ECEn 671: Mathematics of Signals and Systems

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Section 1

Rank Reducing Approximations

Rank Reducing Approximations

Problem: Given A with $rank(A) = r$, find a matrix B that is "close" to A in some sense, but with lower rank.

Theorem (Moon Theorem 7.2)

Given $A \in \mathbb{C}^{m \times n}$ with $rank(A) = r$, then

$$A = U_1 \Sigma_1 V_1^H = \sum_{j=1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^H$$

Let $k < r$ and let

$$A_k \triangleq \sum_{j=1}^{k} \sigma_j \mathbf{u}_j \mathbf{v}_j^H \qquad (rank(A_k) = k)$$

Then $\|A - A_k\|_2 = \sigma_{k+1}$ and A_k is the nearest rank k matrix to A, in the matrix 2-norm, i.e.

$$A_k = \arg \min_{rank(B)=k} \|A - B\|_2 .$$

Rank Reducing Approximations, Proof

Remark: In the previous section, we saw that we could reduce the rank by zeroing small singular values. This theorem shows that this is the best way to reduce the rank in the matrix 2-norm sense.

Proof.

$$\|A - A_k\|_2 = \left\| \sum_{j=k+1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^H \right\|_2$$
$$= \max_{\|x\|=1} \left\| \sum_{j=k+1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^H x \right\|_2$$

Note that we maximize by letting $x^* = \mathbf{v}_{k+1}$ since any other x will be a linear combination of smaller singular values.

Rank Reducing Approximations, Proof

Therefore

$$\|A - A_k\| = \|\sigma_{k+1}\mathbf{u}_{k+1}\| = \sigma_{k+1}$$

since $\|\mathbf{u}_{k+1}\| = 1$.

Because $\|A - A_k\|_2 = \sigma_{k+1}$ we know that

$$\min_{rank(B)=k} \|A - B\| \leq \sigma_{k+1}.$$

To complete the proof we need to show that

$$\sigma_{k+1} \leq \min_{rank(B)=k} \|A - B\|.$$

Rank Reducing Approximations, Proof

Let B be any rank-k matrix. Then

$$rank(B) = k \quad \Longrightarrow \quad dim(\mathcal{N}(B)) = n - k.$$

Therefore, there exists $\{x_{k+1}, \ldots, x_n\}$ such that

$$\mathcal{N}(B) = span\{x_{k+1}, \ldots, x_n\}$$

The columns of V_1 are $\{\mathbf{v}_1 \ldots \mathbf{v}_k, \mathbf{v}_{k+1} \ldots \mathbf{v}_r\}$ where $\mathbf{v}_i \in \mathbb{C}^n$. Let

$$z \in \underbrace{span\underbrace{\{x_{k+1}, \ldots, x_n\}}_{dim=n-k} \quad \cap \quad span\underbrace{\{\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}\}}_{dim=k+1}}_{\text{dimension at least one since there are } n+1 \text{ vectors}}$$

Therefore $z \neq 0$.

Rank Reducing Approximations, Proof

Let

$$\|A - B\|_2 = \max_{\|x\| \neq 0} \frac{\|(A - B)x\|}{\|x\|} \leq \frac{\|(A - B)z\|}{\|z\|}$$
$$= \frac{\|Ax\|}{\|z\|} \text{ since } z \in \mathcal{N}(B)$$
$$= \frac{\left\|\sum_{j=1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^H z\right\|}{\|z\|} = \frac{\left\|\sum_{j=1}^{k+1} \sigma_j \mathbf{u}_j \mathbf{v}_j^H z\right\|}{\|z\|}$$

Since $z \perp span\{\mathbf{v}_{k+2}, \ldots, \mathbf{v}_r\}$ the smallest we can make the numerator is σ_{k+1} by a choice of $z = \mathbf{v}_{k+1}$. So

$$\|A - B\|_2 \geq \frac{\|\sigma_{k+1}\mathbf{v}_{k+1}\|}{\|\mathbf{v}_{k+1}\|} = \sigma_{k+1}$$

for any B such that $rank(B) = k$ so that

$$\min_{rank(B)=k} \|A - B\|_2 \geq \sigma_{k+1}.$$