ECEn 671: Mathematics of Signals and Systems

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Section 1

Matrix Norms

Matrix Norms

For matrices $A : \mathbb{C}^m \to \mathbb{C}^n$ we have the following induced norm:

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty$$

(Why max not sup?)

Lemma

$$\|A\|_\infty = \max_{i=1:m} \sum_{j=1:n} |a_{ij}|$$

i.e., the largest row sum.

Proof

First show that $\|A\|_\infty \leq \max_{i=1:m} \sum_{i=1:n} |a_{ij}|$:

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \left\| \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_\infty$$

$$= \max_{\|x\|_\infty=1} \left[\max \begin{pmatrix} \left| \sum_{j=1}^n a_{1j} x_j \right| \\ \vdots \\ \left| \sum_{j=1}^n a_{mj} x_j \right| \end{pmatrix} \right]$$

$$\leq \max_{x \text{ s.t. } \max|x_i|=1} \left[\max \left(\sum_{j=1}^n |a_{1j}|\,|x_j|, \quad \cdots, \quad \sum_{j=1}^n |a_{mj}|\,|x_j| \right) \right]$$

$$\leq \max_{\|x\|_\infty=1} \left[\max \left(\|x\|_\infty \sum_{j=1}^n |a_{1j}|, \quad \cdots, \quad \|x\|_\infty \sum_{j=1}^m |a_{mj}| \right) \right]$$

$$= \max_{i=1:m} \sum_{j=1}^m |a_{ij}|$$

Proof, cont.

Now we need to show that $\max_{i=1:m} \sum_{j=1:n} |a_{ij}| \leq \|A\|_\infty$:

Let $k = \underset{i=1:m}{\arg\max} \sum_{j=1:n} |a_{ij}|$
and let $\hat{x}$ be such that

$$\hat{x}_j = \begin{cases} 1 & \text{if} \quad a_{kj} \geq 0 \\ -1 & \quad \text{otherwise} \end{cases}$$

then $\|\hat{x}\|_\infty = 1$ and then

$$\|A\hat{x}\|_\infty = \max_{i=1:m} \sum_{j=1:n} |a_{ij}| \leq \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \|A\|_\infty .$$

Other Matrix Norms

Lemma

$$
\begin{aligned}
\|A\|_1 &= \max_{\|x\|_1=1} \|Ax\|_1 \\
&= \max_{j=1:n} \sum_{i=1}^{m} |a_{ij}| \quad \textit{(largest column sum)}
\end{aligned}
$$

Lemma

$$\|A\|_2 = \max_i \sqrt{\lambda_i(A^H A)} = \textit{ largest singular value of } A$$

More discussion of this in Chapter 7.

Norm of A^{-1}

Theorem

For induced matrix norms, where A^{-1} exists we have

$$\|A^{-1}\| = \frac{1}{\min\limits_{x\neq 0} \frac{\|Ax\|}{\|x\|}} = \frac{1}{\min\limits_{\|x\|=1} \|Ax\|}$$

Proof.

Let $Ax = b \Rightarrow x = A^{-1}b$ then

$$\|A^{-1}\| = \max_{b\neq 0} \frac{\|A^{-1}b\|}{\|b\|} = \max_{x\neq 0} \frac{\|x\|}{\|Ax\|} = \max_{x\neq 0} \frac{1}{\frac{\|Ax\|}{\|x\|}}$$

$$= \frac{1}{\min\limits_{x\neq 0} \frac{\|Ax\|}{\|x\|}} = \frac{1}{\min\limits_{\|x\|=1} \|Ax\|}$$

$\square$

Frobenius Norm

Definition

The Frobenius norm of a matrix is given by

$$\begin{aligned}\|A\|_F &= \left(\sum_{i=1}^{m}\sum_{j=1}^{m}|a_{ij}|^2\right)^{\frac{1}{2}} \\ &= \sqrt{tr(A^H A)}\end{aligned}$$

Fact: The Frobenius norm is NOT an induced norm.

Matrix Convergence

For matrices: convergence in any norm implies convergence in any other norm. In particular

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2$$
$$\max|a_{ij}| \leq \|A\|_2 \leq \sqrt{mn}\max|a_{ij}|$$
$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$$
$$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\,\|A\|_1$$