Spectral Graph Theory

Why are graphs required?

It is a mathematical structures used to model pairwise didactic relationship between objects.

Examples of Graphs

Social Networks

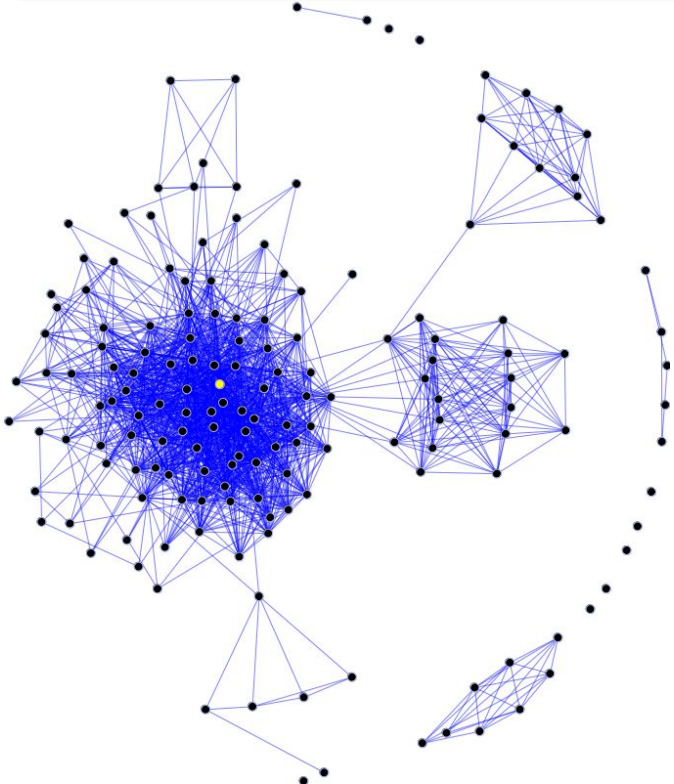
Road Networks

Robotics

Communication networks

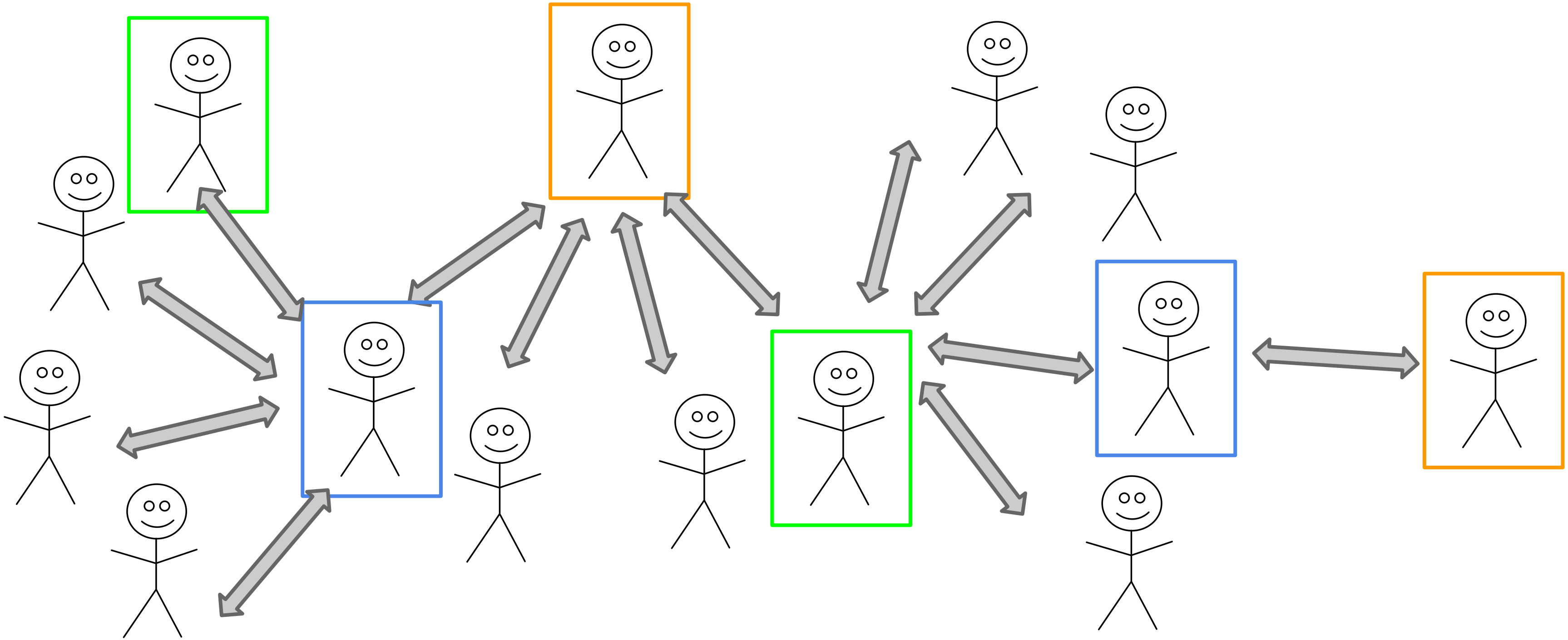
Social Networks

A drawing of a graph in which each person is represented by a dot called node and the friendship relationship is represented by a line called edge



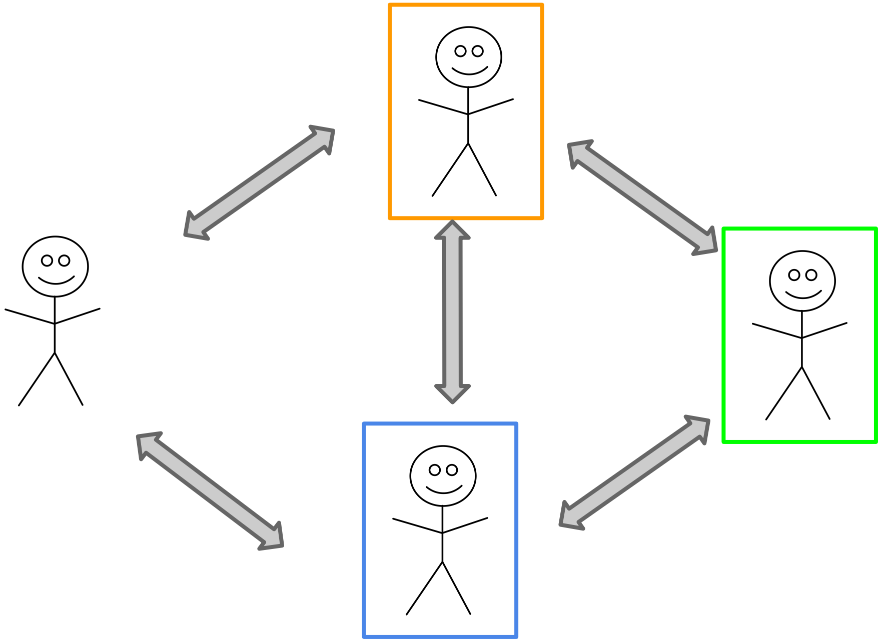
Social Networks

Friends have to be re-represented in trees.



Social Networks

Friends represented only once in graphs



Social Networks

Nodes can be:

Best Friends

Colleagues

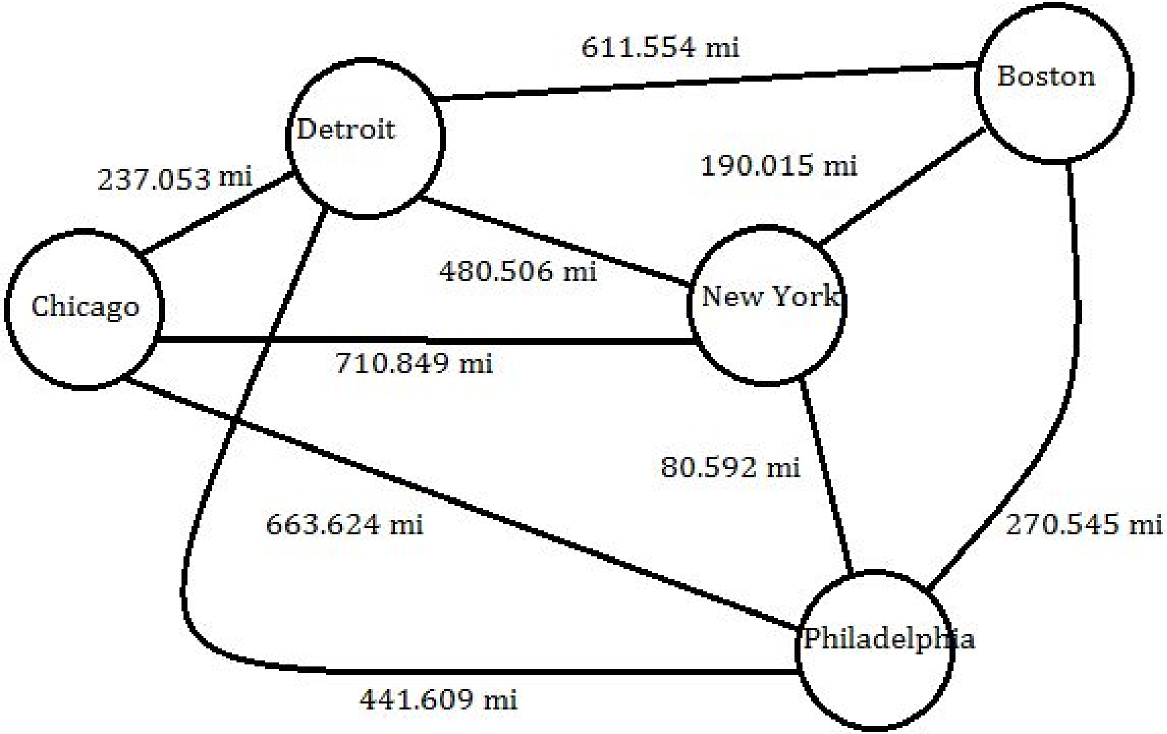
Home town

Edges can be:

Friendship

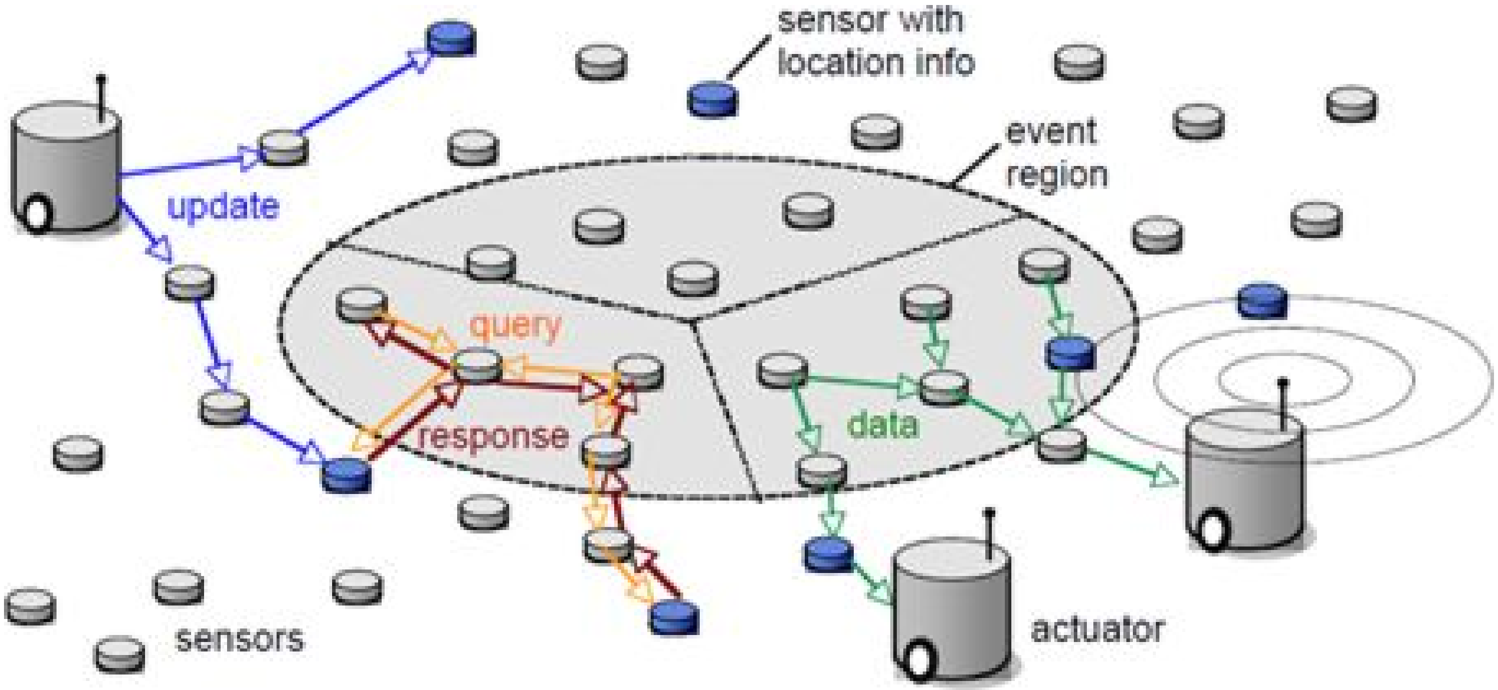
Weighted edges can be interactions between friends.

Road Networks



Robotics

https://raweb.inria.fr/rapportsactivite/RA2014/non-a/uid22.html



Representing a Graph

VARIOUS MATRIX REPRESENTATIONS OF A GRAPH

Adjacency Matrix

Adjacency Matrix represents which vertices (or nodes) of a graph are connected to one another.

Consider a Graph G = {V,E}. And G has *n* number of vertices, then the Adjacency Matrix (A) is a *nxn* Matrix.

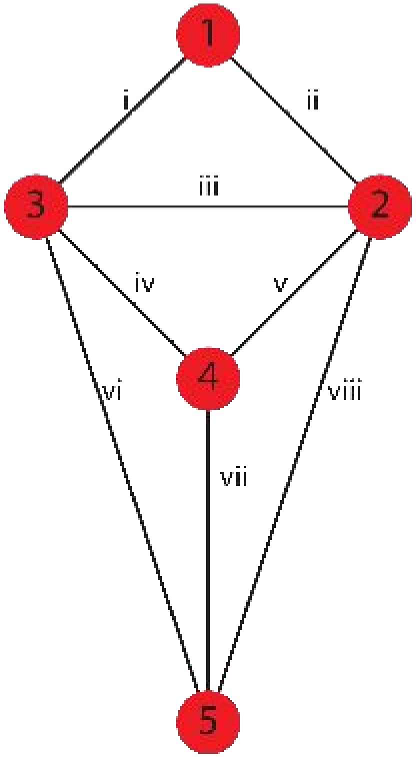
An element a_{ij} in the matrix A, is 1 when the vertices v_i and v_j are connected. For undirected graphs, aji is the same value as aij.

For weighted Graphs, a_{ij} is equal to the weight of the edge between v_i and v_j. It is also usually referred to as w_{ij}.

Adjacency Matrix (Contd...)

The graph is undirected, and each edge weight of the graph is 1.

n = 5; m = 8.



Adjacency Matrix (Contd...)

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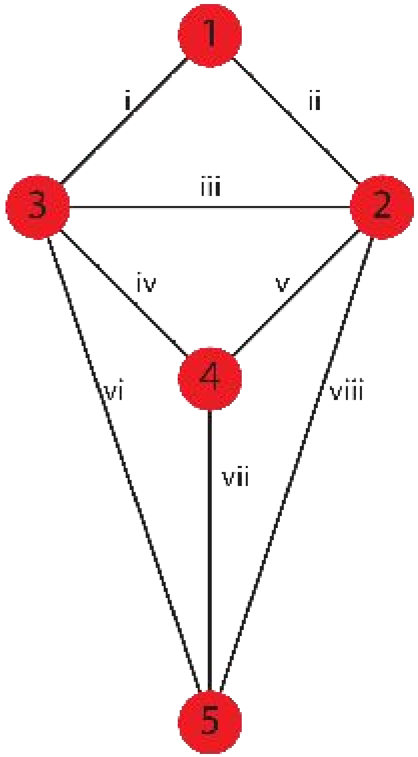


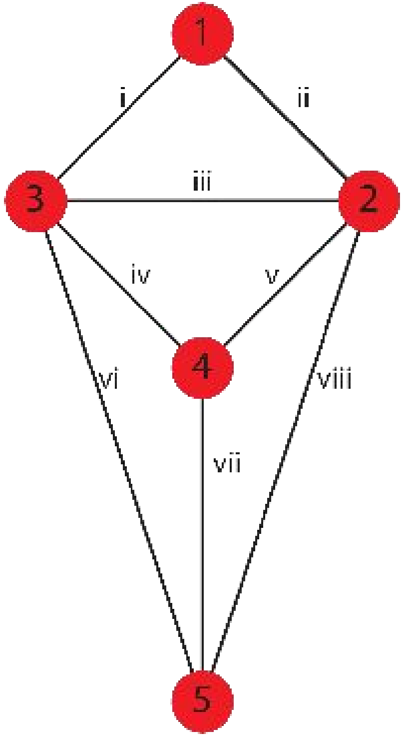
Degree Matrix

It is a diagonal n x n matrix.

Each value dii denotes the degree of the vi.

Degree Matrix (Contd...)

2	0	0	0	0
0	4	0	0	0
0	0	4	0	0
0	0	0	3	0
0	0	0	0	3



Incidence Matrix

In general, it is a matrix showing the relationship between two classes of objects.

The first class is X and the second class is Y, the matrix has one row for each element of X and one column for each element of Y.

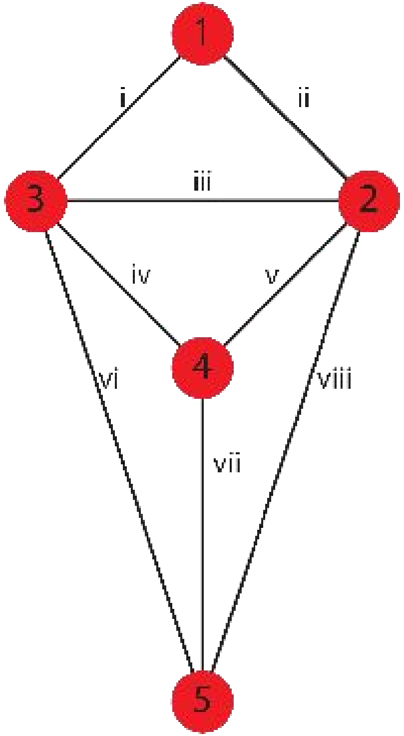
For a graph, the first class is the vertices and the second class is the edges. A n x m matrix.

For an undirected graph, for a given edge (I.e. for a column), two row values are 1, rest are 0.

For a directed graph, the vertex of the origin of edge is -1, and the destination vertex of the edge is 1.

Incidence Matrix (Contd…)

1	1	0	0	0	0	0	0
0	1	1	0	1	0	0	1
1	0	1	1	0	1	0	0
0	0	0	1	1	0	1	0
0	0	0	0	0	1	1	1



Graph Information

d dimensional.

Can be compared to a manifold.

No standard basis vector. (X and Y vectors in images).

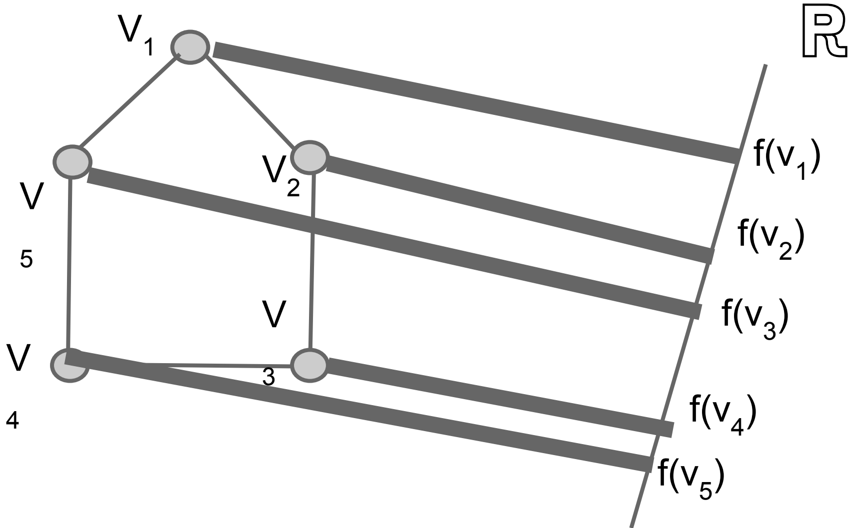
Relation to Manifold

Consider local area

Consider euclidean distance between these distances

Function on a graph

A function on a graph maps the vertices of the graph to real numbers. f : V -> ℝ



$$f = \begin{bmatrix} f(v_1) \\ f(v_2) \\ f(v_3) \\ f(v_4) \\ f(v_5) \end{bmatrix}$$

What is Laplacian?

A differential operator given by the divergence of a function in euclidean space.

Laplacian in Images

0	-1	0
-1	4	-1
0	-1	0

$$\Delta f = (\partial^2 f / \partial x^2) + (\partial^2 f / \partial y^2)$$

$$\Delta f = \nabla^2 f = (\nabla . \nabla) f = \nabla^T \nabla f$$

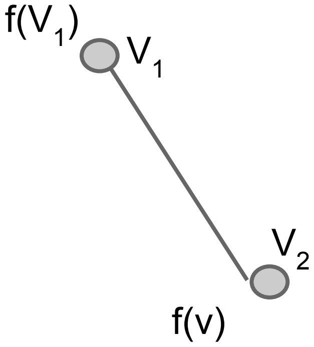
Laplacian of n dimensions

2 Dimensions: $\Delta f = (\partial^2 f / \partial x^2) + (\partial^2 f / \partial y^2)$

3 Dimensions: $\Delta f = (\partial^2 f / \partial x^2) + (\partial^2 f / \partial y^2) + (\partial^2 f / \partial z^2)$

n Dimensions: $\Delta f = (\partial^2 f / \partial^2 x_1) + (\partial^2 f / \partial^2 x_2) + \ldots\ldots + (\partial^2 f / \partial^2 x_n)$

Finding Laplacian for an edge



$$L_{V_1, V_2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

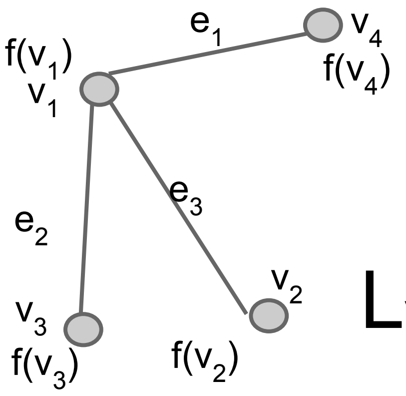
$$L_{V_1, V_2} f = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f(V_1) \\ f(V_2) \end{bmatrix} = \begin{bmatrix} f(V_1) - f(V_2) \\ f(V_2) - f(V_1) \end{bmatrix}$$

$$f^T L_{V_1, V_2} f = \begin{bmatrix} f(V_1) & f(V_2) \end{bmatrix} \begin{bmatrix} f(V_1) - f(V_2) \\ f(V_2) - f(V_1) \end{bmatrix} = (f(V_1) - f(V_2))^2$$

The laplacian of an edge is the eigenvector **f,**

s.t, $L_{V_1, V_2} f = (f(V_1) - f(V_2))^2$

Finding Laplacian for an edge



$$L_{v_1, v_2} = \begin{bmatrix} 1 & -w \\ -1 & 1 \end{bmatrix}$$

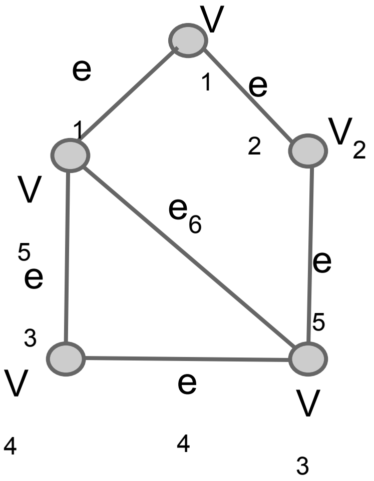
$$L_{v_1, v_2}\, f = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f(v_1) \\ f(v_2) \end{bmatrix} = \begin{bmatrix} f(v_1) - f(v_2) \\ f(v_2) - f(v_1) \end{bmatrix}$$

$$f^T L_{v_1, v_2}\, f = \begin{bmatrix} f(1) & f(2) \end{bmatrix} \begin{bmatrix} f(1) - f(2) \\ f(2) - f(1) \end{bmatrix} = (\ f(1) - f(2)\ )^2$$

The laplacian of an edge is the eigenvector **f,**

s.t, $L_{u,v}f = (\ f(1) - f(2)\ )^2$

Finding Laplacian of Graphs



$L =$

2	-1	0	0	-1
-1	2	-1	0	0
0	-1	2	-1	0
0	0	-1	2	-1
-1	0	-1	-1	3

$\nabla =$

	V_1	V_1	V_3	V_4	V_5
e_1	1	0	0	0	-1
e_2	1	-1	0	0	0
e_3	0	0	0	1	-1
e_4	0	0	1	-1	0
e_5	0	1	-1	0	0
e_6	0	0	1	0	-1

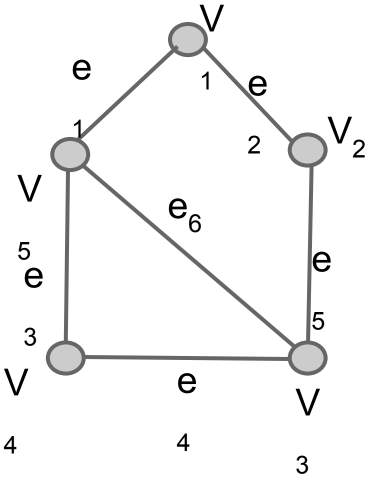
$$f\,L\,f = f\ \nabla^T \nabla\ f$$

$$\nabla^T \nabla = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

$$L = \nabla^T \nabla$$

$$f\, L\, f = f\, \nabla^T \nabla\, f$$

$$\nabla^T \nabla f = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} f(V_1) \\ f(V_2) \\ f(V_3) \\ f(V_4) \\ f(V_5) \end{bmatrix} = \begin{bmatrix} 2f(V_1) - f(V_2) - f(V_5) \\ 2f(V_2) - f(V_1) - f(V_3) \\ 2f(V_3) - f(V_2) - f(V_4) \\ 2f(V_4) - f(V_3) - f(V_5) \\ 3f(V_5) - f(V_4) - f(V_1) \end{bmatrix}$$

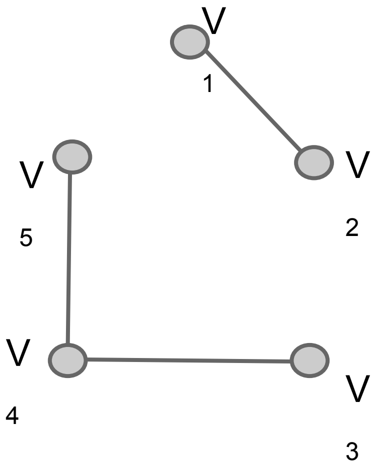


$$f \nabla^T \nabla f = \begin{bmatrix} [\, 2f(V_1) - f(V_2) - f(V_5) \,]^2 \\ [\, 2f(V_2) - f(V_1) - f(V_3) \,]^2 \\ [2f(V_3) - f(V_2) - f(V_4) \,]^2 \\ [\, 2f(V_4) - f(V_3) - f(V_5) \,]^2 \\ [\, 3f(V_5) - f(V_4) - f(V_1) \,]^2 \end{bmatrix}$$

Properties of the Laplacian

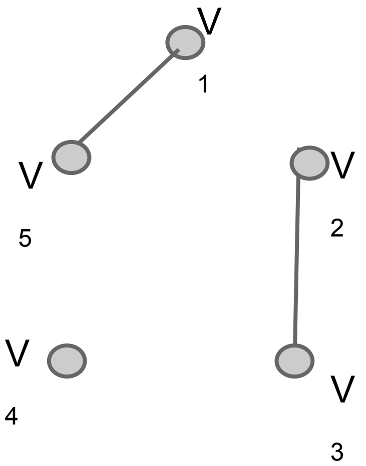
Edge Union - If G and H are 2 graphs on the same vertex set with disjoint edge sets,

$$L_{G \cup H} = L_G + L_H \text{ (Additivity)}$$



1	-1	0	0	0
-1	1	0	0	0
0	0	1	-1	0
0	0	-1	2	-1
0	0	0	-1	1

G



1	0	0	0	-1
0	1	-1	0	0
0	-1	1	0	0
0	0	0	0	0
-1	0	0	0	1

H

Properties of the Laplacian

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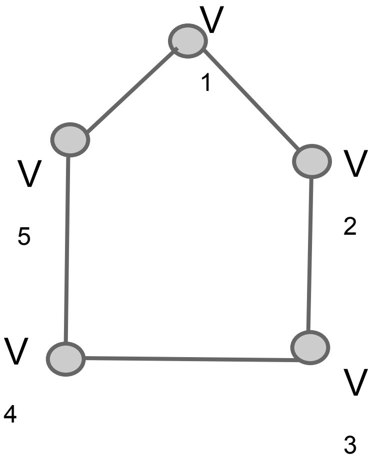


1	-1	0	0	0
-1	1	0	0	0
0	0	1	-1	0
0	0	-1	2	-1
0	0	0	-1	1

+

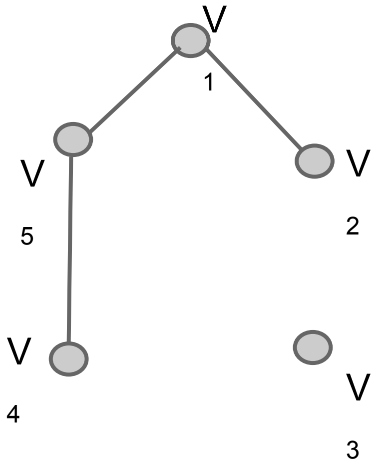
1	0	0	0	-1
0	1	-1	0	0
0	-1	1	0	0
0	0	0	0	0
-1	0	0	0	1

=

2	-1	0	0	-1
-1	2	-1	0	0
0	-1	2	-1	0
0	0	-1	2	-1
-1	0	0	-1	2

Properties of the Laplacian

Disjoint Union - If a vertex i∈G is isolated, then the corresponding row and column of the laplacian are 0.



2	-1	0	0	-1
-1	2	0	0	0
0	0	0	0	0
0	0	0	2	-1
-1	0	0	-1	2

Properties of the Laplacian

Isolated Vertex - The properties of Isolated Vertices and Disjoint Union together implies that the disjoint union of G and H is the sum of L_G and L_H

$$L_{G \cup H} = L_G + L_H = \begin{bmatrix} L_G & 0 \\ 0 & L_H \end{bmatrix}$$

Properties of the Laplacian

Disjoint Union Spectrum - If L_G has eigenvectors $v_1, v_2, \ldots\ldots, v_n$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and L_H has eigenvectors $w_1, w_2, \ldots\ldots, w_n$ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ then $L_{G \cup H}$ has eigenvectors

$v_1 + \mathbf{0}$, ……, $v_n + \mathbf{0}$, $\mathbf{0} + w_1$, ……., $\mathbf{0} + w_n$

and the corresponding eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n$

Fundamental Theorem of Spectral Graph Theory

$L = I - D^{-1/2}AD^{-½}$

L is symmetric, all its eigenvalues are real.

A = Adjacency matrix

D the diagonal martix of degree of each vertex.

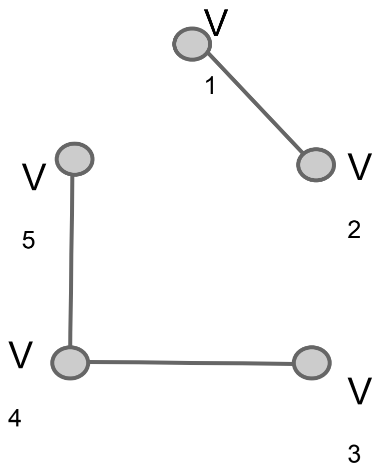
Fundamental Theorem of Spectral Graph Theory

Let $0 = \lambda_1 <= \lambda_2 <= \ldots\ldots. \lambda_n$ be the eigenvalues of L in sorted order with multiplicities. Then,

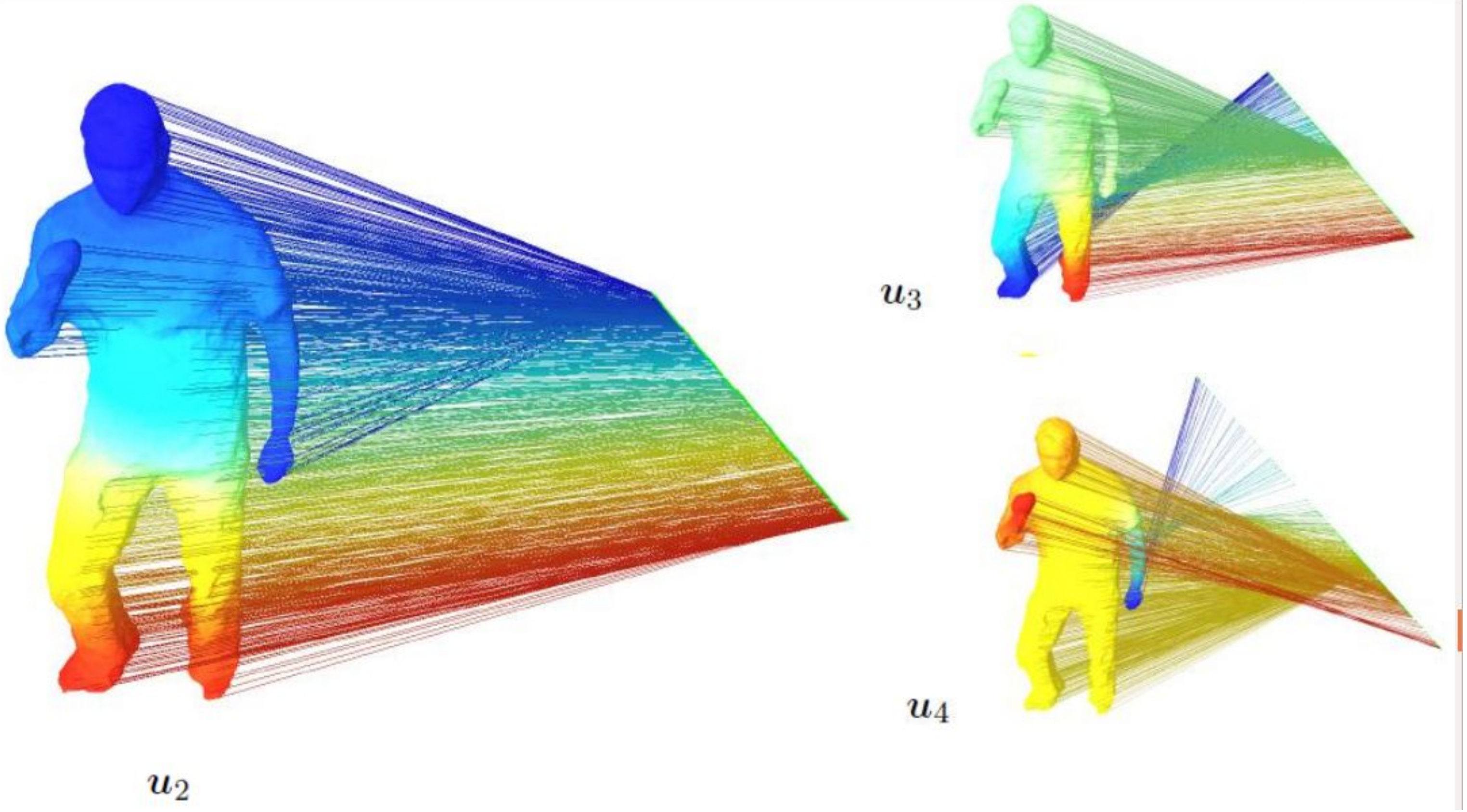
$\lambda_1 = 0$ and $\lambda_n <= 2$

$\lambda_k = 0$ iff G has >= k connected components

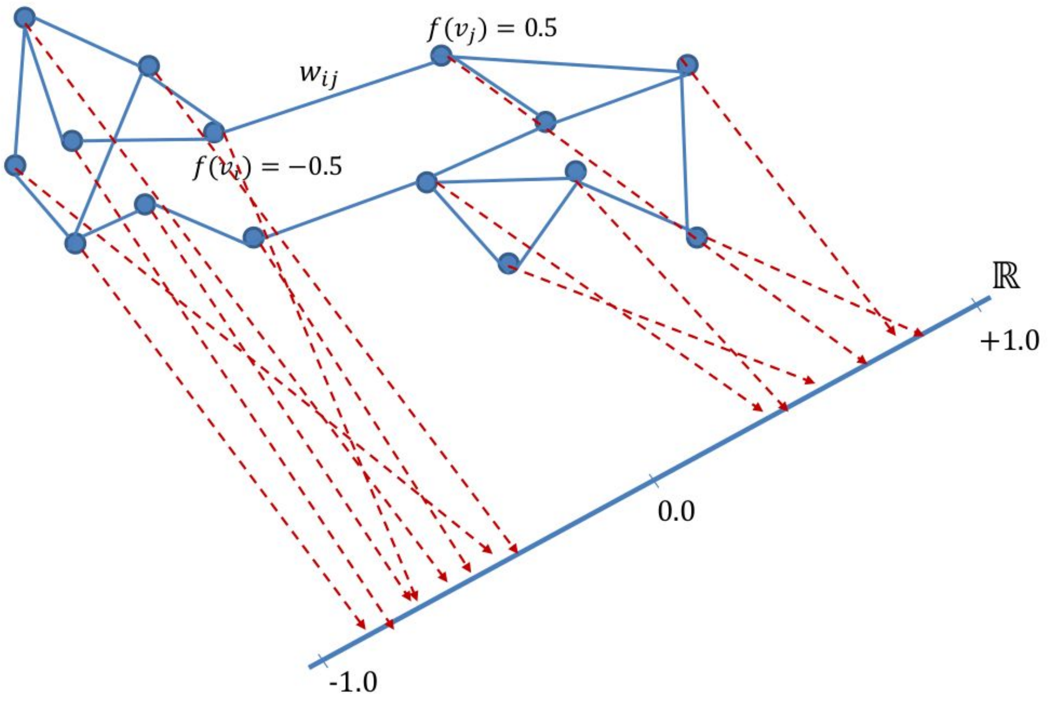
$\lambda_n = 2$ iff G has a bipartitite connected component.



$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 > 0..$



The Laplacian allows a natural link between discrete representations (graphs), and continuous representations, such as metric spaces and manifolds.



Laplacian embedding consists in representing the vertices of a graph in the space spanned by the smallest eigenvectors of the Laplacian.

The Fiedler vector of the Laplacian

The first non-null eigenvalue λ_{k+1} is called the Fiedler value.

The corresponding eigenvector u_{k+1} is called the Fiedler vector.

The multiplicity of the Fiedler eigenvalue depends on the graph's structure and it is difficult to analyse.

The Fiedler value is the algebraic connectivity of a graph, the further from 0, the more connected.