Exercise 4 Solutions
Fuid Dynamics II SS 2022

11.5.2022

1 Exercise: Reynolds-averaged Navier-Stokes equation

We write the Navier-Stokes equation in component form

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}u_i = -\frac{\partial}{\partial x_i}p + \nu\Delta u_i \tag{1}$$

We can now decompose the velocity field into mean and fluctuating parts as

$$u_i = \bar{u}_i + u_i' \tag{2}$$

where the mean of the fluctuations

$$\left\langle u_i'(\mathbf{x},t)\right\rangle = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\mathrm{d}t' u_i'(\mathbf{x},t') = 0. \tag{3}$$

is assumed to vanish. Inserting Eq. (2) into Eq. (1) yields

$$\frac{\partial}{\partial t}(\bar{u}_i + u_i') + (\bar{u}_j + u_j')\frac{\partial}{\partial x_j}(\bar{u}_i + u_i') = -\frac{\partial}{\partial x_i}(\bar{p} + p') + \nu\Delta(\bar{u}_i + u_i') \tag{4}$$

- Averaging temporal derivative

$$\begin{aligned}
\left\langle\frac{\partial}{\partial t}(\bar{u}_i + u_i')\right\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}dt'\frac{\partial}{\partial t'}\left[\bar{u}_i(\mathbf{x},t') + u_i'(\mathbf{x},t')\right] = \lim_{T\to\infty}\frac{1}{T}\left[\bar{u}_i(\mathbf{x},t') + u_i'(\mathbf{x},t')\right]_t^{t+T} \\
&= \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}dt'\left[\bar{u}_i(\mathbf{x},t') + u_i'(\mathbf{x},t')\right] \\
&= \frac{\partial}{\partial t}\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}dt'\bar{u}_i(\mathbf{x},t') + \frac{\partial}{\partial t}\underbrace{\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}dt' u_i'(\mathbf{x},t')}_{=0,\text{ Eq.(3)}} \\
&= \frac{\partial}{\partial t}\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}dt'\bar{u}_i = \frac{\partial}{\partial t}\left\langle\bar{u}_i\right\rangle = \frac{\partial}{\partial t}\bar{u}_i
\end{aligned} \tag{5}$$

- Averaging nonlinearity

$$\begin{aligned}
\left\langle(\bar{u}_j + u_j')\frac{\partial}{\partial x_j}(\bar{u}_i + u_i')\right\rangle &= \left\langle\bar{u}_j\frac{\partial}{\partial x_j}\bar{u}_i\right\rangle + \left\langle u_j'\frac{\partial}{\partial x_j}\bar{u}_i\right\rangle + \left\langle\bar{u}_j\frac{\partial}{\partial x_j}u_i'\right\rangle + \left\langle u_j'\frac{\partial}{\partial x_j}u_i'\right\rangle \\
&= \bar{u}_j\frac{\partial}{\partial x_j}\bar{u}_i + \underbrace{\left\langle u_j'\right\rangle}_{=0}\frac{\partial}{\partial x_j}\bar{u}_i + \bar{u}_j\frac{\partial}{\partial x_j}\underbrace{\left\langle u_i'\right\rangle}_{=0} + \frac{\partial}{\partial x_j}\left\langle u_j'u_i'\right\rangle
\end{aligned} \tag{6}$$

Here, in the last step we made use of the incompressibility of the fluctuating fields $\frac{\partial}{\partial x_j}u_j' = 0$.

- Averaging pressure

$$\left\langle -\frac{\partial}{\partial x_i}(\bar{p}+p')\right\rangle = -\frac{\partial}{\partial x_i}\left\langle(\bar{p}+p')\right\rangle = -\frac{\partial}{\partial x_i}\langle\bar{p}\rangle - \frac{\partial}{\partial x_i}\underbrace{\langle p'\rangle}_{=0} \tag{7}$$

Hence, we obtain the RANS equation in component form according to

$$\frac{\partial}{\partial t}\bar{u}_i + \bar{u}_j\frac{\partial}{\partial x_j}\bar{u}_i + \frac{\partial}{\partial x_j}\left\langle u_j' u_i'\right\rangle = -\frac{\partial}{\partial x_i}\bar{p} + \nu\Delta\bar{u}_i \tag{8}$$

In order to derive an evolution equation for the kinetic energy of the mean field

$$\bar{E}_{kin}(t) = \frac{1}{2}\int \mathrm{d}\mathbf{x}\ \bar{u}^2(\mathbf{x},t) \tag{9}$$

we scalar multiply Eq. (8) by $\bar{u}_i(\mathbf{x},t)$.

$$\begin{aligned}&\bar{u}_i(\mathbf{x},t)\frac{\partial}{\partial t}\bar{u}_i(\mathbf{x},t) + \bar{u}_i(\mathbf{x},t)\bar{u}_j(\mathbf{x},t)\frac{\partial}{\partial x_j}\bar{u}_i(\mathbf{x},t) + \bar{u}_i(\mathbf{x},t)\frac{\partial}{\partial x_j}\left\langle u_j'(\mathbf{x},t)u_i'(\mathbf{x},t)\right\rangle \\ &= -\bar{u}_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\bar{p}(\mathbf{x},t) + \bar{u}_i(\mathbf{x},t)\nu\Delta\bar{u}_i(\mathbf{x},t)\end{aligned} \tag{10}$$

We thus obtain a balance equation for the kinetic energy density of the mean field $\bar{e}_{kin}(\mathbf{x},t) = \frac{1}{2}\bar{u}^2(\mathbf{x},t)$ as

$$\frac{\partial}{\partial t}\bar{e}_{kin}(\mathbf{x},t) + \nabla\cdot\bar{\mathbf{J}}^{kin}(\mathbf{x},t) = \bar{q}(\mathbf{x},t) \tag{11}$$

where

$$\begin{aligned}\bar{J}_j^{kin}(\mathbf{x},t) =&\bar{u}_j(\mathbf{x},t)\left[\frac{\bar{\mathbf{u}}^2(\mathbf{x},t)}{2} + \bar{p}(\mathbf{x},t)\right] - \frac{\nu}{2}\frac{\partial}{\partial x_j}\bar{\mathbf{u}}^2(\mathbf{x},t) \\ &- \nu\bar{u}_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\bar{u}_j(\mathbf{x},t) + \bar{u}_i(\mathbf{x},t)\left\langle u_i'(\mathbf{x},t)u_j'(\mathbf{x},t)\right\rangle,\end{aligned} \tag{12}$$

$$\bar{q}(\mathbf{x},t) = -\bar{\varepsilon}(\mathbf{x},t) + \left\langle u_i'(\mathbf{x},t)u_j'(\mathbf{x},t)\right\rangle\frac{\partial\bar{u}_i(\mathbf{x},t)}{\partial x_j}\ . \tag{13}$$

where

$$\bar{\varepsilon}(\mathbf{x},t) = \frac{\nu}{2}\left(\frac{\partial\bar{u}_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial\bar{u}_j(\mathbf{x},t)}{\partial x_i}\right)^2 \tag{14}$$

By taking a spatial average $\int \mathrm{d}\mathbf{x}$, we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{E}_{kin}(t) = \bar{Q}(t) \tag{15}$$

where

$$\bar{Q}(t) = \int \mathrm{d}\mathbf{x}\bar{q}(\mathbf{x},t) \tag{16}$$

In comparison to the ordinary evolution equation for the kinetic energy (see first lecture), we thus observe an additional soure term in Eq. (13), which is due to the presence of the Reynolds stress tensor.