Exercise 2 Solutions
Fuid Dynamics II SS 2022

27.4.2022

1 Exercise: Hopf bifurcation

We can derive $x = r\cos\varphi$ and $y = r\sin$ with respect to time

$$\dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \tag{1}$$
$$\dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \tag{2}$$

Inserting this into the Hopf equations yields

$$\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi = -\omega r\sin\varphi + r\cos\varphi[(\mu - \mu_{crit}) - r^2] \tag{3}$$
$$\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi = \omega r\cos\varphi + r\sin\varphi[(\mu - \mu_{crit}) - r^2] \tag{4}$$

We can now multiply (3) by $\cos\varphi$ and (4) by $\sin\varphi$ and add these two equations together

$$\dot{r}\underbrace{\left[\cos^2\varphi + \sin^2\varphi\right]}_{=1} = r\underbrace{\left[\cos^2\varphi + \sin^2\varphi\right]}_{=1}[(\mu - \mu_{crit}) - r^2] \tag{5}$$

Similarly, we can multiply (3) by $\sin\varphi$ and (4) by $\cos\varphi$ and subtract (3) from (4) together

$$r\dot{\varphi} = \omega r \tag{6}$$

Hence, we obtain

$$\dot{r} = (\mu - \mu_{crit})r - r^3 \tag{7}$$
$$\dot{\varphi} = \omega \tag{8}$$

The stationary solutions, =0, of Eq. (7) read $r_0 = 0$ and $r_1 = \sqrt{\mu - \mu_{crit}}$. Nonetheless, only r_0 is an actual stationary solution $\dot{x} = \dot{y} = 0$ of the Hopf equations. The second branch r_1 actually is a so-called limit cycle where $x = r_1\cos\omega t$ and $y = r_1\sin\omega t$.

The stability of these two solutions can be investigated by linearization of Eq. (7)

$$\begin{aligned}\underbrace{\dot{r}_i}_{=0} + \dot{\tilde{r}}_i(t) &= (\mu - \mu_{crit})(r_i + \tilde{r}_i(t)) - \underbrace{(r_i + \tilde{r}_i(t))^3}_{=r_i(t)^3 + 3r_i(t)^2\tilde{r}(t) + \mathcal{O}(\tilde{r}_i(t)^2)} \\ &= (\mu - \mu_{crit})\underbrace{(r_i(t) - r_i(t)^3)}_{=0} + (\mu - \mu_{crit})\tilde{r}_i(t) - 3r_i(t)^2\tilde{r}(t) + \mathcal{O}(\tilde{r}_i(t)^2)\end{aligned} \tag{9}$$

Discarding higher order terms thus yields

$$\dot{\tilde{r}}_i(t) = (\mu - \mu_{crit})\tilde{r}_i(t) - 3r_i^2\tilde{r}(t) \tag{10}$$

This yields for $r_0 = 0$ an $r_1 = \sqrt{\mu - \mu_{crit}}$

$$\dot{\tilde{r}}_0(t) =(\mu - \mu_{crit})\tilde{r}_0(t) \tag{11}$$
$$\dot{\tilde{r}}_1(t) =(\mu - \mu_{crit})\tilde{r}_1(t) - 3(\mu - \mu_{crit})\tilde{r}_1(t) = -2(\mu - \mu_{crit})\tilde{r}_1(t) \tag{12}$$

These equations can be solved according to

$$\tilde{r}_0(t) =R_0 e^{(\mu-\mu_{crit})t} \tag{13}$$
$$\tilde{r}_1(t) =R_1 e^{-2(\mu-\mu_{crit})t} \tag{14}$$

Hence, we observe that for $\mu < \mu_{crit}$, the fixed point r_0 is stable whereas it becomes unstable for $\mu > \mu_{crit}$, and the limit cycle becomes stable.

Directly solving the (uncoupled) equations (7-8) is easy for the second equation which yields $\varphi(t) = \omega t + \varphi_1$. Eq. (7) can be solved by separation of variables

$$\frac{\mathrm{d}r}{(\mu - \mu_{crit})r - r^3} = \mathrm{d}t \tag{15}$$

which can be integrated according to

$$\int_{r_{init}}^{r} \frac{\mathrm{d}r'}{(\mu - \mu_{crit})r' - r'^3} = \int_0^t \mathrm{d}t' = t \tag{16}$$

The integral can be solved by a partial fraction decomposition

$$\begin{aligned}\int_{r_{init}}^{r} \frac{\mathrm{d}r'}{(\mu - \mu_{crit})r' - r'^3} &= \int_{r_{init}}^{r} \mathrm{d}r' \left[\frac{1}{(\mu - \mu_{crit})r'} + \frac{1}{\mu - \mu_{crit}} \frac{r'}{(\mu - \mu_{crit}) - r'^2}\right] \\ &= \left.\frac{\ln r'}{\mu - \mu_{crit}}\right|_{r_{init}}^{r} - \frac{1}{2(\mu - \mu_{crit})} \ln\left[(\mu - \mu_{crit}) - r'^2\right]\Big|_{r_{init}}^{r} \\ &= \frac{1}{2(\mu - \mu_{crit})} \ln\left[\frac{r'^2}{(\mu - \mu_{crit}) - r'^2}\right]\Bigg|_{r_{init}}^{r} \\ &= \frac{1}{2(\mu - \mu_{crit})} \ln\left[\frac{r^2}{r_{init}^2} \frac{(\mu - \mu_{crit}) - r_{init}^2}{(\mu - \mu_{crit}) - r^2}\right]\end{aligned} \tag{17}$$

Here, we made use of the following rules for the logarithm:

$$\ln a - \ln b = \ln\left[\frac{a}{b}\right] \tag{18}$$
$$\ln x^2 =2 \ln x \tag{19}$$

Eq. (16) can now be solved for r according to

$$r = \sqrt{(\mu - \mu_{crit})}\left[1 + \left(\frac{\mu - \mu_{crit}}{r_{init}^2} - 1\right) e^{-2(\mu-\mu_{crit})t}\right]^{-\frac{1}{2}} \tag{20}$$

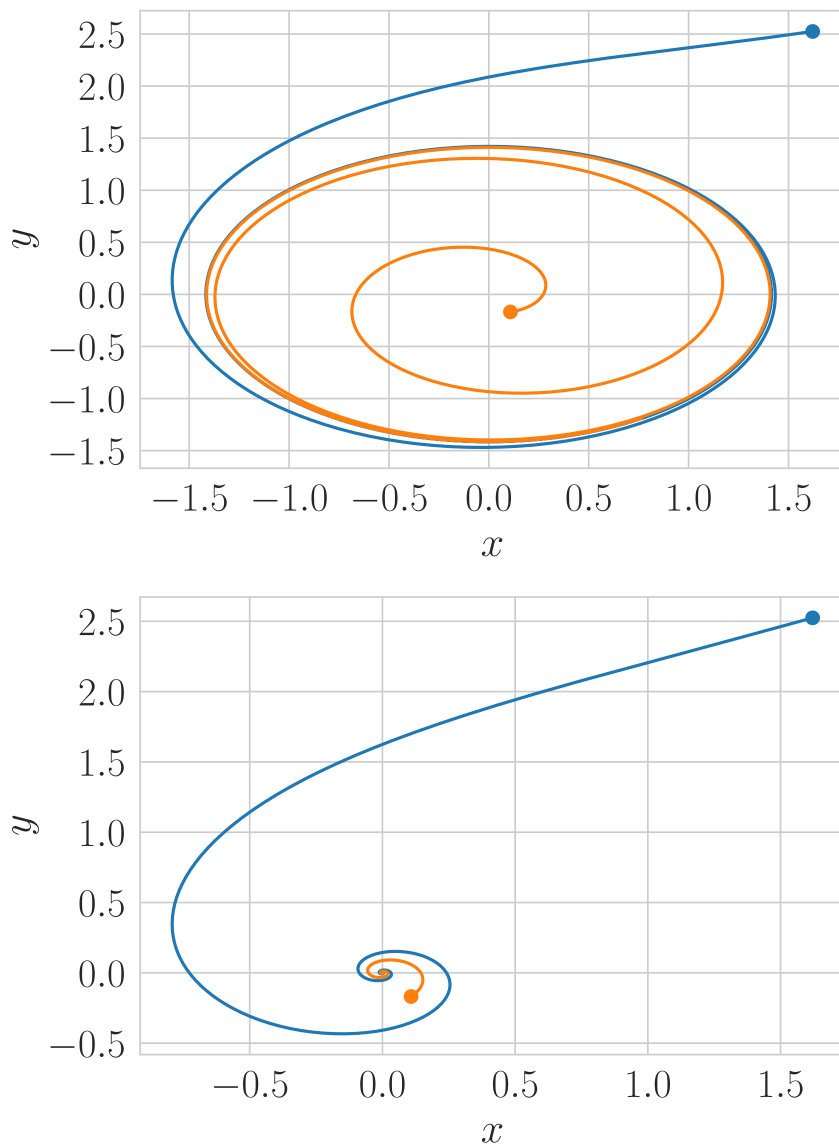

Figure 1: *Top:* Hopf equations for $\mu > \mu_{crit}$ and two different initial conditions marked by the dots. Both trajectories approach the limit cycle with $r = \sqrt{\mu - \mu_{crit}}$
Bottom: Hopf equations for $\mu < \mu_{crit}$ and same initial conditions as in the top figure. This time the limit cycle is unstable and the stable fixed point is $x = y = 0$. The solution is stationary whereas the limit cycle in the top figure is a periodic motion.

2 Exercise: Gaussian distribution

The first moment of a Gaussian distribution

$$g(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-U)^2}{2\sigma^2}} \tag{21}$$

can be calculated as

$$\begin{aligned}\langle u\rangle &= \int_{-\infty}^{\infty} \mathrm{d}u u g(u) = \int_{-\infty}^{\infty} \mathrm{d}u u \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-U)^2}{2\sigma^2}} \\ &= -\int_{-\infty}^{\infty} \mathrm{d}u \frac{\sigma^2}{2\sqrt{2\pi}\sigma} \frac{\mathrm{d}}{\mathrm{d}u} e^{-\frac{(u-U)^2}{2\sigma^2}} + U \int_{-\infty}^{\infty} \mathrm{d}u \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-U)^2}{2\sigma^2}} \\ &= \frac{\sigma^2}{2\sqrt{2\pi}\sigma} \underbrace{e^{-\frac{(u-U)^2}{2\sigma^2}}\Big|_{-\infty}^{\infty}}_{=0} + U \underbrace{\int_{-\infty}^{\infty} \mathrm{d}u \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-U)^2}{2\sigma^2}}}_{=1} = U \end{aligned} \tag{22}$$

Therefore, the mean is just the shift of the Gaussian U. Next, we calculate the second moment

$$\langle u^2\rangle = \int_{-\infty}^{\infty} \mathrm{d}u u^2 g(u) = \int_{-\infty}^{\infty} \mathrm{d}u u^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-U)^2}{2\sigma^2}} \tag{23}$$

It is convenient to introduce the substitution $\tilde{u} = u - U$

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{d}\tilde{u} (\tilde{u} - U)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\tilde{u}^2}{2\sigma^2}} \\ =&\int_{-\infty}^{\infty} \mathrm{d}\tilde{u} \tilde{u}^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\tilde{u}^2}{2\sigma^2}} - 2U \underbrace{\int_{-\infty}^{\infty} \mathrm{d}\tilde{u} \tilde{u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\tilde{u}^2}{2\sigma^2}}}_{=0,\text{ see above}} + U^2 \underbrace{\int_{-\infty}^{\infty} \mathrm{d}\tilde{u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\tilde{u}^2}{2\sigma^2}}}_{=1}\end{aligned}$$

The remaining first integral can be treated as follows

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}\tilde{u} \tilde{u}^2 e^{-\frac{\tilde{u}^2}{2\sigma^2}} &= -\frac{\partial}{\partial\alpha} \int_{-\infty}^{\infty} \mathrm{d}\tilde{u} e^{-\alpha\tilde{u}^2}\Big|_{\alpha=\frac{1}{2\sigma^2}} \\ &= -\frac{\partial}{\partial\alpha}\sqrt{\frac{\pi}{\alpha}}\Big|_{\alpha=\frac{1}{2\sigma^2}} = \frac{\sqrt{\pi}}{2}\alpha^{-\frac{3}{2}}\Big|_{\alpha=\frac{1}{2\sigma^2}} = \frac{\sqrt{\pi}}{2} 2^{\frac{3}{2}}\sigma^3 = \sqrt{2\pi}\sigma^3\end{aligned}$$

Collecting all the terms yields

$$\langle u^2\rangle = \sigma^2 + U^2 \tag{24}$$

Now, the variance can be calculated according to

$$\begin{aligned}\langle (u - \langle u\rangle)^2\rangle &= \langle u^2\rangle - 2\langle u\langle u\rangle\rangle + \langle(\langle u\rangle)^2\rangle \\ &= \underbrace{\langle u^2\rangle}_{\text{Eq.24}} - 2\langle u\rangle\langle u\rangle + \langle u\rangle^2 = \langle u^2\rangle - \underbrace{\langle u\rangle}_{\text{Eq.22}}^2 \\ &= \sigma^2 + U^2 - U^2 = \sigma^2\end{aligned}$$

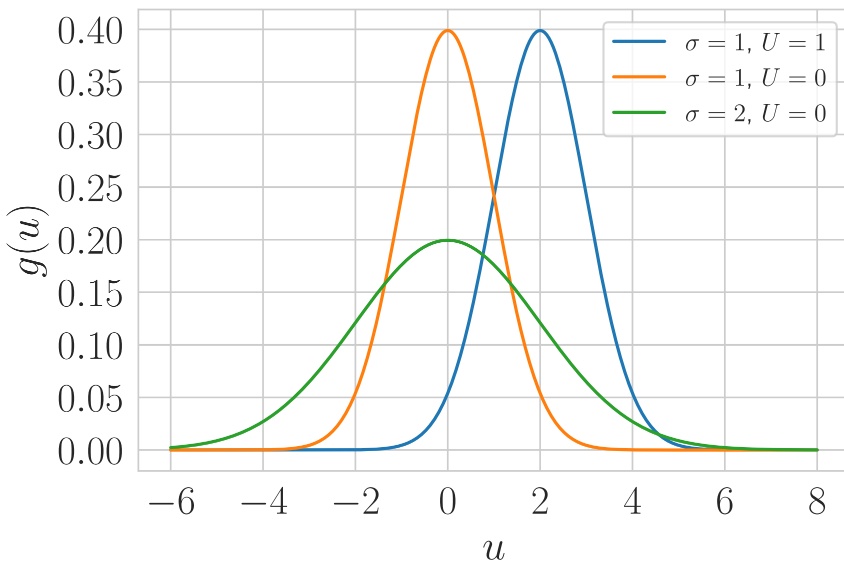

Figure 2: Gaussian distributions for different means and standard deviations.