ECEn 671: Mathematics of Signals and Systems
Moon: Chapter 2.

Randal W. Beard

Brigham Young University

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Section 1

Metric Spaces

Spaces

- One of the objectives of this course is to develop tools that work in a wide variety of settings.
- We will mostly focus on finite dimensional Hilbert spaces, which include:
  - $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{C}^{m\times n}$,
  - the set of all functions with finite integral,
  - the set of all finitely summable sequences,
  - binary vectors, binary sequences.
- But does not include important objects like
  - rotations matrices, quaternions, homogeneous transformations.
- To make things clear, we will develop the theory systematically in the following order:
  1. Metric space
  2. Norm space / Banach space
  3. Inner product space / Hilbert space

Metric Spaces

Definition (Metric Space)

A metric space is a pair $(\mathbb{X}, d)$ where $\mathbb{X}$ is a set and

$$d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$$

is a metric defined over $\mathbb{X}$.

A metric is a measure of distance between elements in a set.

Metric Spaces

Definition (Metric)

Let $\mathbb{X}$ be a set. Then $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is a metric if:

$$
\begin{array}{ll}
(M1) & d(x,y) = d(y,x), \quad \forall x,y \in \mathbb{X} \\
(M2) & d(x,y) \geq 0, \quad \forall x,y \in \mathbb{X} \\
(M3) & d(x,y) = 0, \quad \iff x = y \\
(M4) & d(x,z) \leq d(x,y) + d(y,z), \quad \forall x,y,z \in \mathbb{X}
\end{array}
$$

(M4) is called the Triangle inequality.

Examples of Metric Spaces

Example (E1)

$(\mathbb{R}, d)$ where $d(x, y) \overset{\triangle}{=} |x - y|$ is a metric space.
Note that

- (M1) $|x - y| = |y - x|$, $\forall x, y \in \mathbb{R}$.
- (M2) $|x - y| \geq 0$, $\forall x, y \in \mathbb{R}$.
- (M3) $|x - y| = 0$, if $x = y$.
- (M4) $|x - z| \leq |x - y| + |y - z|$ $\forall x, y, z \in \mathbb{R}$.

To convince yourself (M4), draw a picture. Note, a picture is not a proof.

Examples of Metric Spaces

Example (E2)

$(\mathbb{R}^n, d)$ where

$$d(x, y) \triangleq \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

where $x = (x_1, \ldots, x_n)^\top$ and $y = (y_1, \ldots, y_n)^\top$.
Verify that $d(\cdot, \cdot)$ satisfies (M1)-(M4).

Examples of Metric Spaces

Example (E3)

$(\mathbb{R}^n, d)$ where

$$d(x,y) \triangleq \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

where $p \geq 1$.

For general $p \geq 1$, the triangle inequality is a nontrivial and famous results.

Examples of Metric Spaces

Example (E4 bounded sequence space)

Let ℓ^∞ be the set of all sequences of complex numbers where each number is bounded, i.e.,

$$x = (x_1, x_2, x_3, \dots) \in \ell$$

if $x_i \in \mathbb{C}$ and $|x_i| < \infty$.
(ℓ, d) is a metric space where

$$d(x, y) = \sup_{j \in \mathbb{N}} |x_i - y_i| .$$

Verify (M1)-(M4).

Examples of Metric Spaces

Example (E5 continuous function space)

- Let $C[a,b]$ be the set of all continuous functions on $[a,b]$, i.e., i.e. $x \in C[a,b] \Rightarrow x(t)$ is continuous on $[a,b]$.
  Let
  $$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$
  then $(C[a,b], d)$ is a metric space.
- This is a different perspective than calculus. In calculus you consider one function at a time. In this class, a function is one point in a larger metric space.

Examples of Metric Spaces

Example (E6 discrete metric space)

Let $\mathbb{X}$ be any set, e.g., the set of three legged dogs, and let

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}.$$

Then $(\mathbb{X}, d)$ is a metric space since

- (M1) $d(x, y) = d(y, x)$, $\forall x, y \in \mathbb{X}$.
- (M2) $d(x, y) \geq 0$, $\forall x, y \in \mathbb{X}$.
- (M3) $d(x, y) = 0$, if $x = y$.
- (M4) $d(x, z) \leq d(x, y) + d(y, z)$ $\forall x, y, z \in \mathbb{X}$.

Examples of Metric Spaces

Example (E7 binary vector space)

Let $\mathbb{X} = \{0,1\}^n$ be the set of binary vectors, i.e $x \in \mathbb{X} \Rightarrow x = (x_1, x_2, \ldots, x_n)$ where $x_i \in \{0,1\}$. Let

$$d(x,y) = \sum_{i=1}^{n} h(x_i - y_i)$$

where

$$h(w) = \begin{cases} 1 & \text{if} \qquad w \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

h is called the hamming distance, and simply counts the number of elements in x and y that are different.

Metric Spaces / Norm Spaces / Inner Product Spaces

- Later in the chapter, we will later introduce the concepts of a norm and a norm space, and an inner product and inner product spaces.
- Many of the metric spaces introduced above are also norm spaces and inner product spaces, but not all.
- Metric spaces are the most general of the three.
- Before introducing the concept of a norm and a normed space, we develop general tools that also work for metric spaces.

Section 2

Topology

Topology

- In this next section, we develop a set of tools that fall under that category of topology.
- These tools hold for metric spaces (including norm and inner product spaces).
- WARNING: There are a lot of definitions. These definitions will help talk formally about things in the future.

Topology: Open and Closed Sets

Definition (Ball)

Given a metric space $(\mathbb{X}, d)$ a δ-ball around x_0 is defined to be $B(x_0, \delta) = \{x \in \mathbb{X} : d(x, x_0) < \delta\}$

Definition (Interior Point)

A point $x_o \in \mathbb{X}$ is interior to $S \subset \mathbb{X}$ if
$\exists \delta > 0$ such that $B(x_o, \delta) \subset S$.

Definition (Open Set)

A set $\mathbb{X}$ is <u>open</u> if all points in $\mathbb{X}$ are interior.

Definition (Closed Set)

A set S is closed in $\mathbb{X}$ if $\mathbb{X} \setminus S$ is open.

Topology: Convergence

Let $(\mathbb{X}, d)$ be a metric space.

Definition (Convergence)

Given a sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n \in \mathbb{X}$, the following are equivalent

- $\lim_{n\to\infty} x_n = x^*$
- $x_n \to x^*$
- $\forall \epsilon > 0, \exists N(\epsilon)$ such that $n \geq N \Rightarrow d(x_n, x^*) < \epsilon$

A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{X}$ with a limit $x^* \in \mathbb{X}$ is said to converge.

Topology: Convergence

Note that a limit may not always exist (similar to min, max)
For example, $\lim_{t\to\infty} sin(t)$ does not exist.

Definition (lim sup)

Define lim sup as the largest limit (possibly infinity) of any subsequence.

Definition (lim inf)

Define lim inf is the smallest limit of all possible subsequences.

Example

- $\limsup_{t\to\infty} \sin(t) = 1$ since the subsequence $t_n = \frac{k\pi}{2}, k = 1, 5, 9, \cdots$ converges to 1
- $\liminf_{t\to\infty} \sin(t) = -1$ since the subsequence $t_n = \frac{k\pi}{2}, k = 3, 7, 11, \cdots$ converges to -1

Topology: Cauchy Sequence

Definition (Cauchy Sequence)

A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space $(\mathbb{X}, d)$ is said to be a Cauchy sequence if $\forall \epsilon > 0, \exists N(\epsilon) > 0$ such that $n, m > N \Rightarrow d(x_n, x_m) < \epsilon$

A sequence is Cauchy if elements in its tail get increasingly closer together. Note that we have not said anything about an element of convergence.

Theorem

If a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{X}$ converges to an element $x^* \in \mathbb{X}$ then it is a Cauchy sequence.

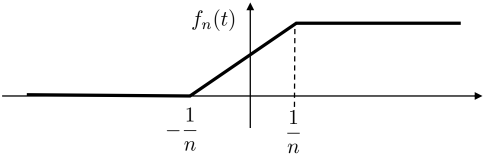
The converse is not true!! I.e., not all Cauchy sequences converge.

Topology: Cauchy Sequence

Example (from book)

Let $\mathbb{X} = C[-1, 1]$ and $d(f, g) = \left(\int_{-1}^{1} (f(t) - g(t))^2 dt \right)^{\frac{1}{2}}$

let f_n :



By integration we get:

$$d(f_n, f_n) = \frac{1}{6m^3 n}(m^3 + 4m^2 n + mn^2 + 2n^3)$$

$$\to 0 \text{ for } n, m \text{ large } (m > n)$$

but f_n converges to a discontinuous function which is not in $\mathbb{X}$.
This is undesirable.

Topology: Complete Metric Space

Definition (Complete metric space)

A metric space $(\mathbb{X}, d)$ is complete if every Cauchy sequence in $\mathbb{X}$ converges to a value in $\mathbb{X}$.

Implication

$C[a,b]$ with metric $\left(\int_a^b |f-g|^2\,dt\right)^{1/2}$ is not complete.

- Banach spaces are complete normed spaces (discussed later).
- Hilbert spaces (extremely important in signal processing and control) are complete inner product spaces (discussed later).
- The importance of L_p and ℓ_p are that they are complete spaces.

Section 3

Vector Spaces

Vector Spaces

A field is a set of scalars with well defined addition and multiplication operations.

Example of fields:

- $\mathbb{R}$ with normal addition and multiplication operations
- $\mathbb{C}$ with complex addition and complex multiplication
- The set of quaternions, with addition and quaternion multiplication
- Binary numbers $\{0, 1\}$ where addition is the "or" operator and multiplication is the "and" operator.

Vector Spaces

Definition (Linear Vector Space)

A linear vector space is a pair $(\mathbb{X}, \mathbb{F})$, where $\mathbb{X}$ is a set of objects, and $\mathbb{F}$ is a field, this is closed under addition and scalar multiplication. i.e.,

- $x \in \mathbb{X}, \alpha \in \mathbb{F} \Rightarrow \alpha x \in \mathbb{X}$
- $x, y \in \mathbb{X} \Rightarrow x + y \in \mathbb{X}$.

By implication

- $x \in \mathbb{X}, \alpha, \beta \in \mathbb{F} \Rightarrow (\alpha + \beta)x = \alpha x + \beta x \in \mathbb{X}$
- $x, y \in \mathbb{X}, \alpha \in \mathbb{F} \Rightarrow \alpha(x + y) = \alpha x + \alpha y \in \mathbb{X}$
- $x, y \in \mathbb{X}, \alpha, \beta \in \mathbb{F} \Rightarrow \alpha x + \beta y \in \mathbb{X}$.

Vector Spaces: Subspace

Definition (Subspace)

A subspace $V \subset \mathbb{X}$ is a subset of $\mathbb{X}$ that is also a linear vector space, in particular it contains zero.

Important property: A vector space contains a zero element.

Vector Spaces: Examples

The following are vector spaces:

- $(\mathbb{R}^n, \mathbb{R})$, $(\mathbb{C}^n, \mathbb{C})$, $(\mathbb{R}^{m \times n}, \mathbb{R})$, $(C[a,b], \mathbb{R})$, $(\ell^\infty, \mathbb{R})$, $(L^\infty, \mathbb{R})$.

The following are NOT vector spaces:

- The set $\mathbb{X} = \mathbb{R} \times [0, 2\pi]$, (a cylinder) is not a vector space for any field $\mathbb{F}$. This is the state space for an inverted pendulum.
- The set of rotation matrices is not a vector space for any field $\mathbb{F}$. This is in the configuration space for robots and satellites.
- The set of unit quaternions is not a vector space for any field. Quaternions are used extensively in robotics, quantum mechanics, and computer graphics.
- There are many useful spaces that are NOT linear vector spaces.

Vector Spaces: Linear Independence

Let S be a vector space and let $T \subset S$. (T may have uncountable infinite members). T is linearly independent if for each <u>finite</u> nonempty subset of T. i.e., $\{p_1, \cdots, p_n\}$ where $p_i \in T$, we have that

$$c_1 p_1 + \cdots + c_n p_n = 0 \qquad \Longleftrightarrow \qquad c_1 = c_2 = \cdots = c_n = 0.$$

Otherwise T is linearly dependent.

Vector Spaces: Linear Independence

Example

Let $S = \mathbb{R}^3$ then the set $T = \{(1,0,0)^\top, (0,1,0)^\top\} \subset \mathbb{R}^3$ is linearly independent since

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if and only if $c_1 = c_2 = 0$.
However, the set $T = \{(1,1,0)^\top, (2,2,0)^\top\} \subset \mathbb{R}^3$ is linearly dependent since

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ c_1 + 2c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

when $c_1 = -2$ and $c_2 = 1$ (as only on example).

Vector Spaces: Span

Definition (Span)

Let S be a vector space, then span(T) is the set of all linear combinations of $T \subseteq S$.

Example

$$\text{span}\left\{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)\right\} = \left\{\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} | \alpha \in \mathbb{R}\right\}$$

Example

$$\text{span}\left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right\} = \left\{\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R}\right\} = \mathbb{R}^2.$$

Vector Spaces: Basis

Definition (Basis)

T is a basis for the vector space S if T is linearly independent and $\mathrm{span}(T) = S$.

Definition (Dimension)

The dimension of the vector space S is the smallest number of linearly independent vectors needed to span S.

Example

One possible basis for $\mathbb{R}^n$ is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

Therefore $\dim(\mathbb{R}^n) = n$.

Vector Spaces: Basis

Example

One possible basis for ℓ^∞ is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \cdots \right\}$$

Therefore $\dim(\ell^\infty) = \infty$.

Vector Spaces: Basis

Example

The set of all polynomials P is a vector space with basis

$$\{1, t, t^2, \cdots\}$$

Therefore $\dim(P) = \infty$.

Example

The set of all polynomials of degree $\leq q$ P^q is a vector space with basis

$$\{1, t, t^2, \cdots, t^q\}$$

Therefore $\dim(P^q) = q$.

Section 4

Normed Spaces

Norms and Normed Spaces

Definition (Norm)

Let S be a vector space, $\|x\|$ is a norm if:

$$
\begin{array}{ll}
(N1) & \|x\| \geq 0 \quad \forall x \in S \\
(N2) & \|x\| = 0 \quad \Leftrightarrow x = 0 \\
(N3) & \|\alpha x\| = \alpha \|x\| \\
(N4) & \|x + y\| \leq \|x\| + \|y\| \quad \text{(triangle inequality)}
\end{array}
$$

Differences between norms and metrics:

- Norms only have one argument (the length of a vector), where metrics are distances between elements of a set.
- Norms are only defined for vector spaces!
  (i.e. there is no norm for rotation matrices, but there are metrics!)
- Norms scale with the vector (N3)
  (there are metrics that don't scale), e.g.
  $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$
- Every norm is also a metric

$$\|x - y\| = d(x, y)$$

$$\|x\| = d(x, 0)$$

Definition: Normed Space

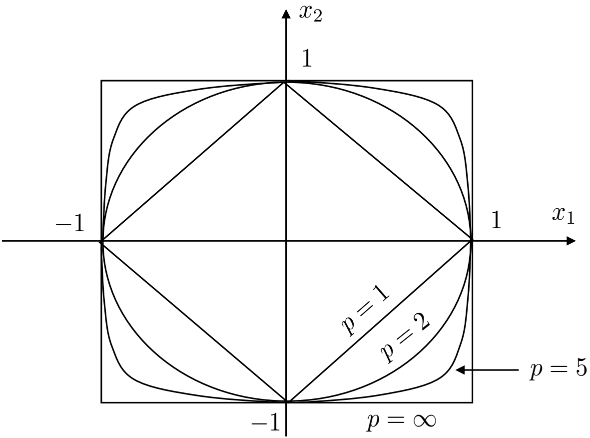
A <u>normed</u> space is a pair $(\mathbb{X}, \|\cdot\|)$ where $\mathbb{X}$ is a vector space and $\|\cdot\|$ is a norm.

Example (Normed Spaces)

$\mathbb{R}^n$ is a vector space. All of the following norms are valid:

- one-norm $\quad \|x\|_1 = \sum_{i=1}^{n} |x_i| \quad$ (power vectors)
- two-norm $\quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \quad$ (energy vectors)
- infinity-norm $\quad \|x\|_\infty = \max_{i=1,\cdots,n} |x_i| \quad$ (bounded vectors)
- p-norm $\quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$

Unit Circle in $\mathbb{R}^2$



Normed Space Example: Sequence Spaces

Let $\boldsymbol{\ell}$ be the set of sequences: $x = (x_1, x_2, x_3, \cdots)$. The following normed vector spaces can be defined:

- $\boldsymbol{\ell}_1$: (power sequences) If $\|x\|_{\ell_1} = \sum_{i=1}^{\infty} |x_i|$ then $\boldsymbol{\ell}_1 \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_1} < \infty\}$
- $\boldsymbol{\ell}_2$: (energy sequences) If $\|x\|_{\ell_2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$ then $\boldsymbol{\ell}_2 \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_2} < \infty\}$
- $\boldsymbol{\ell}_\infty$: (bounded sequences) If $\|x\|_{\ell_\infty} = \sup_{j \in \mathbb{N}} |x_j|$ then $\boldsymbol{\ell}_\infty \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_\infty} < \infty\}$
- $\boldsymbol{\ell}_p$: If $\|x\|_{\ell_p} = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ then $\boldsymbol{\ell}_p \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_p} < \infty\}$ for $1 \leq p \leq \infty$

Normed Space Examples

Example

Consider the sequence $x = (1, 1, 1, \ldots)$:

- $x \in \boldsymbol{\ell}_\infty$, but
- $x \notin \boldsymbol{\ell}_p$ for $1 \leq p < \infty$.

Example

Consider the sequence $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

- $x \notin \boldsymbol{\ell}_1$ (prove this), but
- $x \in \boldsymbol{\ell}_p \quad p > 1$ (prove this)

Normed Space Example: Function Spaces

Let $L^n(\Omega)$ be the set of functions on Ω. $x \in L^n(\Omega)$ is an equivalent classes of functions, i.e. equal except on a set of measure zero. (picture) The following norms are valid:

- $L_1^n(\Omega)$ (power signals). If $\|x\|_{L_1^n(\Omega)} = \int_\Omega \|x(t)\|\, dt$, then $L_1^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_1^n(\Omega)} < \infty\}$.
- $L_2^n(\Omega)$ (energy signals). If $\|x\|_{L_2^n(\Omega)} = \left(\int_\Omega \|x(t)\|^2\, dt\right)^{1/2}$, then $L_2^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_2^n(\Omega)} < \infty\}$.
- $L_p^n(\Omega)$. If $\|x\|_{L_p^n(\Omega)} = \left(\int_\Omega \|x(t)\|^p\, dt\right)^{1/p}$, then $L_p^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_p^n(\Omega)} < \infty\}$, $1 \leq p \leq \infty$.
- $L_\infty^n(\Omega)$ (bounded signals). If $\|x\|_{L_\infty^n(\Omega)} = \sup_{t \in \Omega} \|x(t)\|$, then $L_\infty^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_\infty^n(\Omega)} < \infty\}$.

Section 5

Inner Product Spaces

Inner Product Spaces

Definition (Inner Product)

Let S be a vector space over $\mathbb{R}$. An inner product $< \cdot, \cdot >: S \times S \to \mathbb{R}$ has the following properties:

$$
\begin{aligned}
&(IP1) \qquad \langle x, y \rangle = \overline{\langle y, x \rangle} \\
&(IP2) \qquad \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \\
&(IP3) \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \\
&(IP4) \qquad \langle x, x \rangle > 0 \quad \text{if } x \neq 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0
\end{aligned}
$$

Definition (Inner Product Space)

A vector space with an inner product defined is called an inner-product space.

Definition (Hilbert Space)

A complete inner-product space is called a <u>Hilbert space</u>.

Inner Product Spaces: Examples

- $\mathbb{R}^n$: $\langle x, y\rangle = \sum_{i=1}^n x_i y_i = x^T y$ is called the Euclidean inner product.
- $\mathbb{C}^n$: $\langle x, y\rangle = \sum_{i=1}^n x_i \overline{y_i} = y^H x$

- $\mathbb{R}^n$ with the Euclidean inner product is a Hilbert space .
- $\mathbb{C}^n$ with the Euclidean inner product is a Hilbert space.
- All finite-dimensional inner-product spaces are Hilbert spaces.

Inner Product Spaces: Examples

- Real sequences ℓ_2: $\langle x, y \rangle_{\ell_2} = \sum_{i=1}^{\infty} x_i y_i$
- Complex sequences ℓ_2: $\langle x, y \rangle_{\ell_2} = \sum_{i=1}^{\infty} x_i \overline{y_i}$
- Both of these examples are Hilbert spaces.

Inner Product Spaces: Examples

- Complex function space $L_2^n(\Omega)$ with inner product:

$$\langle x, y \rangle = \int_{\infty} y^H(t) x(t)\, dt$$

  is a Hilbert space, but
- Continuous function $C[a, b]$ with the same inner product is NOT a Hilbert space.

Norms vs Inner Products

Every inner product defines a norm (but not vice-versa)

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}$$

where $\|\cdot\|$ is called the norm induced by the inner product $\langle \cdot, \cdot \rangle$.

Examples of inducted norms

$$\|\cdot\|_2 : \ \langle x, x\rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \|x\|_2$$

$$\|\cdot\|_{\ell_2} : \ \langle x, x\rangle^{1/2} = \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2} = \|x\|_{\ell_2}$$

$$\|\cdot\|_{L_2} : \ \langle x, x\rangle^{1/2} = \left(\int_{\Omega} x^T(t)x(t)dt\right)^{1/2} = \left(\int_{\Omega} \|x(t)\|_2^2 \, dt\right)^{1/2} = \|x\|_{L_2}$$

Note that induced norms are all $2-$norms.

Cauchy-Schwartz Inequality

Theorem (Cauchy-Schwartz)

Let S be any inner product space (doesn't need to be Hilbert) and let $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ then $\forall x, y \in S$

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$

with equality iff $y = \alpha x$ where $\alpha \in \mathbb{F}$ is any scalar in the field $\mathbb{F}$.

Cauchy-Schwartz Inequality: Proof

The inequality clearly holds if either $x = 0$ or $y = 0$. Therefore assume that $x \neq 0$ and $y \neq 0$. Then

$$
\begin{aligned}
\|x - \alpha y\|^2 &= \langle x - \alpha y, x - \alpha y \rangle \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \langle x, \alpha y \rangle + \langle \alpha y, \alpha y \rangle \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\overline{\langle x, \alpha y \rangle}} + \alpha \overline{\overline{\langle y, \alpha y \rangle}} \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\langle \alpha y, x \rangle} + \alpha \overline{\langle \alpha y, y \rangle} \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\alpha} \overline{\langle y, x \rangle} + \alpha \overline{\alpha} \overline{\langle y, \alpha y \rangle} \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\alpha} \langle x, y \rangle + |\alpha|^2 \langle y, y \rangle \\
&= \|x\|^2 - \alpha \langle y, x \rangle - \overline{\alpha} \langle x, y \rangle + |\alpha|^2 \|y\|^2
\end{aligned}
$$

Cauchy-Schwartz Inequality: Proof

Recall the technique of completing the square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x + \frac{b}{2a})^2 - \frac{b}{4a}^2 + c. \end{aligned}$$

Complete the square in α:

$$\begin{aligned} \|x - \alpha y\|^2 &= \|y\|^2 \left(\alpha\overline{\alpha} - \alpha \frac{\overline{\langle x, y \rangle}}{\|y\|^2} - \bar{\alpha} \frac{\langle x, y \rangle}{\|y\|^2} \right) + \|x\|^2 \\ &= \|y\|^2 \left(\alpha - \frac{\langle x, y \rangle}{\|y\|^2} \right) \left(\bar{\alpha} - \frac{\overline{\langle x, y \rangle}}{\|y\|^2} \right) - \frac{|\langle x, y \rangle|^2}{\|y\|^2} + \|x\|^2 \end{aligned}$$

Cauchy-Schwartz Inequality: Proof

Let $\alpha^* = \frac{\langle x,y \rangle}{\|y\|^2}$ to get

$$
\begin{aligned}
0 \leq \|x - \alpha^* y\|^2 &= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} \\
\Rightarrow \qquad |\langle x, y\rangle|^2 &\leq \|x\|^2 \, \|y\|^2 \\
\Rightarrow \qquad |\langle x, y\rangle| &\leq \|x\| \, \|y\|
\end{aligned}
$$

Section 6

Notions of Convergence

Notions of Convergence

Definition (Strong Convergence/ Convergence in norm)

x_n converges strongly to x, i.e. $x_n \xrightarrow{s} x$ iff

$$\|x_n - x\| \to 0 \quad as \quad n \to \infty$$

Definition (Weak Convergence / Convergence in inner product)

x_n converges weakly to x, i.e. $x_n \xrightarrow{w} x$ iff

$$\langle x_n, y \rangle \to \langle x, y \rangle, \forall y \in S,$$

Note that this must hold for all $y \in S$, therefore Example 2.4.4 in the book is bogus!

Notions of Convergence (cont.)

Theorem (Strong vs. Weak Convergence)

Let (x_n) be a sequence in a normed space $\mathbb{X}$. Then

- A. *Strong convergence $\Rightarrow$ weak convergence with the same limit*
- B. *The converse of (A.) is not generally true*
- C. *If dim $\mathbb{X} < \infty$, then weak convergence $\Rightarrow$ strong convergence.*

Proof:

(A) By definition of strong convergence,

$$x_n \xrightarrow{s} x^* \quad \Rightarrow \quad \|x_n - x^*\| \to 0$$

so let y be <u>any</u> element in $\mathbb{X}$ then

$$|\langle x_n, y\rangle - \langle x^*, y\rangle| = |\langle x_n - x^*, y\rangle| \leq \|x_n - x^*\| \, \|y\|$$

but the RHS $\to 0$ which implies that the LHS $\to 0$ which implies weak convergence.

Proof:

(B) Before proving part (B) lets first understand what is wrong with Example 2.4.4 in the book.

$$x_n = (0, 0, 0, \ldots, 1, 0, \ldots)$$
$$y = (1, 1/2, 1/4, 1/8, \ldots)$$

Then $\langle x_n, y \rangle \to 0$ but this does not imply weak convergence since it must hold for all $y \in \mathbb{X}$.

Proof:

To prove part (B) we need a counter example. Again let $x_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ and let $\mathbb{X} = \ell_2$ i.e.

$$y \in \mathbb{X} \Rightarrow \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{\frac{1}{2}} < \infty$$
$$\Rightarrow y_i \to 0 \quad as \quad i \to \infty$$

so

$$\langle x_n, y \rangle = y_n \to 0 \quad as \quad n \to \infty \qquad \forall y \in \mathbb{X}$$
$$\Rightarrow \{x_n\} \xrightarrow{w} 0$$

but there is no x^* such that $\|x_n - x^*\| \to 0$.

Proof:

(C) Suppose that $x_n \stackrel{w}{\to} x$ and $dim(\mathbb{X}) = k$ then

$$\forall y \in \mathbb{X} \qquad \langle x_n, y \rangle \to \langle x, y \rangle \, .$$

Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis for $\mathbb{X}$, i.e. $\langle e_i, e_j \rangle = \delta_{ij}$, then

$$x_n = a_1^{(n)} e_1 + \cdots + a_k^{(n)} e_k$$
$$x = a_1 e_1 + \cdots + a_k e_k .$$

Proof:

Then since $\langle x_n, y\rangle \to \langle x, y\rangle\ \forall y$, let $y = e_j$

$$\Rightarrow \left\langle a_1^{(n)} e_1 + \cdots + a_k^n e_k, e_j \right\rangle = a_j^{(n)}$$

and

$$\langle a_1 e_1 + \cdots + a_k e_k, e_j\rangle = a_j$$

so

$$\langle x_n, e_j\rangle \to \langle x, e_j\rangle \Rightarrow a_j^{(n)} \to a_j \qquad \forall y = 1, \ldots, k$$

Also,

$$\|x_n - x\| = \left\| \sum_{j=1}^{k} a_j^{(n)} e_j - \sum_{j=1}^{k} a_j e_j \right\| = \left\| \sum_{j=1}^{k} (a_j^{(n)} - a_j) e_j \right\|$$

$$\leq \sum_{j=1}^{k} |a_j^{(n)} - a_j|\, \|e_j\| \to 0$$

$$\Rightarrow \text{ strong convergence}$$

Equivalence of Norms

Theorem

Let $dim(\mathbb{X}) = k$ *and let* $\|\cdot\|$ *and* $\|\cdot\|_0$ *be two different norms on* $\mathbb{X}$ *then* $\exists a, b$ *such that*

$$a \|x\|_0 \leq \|x\| \leq b \|x\|_0$$

Proof.

(in book page 96) $\square$

Implication: For convergence proofs, it doesn't matter which norm you use, therefore, use the one that simplifies the proof.

Section 7

Orthogonality

Orthogonality

Let $x, y \in \mathbb{X}$ where $\mathbb{X}$ is an inner product space. Then the angle between x and y is

$$\theta = cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\| \, \|y\|}\right).$$

i.e.

$$\langle x, y\rangle = \|x\| \, \|y\| \, cos\theta$$

Orthogonality, cont.

Definition (Colinear)

Two vectors $x, y \in \mathbb{X}$ are said to be colinear if

$$\theta = 180 * n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Definition (Orthogonal)

Two vectors $x, y \in \mathbb{X}$ are said to be orthogonal if

$$\theta = 90 * n \qquad n = \pm 1, \pm 3, \pm 5, \ldots$$

i.e., $\langle x, y \rangle = 0$.

If $\langle x, y \rangle = 0$ we write $x \perp y$.

Orthogonality, cont.

Example (Vectors in $L_2[0, 2\pi)$)

The functions $x = sin(t)$ and $y = cos(t)$ are orthogonal since

$$\langle x, y \rangle = \int_0^{2\pi} sin(t)cos(t)dt = 0.$$

Example (Vectors in ℓ)

The sequences

$$x = (1, 1, 1, 1, 0, 0, \dots)$$
$$y = (1, -1, 1, -1, 1, \dots)$$

are orthogonal since

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i = 0.$$

Other useful inner products and norms: Weighting

Definition (Positive Definite Matrix)

A matrix $W : \mathbb{R}^k \to \mathbb{R}^k$ is positive definite (PD) if
$\forall x \in \mathbb{R}^k \qquad x^T W x > 0$

- W is positive semi-definite (PSD) if $x^T W x \geq 0$
- W is negative definite (ND) if $x^T W x < 0 \qquad \forall x \in \mathbb{R}^k$
- W is negative semi-definite (NSD) if $x^T W x \leq 0 \qquad \forall x \in \mathbb{R}^k$
- Otherwise it is indefinite

Examples of positive definiteness

- $W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is PD since

$$x^T W x = x_1^2 + x_2^2 > 0 \qquad \forall x \in \mathbb{R}^2$$

- $W = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is PSD since

$$x^T W x = x_1^2 = 0 \qquad \forall x = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \neq 0$$

- $W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is indefinite since

$$x^T W x = -x_1^2 + x_2^2$$

which can be positive or negative depending on x.

Examples of Inner Products

Weighted Inner Products and Norms

If $W > 0$ then $\langle x, y \rangle_W = x^H W y$ is a valid inner product which induces the weighted norm $\|x\|_W = (x^H W x)^{\frac{1}{2}}$

We can define weighted inner products for functions:

$$\langle f, g \rangle_W = \int f(t) g(t) w(t) dt$$

where $w(t) > 0$ except on a set of measure zero.

Examples of Inner Products

Definition (Expectation)

Expectation is a weighted inner product with weight $f_{\mathbb{X}\mathbb{Y}}(x, y)$

$$\langle \mathbb{X}, \mathbb{Y} \rangle = \int xyf_{\mathbb{X}\mathbb{Y}}(x, y)dxdy = E[\mathbb{X}\mathbb{Y}]$$

if $\mathbb{X}$ is a zero mean then

$$\langle x, x \rangle = var(x)$$

is the norm induced by $E[\cdot\cdot]$

Examples of Inner Products

- Let $\mathbb{I}(m, n)$ be the set of grayscale images with $m \times n$ pixels, each valued between $[0, 255]$.
- A valid inner on $\mathbb{I}(m, n)$ is given by

$$\langle I, J \rangle = \sum_{u=1}^{m} \sum_{v=1}^{n} I(u, v) J(u, v), \qquad \forall I, J \in \mathbb{I}(m, n).$$

Orthogonal Subspaces

Definition (Orthogonal Subspaces)

Let V, W be subspaces of S. $V \perp W$ if

$$\forall v \in V \text{ and } \forall w \in W, \qquad \langle v, w \rangle = 0$$

Definition (Orthogonal Complement)

$V^\perp$, called the orthogonal complement of V, is the set

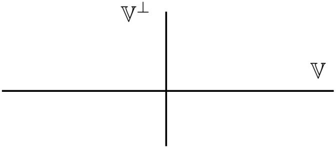
$$V^\perp = \{x \in S : \forall v \in V, \langle x, v \rangle = 0\}$$

Orthogonal Subspaces, cont.

Example

Let $S = \mathbb{R}^2$ and $V = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$ then

$V^{\perp} = \left\{ \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\}$



Orthogonal Subspaces, cont.

Theorem

Let V and W be subspaces of an inner product space S (not necessarily Hilbert). Then

1. $V^{\perp}$ *is a closed subspace of* S
2. $V \subset V^{\perp\perp}$ ($V = V^{\perp\perp}$ *if* S *is complete)*
3. *If* $V \subset W$ *then* $W^{\perp} \subset V^{\perp}$
4. $V^{\perp\perp\perp} = V^{\perp}$
5. *If* $x \in V \cap V^{\perp}$ *then* $x = 0$
6. $\{0\}^{\perp} = S$ *and* $S^{\perp} = \{0\}$

Prove one in class.

Inner Sum and Direct Sum

Definition (Inner Sum)

If V and W are linear subspaces then

$$V + W = \{x : x = v + w, v \in V \text{ and } w \in W\}$$

is the inner sum.

Definition (Orthogonal Sum)

If V and W are orthogonal subspaces then the sum

$$V \oplus W = \{x : x = v + w, v \in V \text{ and } w \in W\}$$

is called the orthogonal sum.

Definition (Disjoint Subspaces)

Two subspaces are said to be disjoint if

$$V \cap W = \{0\}$$

Inner Sum and Direct Sum, cont.

Lemma

Let $V + W$ be subspaces of S and let $x \in V + W$ then the representation $x = v + w$ is unique iff $V + W$ are disjoint.

Proof.

($\Leftarrow$) Assume V, W are disjoint but $x = v + w$ is not unique i.e. $x = v_1 + w_1 = v_2 + w_2$ where $v_1 \neq v_2$ and $w_1 \neq w_2$. This implies that $v_1 - v_2 = w_2 - w_1$ but $v_1 - v_2 \in V$ and $w_2 - w_1 \in W$ since V, W are subspaces. Since $V \cap W = \{0\}$ we must have that $v_1 - v_2 = w_2 - w_1 = 0$ or $v_1 = v_2$ and $w_1 = w_2$ which is a contradiction. $\square$

Inner Sum and Direct Sum, cont.

Lemma

If V and W are orthogonal subspaces then the representation of $x \in V \oplus W$ is unique (i.e. $x = v + w$, where $v \in V$ and $w \in W$).

Example

Let $S = \mathbb{R}^2$, let $V = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$, let $W = \left\{ \begin{pmatrix} 0 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$ Then

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

is a <u>unique</u> decomposition.

Difference between a Hamel basis and a Complete basis.

Definition

An orthonormal set of basis vectors $T = \{p_i, p_2, \ldots\}$ is said to be a compelte basisfor a Hilbert space S if every $x \in S$ can be represented as

$$x = \sum_{j=1}^{\infty} c_j p_j$$

Examples of complete bases: Fourier functions: $e^{j\omega t}$
Legendre & Chebyshev polynomials

Difference: A Hamel basis $\Rightarrow$ every x can be represented by a finite representation. A complete basis allows functions through a limiting process.

Section 8

Linear Operators

Operators and Transformations

Definition (Linear Operator)

Let $\mathcal{L} : \mathbb{X} \to \mathbb{Y}$ be an operator from $\mathbb{X}$ to $\mathbb{Y}$. $\mathcal{L}$ is a linear operator if

1. $\mathcal{L}[\alpha x] = \alpha \mathcal{L}[x] \qquad \forall x \in \mathbb{X} \qquad \forall \alpha \in \mathbb{F}$
2. $\mathcal{L}[x_1 + x_2] = \mathcal{L}[x_1] + \mathcal{L}[x_2], \qquad \forall x_1, x_2 \in \mathbb{X}$

Examples of Linear Operators

Example (Matrices)

Operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ are $m \times n$ matrices.

$$A(\alpha x + \beta y) = \alpha A x + \beta A y$$

A is a linear operator.

Example (Differential Equations with no input)

The differential equation $\dot{x} = Ax; \quad x(0) = x_0$ defines a linear operator from $\mathbb{R}^n$ to $L_2[0, T]$

$$y(t) = \mathcal{L}[x_0] \text{ where } \mathcal{L}[x_0] = e^{At} x_0$$

$\mathcal{L}$ is linear since

$$e^{At}(\alpha x_{01} + \beta x_{01}) = \alpha e^{At} x_{01} + \beta e^{At} x_{02}$$

Examples of Linear Operators

Example (Convolution)

Convolution is a linear operator from L_∞ to L_∞ if $h(t) \in L_1[-\infty, \infty]$, i.e.

$$y(t) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

(Recall: for a system to be BIBO stable required that $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$ i.e. $h(t) \in L_1[-\infty, \infty]$)

Examples of Linear Operators

Example (Fourier Transform)

(E4) The Fourier transform defines a linear operator from $L_2[-\infty, \infty]$ to $L_2[-\infty, \infty]$.

$$X(j\omega) = \mathcal{L}[x(t)] \triangleq \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

<u>There are many examples of linear operators!</u>

Range and Null Space of an Operator

Definition (Range Space)

Let $\mathcal{L} : \mathbb{X} \to \mathbb{Y}$ be a linear operator. The range space (or image) of $\mathcal{L}$ is

$$\mathcal{R}(\mathcal{L}) = \{y \in \mathbb{Y} : y = \mathcal{L}[x] \text{ and } x \in \mathbb{X}\} \subseteq \mathbb{Y}$$

Definition (Null Space)

The Null space or kernel of $\mathcal{L}$ is

$$\mathcal{N}(\mathcal{L}) = \{x \in \mathbb{X} : \mathcal{L}[x] = 0\} \subseteq \mathbb{X}$$

Example of Range and Null Space

- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which defines a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^2$.
- Note that $y = Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$.
- Therefore, the range space is

$$\mathcal{R}(A) = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\} \subset \mathbb{R}^2.$$

- Similarly, the null space is

$$\mathcal{N}(A) = \left\{ \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

Section 9

Projections

Projections

- Suppose that V and W are disjoint subspaces of S such that $V + W = S$, i.e.

$$x \in S \Rightarrow x = v + w$$

  where $v \in V$ and $w \in W$ is a unique decomposition.
- Define the linear operator $P : S \to V \subset S$ as

$$Px = P(v + w) = v$$

- Note that $P(Px) = Pv = v$

Projections, cont.

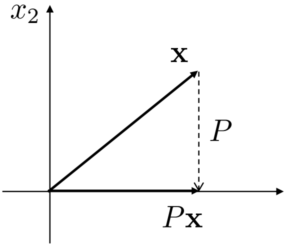
Definition (Projection Operator)

Let $P : S \to S$ such that $P^2 = P$, then P is called a projection operator or idempotent.

Example

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

i.e. P projects elements of P onto the x_1 axis:



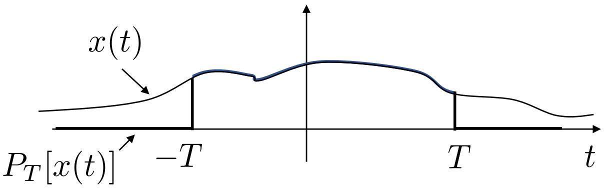
Projections, cont.

Example

Truncation: let

$$(P_T x)(t) = \begin{cases} x(t), & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Then P_T projects $x(t)$ onto its truncated function:



Projections, cont.

Theorem (Moon 2.7)

Let $P : S \to S$ be a projection operator, then

$$S = R(P) + N(P)$$

Proof.

Homework problem. $\square$

Projections, cont.

Theorem

If $P : S \to S$ is a projection operator then so is $(I - P) : S \to S$

Proof.

$$
\begin{aligned}
(I - P)^2 &= (I - P)(I - P) = \\
&= I - P - P - P^2 \\
&= I - P - P + P \\
&= I - P
\end{aligned}
$$

$\square$

Projections, cont.

- Note that if $P : S \to V$ and $I - P : S \to W$ then V and W are disjoint and $S = V + W$ since

$$x = \underbrace{Px}_{\in V} + \underbrace{(I - P)x}_{\in W}.$$

- V and W are disjoint. If not, then $\exists x_0 (\neq 0) \in S$ such that

$$\begin{aligned} Px_0 &= (I - P)x_0 = x_0 - Px_0 \\ 2Px_0 &= x_0 \\ \Rightarrow Px_0 &= \frac{1}{2}x_0 \\ \text{and } P^2 x_0 &= \frac{1}{4}x_0 = \frac{1}{2}x_0 \Leftrightarrow x_0 = 0 \end{aligned}$$

Projections, cont.

Definition (Orthogonal Projection)

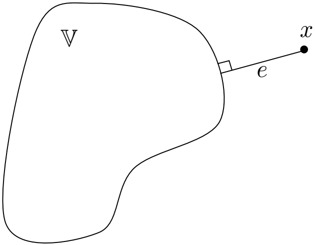
If V and W are orthogonal then P is an orthogonal projection

Theorem

P is an orthogonal projection iff $R(P) \perp N(P)$

Applications to Engineering

Given a point $x \in S$, suppose that we want to approximate x by a point in $\mathbb{V} \subset S$ (assuming $x \notin \mathbb{V}$) then we want to find the point in $\mathbb{V}$ that is closest to x.



This is given by the orthogonal projection of x onto $\mathbb{V}$. i.e.
$\langle e, v \rangle = 0 \qquad \forall v \in \mathbb{V}$

Applications to Engineering

Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$ (i.e., $\|\mathbf{n}\| = 1$), then

$$\Pi_{\mathbf{n}}^{\perp} \triangleq \mathbf{n}\mathbf{n}^{\top}$$

is a projection operation. Geometrically $\Pi_{\mathbf{n}}^{\perp} x = \mathbf{n}\mathbf{n}^{\top} x$ find the projection of x along the unit vector $\mathbf{n}$
Also

$$\Pi_{\mathbf{n}} = I - \mathbf{n}\mathbf{n}^{\top}$$

is a projection operator. Geometrically, $\Pi_{\mathbf{n}} x$ projections x onto the 2D space that is orthogonal to $\mathbf{n}$.

The Projection Theorem

Theorem

Let $\mathbb{S}$ be a Hilbert space and let $\mathbb{V}$ be a closed subspace of $\mathbb{S}$. For any $x \in \mathbb{S}$ there exists a unique $v_0 \in \mathbb{V}$ closest to x; i.e.
$\|x - v_0\| \leq \|x - v\| \quad \forall v \in \mathbb{V}.$

Furthermore v_0 minimizes $\|x - v_0\|$ iff $x - v_0$ is orthogonal to $\mathbb{V}$.

The Projection Theorem, proof

Step 1. Show that v_0 exists.
Assume $x \notin \mathbb{V}$ and let $\delta = \inf_{v \in \mathbb{V}} \|x - v\|$ We need to show that in fact $\exists v_0 \in \mathbb{V}$ such that $\|x - v_0\| = \delta$
Let $\{v_i\}$ be a sequence in $\mathbb{V}$ such that $\|x - v_i\| \to \delta$ and show that $\{v_i\}$ is Cauchy.

Need parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

Consider

$$\|(v_j - x) + (x - v_i)\|^2 + \|(v_j - x) - (x - v_i)\|^2 = 2\|(v_j - x)\|^2 + 2\|x - v_i\|^2$$

$$\Rightarrow \|v_j - v_i\|^2 = 2\|v_j - x\|^2 + 2\|x - v_i\|^2 - 4\left\|\frac{(v_j + v_i)}{2} - x\right\|^2$$

The Projection Theorem, proof

$$v_i, v_j \in \mathbb{V} \quad \Rightarrow \quad \frac{v_j + v_i}{2} \in \mathbb{V} \quad \Rightarrow \quad \left\| \frac{(v_j - v_i)}{2} - x \right\|^2 \geq \delta^2$$

$$\Rightarrow \|v_j - v_i\|^2 \leq 2\,\|v_j - x\|^2 + 2\,\|v_i - x\|^2 - 4\delta^2$$

But $\|v_j - x\| \to \delta$

$$\Rightarrow \|v_j - v_i\| \to 0$$

and is therefore Cauchy.

Since $\mathbb{V}$ is a Hilbert space

$$v_i \to v_0 \in \mathbb{V}.$$

Note that this proof is not constructive, i.e. it doesn't tell you how to construct the sequence $\{v_i\}$.

The Projection Theorem, proof

Step 2. Show that $v_0 = \arg\min_{v \in \mathbb{V}} \|x - v\| \Rightarrow x - v_0 \perp \mathbb{V}$.
Proof by contradiction. Suppose that $x - v_0$ is not perpendicular to $\mathbb{V}$. Then there exists a $v \in \mathbb{V}$ such that

$$\langle x - v_0, v \rangle = \delta \neq 0$$

and w.l.o.g. (why?) let $\|v\| = 1$

Let $z = v_0 + \delta v \in \mathbb{V}$ then

$$\begin{aligned} \|x - z\|^2 = \|x - v_0 - \delta v\|^2 &= \|x - v_0\|^2 - 2Re \langle x - v_0, \delta v \rangle + \|\delta v\|^2 \\ &= \|x - v_0\|^2 - 2\delta^2 + \delta^2 < \|x - v_0\|^2 \end{aligned}$$

which is a contradiction since v_0 is the minimizer.

The Projection Theorem, proof

Step 3. Suppose $(x - v_0) \perp \mathbb{V}$ then $\forall v \in \mathbb{V}$ such that $v \neq v_0$

$$
\begin{aligned}
\|x - v\|^2 &= \|x - v_0 + v_0 - v\|^2 \\
&= \|x - v_0\|^2 + 2Re\,\langle x - v_0, v_0 - v\rangle + \|v_0 - v\|^2 \\
&= \|x - v_0\|^2 + \|v_0 - v\|^2 \\
&> \|x - v_0\|^2
\end{aligned}
$$

Step 4. Uniqueness Same as proof on page 25 of notes.

Closed Subspace

Theorem (Moon Theorem 2.10)

Let $\mathbb{V}$ be a closed subspace of a Hilbert space $\mathbb{S}$, then

$$\mathbb{S} = \mathbb{V} \oplus \mathbb{V}^{\perp}$$
$$\mathbb{V} = \mathbb{V}^{\perp\perp}$$

Proof.

In book. $\square$

Section 10

Gram Schmidt Orthogonalization

Application: Gram Schmidt Orthogonalization

Given a set $T = \{p_1, \ldots, p_n\}$
Find a set $T' = \{q_1, \ldots, q'_n\}$ $\quad n' \leq n$ such that

$$span(T') = span(T) \text{ and } < q_i, q_j >= \delta_{ij}$$

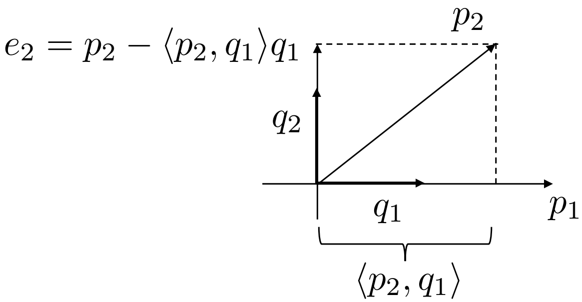
Step 1. Normalize the First Vector

$$q_1 = \frac{p_1}{\|p_1\|} \qquad (\text{i.e. } \langle q_1, q_1 \rangle = 1\ )$$

Application: Gram Schmidt Orthogonalization, cont

Step 2. Let e_2 be p_2 minus the projection of p_2 on q_1 i.e.

$$e_2 = p_2 - \langle p_2, q_1 \rangle \, q_1$$



Then normalize e_2:

$$q_2 = \frac{e_2}{\|e_2\|}$$

Application: Gram Schmidt Orthogonalization, cont

Step 3. Let e_k be p_k minus the projection of p_k on $q_1, \ldots, q_{k-1}$:

$$e_k = p_k - \sum_{j=1}^{k-1} \langle p_k, q_j \rangle \, q_j \Rightarrow q_k = \frac{e_k}{\|e_k\|}$$

Example: Gram Schmidt Orthogonalization

Given the set

$$T = \{p_1, p_2, p_3\} \triangleq \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

find a set $T' = \{q_1, q_2, q_3\}$ where the vectors in T' are orthonormal and $\mathrm{span}(T) = \mathrm{span}(T')$.

$$q_1 = \frac{p_1}{\|p_2\|} = \frac{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Example: Gram Schmidt Orthogonalization, cont.

$$
\begin{aligned}
e_2 &= p_2 - \langle p_2, q_1 \rangle \, q_1 \\
&= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}^\top \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}
\end{aligned}
$$

Therefore $q_2 = \frac{e_2}{\|e_2\|} = \frac{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Example: Gram Schmidt Orthogonalization, cont.

$$
\begin{aligned}
e_3 &= p_3 - \langle p_3, q_1 \rangle \, q_1 - \langle p_3, q_2 \rangle \, q_2 \\
&= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^{\top} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^{\top} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}
\end{aligned}
$$

Therefore $q_3 = \frac{e_3}{\|e_3\|} = \frac{\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}}{3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Example: Gram Schmidt Orthogonalization, cont.

Therefore, the Gram Schmidt orthonormalization of

$$T = \{p_1, p_2, p_3\} \triangleq \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

is

$$T' = \{q_1, q_2, q_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Note that $\text{span}(T) = \text{span}(T')$.