Exercise 1
Fuid Dynamics II SS 2022

20.4.2022

1 Exercise: Brief recapitulation of some vector relations in fluid mechanics

Show that the following relations hold

$$u_j \frac{\partial}{\partial x_j} u_i = \frac{\partial}{\partial x_j} u_j u_i \ , \tag{1}$$

$$\mathbf{u} \times \boldsymbol{\omega} = \frac{1}{2} \nabla u^2 - \mathbf{u} \cdot \nabla \mathbf{u} \ , \text{where the vorticity is defined as } \ \boldsymbol{\omega} = \nabla \times \mathbf{u} \ , \tag{2}$$

$$\nabla \times [\mathbf{u} \cdot \nabla \mathbf{u}] = \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} \ , \tag{3}$$

$$T_{ij} = -\frac{1}{2} \varepsilon_{ijk} \omega_k \tag{4}$$

$$[\boldsymbol{\omega} \cdot \nabla \mathbf{u}]_i = \omega_j \frac{\partial}{\partial x_j} u_i = S_{ij} \omega_j \ , \tag{5}$$

$$S_{ij} \omega_j = 0 \ , \ \text{for a two-dimensional velocity field } \mathbf{u} = [u_x(x,y), u_y(x,y)] \tag{6}$$

$$\nabla \times \nabla p = 0 \ . \tag{7}$$

Hints:
$[\nabla \times \mathbf{A}]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} A_k$, where ε_{ijk} denotes the Levi-Civita tensor.
Decomposition of velocity gradient tensor into symmetric and antisymmetric part
$\frac{\partial u_i}{\partial x_j} = S_{ij} + T_{ij} = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] + \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right]$
$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$, where δ_{ij} denotes the Kronecker delta.

2 Exercise: Biot-Savart law

The velocity field can be decomposed as $\mathbf{u} = \nabla \phi + \nabla \times \mathbf{A}$. First, show that

$$-\Delta \mathbf{A} = \boldsymbol{\omega} \ . \tag{8}$$

where Δ denotes the Laplace operator. The Green function of the Laplace operator is defined by

$$\Delta_{\mathbf{x}} G(\mathbf{x} - \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \ , \tag{9}$$

and has the following form in three dimensions

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \, . \tag{10}$$

Show that Eq. (8) can be solved as $\mathbf{A}(\mathbf{x},t) = \int \mathrm{d}\mathbf{x}' G(\mathbf{x}-\mathbf{x}')\boldsymbol{\omega}(\mathbf{x}',t)$, and that the velocity obeys the so-called Biot-Savart law

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{4\pi}\int \mathrm{d}\mathbf{x}' \boldsymbol{\omega}(\mathbf{x}',t) \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \, . \tag{11}$$

Consider the velocity field that is generated from a single point vortex $\boldsymbol{\omega} = \Gamma\delta(\mathbf{x}-\mathbf{x}_0)\mathbf{e}_z$.

Comment:
Here, $\delta(\mathbf{x}-\mathbf{x}')$ stands for the Dirac-delta-distribution which, amongst others, has the following property $\int \mathrm{d}\mathbf{x}' f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') = f(\mathbf{x})$.