ECEn 671: Mathematics of Signals and Systems

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Section 1

Self Adjoint Matrices

Self Adjoint Matrices

Definition

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be self adjoint (also called Hermitian) if

$$A = A^H.$$

Lemma (Moon 6.2)

If $A = A^H$ then the eigenvalues of A are real.

Proof.

Let (λ, x) be a right eigen-pair, then

$$Ax = \lambda x, \text{ and } x^H A^H = \bar{\lambda} x^H.$$

Therefore

$$\begin{aligned} & x^H A x = \lambda x^H x, \text{ and } x^H A^H x = \bar{\lambda} x^H x \\ \implies & \lambda x^H x = \bar{\lambda} x^H x \implies \bar{\lambda} = \lambda, \\ \implies & \lambda \text{ is real.} \end{aligned}$$

Self Adjoint Matrices

Lemma (Moon 6.3)

If $A = A^H$ and the eigenvalues are distinct, then the eigenvectors are orthogonal.

Proof.

Let (λ_1, x_1) and (λ_2, x_2) be distinct eigenpairs, i.e. $\lambda_1 \neq \lambda_2$, then

$$x_2^H A x_1 = \lambda_1 x_2^H x_1$$
$$\text{and } x_2^H A^H x_1 = \lambda_2 x_2^H x_1$$

Therefore $(\lambda_1 - \lambda_2) x_2^H x_1 = 0$. Because $\lambda_1 \neq \lambda_2$ we must have that

$$x_2^H x_1 = 0$$

which implies that x_1 and x_2 are orthogonal. □

Note the eigenvectors can always be chosen to be orthonormal.

Self Adjoint Matrices

Theorem (Moon Theorem 6.2 (Special Theorem))

If $A \in \mathbb{C}^{n \times n}$ is Hermitian, then $q_i = m_i$ for each eigenvalue λ_i.

Corollary

If $A = A^H$, then $\exists$ a unitary U and real diagonal Λ such that

$$A = U \Lambda U^H.$$

Eigenvalues and Rank

Lemma (Moon Lemma 6.5)

Let $A \in \mathbb{C}^{m \times m}$ be of rank $r < m$. Then at least $m - r$ of the eigenvalues of A are equal to zero