ECEn 671: Mathematics of Signals and Systems

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Section 1

Adjoint Operators

Adjoint Operator

Definition

Let $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator from Hilbert space $\mathbb{X}$ to Hilbert space $\mathbb{Y}$, then the adjoint of $\mathcal{A}$ ($\mathcal{A}^*$) is the linear operator $\mathcal{A}^* : \mathbb{Y} \to \mathbb{X}$ such that

$$\langle \mathcal{A}x, y \rangle_{\mathbb{Y}} = \langle x, \mathcal{A}^* y \rangle_{\mathbb{X}}$$

$\forall x \in \mathbb{X}$ and $\forall y \in \mathbb{Y}$.

$\mathcal{A}$ is self-adjoint if $\mathcal{A}^* = \mathcal{A}$

Adjoint Operator, Example

Example (Complex matrices)

$A : \mathbb{C}^n \to \mathbb{C}^m$

What is A^*?

By definition:

$$
\begin{aligned}
& \langle Ax, y\rangle_{\mathbb{C}^m} = \langle x, A^* y\rangle_{\mathbb{C}^n} \\
\iff & y^H A x = y^H (A^*)^H x \\
\iff & A^* = A^H
\end{aligned}
$$

Note $A^H : \mathbb{C}^m \to \mathbb{C}^n$

Adjoint Operator, Example

Example (Real matrices)

$A : \mathbb{R}^n \to \mathbb{R}^m$

What is A^*?

By definiton,

$$
\begin{aligned}
& \langle Ax, y \rangle_{\mathbb{R}^m} = \langle x, A^* y \rangle_{\mathbb{R}^n} \\
\iff & x^\top A^\top y = x^\top A^* y \\
\iff & A^* = A^\top
\end{aligned}
$$

Adjoint Operator, Example

Example (Convolution)

$$\mathcal{A} : L_2 \to L_2$$

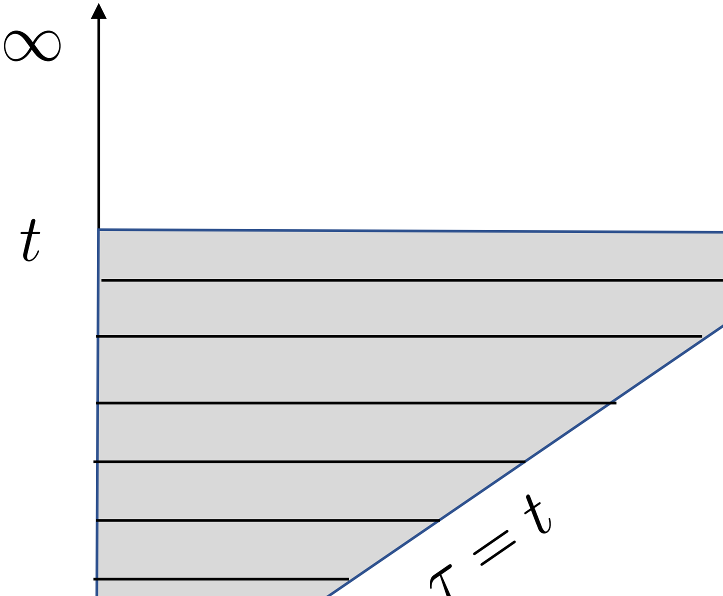
$$\mathcal{A}[x](t) = \int_0^{\top} h(t-\tau)x(\tau)d\tau$$

Let $x \in L_2[0,\infty]$ and $y \in L_2[0,\infty]$ then $\mathcal{A}^*$ is defined by

$$\langle \mathcal{A}x, y \rangle_{L_2} = \langle x, \mathcal{A}^* y \rangle_{L_2}$$

$$\iff \int_{t=0}^{\infty} \left[\int_{\tau=0}^{t} h(t-\tau)x(\tau)d\tau \right] y(t)dt = \int_0^{\infty} x(t)\mathcal{A}^*[y](t)dt$$

Adjoint Operator, Example, Convolution, cont.



Adjoint Operator, Example

Example (linear ode's)

$$\dot{x} = Fx \quad ; \quad x(0) = x_0$$

The solution is $x(t) = e^{Ft}x_0$
Let $\mathcal{A}[x_0](t) = e^{Ft}x_0$, then

$$\mathcal{A} : \mathbb{R}^n \to L_{2[0,T]}$$

What is $\mathcal{A}^*$?

Adjoint Operator, Example, linear ODE, cont.

Let $x \in \mathbb{R}^n$ and let $y \in L_2[0, T]$ then by definition,

$$\begin{aligned}
\langle \mathcal{A}[x_0], y \rangle_{L_2[0,T]} &= \langle x_0, \mathcal{A}^* y \rangle_{\mathbb{R}^n} \\
\iff \int_0^T x_0^\top (e^{Ft})^\top y(t)dt &= x_0^\top \mathcal{A}^* y \\
\iff x_0^\top \int_0^T e^{F^\top t} y(t)dt &= x_0^\top \mathcal{A}^* y \\
\Rightarrow \boxed{\mathcal{A}^*[y] = \int_0^T e^{F^\top t} y(t)dt}
\end{aligned}$$