ECEn 671: Mathematics of Signals and Systems

Randal W. Beard

Brigham Young University

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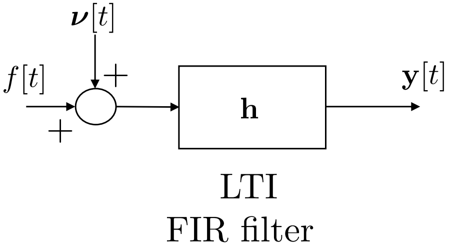
Section 1

Eigenfilters

Eigenfilters for Random Signals

Problem: Given



where ν is white noise with variance σ^2, and f is a stationary, zero-mean random process.

Find **h** to maximize the signal-to-noise ratio.

Eigenfilters for Random Signals

Let

$$\mathbf{f}(t) = \begin{pmatrix} f(t) \\ f(t-1) \\ \vdots \\ f(t-(m-1)) \end{pmatrix}$$

then

$$y(t) = \mathbf{h}^H \mathbf{f}(t).$$

The output power due to the signal **f** is

$$\begin{aligned} P_0 &= E|y(t)|^2 = E|\mathbf{h}^H \mathbf{f}(t)|^2 = E\{\mathbf{h}^H \mathbf{f}(t) \mathbf{h}^H \mathbf{f}(t)\} \\ &= E\{\mathbf{h}^H \mathbf{f}(t) \mathbf{f}^H(t) \mathbf{h}\} = \mathbf{h}^H E\{\mathbf{f}(t)\mathbf{f}^H(t)\}\mathbf{h} \\ &= \mathbf{h}^H R \mathbf{h} \end{aligned}$$

where $R = E\{\mathbf{f}(t)\mathbf{f}^H(t)\}$

Eigenfilters for Random Signals

Let

$$\boldsymbol{\nu}(t) = \begin{pmatrix} v(t) \\ v(t-1) \\ \vdots \\ v(t-m+1) \end{pmatrix}$$

Then the output due to the noise is

$$\mathbf{h}\boldsymbol{\nu}(t)$$

and the average noise power is

$$N_0 = E\{\mathbf{h}^H \boldsymbol{\nu}(t) \boldsymbol{\nu}^H(t) \mathbf{h}\} = \sigma^2 \mathbf{h}^H \mathbf{h}$$

Eigenfilters for Random Signals

The signal-to-noise ratio is

$$\begin{aligned} SNR &= \frac{P_0}{N_0} \\ &= \frac{1}{\sigma^2} \cdot \underbrace{\frac{\mathbf{h}^H R \mathbf{h}}{\sigma^2 \mathbf{h}^H \mathbf{h}}}_{\text{Rayleigh quotient}} \end{aligned}$$

Therefore

$$SNR_{max} = \frac{\lambda_1}{\sigma^2}$$

where λ_1 is the largest eigenvalue of R and $\mathbf{h} = q_1$ the largest eigenvector of R.