ECEn 671: Mathematics of Signals and Systems

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Section 1

Dual Approximation

Dual Approximation

This section develops an approach that allows approximation in infinite dimensional spaces with finite constraints.

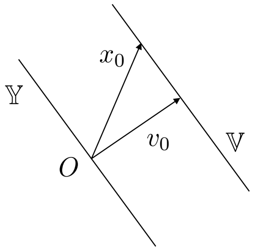
For matrices, we will solve the problem

$$\begin{aligned} \min \quad & \|x\| \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Dual Approximation, cont.

Definition (Affine Space)

Let $\mathbb{Y}$ be a subspace of $\mathbb{S}$ and let $x_o \in \mathbb{S}$. The set $\mathbb{V} = x_0 + \mathbb{Y}$ is called a <u>linear variety</u> or an <u>affine</u> space.



The projection theorem says that there exists a $v_0 \in \mathbb{V}$ such that $v_0 = \arg\min_{v \in \mathbb{V}} \|v\|$ such that $v_0 \perp \mathbb{Y}$.

Dual Approximation, cont.

Let $M = span\{y_1, \ldots, y_m\}$ then $\dim(M) < \infty$.

If $\dim(\mathbb{S}) = \infty$ then $\dim(M^{\perp}) = \infty$ where $M^{\perp}$ is the set of all $x \in \mathbb{S}$ such that

$$\langle x, y_1 \rangle = 0$$
$$\vdots$$
$$\langle x, y_m \rangle = 0$$

Dual Approximation, cont.

Now suppose that there are m inner product constraints:

$$\langle x, y_1 \rangle = a_1$$
$$\vdots$$
$$\langle x, y_m \rangle = a_n$$

If $\exists x_0$ that satisfies the constraints then so does $x_0 + v$ where $v \in M^\perp$ since

$$\begin{aligned} \langle x_0 + v, y_j \rangle &= \langle x_0, y_j \rangle + \langle v, y_j \rangle \\ &= \langle x_0, y_j \rangle \\ &= a_j \end{aligned}$$

Therefore all solutions are in the (infinite dimensional) affine space

$$v = x_0 + M^\perp$$

Dual Approximation, cont.

Theorem (Moon Theorem 3.4)

Let $\{y_1, \cdots, y_m\}$ be linearly independent in a Hilbert space $\mathbb{S}$, and let $M = span\{y_1, \cdots, y_m\}$. The solution of the problem

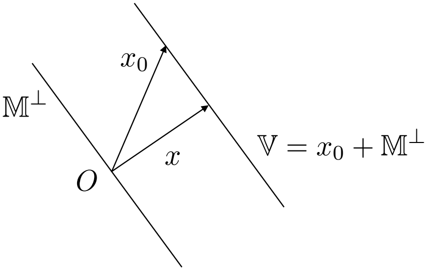
$$\begin{aligned} \min_{x \in \mathbb{S}} \quad & \|x\|^2 \\ \text{s.t.} \quad & \langle x, y_1 \rangle = \alpha_1, \\ & \vdots, \\ & \langle x, y_m \rangle = \alpha_m \end{aligned}$$

is an element of M, i.e., $\hat{x} = \arg\min_{x \in \mathbb{S}} \|x\|^2 = \sum_{i=1}^m c_i y_i$, where $\mathbf{c}$ satisfies $R\mathbf{c} = \boldsymbol{\alpha}$, where R is the Grammian and

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)^\top.$$

Proof:

From the previous discussion, the solution lies in the affine space $\mathbb{V} = x_0 + M^\perp$ for some $x_0 \in \mathbb{S}$.



The minimum norm solution is orthogonal to $M^\perp$ i.e. $\hat{x} \perp M^\perp \Rightarrow \hat{x} \in M^{\perp\perp} = M$

So $\hat{x}$ is of the form $\hat{x} = \sum_{j=1}^{m} c_j y_j$

Proof, cont.

Now projecting x onto M gives

$$\begin{aligned}
\langle \hat{x}, y_1 \rangle &= \left\langle \sum c_j y_j, y_1 \right\rangle &&= \sum c_j \langle y_j, y_1 \rangle = \alpha_1 \\
\vdots \quad &= \quad \vdots &&= \quad \vdots \quad = \quad \vdots \\
\langle \hat{x}, y_m \rangle &= \left\langle \sum c_j y_j, y_m \right\rangle &&= \sum c_j \langle y_j, y_m \rangle = \alpha_m
\end{aligned}$$

rewriting in matrix notation gives

$$R\mathbf{c} = \boldsymbol{\alpha}$$

Dual Approximation, Example

Given the differential equation

$$\ddot{y} + 6\dot{y} + 8y = 4\dot{u} + 10u, \qquad y(0) = \dot{y}(0) = 0$$

Solve the following optimal control problem:

$$\begin{aligned} \min_{u \in L_2} \quad & \|u\|^2 \\ \text{s.t.} \quad & y(1) = 1, \\ & \int_0^1 y(t)dt = 0 \end{aligned}$$

Dual Approximation, Example, cont.

The corresponding transfer function is

$$H(s) = \frac{4s+10}{s^2+6s+8} = \frac{1}{s+2} + \frac{3}{s+4}$$
$$\Rightarrow h(t) = e^{-2t} + 3e^{-4t}$$
$$\Rightarrow y(t) = \int_0^t \left[e^{-2(t-\tau)} + 3e^{-4(t-\tau)}\right] u(\tau) d\tau$$

Define the following inner product

$$\langle f(t), g(t) \rangle = \int_0^1 f(\tau) g(\tau) d\tau$$

then $y(1) = 1$ can be written as

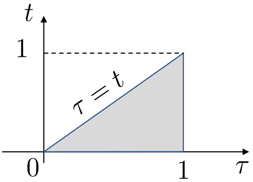
$$\int_0^1 \left[e^{-2(1-\tau)} + 3e^{-4(1-\tau)}\right] u(\tau) d\tau = \langle u, y_1 \rangle = 1$$

where $y_1(t) = e^{-1(1-t)} + 3e^{-4(1-t)}$

Dual Approximation, Example, cont.

The second constraint is of the form

$$\int_0^1 y(t)dt = \int_{t=0}^{t=1} \int_{\tau=0}^{\tau=t} h(t-\tau)u(\tau)d\tau dt = 0$$



Changing order of integration gives

$$= \int_{\tau=0}^{1} \left[\int_{t=\tau}^{1} h(t-\tau)dt \right] u(\tau)d\tau.$$

Dual Approximation, Example, cont.

Letting $\sigma = t - \tau \Rightarrow t = \sigma + \tau \Rightarrow dt = d\sigma$ gives

$$= \int_{\tau=0}^{1} \left[\int_{\sigma=0}^{\sigma=1-\tau} h(\sigma) d\sigma \right] u(\tau) d\tau$$

$$= \int_{\tau=0}^{1} \left(\frac{5}{4} - \frac{3}{4} e^{-4(1-\tau)} - \frac{1}{2} e^{-2(1-\tau)} \right) u(\tau) d\tau$$

$$= \langle u, y_2 \rangle = 0$$

where

$$y_2(t) = \frac{5}{4} - \frac{3}{4} e^{-4(1-\tau)} - \frac{1}{2} e^{-2(1-\tau)}$$

so we have that

$$\langle u, y_1 \rangle = 1$$
$$\langle u, y_2 \rangle = 0$$

and we want to minimize $\|u\|^2_{L_2[0,1]}$

Dual Approximation, Example, cont.

Let $M = span\{y_1, y_2\}$.
By Theorem 3.4

$$u \in M \Rightarrow u(t) = c_1 y_1(t) + c_2 y_2(t)$$

where

$$\begin{pmatrix} \langle y_1, y_1 \rangle & \langle y_2, y_1 \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$