ECEn 671: Mathematics of Signals and Systems

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Section 1

Metric Spaces

Spaces

- One of the objectives of this course is to develop tools that work in a wide variety of settings.
- We will mostly focus on finite dimensional Hilbert spaces, which include:
  - $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{C}^{m \times n}$,
  - the set of all functions with finite integral,
  - the set of all finitely summable sequences,
  - binary vectors, binary sequences.
- But does not include important objects like
  - rotations matrices, quaternions, homogeneous transformations.
- To make things clear, we will develop the theory systematically in the following order:
  1. Metric space
  2. Norm space / Banach space
  3. Inner product space / Hilbert space

Metric Spaces

Definition (Metric Space)

A metric space is a pair $(\mathbb{X}, d)$ where $\mathbb{X}$ is a set and

$$d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$$

is a metric defined over $\mathbb{X}$.

A metric is a measure of distance between elements in a set.

Metric Spaces

Definition (Metric)

Let $\mathbb{X}$ be a set. Then $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is a metric if:

$$
\begin{aligned}
(M1) \qquad & d(x,y) = d(y,x), \quad \forall x, y \in \mathbb{X} \\
(M2) \qquad & d(x,y) \geq 0, \quad \forall x, y \in \mathbb{X} \\
(M3) \qquad & d(x,y) = 0, \quad \iff x = y \\
(M4) \qquad & d(x,z) \leq d(x,y) + d(y,z), \quad \forall x, y, z \in \mathbb{X}
\end{aligned}
$$

(M4) is called the Triangle inequality.

Examples of Metric Spaces

Example (E1)

$(\mathbb{R}, d)$ where $d(x, y) \triangleq |x - y|$ is a metric space.
Note that

- (M1) $|x - y| = |y - x|$, $\forall x, y \in \mathbb{R}$.
- (M2) $|x - y| \geq 0$, $\forall x, y \in \mathbb{R}$.
- (M3) $|x - y| = 0$, if $x = y$.
- (M4) $|x - z| \leq |x - y| + |y - z|$ $\forall x, y, z \in \mathbb{R}$.

To convince yourself (M4), draw a picture. Note, a picture is not a proof.

Examples of Metric Spaces

Example (E2)

$(\mathbb{R}^n, d)$ where

$$d(x, y) \triangleq \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

where $x = (x_1, \ldots, x_n)^\top$ and $y = (y_1, \ldots, y_n)^\top$.
Verify that $d(\cdot, \cdot)$ satisfies (M1)-(M4).

Examples of Metric Spaces

Example (E3)

$(\mathbb{R}^n, d)$ where

$$d(x, y) \triangleq \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

where $p \geq 1$.

For general $p \geq 1$, the triangle inequality is a nontrivial and famous results.

Examples of Metric Spaces

Example (E4 bounded sequence space)

Let ℓ^∞ be the set of all sequences of complex numbers where each number is bounded, i.e.,

$$x = (x_1, x_2, x_3, \dots) \in \ell$$

if $x_i \in \mathbb{C}$ and $|x_i| < \infty$.
(ℓ, d) is a metric space where

$$d(x, y) = \sup_{j \in \mathbb{N}} |x_i - y_i| .$$

Verify (M1)-(M4).

Examples of Metric Spaces

Example (E5 continuous function space)

- Let $C[a, b]$ be the set of all continuous functions on $[a, b]$, i.e., i.e. $x \in C[a, b] \Rightarrow x(t)$ is continuous on $[a, b]$.
  Let
  $$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|$$
  then $(C[a, b], d)$ is a metric space.
- This is a different perspective than calculus. In calculus you consider one function at a time. In this class, a function is one point in a larger metric space.

Examples of Metric Spaces

Example (E6 discrete metric space)

Let $\mathbb{X}$ be any set, e.g., the set of three legged dogs, and let

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}.$$

Then $(\mathbb{X}, d)$ is a metric space since

- (M1) $d(x,y) = d(y,x)$, $\forall x, y \in \mathbb{X}$.
- (M2) $d(x,y) \geq 0$, $\forall x, y \in \mathbb{X}$.
- (M3) $d(x,y) = 0$, if $x = y$.
- (M4) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x, y, z \in \mathbb{X}$.

Examples of Metric Spaces

Example (E7 binary vector space)

Let $\mathbb{X} = \{0,1\}^n$ be the set of binary vectors, i.e $x \in \mathbb{X} \Rightarrow x = (x_1, x_2, \ldots, x_n)$ where $x_i \in \{0,1\}$. Let

$$d(x,y) = \sum_{i=1}^{n} h(x_i - y_i)$$

where

$$h(w) = \begin{cases} 1 & \text{if} \qquad w \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

h is called the hamming distance, and simply counts the number of elements in x and y that are different.

Metric Spaces / Norm Spaces / Inner Product Spaces

- Later in the chapter, we will later introduce the concepts of a norm and a norm space, and an inner product and inner product spaces.
- Many of the metric spaces introduced above are also norm spaces and inner product spaces, but not all.
- Metric spaces are the most general of the three.
- Before introducing the concept of a norm and a normed space, we develop general tools that also work for metric spaces.