Exercise 1
Fuid Dynamics II SS 2022

27.4.2022

1 Exercise: Hopf bifurcation

As discussed in the lecture, the so-called Landau scenario of turbulence treats the transition to a turbulent regime as a series of instabilities with newly emerging frequencies ω_i. Here, we consider a system of ordinary differential equations

$$\dot{x} = -\omega y + x\left[(\mu - \mu_{crit}) - (x^2 + y^2)\right] \tag{1}$$

$$\dot{y} = \omega x + y\left[(\mu - \mu_{crit}) - (x^2 + y^2)\right] \tag{2}$$

where $\mu_{crit} > 0$.

- Show that this system transforms in polar coordinates $x = r\cos\varphi$ and $y = r\varphi$ according to

$$\dot{r} = (\mu - \mu_{crit})r - r^3 \tag{3}$$

$$\dot{\varphi} = \omega \tag{4}$$

- Determine the stationary solutions r_0 and r_1 of the first equation for the radius r. Which of the two solutions is the actual stationary solution of (1) and (2)?
- Determine the stability of those solutions, i.e., derive a linear equation for the perturbations $\tilde{r}_i(t)$ where $r_i(t) = r_i + \tilde{r}_i(t)$, where $i = 0, 1$. Solve these linear equations with the ansatz $\tilde{r}_i(t) \sim e^{\lambda t}$ and determine the parameter regime μ for which perturbations grow and decay.
- Solve Eqs. (3) and (4) directly.
  Hint: Perform a separation of variables and make use of $\int \mathrm{d}r \frac{1}{r(a-r^2)} = \frac{1}{2a}\ln\frac{r^2}{a^2-r^2}$
- What happens in the limit of $t \to \infty$.
- Plot trajectories for different initial conditions and parameters μ.

2 Exercise: Gaussian distribution

Consider a Gaussian distribution of a one-dimensional velocity field

$$g(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-U)^2}{2\sigma^2}} \tag{5}$$

Calculate the mean $\langle u \rangle = \int_{-\infty}^{\infty} \mathrm{d}u u g(u)$ and the standard deviation $\left\langle (u - \langle u \rangle)^2 \right\rangle$