ECEn 671: Mathematics of Signals and Systems

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Section 1

Invariant Subspaces

Invariant Subspaces

Definition

Let A be a square matrix. If $\mathbb{S} \subset \mathcal{R}(A)$ is such that $x \in \mathbb{S} \implies Ax \in \mathbb{S}$ then $\mathbb{S}$ is an invariant subspace of A.

Example

An eigenvector forms an invariant subspace i.e.

$$\mathbb{S} = \{\alpha x \mid x \text{ is an eigenvector}\}$$

is invariant since $\hat{x} \in \mathbb{S} \Rightarrow A\hat{x} = \lambda\hat{x} \in \mathbb{S}$.

Invariant Subspaces

Example

The span of any subset of eigenvectors is invariant: Let $x_1 \dots x_p$ be eigenvectors with associated eigenvalues $\lambda_1 \dots \lambda_p$.
Let

$$\mathbb{S} = \text{span}\{x_1 \dots x_p\}$$

then

$$\begin{aligned} & \hat{x} \in \mathbb{S} \\ \implies & \hat{x} = \alpha_1 x_1 + \cdots + \alpha_p x_p \\ \implies & A\hat{x} = \alpha_1 A x_1 + \cdots + \alpha_p A x_p \\ \implies & A\hat{x} = \alpha_1 \lambda_1 x_1 + \cdots + \lambda_p \alpha_p x_p \\ \implies & A\hat{x} \in \mathbb{S}. \end{aligned}$$

Applications to Differential Equations

Consider the differential equation $\dot{x} = Ax$ with initial condition $x(0) = x_0$.

Lemma

The solution is given by $x(t) = e^{At}x_0$ where $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$.

Proof.

Plug into equation

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \frac{d}{dt}(e^{At})x_0 + e^{At}\frac{d}{dt}(x_0) = \frac{d}{dt}e^{At}x_0 \\
&= \frac{d}{dt}(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \cdots x_0 \\
&= (A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots)x_0 \\
&= A(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots)x_0 \\
&= Ae^{At}x_0 = Ax(t)
\end{aligned}
$$

so $x(t) = e^{At}x_0$ satisfies $\dot{x} = Ax$ with initial condition x_0. □

Applications to Differential Equations

Lemma

If $\mathbb{S}$ is an invariant subspace of A then $\mathbb{S}$ is an invariant subspace of e^{At}

Proof.

Let $x_0 \in \mathbb{S}$ then

$$\begin{aligned} Ax_0 \in \mathbb{S} &\implies tAx_0 \in \mathbb{S} \\ &\implies A(Ax_0) \in \mathbb{S} \\ &\implies \frac{A^2 t^2}{2!} x_0 \in \mathbb{S} \\ &\cdots \end{aligned}$$

Therefore

$$x(t) = Ix_0 + Atx_0 + \frac{A^2 t^2}{2!} x_0 + \frac{A^3 t^3}{3!} x_0 + \cdots \in \mathbb{S}.$$

Applications to Differential Equations: Example

Consider the differential equation

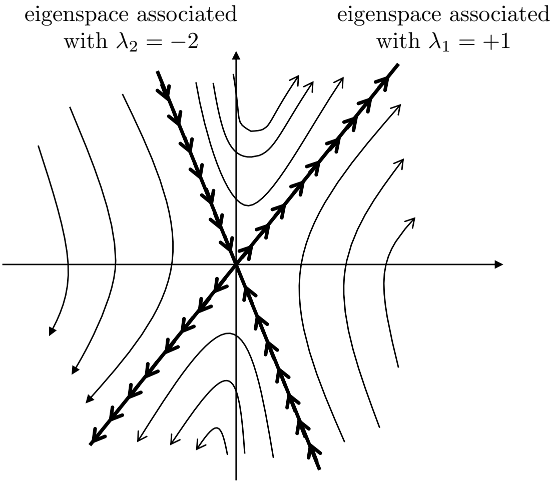
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} x.$$

The eigenvalues of A are given by $\det(\lambda I - \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}) = (\lambda - 1)(\lambda + 2) = 0$ and so $\lambda_1 = 1$ and $\lambda_2 = -2$.

The associated eigenvector are

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Applications to Differential Equations: Example



- If the initial condition is on $span\{x_1\}$, then the solution remains on $span\{x_1\}$.
- If the initial condition is on $span\{x_2\}$, then the solution remains on $span\{x_2\}$.

Applications to Difference Equations: Example

RWB:Change system to eigenvalues in unit circle. Show that eigenspaces are invariant. Provide example.

Consider the differential equation

$$x[k+1] = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} x[k].$$

Again the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -2$, and the eigenvectors are

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$