ECEn 671: Mathematics of Signals and Systems

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Section 1

Inner Product Spaces

Inner Product Spaces

Definition (Inner Product)

Let S be a vector space over $\mathbb{R}$. An inner product $< \cdot, \cdot >: S \times S \to \mathbb{R}$ has the following properties:

$$
\begin{array}{ll}
(IP1) & \langle x, y \rangle = \overline{\langle y, x \rangle} \\
(IP2) & \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \\
(IP3) & \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \\
(IP4) & \langle x, x \rangle > 0 \quad \text{if } x \neq 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0
\end{array}
$$

Definition (Inner Product Space)

A vector space with an inner product defined is called an inner-product space.

Definition (Hilbert Space)

A complete inner-product space is called a Hilbert space.

Inner Product Spaces: Examples

- $\mathbb{R}^n$: $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y$ is called the Euclidean inner product.
- $\mathbb{C}^n$: $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} = y^H x$

- $\mathbb{R}^n$ with the Euclidean inner product is a Hilbert space .
- $\mathbb{C}^n$ with the Euclidean inner product is a Hilbert space.
- All finite-dimensional inner-product spaces are Hilbert spaces.

Inner Product Spaces: Examples

- Real sequences ℓ_2: $\langle x, y \rangle_{\ell_2} = \sum_{i=1}^{\infty} x_i y_i$
- Complex sequences ℓ_2: $\langle x, y \rangle_{\ell_2} = \sum_{i=1}^{\infty} x_i \overline{y_i}$
- Both of these examples are Hilbert spaces.

Inner Product Spaces: Examples

- Complex function space $L_2^n(\Omega)$ with inner product:

$$\langle x, y \rangle = \int_{\infty} y^H(t)x(t)\,dt$$

  is a Hilbert space, but
- Continuous function $C[a, b]$ with the same inner product is NOT a Hilbert space.

Norms vs Inner Products

Every inner product defines a norm (but not vice-versa)

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

where $\|\cdot\|$ is called the norm induced by the inner product $\langle \cdot, \cdot \rangle$.

Examples of inducted norms

$$\|\cdot\|_2 : \ \langle x, x\rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \|x\|_2$$

$$\|\cdot\|_{\ell_2} : \ \langle x, x\rangle^{1/2} = \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2} = \|x\|_{\ell_2}$$

$$\|\cdot\|_{L_2} : \ \langle x, x\rangle^{1/2} = \left(\int_{\Omega} x^T(t)x(t)dt\right)^{1/2} = \left(\int_{\Omega} \|x(t)\|_2^2 \, dt\right)^{1/2} = \|x\|_{L_2}$$

Note that induced norms are all $2-$norms.

Cauchy-Schwartz Inequality

Theorem (Cauchy-Schwartz)

Let S be any inner product space (doesn't need to be Hilbert) and let $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ then $\forall x, y \in S$

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

with equality iff $y = \alpha x$ where $\alpha \in \mathbb{F}$ is any scalar in the field $\mathbb{F}$.

Cauchy-Schwartz Inequality: Proof

The inequality clearly holds if either $x = 0$ or $y = 0$. Therefore assume that $x \neq 0$ and $y \neq 0$. Then

$$
\begin{aligned}
\|x - \alpha y\|^2 &= \langle x - \alpha y, x - \alpha y \rangle \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \langle x, \alpha y \rangle + \langle \alpha y, \alpha y \rangle \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\overline{\langle x, \alpha y \rangle}} + \alpha \overline{\overline{\langle y, \alpha y \rangle}} \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\langle \alpha y, x \rangle} + \alpha \overline{\langle \alpha y, y \rangle} \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\alpha} \overline{\langle y, x \rangle} + \alpha \overline{\alpha} \overline{\langle y, \alpha y \rangle} \\
&= \langle x, x \rangle - \alpha \langle y, x \rangle - \overline{\alpha} \langle x, y \rangle + |\alpha|^2 \langle y, y \rangle \\
&= \|x\|^2 - \alpha \langle y, x \rangle - \overline{\alpha} \langle x, y \rangle + |\alpha|^2 \|y\|^2
\end{aligned}
$$

Cauchy-Schwartz Inequality: Proof

Recall the technique of completing the square:

$$
\begin{aligned}
ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\
&= a(x + \frac{b}{2a})^2 - \frac{b}{4a}^{2} + c.
\end{aligned}
$$

Complete the square in α:

$$
\begin{aligned}
\|x - \alpha y\|^2 &= \|y\|^2 \left(\alpha\overline{\alpha} - \alpha \frac{\overline{\langle x, y \rangle}}{\|y\|^2} - \bar{\alpha} \frac{\langle x, y \rangle}{\|y\|^2} \right) + \|x\|^2 \\
&= \|y\|^2 \left(\alpha - \frac{\langle x, y \rangle}{\|y\|^2} \right) \left(\bar{\alpha} - \frac{\overline{\langle x, y \rangle}}{\|y\|^2} \right) - \frac{|\langle x, y \rangle|^2}{\|y\|^2} + \|x\|^2
\end{aligned}
$$

Cauchy-Schwartz Inequality: Proof

Let $\alpha^* = \frac{\langle x,y \rangle}{\|y\|^2}$ to get

$$
\begin{aligned}
0 \leq \|x - \alpha^* y\|^2 &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\
\Rightarrow \qquad |\langle x, y \rangle|^2 &\leq \|x\|^2 \, \|y\|^2 \\
\Rightarrow \qquad |\langle x, y \rangle| &\leq \|x\| \, \|y\|
\end{aligned}
$$