ECEn 671: Mathematics of Signals and Systems

Randal W. Beard

Brigham Young University

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Section 1

LU Factorization

LU Factorization

- Suppose that $A \in \mathbb{C}^{n \times n}$ is full rank. What is a numerically efficient method for computing the solution to $Ax = b$, i.e. $x = A^{-1}b$?
- An explicit formula is:

$$x = \frac{adj(A)b}{det(A)}$$

  but this requires numerical computation of determinants.
- LU factorization is more efficient.

LU Factorization: Basic Idea

- Find a permutation matrix P, a lower diagonal matrix with ones on the diagonal L, and an upper diagonal matrix U such that
$$PA = LU.$$
- How? Will illustrate by example:

LU Factorization: cont.

Let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix}$$

The idea is to perform row reductions to get a triangular matrix.
Key Idea: Reduce the row with the largest element.

LU Factorization: cont.

First, permute A to get the third row on top:

$$P_{13}A = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_{13}} \begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -8 & 9 \\ -4 & 5 & -6 \\ 1 & -2 & 3 \end{pmatrix}$$

The idea is that you always want to divide by the largest element (in absolute value) in the row to avoid numerical problems.

LU Factorization: cont.

Now zero out the -4 and 1 by multiplying the first row by $+\frac{4}{7}$ and adding to the second row and multiplying the first row by $-\frac{1}{7}$ and adding to the third row:

$$E_1 P_{13} A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{7} & 1 & 0 \\ \frac{-1}{7} & 0 & 1 \end{pmatrix}}_{E_1} \begin{pmatrix} 7 & -8 & 9 \\ -4 & 5 & -6 \\ 1 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -8 & 9 \\ 0 & 0.4286 & -5.4286 \\ 0 & -0.8571 & 2.8571 \end{pmatrix}$$

LU Factorization: cont.

Now permute (or "pivot") to get the largest (in absolute value) number in the second column in the second row:

$$P_{23}E_1P_{13}A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P_{23}} \begin{pmatrix} 7 & -8 & 9 \\ 0 & 0.4286 & -5.4286 \\ 0 & -0.8571 & 2.8571 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -8 & 9 \\ 0 & -0.8571 & 2.8571 \\ 0 & 0.4286 & -5.4286 \end{pmatrix}$$

LU Factorization: cont.

Zero out the 0.4286 by multiplying the second row by $\frac{0.4286}{0.8571}$ and adding to the third row:

$$
\begin{aligned}
E_2 P_{23} E_1 P_{13} A &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{0.4286}{0.8571} & 1 \end{pmatrix}}_{E_2} \begin{pmatrix} 7 & -8 & 9 \\ 0 & -0.8571 & 2.8571 \\ 0 & 0.4286 & -5.4286 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -8 & 9 \\ 0 & -0.8571 & 2.8571 \\ 0 & 0 & -4 \end{pmatrix} \\
&= U
\end{aligned}
$$

Therefore

$$
\begin{aligned}
A &= (E_2 P_{23} E_1 P_{13})^{-1} U \\
&= P_{13}^{-1} E_1^{-1} P_{23}^{-1} E_2^{-1} U
\end{aligned}
$$

LU Factorization: cont.

Note that if $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{7} & 1 & 0 \\ -\frac{1}{7} & 0 & 1 \end{pmatrix}$, then $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{7} & 1 & 0 \\ \frac{1}{7} & 0 & 1 \end{pmatrix}$

since

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{7} & 1 & 0 \\ -\frac{1}{7} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{7} & 1 & 0 \\ \frac{1}{7} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the inverse of any lower diagonal matrix formed by multiplying and adding rows is found by negating the off-diagonal terms.

Therefore E_1^{-1} and E_2^{-1} are easy to compute.

LU Factorization: cont.

Also note that for permutation matrices

$$P_{ij}^{-1} = P_{ji}$$

since

$$\underbrace{P_{ij}}_{\text{switch } ij \text{ rows}} \quad \underbrace{P_{ij}^{-1}}_{\text{switch back}} = I.$$

For example

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

LU Factorization: cont.

So we have $A = VU$ where
$V = P_{13}E_1^{-1}P_{23}E_2^{-1} = \begin{pmatrix} 0.1429 & 1 & 0 \\ -0.5714 & -0.5 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Note that V is not lower triangular but

$$L = P_{23}P_{13}V = P_{23}\begin{pmatrix} 1 & 0 & 0 \\ -0.5714 & -0.5 & 1 \\ 0.1429 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0.1429 & 1 & 0 \\ -0.5714 & -0.5 & 1 \end{pmatrix}$$

is, so $P_{23}P_{13}A = P_{23}P_{13}VU$. Therefore

$$PA = LU$$

where $P = P_{23}P_{13}$.

LU Factorization: cont.

For our example we have

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0.1429 & 1 & 0 \\ -0.5714 & -0.5 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 7 & -8 & 9 \\ 0 & -0.8571 & 2.8571 \\ 0 & 0 & -4 \end{pmatrix}}_{U}$$

How do we solve the equation $Ax = b$?

Suppose $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Note that

$$PAx = Pb = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

So that

$$LUx = Pb.$$

LU Factorization: cont.

Let $y = Ux$ then

$$Ly = Pb$$

$$\Longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0.1429 & 1 & 0 \\ -0.5714 & -0.5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\Longrightarrow \quad \begin{cases} y_1 & = 3 \\ y_2 & = 1 - 0.1429 y_1 \\ y_3 & = 2 + 0.5714 y_1 + 0.5 y_2 \end{cases} \qquad \text{(easy to solve)}$$

$$\Longrightarrow \quad \begin{cases} y_1 & = 3 \\ y_2 & = 0.5741 \\ y_3 & = 4 \end{cases} \qquad \text{(easy to solve)}$$

LU Factorization: cont.

Next solve $Ux = y$ for x:

$$
\begin{aligned}
& Ux = y \\
\Longrightarrow \quad & \begin{pmatrix} 7 & -8 & 9 \\ 0 & -0.8571 & 2.8571 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0.5714 \\ 4 \end{pmatrix} \\
\Longrightarrow \quad & \left\{ \begin{aligned} -4x_3 &= 4 \\ -0.8571x_2 &= 0.5714 - 2.8571x_2 \\ 7x_1 &= 3 + 8x_2 - x_3 \end{aligned} \right. \qquad \text{(easy to solve)} \\
\Longrightarrow \quad & \left\{ \begin{aligned} x_1 &= -4 \\ x_2 &= -4 \\ x_3 &= -1 \end{aligned} \right.
\end{aligned}
$$

LU Factorization: cont.

In Matlab:

```
A = [1, 2, 3; 4, 5, 6; 7, 8, 0];
[L, U, P] = lu(A)
```

In Python:

```
import numpy as np
import scipy.linalg as linalg

A = np.array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
P, L, U = linalg.lu(A)
```

Homework problem: Write your own custom `lu` function and compare to the built in `lu` function on 100 randomly generated matrices.