Exercise 1 Solutions
Fuid Dynamics II SS 2022

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1 Exercise: Brief recapitulation of some vector relations in fluid mechanics

Product rule: $\frac{\partial}{\partial x_j} u_j u_i = u_j \frac{\partial}{\partial x_j} u_i + \underbrace{u_i \frac{\partial}{\partial x_j} u_j}_{0\text{ , incompressibility}} \quad ,$ (1)

$[\mathbf{u} \times \boldsymbol{\omega}]_i = \varepsilon_{ijk} u_j \omega_k = \varepsilon_{ijk} u_j \varepsilon_{klm} \frac{\partial}{\partial x_l} u_m = \varepsilon_{ijk} u_j \omega_k = \underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}} u_j \frac{\partial}{\partial x_l} u_m \ ,$ (2)

$= u_j \frac{\partial}{\partial x_i} u_j - u_j \frac{\partial}{\partial x_j} u_i = \frac{1}{2} \frac{\partial}{\partial x_i} u_j^2 - u_j \frac{\partial}{\partial x_j} u_i = \left[\frac{1}{2}\nabla u^2 - \mathbf{u} \cdot \nabla \mathbf{u}\right]_i$

Insert result from (2): $\nabla \times [\mathbf{u} \cdot \nabla \mathbf{u}] = \nabla \times \left[\frac{1}{2}\nabla u^2 - \mathbf{u} \times \boldsymbol{\omega}\right] \ ,$ (3)

The rotation of a gradient is zero (see also (7)) so only the second term remains

$[\nabla \times [\mathbf{u} \cdot \nabla \mathbf{u}]]_i = -\left[\nabla \times [\mathbf{u} \times \boldsymbol{\omega}]\right]_i = -\varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} u_l \omega_k = -[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}] \frac{\partial}{\partial x_j} u_l \omega_m$

$= -\frac{\partial}{\partial x_j} u_i \omega_j + \frac{\partial}{\partial x_j} u_j \omega_i = [\mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u}]_i$

here, we made use of divergence of a rotation is zero $\frac{\partial}{\partial x_j}\omega_j = 0$, and incompressibility

$-\varepsilon_{ijk}\omega_k = -\varepsilon_{ijk}\varepsilon_{klm} \frac{\partial u_m}{\partial x_l} = \left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right] = 2T_{ij}$ (4)

$[\boldsymbol{\omega} \cdot \nabla \mathbf{u}]_i = \omega_j \frac{\partial}{\partial x_j} u_i = \omega_j [S_{ij} + T_{ij}] \underbrace{=}_{\text{Eq.(4)}} S_{ij}\omega_j \underbrace{-\frac{1}{2}\varepsilon_{ijk}\omega_k\omega_j}_{=0} \ ,$ (5)

For a two-dimensional velocity field $\mathbf{u} = [u_x(x,y,t), u_y(x,y,t)]$ $\boldsymbol{\omega} = \omega(x,y,t)\mathbf{e}_z$ (6)

$S_{ij}\omega_j = S_{i3}\omega_3 = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right]\omega_3 = 0$

Rotation of gradient field is zero, here a short proof: (7)

$[\nabla \times \nabla p]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} p$, however, we can also re-label the summation

$[\nabla \times \nabla p]_i = \varepsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} p = -\varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} p$, same expression as above with the opposite sign hence, it must vanish. (8)

2 Exercise: Biot-Savart law

We apply the Laplace operator on $\mathbf{A}$

$$\Delta_{\mathbf{x}} \mathbf{A}(\mathbf{x}, t) = \int \mathrm{d}\mathbf{x}' \underbrace{\Delta_{\mathbf{x}} G(\mathbf{x} - \mathbf{x}')}_{-\delta(\mathbf{x}-\mathbf{x}')} \boldsymbol{\omega}(\mathbf{x}', t) = -\omega(\mathbf{x}, t) \ . \tag{9}$$

We thus have

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{4\pi} \int \mathrm{d}\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \boldsymbol{\omega}(\mathbf{x}', t) \ . \tag{10}$$

We now calculate the quantity

$$\nabla_{\mathbf{x}} \times \left[\frac{\boldsymbol{\omega}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \right] \tag{11}$$

In component form we obtain

$$\varepsilon_{ijk} \frac{\partial}{\partial x_j} \left[\frac{\omega_k(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \right] = \omega_k(\mathbf{x}', t) \varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \omega_k(\mathbf{x}', t) \varepsilon_{ijk} \frac{x_j - x'_j}{|\mathbf{x} - \mathbf{x}'|^3} \tag{12}$$

We obtain

$$\nabla_{\mathbf{x}} \times \left[\frac{\boldsymbol{\omega}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \right] = -\boldsymbol{\omega}(\mathbf{x}', t) \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{13}$$

and has the following form in three dimensions

$$G(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \ . \tag{14}$$

Since $\mathbf{u} = \nabla \times \mathbf{A}$, we obtain

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{4\pi} \int \mathrm{d}\mathbf{x}' \boldsymbol{\omega}(\mathbf{x}', t) \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \ . \tag{15}$$

A single point vortex $\boldsymbol{\omega} = \Gamma \delta(\mathbf{x} - \mathbf{x}_0) \mathbf{e}_z$, reduces the Biot-Savart law to

$$\mathbf{u}(\mathbf{x}, t) = \frac{\Gamma}{4\pi} \mathbf{e}_z \times \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x_0}|^3} \tag{16}$$

What is the circulation induced by this point vortex?