Exercise 10 Timo Marks 26.06.22

1a) $1 = g(z) + g(-z) \quad , \quad g(z) = \frac{1}{1+e^{-z}}$

$$g(z) + g(-z) = \frac{1}{1+e^{-z}} + \frac{1}{1+e^{z}}$$

$$= \frac{1+e^{z} + 1 + e^{-z}}{(1+e^{-z})(1+e^{z})}$$

$$= \frac{2 + e^{z} + e^{-z}}{1+e^{-z} + e^{z} + e^{z}e^{-z}}$$

$$= \cancel{\frac{2+e^{z}+e^{-z}}{2+e^{z}+e^{-z}}} = 1 \quad //$$

1b) $\vec{z} = \vec{W}\vec{x} = \begin{pmatrix} \vec{w}^T \\ -\vec{w}^T \end{pmatrix} \cdot \vec{x} \Rightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \vec{w}^T\vec{x} \\ -\vec{w}^T\vec{x} \end{pmatrix} \equiv \begin{pmatrix} z \\ -z \end{pmatrix}$

We want to show that we get the same probabilities for $P(\text{class } i \mid \vec{w}, \vec{x}) = g(z_i) = [\sigma(\vec{z})]_i$ by rescaling $\vec{w}$ in the softmax function σ.

$$\sigma(z_1) = \frac{e^{z_1}}{e^{z_1} + e^{z_2}} = \frac{e^{z}}{e^{z} + e^{-z}} = \frac{1}{1+e^{-2z}}$$

$$= \frac{1}{1+e^{-2\vec{w}^T\vec{x}}} \overset{!}{=} \frac{1}{1+e^{-\vec{w}^T\vec{x}}} = g(z)$$

$$\Rightarrow \vec{w}' = \frac{\vec{w}}{2}$$

So with $\bar{w}' = \bar{w} \cdot \frac{1}{2} = \frac{1}{2} \cdot \begin{pmatrix} w^T \\ -w^T \end{pmatrix}$ we get the same probability-results with the sigmoid and softmax function.

2) a) Joint probability $P(x,y)$ means that we want to know the probability that both events x and y occur together.

Marginal probability $P(x)$ means that we want to know the probability that the event x occurs regardless of the other outcome of y. So

$$P(x) = \sum_{y} P(x,y)$$

Conditional probability $P(x|y)$ is the probability of the event x occuring under the condition that the event y already happend.

b) We know: $P(a,b) = P(b,a)$

$$\downarrow \text{ prob. chain rule}$$

$$P(a|b) \cdot P(b) = P(b|a) \cdot P(a)$$

$$\Rightarrow P(a|b) = \frac{P(b|a) \cdot P(a)}{P(b)} \quad \text{- Bayes' eq.}$$

$$P(b|a) = \frac{P(a|b) \cdot P(b)}{P(a)}$$

c) $P(a|b)$ is the posterior probability, that's what we want to know.

P(a) is the prior-probability. The probability of the event a occuring that we know beforehand. And P(b | a) is the likelihood, so the information of our current situation that we combine with the prior-prob. to get the outcome (posterior-prob.).

d)

P(R, S)	R=0	R=1
S=0	0.4	0.2
S=1	0.3	0.1

$$\sum_{R\in\{0,1\}} \sum_{S\in\{0,1\}} P(R,S) = P(0,0) + P(0,1) + P(1,0) + P(1,1)$$
$$= 0.4 + 0.2 + 0.3 + 0.1$$
$$= 1$$

$$P(R=0) = \sum_S P(0,S) = P(0,0) + P(0,1) = 0.4 + 0.3 = 0.7$$

$$P(R=1) = \sum_S P(1,S) = P(1,0) + P(1,1) = 0.2 + 0.1 = 0.3$$

$$P(S=0) = \sum_R P(R,0) = P(0,0) + P(1,0) = 0.4 + 0.2 = 0.6$$

$$P(S=1) = \sum_R P(R,1) = P(0,1) + P(1,1) = 0.3 + 0.1 = 0.4$$

$$P(R|S) = \frac{P(R,S)}{P(S)}$$

$$P(0|0) = \frac{0.4}{0.6} = \frac{2}{3}$$

$$P(0|1) = \frac{0.3}{0.4} = \frac{3}{4}$$

$$P(1|0) = \frac{0.2}{0.6} = \frac{1}{3}$$

$$P(1|1) = \frac{0.1}{0.4} = \frac{1}{4}$$

sums to 1 ✓

sums to 1 ✓

$P(S \mid R) = \frac{P(S,R)}{P(R)}$ $P(0 \mid 0) = \frac{0.4}{0.7} = \frac{4}{7}$

$P(0 \mid 1) = \frac{0.2}{0.3} = \frac{2}{3}$

$P(1 \mid 0) = \frac{0.3}{0.7} = \frac{3}{7}$

$P(1 \mid 1) = \frac{0.1}{0.3} = \frac{1}{3}$

$R=1, S=1$ with Bayes eq.

$$P(R=1 \mid S=1) = \frac{P(S=1 \mid R=1) \cdot P(R=1)}{P(S=1)} = \frac{\frac{1}{3} \cdot 0.3}{0.4} = \frac{1}{4} \quad // \checkmark$$

$$P(S=1 \mid R=1) = \frac{P(R=1 \mid S=1) \cdot P(S=1)}{P(R=1)} = \frac{1/4 \cdot 0.4}{0.3} = \frac{1}{3} \quad // \checkmark$$

3)

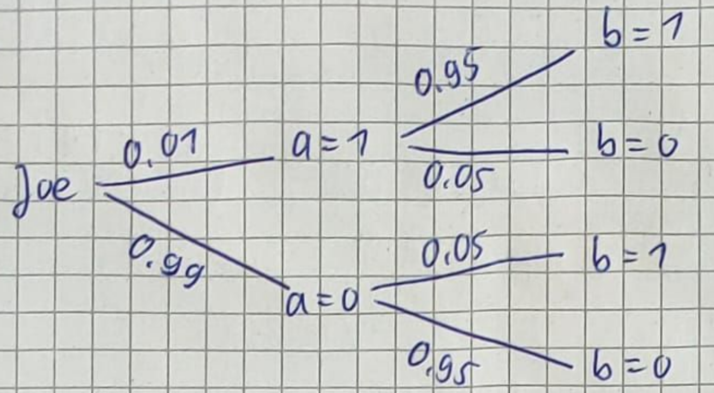
		disease $a=1$	$a=0$
test	$b=1$	0.0095	0.0495
	$b=0$	0.0005	0.9405

$P(a=1) = 0.01$
$P(b=1 \mid a=1) = 0.95$
$P(b=0 \mid a=0) = 0.95$

Test is positive $b=1$, what is the probability that Joe has the disease? $P(a=1 \mid b=1)$

$$P(a \mid b) = \frac{P(b \mid a) \cdot P(a)}{P(b)} = \frac{0.95 \cdot 0.01}{0.059} \approx 16.10\ \%$$

3)



$P(a=1, b=1) = 0.0095$

$P(a=1, b=0) = 0.0005$

$P(a=0, b=1) = 0.0495$

$P(a=0, b=0) = 0.9405$