ECEn 671: Mathematics of Signals and Systems

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Section 1

Notions of Convergence

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Definition (Strong Convergence/ Convergence in norm)

x_n converges strongly to x, i.e. $x_n \xrightarrow{s} x$ iff

$$\|x_n - x\| \to 0 \quad as \quad n \to \infty$$

Definition (Weak Convergence / Convergence in inner product)

x_n converges weakly to x, i.e. $x_n \xrightarrow{w} x$ iff

$$\langle x_n, y \rangle \to \langle x, y \rangle, \forall y \in S,$$

Note that this must hold for all $y \in S$, therefore Example 2.4.4 in the book is bogus!

Notions of Convergence (cont.)

Theorem (Strong vs. Weak Convergence)

Let (x_n) be a sequence in a normed space $\mathbb{X}$. Then

A. *Strong convergence $\Rightarrow$ weak convergence with the same limit*
B. *The converse of (A.) is not generally true*
C. *If dim $\mathbb{X} < \infty$, then weak convergence $\Rightarrow$ strong convergence.*

Proof:

(A) By definition of strong convergence,

$$x_n \xrightarrow{s} x^* \quad \Rightarrow \quad \|x_n - x^*\| \to 0$$

so let y be <u>any</u> element in $\mathbb{X}$ then

$$|\langle x_n, y\rangle - \langle x^*, y\rangle| = |\langle x_n - x^*, y\rangle| \leq \|x_n - x^*\| \, \|y\|$$

but the RHS $\to 0$ which implies that the LHS $\to 0$ which implies weak convergence.

Proof:

(B) Before proving part (B) lets first understand what is wrong with Example 2.4.4 in the book.

$$x_n = (0, 0, 0, \ldots, 1, 0, \ldots)$$
$$y = (1, 1/2, 1/4, 1/8, \ldots)$$

Then $\langle x_n, y \rangle \to 0$ but this does not imply weak convergence since it must hold for all $y \in \mathbb{X}$.

Proof:

To prove part (B) we need a counter example. Again let $x_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ and let $\mathbb{X} = \ell_2$ i.e.

$$y \in \mathbb{X} \Rightarrow \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{\frac{1}{2}} < \infty$$

$$\Rightarrow y_i \to 0 \quad as \quad i \to \infty$$

so

$$\langle x_n, y \rangle = y_n \to 0 \quad as \quad n \to \infty \qquad \forall y \in \mathbb{X}$$

$$\Rightarrow \{x_n\} \xrightarrow{w} 0$$

but there is no x^* such that $\|x_n - x^*\| \to 0$.

Proof:

(C) Suppose that $x_n \stackrel{w}{\to} x$ and $dim(\mathbb{X}) = k$ then

$$\forall y \in \mathbb{X} \qquad \langle x_n, y \rangle \to \langle x, y \rangle \,.$$

Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis for $\mathbb{X}$, i.e. $\langle e_i, e_j \rangle = \delta_{ij}$, then

$$x_n = a_1^{(n)} e_1 + \cdots + a_k^{(n)} e_k$$
$$x = a_1 e_1 + \cdots + a_k e_k.$$

Proof:

Then since $\langle x_n, y\rangle \to \langle x, y\rangle \ \forall y$, let $y = e_j$

$$\Rightarrow \left\langle a_1^{(n)} e_1 + \cdots + a_k^n e_k, e_j \right\rangle = a_j^{(n)}$$

and

$$\langle a_1 e_1 + \cdots + a_k e_k, e_j\rangle = a_j$$

so

$$\langle x_n, e_j\rangle \to \langle x, e_j\rangle \Rightarrow a_j^{(n)} \to a_j \qquad \forall y = 1, \ldots, k$$

Also,

$$\|x_n - x\| = \left\| \sum_{j=1}^{k} a_j^{(n)} e_j - \sum_{j=1}^{k} a_j e_j \right\| = \left\| \sum_{j=1}^{k} (a_j^{(n)} - a_j) e_j \right\|$$

$$\leq \sum_{j=1}^{k} |a_j^{(n)} - a_j| \, \|e_j\| \to 0$$

$$\Rightarrow \text{ strong convergence}$$

Equivalence of Norms

Theorem

Let $dim(\mathbb{X}) = k$ *and let* $\|\cdot\|$ *and* $\|\cdot\|_0$ *be two different norms on* $\mathbb{X}$ *then* $\exists a, b$ *such that*

$$a \|x\|_0 \leq \|x\| \leq b \|x\|_0$$

Proof.

(in book page 96) $\square$

Implication: For convergence proofs, it doesn't matter which norm you use, therefore, use the one that simplifies the proof.