ECEn 671: Mathematics of Signals and Systems

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September 1, 2023

Section 1

Topology

Topology

- In this next section, we develop a set of tools that fall under that category of topology.
- These tools hold for metric spaces (including norm and inner product spaces).
- WARNING: There are a lot of definitions. These definitions will help talk formally about things in the future.

Topology: Open and Closed Sets

Definition (Ball)

Given a metric space $(\mathbb{X}, d)$ a δ-ball around x_0 is defined to be $B(x_0, \delta) = \{x \in \mathbb{X} : d(x, x_0) < \delta\}$

Definition (Interior Point)

A point $x_o \in \mathbb{X}$ is interior to $S \subset \mathbb{X}$ if $\exists \delta > 0$ such that $B(x_o, \delta) \subset S$.

Definition (Open Set)

A set $\mathbb{X}$ is <u>open</u> if all points in $\mathbb{X}$ are interior.

Definition (Closed Set)

A set S is closed in $\mathbb{X}$ if $\mathbb{X} \setminus S$ is open.

Topology: Convergence

Let $(\mathbb{X}, d)$ be a metric space.

Definition (Convergence)

Given a sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n \in \mathbb{X}$, the following are equivalent

- $\lim_{n\to\infty} x_n = x^*$
- $x_n \to x^*$
- $\forall \epsilon > 0, \exists N(\epsilon)$ such that $n \geq N \Rightarrow d(x_n, x^*) < \epsilon$

A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{X}$ with a limit $x^* \in \mathbb{X}$ is said to converge.

Topology: Convergence

Note that a limit may not always exist (similar to min, max)
For example, $\lim_{t\to\infty} sin(t)$ does not exist.

Definition (lim sup)

Define lim sup as the largest limit (possibly infinity) of any subsequence.

Definition (lim inf)

Define lim inf is the smallest limit of all possible subsequences.

Example

- $\limsup_{t\to\infty} \sin(t) = 1$ since the subsequence $t_n = \frac{k\pi}{2}, k = 1, 5, 9, \cdots$ converges to 1
- $\liminf_{t\to\infty} \sin(t) = -1$ since the subsequence $t_n = \frac{k\pi}{2}, k = 3, 7, 11, \cdots$ converges to -1

Topology: Cauchy Sequence

Definition (Cauchy Sequence)

A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space $(\mathbb{X}, d)$ is said to be a Cauchy sequence if $\forall \epsilon > 0, \exists N(\epsilon) > 0$ such that $n, m > N \Rightarrow d(x_n, x_m) < \epsilon$

A sequence is Cauchy if elements in its tail get increasingly closer together. Note that we have not said anything about an element of convergence.

Theorem

If a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{X}$ converges to an element $x^* \in \mathbb{X}$ then it is a Cauchy sequence.

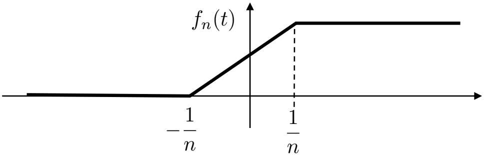
The converse is not true!! I.e., not all Cauchy sequences converge.

Topology: Cauchy Sequence

Example (from book)

Let $\mathbb{X} = C[-1,1]$ and $d(f,g) = \left(\int_{-1}^{1} (f(t) - g(t))^2 dt\right)^{\frac{1}{2}}$

let f_n :



By integration we get:

$$d(f_n, f_n) = \frac{1}{6m^3 n}(m^3 + 4m^2 n + mn^2 + 2n^3)$$

$$\to 0 \text{ for } n, m \text{ large } (m > n)$$

but f_n converges to a discontinuous function which is not in $\mathbb{X}$.
This is undesirable

Topology: Complete Metric Space

Definition (Complete metric space)

A metric space $(\mathbb{X}, d)$ is complete if every Cauchy sequence in $\mathbb{X}$ converges to a value in $\mathbb{X}$.

Implication

$C[a, b]$ with metric $(\int_a^b |f - g|^2 \, dt)^{1/2}$ is not complete.

- Banach spaces are complete normed spaces (discussed later).
- Hilbert spaces (extremely important in signal processing and control) are complete inner product spaces (discussed later).
- The importance of L_p and ℓ_p are that they are complete spaces.