Exercise 1 Linear Algebra Recap

Timo Markus
Munther Odeh

1) The scalar product is the same as the dot product and the inner product is a generalization of them.

In Euclidean space $\vec{u}, \vec{v} \in \mathbb{R}^n$ the inner product is given by the dot product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

The outer product, dyadic product and tensor product are equivalent terms for a product of two vectors: $\vec{u}$ of size $m \times 1$ and $\vec{v}$ of size $n \times 1$ which is defined as

$$\vec{u} \otimes \vec{v} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & & & \\ u_m v_1 & u_m v_2 & \dots & u_m v_n \end{pmatrix}$$

or in index notation $(\vec{u} \otimes \vec{v})_{ij} = u_i v_j$

The Hadamard product is defined for matrices of a general size $m \times n$ in which the components are multiplied componentwise. For $\vec{u}, \vec{v}$ of size $n \times 1$ it is defined as

$$\vec{u} \circ \vec{v} = \begin{pmatrix} u_1 v_1 \\ u_2 v_2 \\ \vdots \\ u_n v_n \end{pmatrix}$$ or in index notation

$$(\vec{u} \circ \vec{v})_i = u_i v_i$$

The cross product is defined for vectors in Euclidian space $\mathbb{R}^3$ as

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \text{ or also}$$

$$= \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_i v_j \vec{e}_k$$

with Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & , (i,j,k) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2) \\ -1 & , (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3) \\ 0 & , i=j \text{ or } j=k \text{ or } k=i \end{cases}$$

3) The vector projection of $\vec{u}$ on $\vec{v}$ is the scalar projection of $\vec{u}$ onto $\vec{v}$, $\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$, in direction of $\vec{v}$ as a multiple of the unit vector $\frac{\vec{v}}{\|\vec{v}\|}$.

Therefore: $\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|}$ //

2. Norm and distance

(a) the norm of a vector v we called as the length of a vector v

$\|v\| = \sqrt{v \cdot v}$ or $= \sqrt{v_1^2 + v_2^2 + v_3^2 + \dots}$

$x \in \mathbb{R}^n$

also as a $\|v\|^2 = v \cdot v$

for example we can a ~~a vector~~ x as a vectornorm if its

- $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$
- $\|ax\| = |a|\|x\|$ $\rightarrow a \in \mathbb{R}$ a scalar
- $\|x + y\| \leq \|x\| + \|y\|$ $x, y \in \mathbb{R}^n$

three specific examples are:

$\|x\|_p$, $p > 0$

$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2} = \sqrt{x^T x}$

$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

(b) the distance between $\vec{x}$ and y vectors is the length of the difference between $x - y$

$d(x, y) = \|x - y\|$

$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$

also $= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2}$

$= \|y - x\| = d(y, x)$

BRUNNEN

~~the distance is simply for two~~

the distance is simply a two vector function $d(x,y)$ but the norm is a one vector function $\|x\|$

Also, all norms can create a distance function but not all distance function have a norm.

ⓒ The Frobenius norm is defined for $n \times m$ matrix A by:

$$\|A\|_F = \underbrace{\sqrt{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)}}_{C1} = \underbrace{\sqrt{tr(A^H A)}}_{C2}$$

take C2 $\|A\| = \sqrt{tr(A^H A)}$

$$\|A\|^2 = tr\, A^T A = \sum_i [A^T A]_{ii}$$

$$\|A\|^2 = \sum_i^m \left(\sum_j^n A^T_{ij} A_{ji}\right) = \sum_i^m \sum_j^n A_{ij}^2$$

$$\|A\| = \sqrt{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right.} = C1$$