ECEn 671: Mathematics of Signals and Systems

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September 1, 2023

Section 1

SVD and Numerically Sensitive Problems

Numerically Sensitive Problems

Suppose that we would like to solve

$$Ax = b$$

where $A \in \mathbb{R}^{n\times n}$ and $rank(A) = n$ but the condition number $\mathcal{K}(A)$ is large. Let $A = U\Sigma V^H$, then

$$\begin{aligned} A^{-1} &= V\Sigma^{-1}U^H \\ &= \sum_{j=1}^{n} \frac{\mathbf{v}_j \mathbf{u}_j^H}{\sigma_j} \end{aligned}$$

so the solution to $Ax = b$ is

$$x = A^{-1}b = \sum_{j=1}^{n} \frac{\mathbf{v}_j \mathbf{u}_j^H}{\sigma_j}.$$

Numerically Sensitive Problems

Recall that $\mathcal{K}(A) = \|A\| \left\|A^{-1}\right\|$ where $\|A\| = \sigma_{\max}(A)$ and $\left\|A^{-1}\right\| = \dfrac{1}{\min_{\|x\|} \|A\|} = \dfrac{1}{\sigma_{\min}(A)}$. Therefore

$$\mathcal{K}(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}$$

Therefore a large $\mathcal{K}(A)$ implies there is significant difference between the largest and smallest singular values.

Numerically Sensitive Problems

For example $\sigma_{\min}(A)$ may be very small, therefore given

$$x = \sum_{j=1}^{n} \frac{\mathbf{v}_j \mathbf{u}_j^H}{\sigma_j} b$$

x is very sensitive to small change in b due to the terms in the sum that have very small singular values.

Solution: Zero out small singular values to get the approximate solution

$$Ax = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_2^H \end{pmatrix} x \approx \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_2^H \end{pmatrix} x$$

so

$$x = V_1 \Sigma_1^{-1} U_1^H b$$

is an approximate solution that is numerically stable.

Numerically Sensitive Problems

- Moon Example 7.4.1 shows that if σ_j-small then the vector $\mathbf{u}_j \in \mathbb{R}^m$ defines a sensitive direction for b. i.e. if b is almost parallel with $\mathbf{u}_j$ then $x = \frac{\mathbf{v}_j \mathbf{u}_j^H}{\sigma_j} b$ is clearly sensitive to small changes in b. If b is perpendicular to $\mathbf{u}_j$ then $\mathbf{u}_j^H b = 0$ and we are ok.
- If A comes from noisy data (almost always) then A will usually be full rank, even if the original data that produced A would have resulted in a lower rank A if it wasn't corrupted by noise.
- But the nonzero singular values added by noise will usually be small.
- Therefore, an effective way to reduce the rank of A to get rid of the effect of noise is to zero the "small" singular values.