ECEn 671: Mathematics of Signals and Systems

Randal W. Beard

Brigham Young University

September 1, 2023

Section 1

Singular Value Decomposition

Singular Value Decomposition

Theorem (Moon Theorem 7.1)

Every matrix $A \in \mathbb{C}^{m \times n}$ can be factored as $A = U \Sigma V^H$ where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary and $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with diagonal elements $\sigma_1 \geq \sigma_2 \geq cdots \geq \sigma_p \geq 0$.

The diagonal elements are called the singular values of A. If A is real then U and V are real and orthogonal.

Singular Value Decomposition, Proof

Note that the A^HA is Hermitian, and positive definite because $x^HA^HAx = \|Ax\|^2 \geq 0 \qquad \forall x \in \mathbb{C}^n$.

So, from Chapter 6 we know that the eigenvalues of A^HA are real with $m_i = q_i$ for each λ_i.

Let $(\lambda_i, \mathbf{v}_i)$ be an eigenpairs of A^HA then

$$A^HAV = V\Lambda \qquad\qquad V\text{-unitary}$$

where

$$V = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}, \qquad\qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Singular Value Decomposition, Proof

Since the $rank(A^H A) \leq \min(m, n) = p$, then number of non-zero eigenvalues is $r \leq p$.
For $1 \leq i \leq r$ let $\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sqrt{\lambda_i}}$.
Then

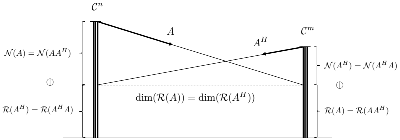
$$\begin{aligned}
\langle \mathbf{u}_i, \mathbf{u}_j \rangle &= \left\langle \frac{A\mathbf{v}_i}{\sqrt{\lambda_i}}, \frac{A\mathbf{v}_j}{\sqrt{\lambda_j}} \right\rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} \mathbf{v}_i^H A^H A \mathbf{v}_j \\
&= \frac{\lambda_j}{\sqrt{\lambda_i \lambda_j}} \mathbf{v}_i^H \mathbf{v}_j = \delta_{ij}
\end{aligned}$$

Use Gram-Schmidt to extend $\mathbf{u}_1, \ldots, \mathbf{u}_r$ to $[\mathbf{u}_1, \ldots, \mathbf{u}_m]$ such that $U = [\mathbf{u}_1, \ldots \mathbf{u}_m]$ is unitary.

Singular Value Decomposition, Proof

Lemma

If $(\lambda_i, \mathbf{v}_i)$ *is an eigenpair of* $A^H A$*, then* $\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sqrt{\lambda_i}}$ *are eigenvectors of* AA^H.



Proof.

Note that since $\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A\mathbf{v}_i \qquad i = 1, \ldots, p$ then

$$\begin{aligned}
& \mathbf{u}_i \in \mathcal{R}(A) \qquad i = 1, \ldots, p \\
\Longrightarrow\ & \mathbf{u}_i \in \mathcal{N}(A^H) \qquad i = p+1, \ldots, m \\
\Longrightarrow\ & \mathbf{u}_i \in \mathcal{N}(AA^H) \qquad i = p+1, \ldots, m \\
\Longrightarrow\ & AA^H \mathbf{u}_i = 0 \cdot \mathbf{u}_i = 0 \\
\Longrightarrow\ & (0, \mathbf{u}_i) \text{ is an eigenpair of } AA^H \qquad i = p+1, \ldots, m
\end{aligned}$$

Singular Value Decomposition, Proof

Now lets look at

$$U^H AV = \begin{pmatrix} \mathbf{u}_1^H \\ \vdots \\ \mathbf{u}_m^H \end{pmatrix} A \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^H A\mathbf{v}_1 & \cdots & \mathbf{u}_1^H A\mathbf{v}_n \\ \vdots & \ddots & \vdots \\ \mathbf{u}_m^H A\mathbf{v}_1 & \cdots & \mathbf{u}_m^H \mathbf{v}_n \end{pmatrix}.$$

The $(i,j)^{th}$ element of $U^H AV$ is $\mathbf{u}_i^H A\mathbf{v}_j$.

If $i \leq p$ then

$$\begin{aligned} \mathbf{u}_i^H A\mathbf{v}_j &= \frac{1}{\sqrt{\lambda_i}} \mathbf{v}_i^H A^H A\mathbf{v}_j \\ &= \frac{\lambda_j}{\sqrt{\lambda_i}} \mathbf{v}_i^H \mathbf{v}_j = \sqrt{\lambda_j}\delta_{ij} \end{aligned}$$

Singular Value Decomposition, Proof

If $i > p$, then

$$\begin{aligned} \mathbf{u}_i \in \mathcal{N}(A^H) &\Rightarrow A^H \mathbf{u}_i = 0 \\ &\Rightarrow \mathbf{u}_i^H A = 0 \\ &\Rightarrow \mathbf{u}_i^H A \mathbf{v}_j = 0 \end{aligned}$$

Therefore

$$U^H A V = \Sigma$$

where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$ is real and diagonal, where $\sigma_j = 0$ when $j > p$. Therefore

$$A = U \Sigma V^H$$

as required. $\square$

Singular Value Decomposition

Note that the singular values of A are the square root of the eigenvalues of $A^H A$ and AA^H.

Also note that we can write

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\Sigma_1 = \underbrace{\mathrm{diag}(\sigma_1, \ldots, \sigma_p)}_{\mathbb{R}^{r \times r}}$$

$$\Sigma_2 = 0$$

Singular Value Decomposition

Then

$$\begin{aligned} A &= \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_2^H \end{pmatrix} \\ &= \underbrace{U_1}_{m\times p} \underbrace{\Sigma_1}_{p\times p} \underbrace{V_1^H}_{n\times p} \qquad \leftarrow \text{alternate form of SVD} \\ &= \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{v}_i^H \qquad \leftarrow \text{alternate form of SVD} \end{aligned}$$

where $\mathbf{u}_i$'s are orthonormal and $\mathbf{v}_i$'s are orthonormal.

Singular Value Decomposition and Matrix Norm

Note that

$$
\begin{aligned}
\|A\|_2 &= \sup_{\|x\|_2=1} \|Ax\|_2 = \sup_{\|x\|_2=1} \sqrt{x^H A^H A x} \\
&= \sup_{\|x\|_2=1} \sqrt{x^H V_1 \Sigma_1 U_1^H U_1 \Sigma_1 V_1^H x} \\
&= \sup_{\|x\|_2=1} \sqrt{x^H V_1 \Sigma_1^2 V_1^H x} \\
&= \sup_{\|x\|_2=1} \sqrt{\begin{pmatrix} x^H \mathbf{v}_1 & \cdots & x^H \mathbf{v}_r \end{pmatrix} \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_p^2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^H x \\ \vdots \\ \mathbf{v}_p^H x \end{pmatrix}} \\
&= \sigma_1,
\end{aligned}
$$

where the minimizer is $x = \mathbf{v}_1$.

Singular Value Decomposition and Rank

Lemma

If $A \in \mathbb{C}^{m \times n}$, then rank$(A) = p$ where p is the number of non-zero singular values.

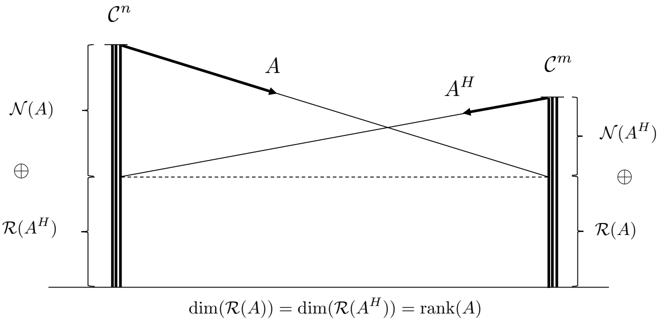
Proof.

$$\text{rank}(A) = \text{rank}(U\Sigma V^H) = \text{rank}(\Sigma)$$

since U and V are both full rank. Clearly $\text{rank}(\Sigma) = p$. $\square$

Singular Value Decomposition and Fundamental Subspaces

Fundamental subspace diagram:



Question: What information does the SVD provide?

Answer: The SVD completely characterizes all of the spaces.

Singular Value Decomposition and Fundamental Subspaces

Given that

$$A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_2^H \end{pmatrix} = U_1 \Sigma_1 V_1^H.$$

Let $y \in \mathcal{R}(A)$, then $\exists x \in \mathbb{C}^n$ such that $y = Ax$. Which implies that

$$\begin{aligned} y &= U_1 \Sigma_1 V_1^H x \\ &= U_1 z \text{ where } z = \Sigma_1 V_1^H x \\ &= [\mathbf{u}_1 \cdots \mathbf{u}_p] \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} = \mathbf{u}_1 z_1 + \cdots + \mathbf{u}_p z_p \\ \implies & y \in span\{\mathbf{u}_1, \cdots, \mathbf{u}_p\} \\ \implies & \boxed{\mathcal{R}(A) = span(U_1)} \end{aligned}$$

Singular Value Decomposition and Fundamental Subspaces

Since the columns of U_2 are orthonormal to U_1 and $\text{span}(U) = \mathbb{C}^m$ and $\mathcal{R}(A) \oplus \mathcal{N}(A^H) = \mathbb{C}^m$ we must have that

$$\boxed{\mathcal{N}(A^H) = \text{span}(U_2)}$$

A similar argument shows that

$$\boxed{\mathcal{R}(A^H) = \text{span}(V_1)}$$

$$\boxed{\mathcal{N}(A) = \text{span}(V_2)}$$

Singular Value Decomposition and Fundamental Subspaces

Therefore, the fundamental subspace diagram becomes