,t')r(r_0,t'+\tau)\\
=\lim_{T\to\infty}\frac{r_0^2}{T}\int_t^{t+T}\mathrm{d}t' e^{-\frac{a}{2}t'}e^{-\frac{a}{2}(t'+\tau)} &= r_0^2 e^{-\frac{a}{2}\tau}\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\mathrm{d}t' e^{-at'}\\
= r_0^2 e^{-\frac{a}{2}\tau}\lim_{T\to\infty}\frac{1}{T}\left[-\frac{e^{-at'}}{a}\right]_t^{t+T} &= -\frac{r_0^2}{a}e^{-\frac{a}{2}\tau}\lim_{T\to\infty}\frac{1}{T}\left(e^{-a(t+T)}-e^{-at}\right) \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle z(z_0,t)z(z_0,t+\tau)\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\mathrm{d}t' z(z_0,t')z(z_0,t'+\tau)\\
=\lim_{T\to\infty}\frac{z_0^2}{T}\int_t^{t+T}\mathrm{d}t' e^{at'}e^{a(t'+\tau)} &= z_0^2 e^{a\tau}\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\mathrm{d}t' e^{2at'}\\
= z_0^2 e^{a\tau}\lim_{T\to\infty}\frac{1}{T}\left[\frac{e^{2at'}}{2a}\right]_t^{t+T} &= \frac{z_0^2}{2a}e^{a\tau}\lim_{T\to\infty}\frac{1}{T}\left(e^{2a(t+T)}-e^{2at}\right) \qquad (12)
\end{aligned}
$$

Hence, the Lagrangian path in the r-direction becomes uncorrelated over time τ, whereas the z-correlation increases. In the long-time limit $T\to\infty$ we obtain

$$
\langle r(r_0,t)r(r_0,t+\tau)\rangle = 0 \;\text{ and }\; \langle z(z_0,t)z(z_0,t+\tau)\rangle = \infty \qquad (13)
$$

5. Taking the ensemble average would correspond of considering different Lagrangian paths with different initial conditions. As it is depicted in Fig. 2 for the Lagrangian paths of two different particles, particles spiral towards $r=0$. Generally speaking, this implies that Lagrangian particles are trapped inside of a Burgers vortex and cannot explore the "entirety" of phase space. Therefore, the ergodicity hypothesis does not hold.

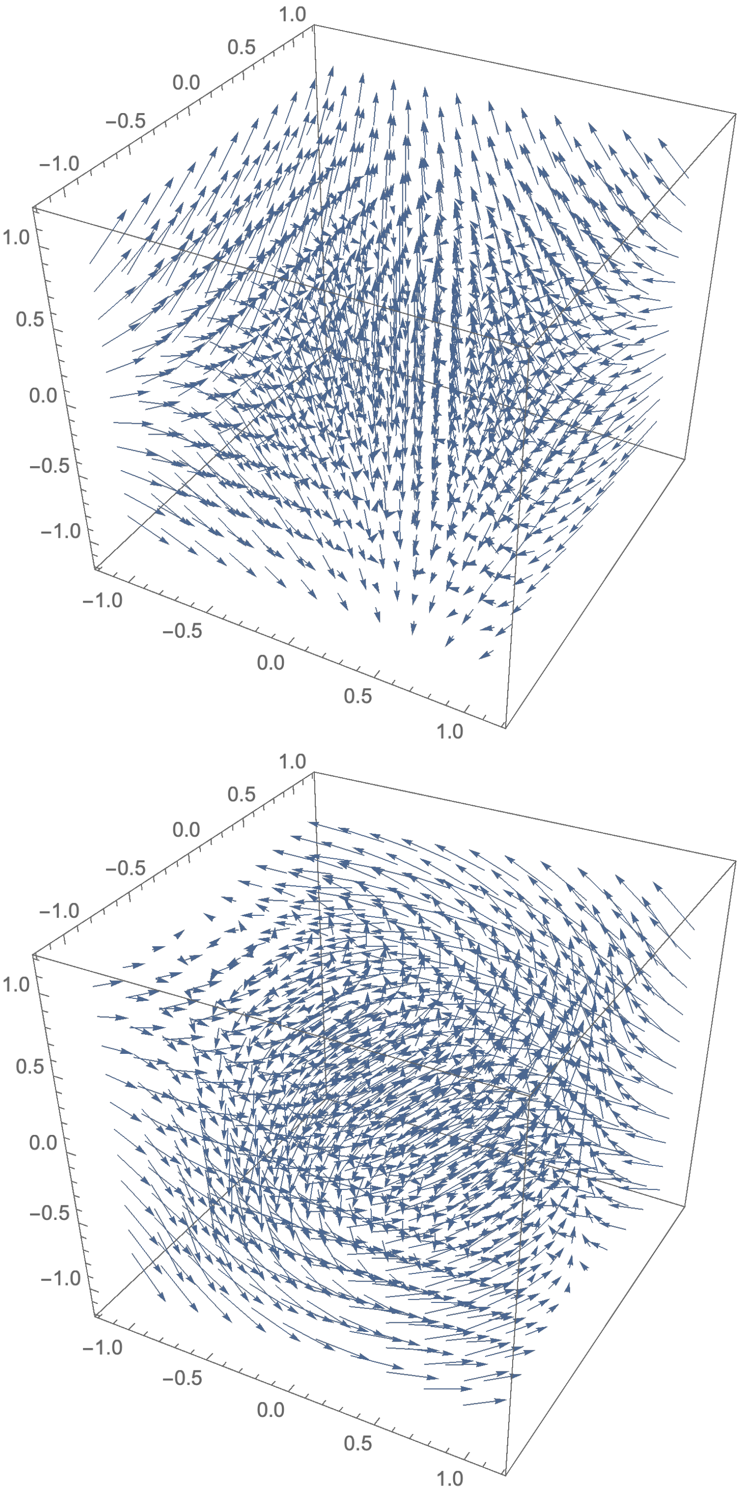

Figure 1: *Top:* Velocity field of Burgers vortex for $a = 0.1$, $\nu = 0.01$ and $\Gamma = 0$, which amounts to the purely linear external strain profile. *Bottom:* Velocity field of Burgers vortex for $a = 0.1$, $\nu = 0.01$ and $\Gamma = 1$.

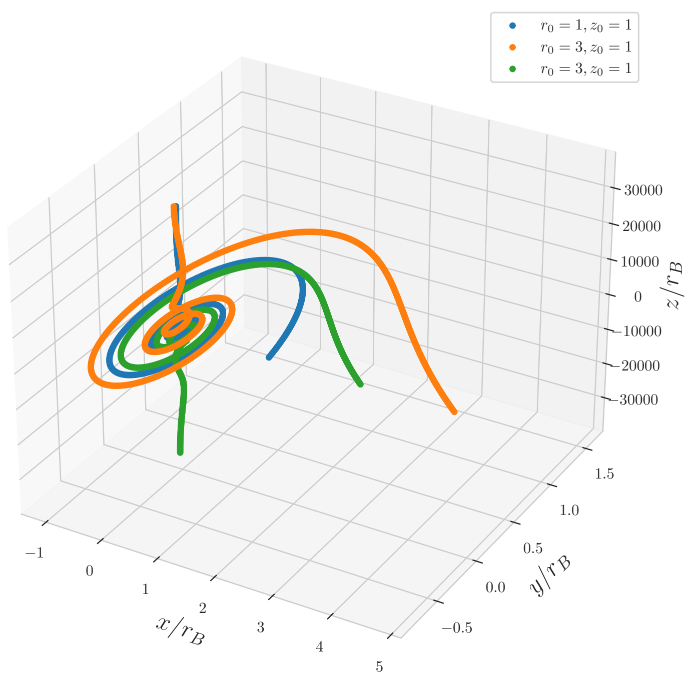

Figure 2: *Top:* Lagrangian tracer particles in a Burgers vortex for different initial conditions but the same parameters as in the bottom of Figure 1.