ECEn 671: Mathematics of Signals and Systems

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Section 1

Pseudo Inverse and the SVD

Pseudo Inverses of A

Least squares solution to $Ax = b$ (i.e. $\min \|Ax - b\|_2$) where A-tall is

$$\hat{x} = (A^H A)^{-1} A^H b \triangleq A^\dagger b.$$

Minimum norm solution to $Ax = b$ (i.e. $\min \|x\|$ for $Ax = b$) where A-fat is

$$x = A^H (A A^H)^{-1} b \triangleq A^\dagger b.$$

How does the SVD help compute the pseudo inverse. We will consider both when A is full rank, and when A is not full rank.

SVD and Least Squared: Full Rank A

Assume $A \in \mathbb{C}^{m\times n}$ is tall, i.e., $m > n$, and that $\operatorname{rank}(A) = n$. Then

$$A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^H = U_1 \Sigma V^H$$

where $U_1 \in \mathbb{C}^{m\times n}$, $\Sigma \in \mathbb{R}^{n\times n}$, and $V \in \mathbb{C}^{n\times n}$.

Then

$$\begin{aligned}
(A^H A)^{-1} A^H &= (V\Sigma U_1^H U_1 \Sigma V^H)^{-1} V \Sigma U_1^H \\
&= (V\Sigma^2 V^H)^{-1} V\Sigma U_1^H \\
&= V\Sigma^{-2} V^H V \Sigma U_1^H \\
&= V\Sigma^{-1} U_1^H
\end{aligned}$$

where $\Sigma^{-1} = \operatorname{diag}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n})$.

SVD and min Norm: Full Rank A

Assume $A \in \mathbb{C}^{m \times n}$ is fat, i.e., $m < n$, and that $\operatorname{rank}(A) = m$. Then

$$\begin{aligned} A &= U \begin{pmatrix} \Sigma & 0 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_2^H \end{pmatrix} \\ &= U \Sigma V_1^H \end{aligned}$$

where $U \in \mathbb{C}^{m \times m}$, $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_m)$, $V_1 \in \mathbb{C}^{n \times m}$.

Then

$$\begin{aligned} A^H (A A^H)^{-1} &= V_1 \Sigma U^H (U \Sigma V_1^H V_1 \Sigma U^H)^{-1} \\ &= V_1 \Sigma U^H (U \Sigma^2 U^H)^{-1} \\ &= V_1 \Sigma U^H U \Sigma^{-2} U^H \\ &= V_1 \Sigma^{-1} U^H \end{aligned}$$

where $\Sigma^{-1} = \operatorname{diag}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_m})$

SVD and Pseudo Inverse: Not Full Rank A

Assume $A \in \mathbb{C}^{m\times n}$ and that $\operatorname{rank}(A) = p < \min(m,n)$. Then

$$A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_2^H \end{pmatrix} = U_1 \Sigma V_1^H$$

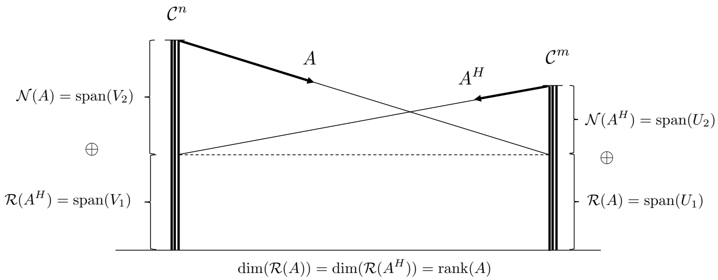
where $U_1 \in \mathbb{C}^{m\times p}$, $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_p) \in \mathbb{R}^{p\times p}$, $V_1 \in \mathbb{C}^{n\times p}$.
Consider the least squares problem

$$\begin{aligned}
\hat{x} &= (A^H A)^{-1} A^H b \\
&= (V_1 \Sigma_1 U_1^H U_1 \Sigma_1 V_1^H)^{-1} V_1 \Sigma_1 U_1^H b \\
&= (V_1 \Sigma_1^2 V_1^H)^{-1} V_1 \Sigma_1 U_1^H b \\
&= V_1 \Sigma_1^{-2} V_1^H V_1 \Sigma_1 U_1^H b \\
&= V_1 \Sigma_1^{-1} U_1^H b
\end{aligned}$$

where $\Sigma_1 = \operatorname{diag}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_p})$.

SVD and Pseudo Inverse: Not Full Rank A

So we can compute it, but what did we do? How do we interpret the solution since the inverse of $A^H A$ does not exist?



Find a solution to $Ax = b$ where $b \in \mathcal{R}(A)$. But $\mathcal{N}(A) \neq \{0\}$ implies that there are more than one solution.

Therefore, find the minimum norm x that minimizes $\|Ax - b\|_2$.

SVD and Pseudo Inverse: Not Full Rank A

Note the following:

$$\underbrace{U_1}_{m \times p} : \mathbb{C}^p \to \mathcal{R}(A) \subset \mathbb{C}^m$$

so that

$$U_1^* = U_1^H : \mathbb{C}^m \to \mathbb{C}^p.$$

Also,

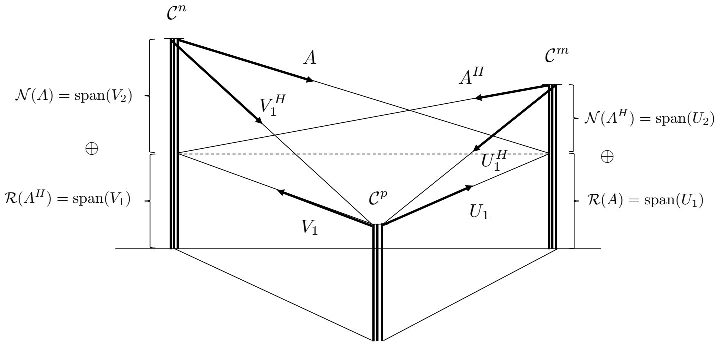
$$\underbrace{V_1}_{n \times p} : \mathbb{C}^p \to \mathcal{R}(A^H) \subset \mathbb{C}^n$$

so that

$$V_1^H : \mathbb{C}^n \to \mathbb{C}^p.$$

SVD and Pseudo Inverse: Not Full Rank A

So we have the following:

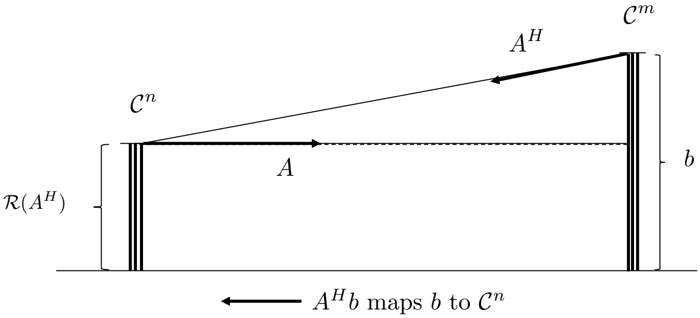


Since $rank(A) = p$ we can only take inverses in $\mathbb{C}^p$. Therefore instead of solving $Ax = b$ directly in $\mathbb{C}^n$ and $\mathbb{C}^m$ we go indirectly through $\mathbb{C}^p$.

SVD and Pseudo Inverse: Not Full Rank A

Step 1: Least Squares

Recall that to solve $\min \|Ax - b\|_2$ when A is full rank:

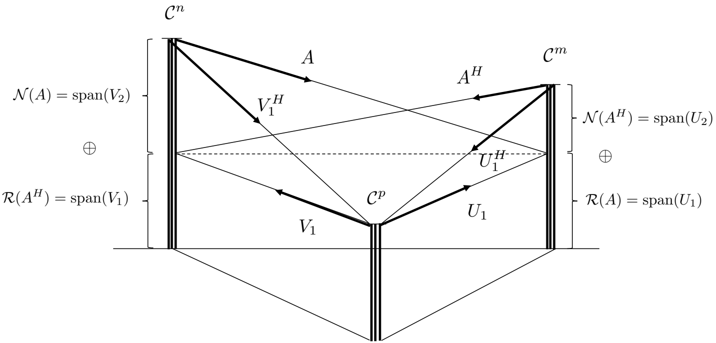


where we can invert things, i.e.

$$A^H A x = A^H b$$
$$\Longrightarrow \hat{x} = (A^H A)^{-1} A^H b.$$

SVD and Pseudo Inverse: Not Full Rank A

So we have the following:



Now instead of A^H we use U_1^H to map to $\mathbb{C}^p$, i.e., given

$$Ax = b$$

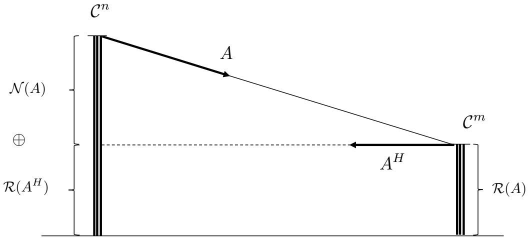
map to $\mathbb{C}^p$ using U_1^H to get:

$$U_1^H Ax = U_1^H b \qquad \in \mathbb{C}^p.$$

SVD and Pseudo Inverse: Not Full Rank A

Step 2: Minimum Norm

Recall that to min $\|x\|$ such that $Ax = b$, A-full rank,



to minimize $\|x\|$ we zero out the part that is in the null space of A, i.e. let

$$x = A^H z \text{ where } z \in \mathbb{C}^m$$

then

$$AA^H z = b \qquad \Rightarrow \qquad z = (AA^H)^{-1} b$$

so that

$$\hat{x} = A^H (AA^H)^{-1} b.$$

SVD and Pseudo Inverse: Not Full Rank A

In our case, again pick x to zero the portion in the null space of A. Let

$$x = V_1 z \quad \text{where} \quad z \in \mathbb{C}^p$$

so that

$$U_1^H A x = (U_1 A V_1)\, z = U_1^H b.$$

Note that

$$U_1 A V_1 : \mathbb{C}^p \to \mathbb{C}^p.$$

In fact,

$$U_1^H A V_1 = U_1^H U_1 \Sigma_1 V_1^H V_1 = \Sigma_1.$$

so we have

$$\begin{aligned} \Sigma_1 z &= U_1^H b \\ \Longrightarrow z &= \Sigma_1^{-1} U_1^H b \\ \Longrightarrow & \boxed{\hat{x} = V_1 \Sigma_1^{-1} U_1^H b} \end{aligned}$$