Exercise_5

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[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd

plt.rcParams["xtick.labelsize"] = 15
plt.rcParams["ytick.labelsize"] = 15
```

1 Task A

[ ]:
```
def generative_model(N, mu, sigma, W):

    data = []
    z_values = []
    for _ in range(N):
        z = np.random.normal(size=1)
        z_values.append(z)
        mean = np.array(W) * z + np.array(mu)
        cov = np.identity(2) * sigma ** 2
        data.append(np.random.multivariate_normal(mean=mean, cov=cov, size=1))

    return np.vstack(np.array(data)), np.array(z_values).flatten()
```

[ ]:
```
mu = [50, 15]
sigma = 3
W = [10, 25]
N = 100
```

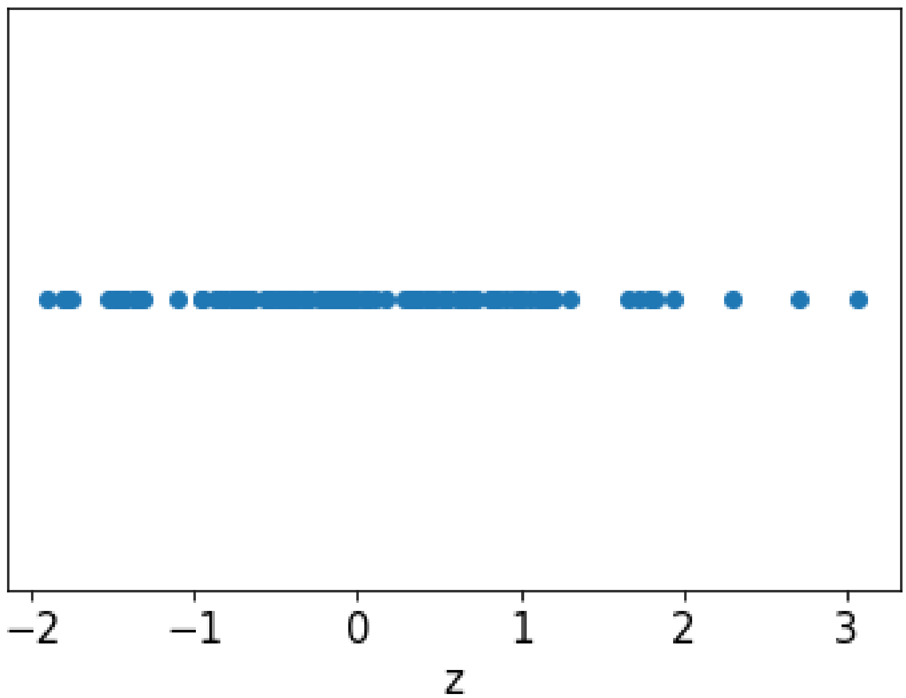
```
generated_data = generative_model(N, mu, sigma, W)
```

2 Task B

[ ]:
```
fig, ax = plt.subplots()

ax.scatter(generated_data[1], np.zeros(generated_data[1].shape))
ax.set_xlabel("z", fontsize=15)
plt.yticks([])
```

[ ]:
```
([], [])
```

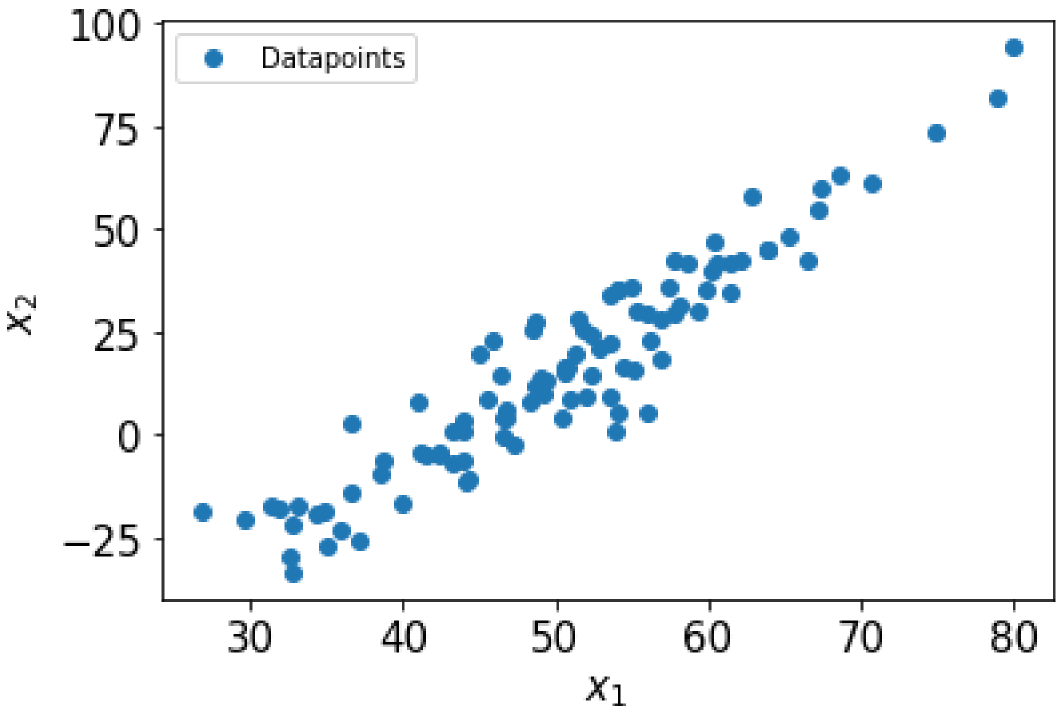


[ ]:
```
fig, ax = plt.subplots()

ax.scatter(generated_data[0][:, 0], generated_data[0][:, 1], label="Datapoints")

ax.legend()
ax.set_xlabel("$x_1$", fontsize=15)
ax.set_ylabel("$x_2$", fontsize=15)
```

[ ]:
```
Text(0, 0.5, '$x_2$')
```



3 Task C

[ ]:
```
class DeterministicPCA:
    """
    Class that implements deterministic PCA.
    """

    def __init__(self, x):
        # data, shape: (number of data points, dimension)
        self.x = x
        # number of data points
        self.N = x.shape[0]

    def run(self):
        """
        Method to run the deterministic PCA process.
        """

        # Calculate mean
        mu = self.x.mean(axis=0)
        x_ = self.x - mu

        # Calculate S
```

```python
        S = 1 / self.N * (x_[:, :, None] * x_[:, None, :]).sum(axis=0)

        # Compute eigenvectors and eigenvalues
        eig_values, eig_vectors = np.linalg.eig(S)

        # Determine eigenvector that belongs to the eigenvalue with the largest␣
↪absolute value
        idx_max_ev = np.argmax(np.abs(eig_values))

        return eig_vectors[:, idx_max_ev], mu
```

[ ]:
```python
det_pca = DeterministicPCA(generated_data[0])
```

[ ]:
```python
eig_vec, comp_mu = det_pca.run()
```

[ ]:
```python
print(comp_mu)
```

```
[50.40632324 15.88282467]
```

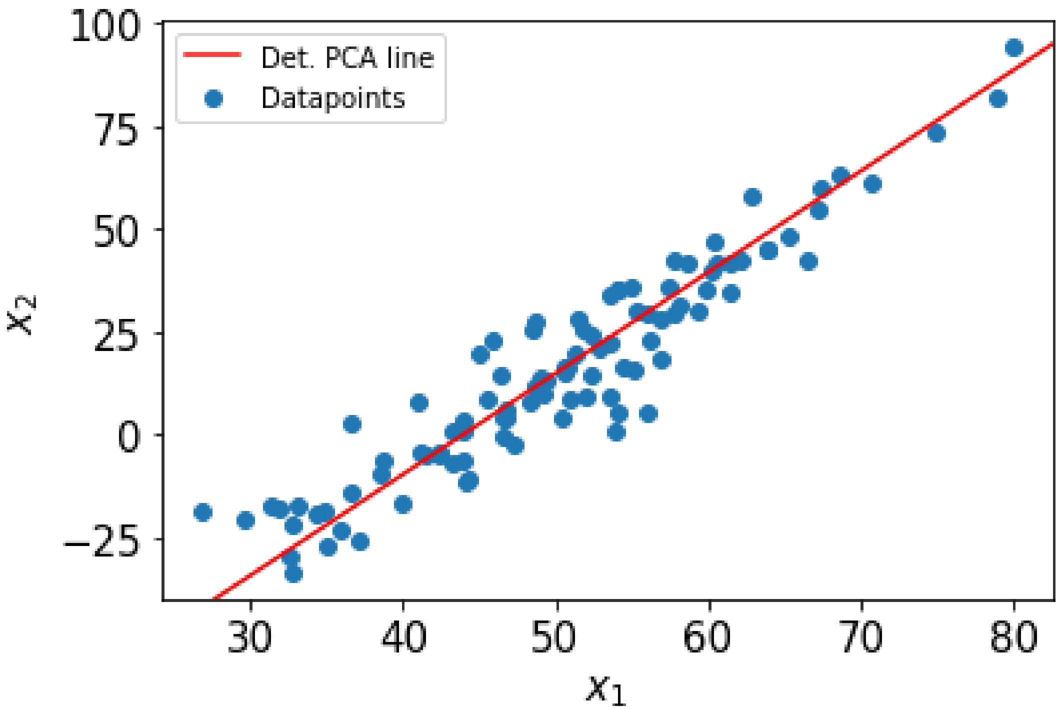
[ ]:
```python
fig, ax = plt.subplots()

p1 = comp_mu
p2 = comp_mu + eig_vec

ax.scatter(generated_data[0][:, 0], generated_data[0][:, 1], label="Datapoints")
ax.axline(p1, p2, color="red", label="Det. PCA line")

ax.legend()
ax.set_xlabel("$x_1$", fontsize=15)
ax.set_ylabel("$x_2$", fontsize=15)
```

[ ]:
```
Text(0, 0.5, '$x_2$')
```



4 Task D

```
[ ]: class ProbabilisticPCA:

        def __init__(self, x, W, sigma, max_steps=100):

            self.x = x
            self.mu = self.x.mean(axis=0)
            self.W = W
            self.sigma = sigma
            self.N = self.x.shape[0] # number of data points
            self.D = self.x.shape[1] # Dimension of observation space
            self.max_steps = max_steps

        def _perform_e_step(self):


            self.M = self.W.transpose() @ self.W + self.sigma ** 2 * np.
    ↪identity(self.W.shape[1])

            # Script equation 139
            self.mom1 = (np.linalg.inv(self.M) @ self.W.transpose()).dot((self.x -␣
    ↪self.mu).transpose())
```

```
        # Script equation 140
        self.mom2 = self.sigma ** 2 * np.linalg.inv(self.M) + self.mom1.
↪transpose()[:, :, None] * self.mom1.transpose()[:, None, :]

    def _perform_m_step(self):

        """
        Performs the m-step of the PCA EM algorithm.
        """

        # Script equation 142
        self.W = ((self.x - self.mu)[:, :, None] * self.mom1.transpose()[:,␣
↪None, :]).sum(axis=0) @ np.linalg.inv(self.mom2.sum(axis=0))

        # Script equation 143
        sigma2 = (
            1 / (self.N * self.D) * (
                ((self.x - self.mu) * (self.x - self.mu)).sum(axis=1) + np.
↪trace((self.W.transpose() @ self.W) @ self.mom2, axis1=1, axis2=2)
                - 2 * (((self.x - self.mu) * (self.W @ self.mom1).transpose()).
↪sum(axis=1))).sum()
        )
        self.sigma = np.sqrt(sigma2)

    def _calculate_log_likelihood(self):

        """
        Calculate the log-likelihood. Script equation 172.
        """

        C = self.W @ self.W.transpose() + self.sigma ** 2 * np.identity(self.W.
↪shape[0])
        S = 1 / self.N * ((self.x - self.mu)[:, :, None] * (self.x - self.mu)[:
↪, None, :]).sum(axis=0)

        log_likelihood = (- self.N / 2 * (
            self.D * np.log(2 * np.pi) + np.log(np.linalg.det(C)) + np.trace(np.
↪linalg.inv(C) @ S)
            )
        )

        return log_likelihood

    def run(self, print_=True, return_=True):

        """
```

```
        Run the EM algorithm.

        parameters:

        print_: Whether to print the log-likelihood in each step
        return_: Whether to return the list of log-likelihoods
        """

        converged = False
        step = 0
        log_likelihood = self._calculate_log_likelihood()
        log_likelihood_list = [log_likelihood]
        mu_parameters = [self.mu]
        sigma_parameters = [self.sigma]
        W_parameters = [self.W]
        if print_:
                print(f"step {step}: {log_likelihood}")

        # The actual algorithm
        while not converged and step < self.max_steps:

            self._perform_e_step()
            self._perform_m_step()

            log_likelihood = self._calculate_log_likelihood()

            # """ Save parameters in array or each iteration """
            log_likelihood_list.append(log_likelihood)
            mu_parameters.append(self.mu)
            sigma_parameters.append(self.sigma)
            W_parameters.append(self.W)
            step += 1

            if print_:
                print(f"step {step}: {log_likelihood}")

        if return_:
            return dict(mu= mu_parameters, sigma=sigma_parameters,␣
↪W=W_parameters, log_likelihood=log_likelihood_list)

    def return_final_parameters(self):

        """
        Return the final parameters for the PCA model.
        """

        return {"mu": self.mu, "sigma": self.sigma, "W": self.W}
```

```
[ ]: prob_pca = ProbabilisticPCA(generated_data[0], np.array([0.0, 10]).reshape(2,␣
      ↪1), 1, max_steps=50)
```

```
[ ]: run_prob_pca = prob_pca.run(print_=False)
```

```
[ ]: prob_pca_opt = prob_pca.return_final_parameters()
     prob_pca_opt
```

```
[ ]: {'mu': array([50.40632324, 15.88282467]),
      'sigma': 3.346054904037953,
      'W': array([[ 9.24571338],
             [22.75449392]])}
```

The parameters of the generative model are quite well approximated. Interesting to notice, the parameter W still changes significantly after 50 iterations (see date frame below). The 'convergence' of sigma is achieved after a few iterations (with respect to the first decimal digit).

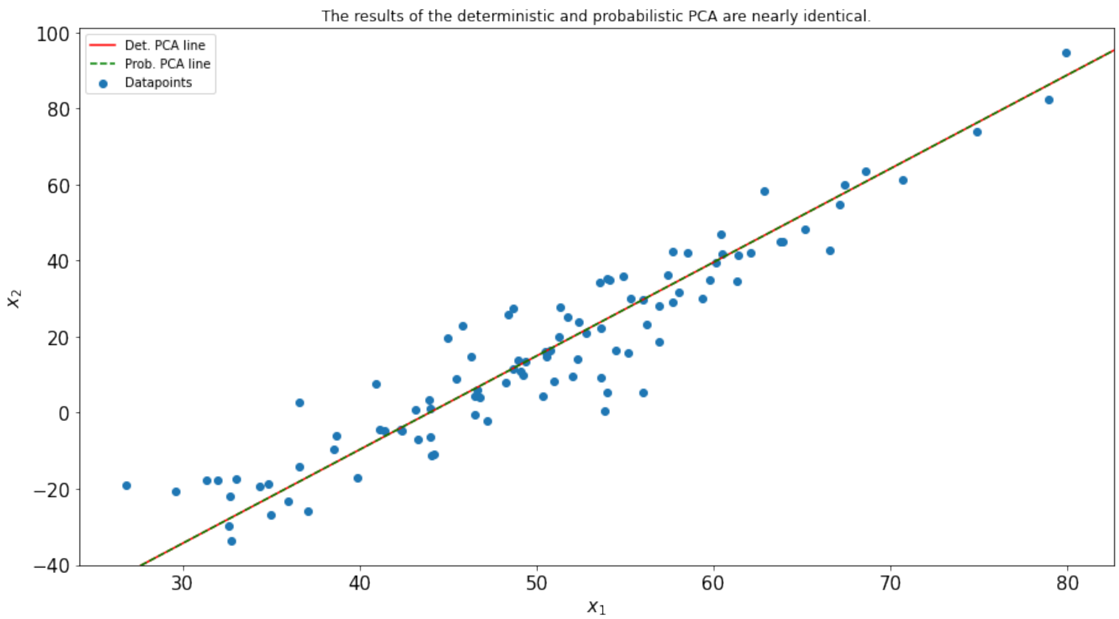
```
[ ]: fig, ax = plt.subplots(figsize=(15, 8))

     p1_det = comp_mu
     p2_det = comp_mu + eig_vec
     p1_prob = prob_pca_opt.get("mu").flatten()
     p2_prob = prob_pca_opt.get("mu").flatten() + prob_pca_opt.get("W").flatten()

     ax.scatter(generated_data[0][:, 0], generated_data[0][:, 1], label="Datapoints")
     ax.axline(p1_det, p2_det, color="red", label="Det. PCA line")
     ax.axline(p1_prob, p2_prob, color="green", label="Prob. PCA line",␣
      ↪linestyle="dashed")

     ax.legend()
     ax.set_xlabel("$x_1$", fontsize=15)
     ax.set_ylabel("$x_2$", fontsize=15)
     ax.set_title("The results of the deterministic and probabilistic PCA are nearly␣
      ↪identical.")
```

```
[ ]: Text(0.5, 1.0, 'The results of the deterministic and probabilistic PCA are
     nearly identical.')
```

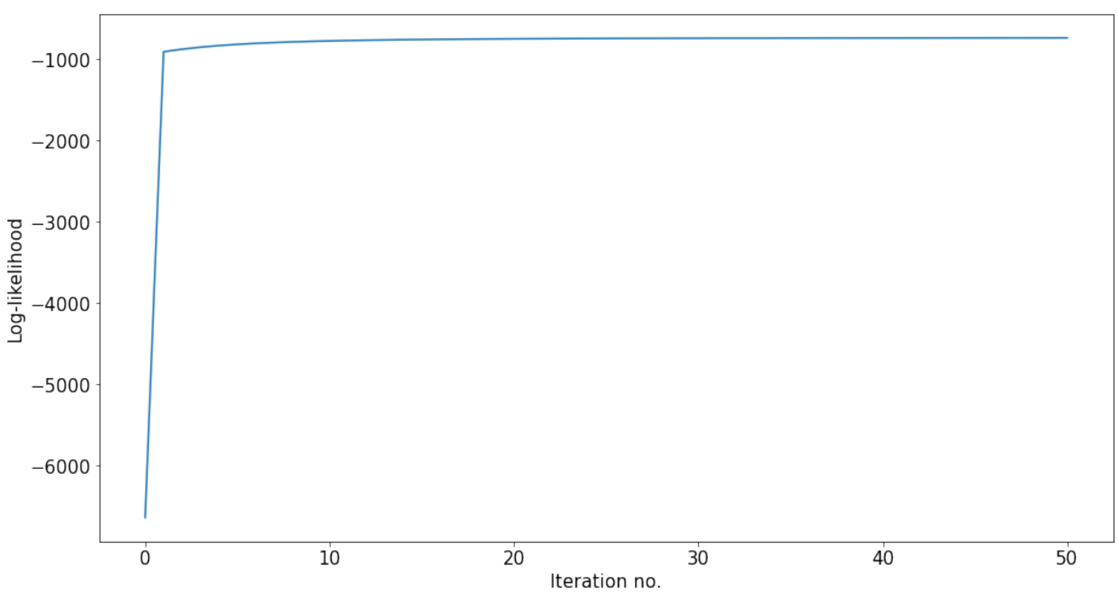


[ ]:
```
fig, ax = plt.subplots(figsize=(15, 8))

log_likelihood = run_prob_pca.get("log_likelihood")

ax.plot(np.arange(len(log_likelihood)), log_likelihood)
ax.set_xlabel("Iteration no.", fontsize=15)
ax.set_ylabel("Log-likelihood", fontsize=15)
```

[ ]: Text(0, 0.5, 'Log-likelihood')



[ ]:
```python
outputs = np.hstack((np.array(run_prob_pca.get("sigma")).reshape(-1, 1), np.
    ↪hstack(run_prob_pca.get("W")).T))
```

[ ]:
```python
pd.DataFrame(outputs, columns=["sigma", "W1", "W2"])
```

[ ]:

	sigma	W1	W2
0	1.000000	0.000000	10.000000
1	2.658207	4.028118	10.084516
2	3.047142	4.289890	10.560406
3	3.238328	4.538377	11.169377
4	3.334150	4.795309	11.801667
5	3.380021	5.045428	12.417230
6	3.399536	5.282345	13.000303
7	3.405518	5.504288	13.546524
8	3.404916	5.711525	14.056553
9	3.401404	5.905112	14.532988
10	3.396856	6.086318	14.978952
11	3.392178	6.256381	15.397490
12	3.387765	6.416418	15.791355
13	3.383764	6.567407	16.162951
14	3.380199	6.710192	16.514358
15	3.377043	6.845503	16.847369
16	3.374250	6.973968	17.163534
17	3.371773	7.096137	17.464201
18	3.369567	7.212488	17.750551
19	3.367591	7.323445	18.023625
20	3.365814	7.429382	18.284346
21	3.364208	7.530635	18.533538
22	3.362750	7.627504	18.771941
23	3.361421	7.720260	19.000221
24	3.360204	7.809149	19.218985
25	3.359086	7.894395	19.428784
26	3.358056	7.976203	19.630121
27	3.357104	8.054761	19.823457
28	3.356222	8.130241	20.009220
29	3.355402	8.202802	20.187800
30	3.354639	8.272593	20.359561
31	3.353927	8.339750	20.524842
32	3.353261	8.404402	20.683954
33	3.352638	8.466666	20.837192
34	3.352053	8.526654	20.984827
35	3.351503	8.584470	21.127116
36	3.350985	8.640211	21.264300
37	3.350497	8.693968	21.396602
38	3.350036	8.745829	21.524235

```
39  3.349601  8.795873  21.647398
40  3.349189  8.844178  21.766280
41  3.348799  8.890815  21.881059
42  3.348429  8.935854  21.991902
43  3.348078  8.979358  22.098970
44  3.347745  9.021389  22.202413
45  3.347428  9.062006  22.302375
46  3.347126  9.101264  22.398990
47  3.346839  9.139214  22.492390
48  3.346565  9.175908  22.582697
49  3.346304  9.211393  22.670027
50  3.346055  9.245713  22.754494
```

[ ]:

```

```

[ ]:

```

```