ECEn 671: Mathematics of Signals and Systems

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Section 1

Schur Complement and the Matrix Inversion Lemma

Schur Complement

Definition

Consider the partitioned matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

1. When A_{11} is non-singular,

$$S_{ch}(A_{11}) \triangleq A_{22} - A_{21}A_{11}^{-1}A_{12}$$

   is called the <u>Schur Complement of A_{11} in A</u>.

2. When A_{22} is non-singular,

$$S_{ch}(A_{22}) \triangleq A_{11} - A_{12}A_{22}^{-1}A_{21}$$

   is called the <u>Schur Complement of A_{22} in A</u>.

Schur Complement, cont.

Lemma

When A_{11} is nonsingular, A is nonsingular if and only if $S_c h(A_{11})$ is nonsingular, in which case

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) \\ -S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & S_{ch}^{-1}(A_{11}) \end{bmatrix}$$

Lemma

When A_{22} is nonsingular, A is nonsingular if and only if $S_c h(A_{22})$ is nonsingular, in which case

$$A^{-1} = \begin{bmatrix} S_{ch}^{-1}(A_{22}) & -S_{ch}^{-1}(A_{22}) A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{12} S_{ch}^{-1}(A_{22}) & A_{22}^{-1} + A_{22}^{-1} A_{21} S_{ch}^{-1}(A_{22}) A_{12} A_{22}^{-1} \end{bmatrix}$$

Proof.

By direct manipulation. $\square$

Matrix Inversion Lemma

Lemma (Matrix Inversion Lemma)

If $A \in \mathbb{R}^{n \times n}$ *and* $R \in \mathbb{R}^{m \times m}$ *are invertible, and* $X \in \mathbb{R}^{n \times m}$ *and* $Y \in \mathbb{R}^{m \times n}$ *then*

$$(A + XRY)^{-1} = A^{-1} - A^{-1}X(R^{-1} + YA^{-1}X)^{-1}YA^{-1}$$

Proof.

Equate the $(2, 2)$ elements of A^{-1} in the previous slide, and re-label matrices. $\square$

Matrix Inversion Lemma, cont.

- A special case of this matrix inversion lemma is the formula

$$(A + xy^H)^{-1} = A^{-1} - \frac{A^{-1}xy^H A^{-1}}{1 + y^H A^{-1} x}$$

  where x and y are vectors.

- Sylvester's inequality gives

$$rank(x) + rank(y) - 1 \leq rank(xy^H) \leq min(rank(x), rank(y)).$$

  But

$$rank(x) + rank(y) - 1 = 1$$
$$min(rank(x), rank(y)) = 1$$

- Therefore $rank(xy^H) = 1$