ECEn 671: Mathematics of Signals and Systems

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Section 1

Generalized Fourier Series

Section 3.17: Generalized Fourier Series

Topic of interest: L_2 function approximation

Definition (Complete Basis)

An orthonormal set $\{p_i, i = 1, \ldots, \infty\}$ in a Hilbert space $\mathbb{S}$ is a <u>complete basis</u> or <u>total basis</u> if $\forall x \in \mathbb{S}$

$$x = \sum_{i=1}^{\infty} \langle x, p_i \rangle \, p_i$$

Note that if $x = \sum_{i=1}^{\infty} c_i p_i$ and $\langle p_i, p_j \rangle = \delta_{ij}$ then

$$\langle x, p_j \rangle = \sum_{i=1}^{\infty} c_i \langle p_i, p_j \rangle = c_j$$

$$\Rightarrow c_j = \langle x, p_j \rangle$$

Generalized Fourier Series, cont.

Therefore we can write

$$x = \sum_{i=1}^{\infty} \langle x, p_i \rangle \, p_i .$$

Most common example: standard Fourier basis

$$P_n(t) = \frac{1}{\sqrt{T}} e^{j\left(\frac{2\pi}{T}\right)nt}$$

Any function $f \in L_2[0, T]$ can be written as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{T} e^{j\left(\frac{2\pi}{T}\right)nt}$$

where the coefficients are given as

$$c_n = \left\langle f, \frac{1}{\sqrt{T}} e^{j\left(\frac{2\pi}{T}\right)nt} \right\rangle \triangleq \frac{1}{\sqrt{T}} \int_0^T f(t) e^{j\left(\frac{2\pi}{T}\right)nt} dt$$

Generalized Fourier Series, cont.

Actually it is common to place the $\frac{1}{\sqrt{T}}$'s together letting $f(t) = \sum_{n=-\infty}^{\infty} b_n e^{j\left(\frac{2\pi}{T}\right)nt}$ where

$$b_n = \left\langle f(t), \frac{1}{T} e^{j\left(\frac{2\pi}{T}\right)nt} \right\rangle = \frac{1}{T}\int_0^T f(t) e^{-j\left(\frac{2\pi}{T}\right)nt} dt$$

Generalized Fourier series hold for <u>any</u> complete basis, i.e.

$$x = \sum_{j=1}^{\infty} \langle x, p_j \rangle \, p_j$$

Generalized Fourier Series, cont.

There are two important relationship between a function and its Fourier transform.

Theorem (Bessel's Inequality)

Suppose $\{p_1, p_2, \ldots\}$ *is orthonormal but not necessarily complete and let*

$$c = \{\langle x, p_1\rangle, \langle x, p_2\rangle, \ldots\} = \{c_1, c_2, \ldots\}$$

then

$$\boxed{\|c\|_{\ell_2} \leq \|x\|_{L_2}}$$

Proof:

$$
\begin{aligned}
0 \leq \left\| x - \sum c_j p_j \right\|_{L_2}^2 &= \left\langle x - \sum c_j p_j , x - \sum c_j p_j \right\rangle_{L_2} \\
&= \langle x, x \rangle_{L_2} - \sum \bar{c}_j \, \langle x, p_j \rangle_{L_2} \\
&- \sum c_j \bar{\langle x, p_j \rangle}_{L_2} + \sum \sum c_j \bar{c}_k \, \langle p_j , p_k \rangle_{L_2} \\
&= \|x\|_{L_2}^2 - \sum \bar{c}_j c_j - \sum c_j \bar{c}_j + \sum c_j \bar{c}_j \\
&= \|x\|_{L_2}^2 - \sum_{j=1}^{\infty} |c_j|^2 \\
&= \|x\|_{L_2}^2 - \|c\|_{\ell_2}^2 \\
&\Rightarrow \|c\|_{\ell_2}^2 \leq \|x\|_{L_2}^2
\end{aligned}
$$

Generalized Fourier Series, cont.

Theorem (Parseval's Equality)

If $T = \{p_1, p_2, \ldots\}$ is complete then

$$\boxed{\|x\|_{L_2}^2 = \|c\|_{\ell_2}^2}$$

Proof.

If T is complete then

$$\left\| x - \sum c_j p_j \right\|^2 = 0$$

and the result follows from the proof of Bessel's inequality . $\square$

Significance of Parseval's Equality

$\|x\|_{L_2}^2 = \|c\|_{\ell_2}^2$ says that the energy in a signal (i.e. $\|x\|_{L_2}$) is equal to the energy in the Fourier coefficients (i.e. $\|c\|_{\ell_2}^2$).

This relationship between x and its transform c is written as

$$x \overset{\mathcal{F}}{\longleftrightarrow} c.$$

Significance of Parseval's Equality, cont.

Lemma (Moon Lemma 3.1)

If $x \overset{\mathcal{F}}{\longleftrightarrow} c$ *and* $y \overset{\mathcal{F}}{\longleftrightarrow} b$ *for the same complete basis* $\{p_1, p_2, \ldots\}$ *then*

$$\langle x, y \rangle_{L_2} = \langle c, b \rangle_{\ell_2} .$$

Proof.

Let $x = \sum_{i=1}^{\infty} c_i p_i$, and $y = \sum_{i=1}^{\infty} b_i p_i$ then

$$\begin{aligned} \langle x, y \rangle_{L_2} &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_i \bar{b}_j \langle p_i, p_j \rangle \\ &= \sum_{i=1}^{\infty} c_i \bar{b}_i \\ &= \langle c, b \rangle_{\ell_2} \end{aligned}$$

$\square$