ECEn 671: Mathematics of Signals and Systems

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Section 1

Cayley-Hamilton Theorem

Functions of Matrices

Lemma

A square matrix can always be put in Jordan form.

This implies that we can always write

$$A = SJS^{-1}$$

This implies that

$$\begin{aligned} A^k &= \underbrace{AA\cdots A}_{k \text{ times}} \\ &= SJS^{-1}SJS^{-1}\cdots SJS^{-1} \\ &= SJ^kS^{-1} \end{aligned}$$

This is particularly simple if $J = \Lambda$ since

$$A^k = S\Lambda^k S^{-1} \text{ where } \Lambda^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

Functions of Matrices, cont.

For square matrices we can define analytic functions of matrices. Analytic functions are functions that can be expanded as a Taylor seires, e.g.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The corresponding Matrix function is defined in terms of its Taylor series, e.g.,

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$
$$\cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \cdots$$
$$\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots$$

Cayley-Hamilton Theorem

Computing infinite series of matrices is a pain. Fortunately we have the following theorem:

Theorem (Cayley-Hamilton Theorem)

Every matrix satisfies its own characteristic polynomial, i.e.

$$\chi_A(A) = 0.$$

Cayley-Hamilton Theorem, proof

The proof holds for all A but we will only prove the case when $q_i = m_i$ for each λ_i. In this case $A = S\Lambda S^{-1}$. Note that for each eigenvalue $\chi_A(\lambda_i) = 0$ since $\chi_A(\lambda_i) = \det(\lambda_i I - A)$
Let

$$\chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

then

$$\begin{aligned}
\chi_A(A) &= A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I \\
&= S\Lambda^n S^{-1} + a_{n-1}S\Lambda^{n-1}S^{-1} + \cdots + a_1 S\Lambda S^{-1} + a_0 SS^{-1} \\
&= S(\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1\Lambda + a_0 I)S^{-1}
\end{aligned}$$

Note that the matrix

$$\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1\Lambda + a_0 I$$

is diagonal with each element on the diagonal equal to $\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \cdots + a_1\lambda_i + a_0 = 0$.
Therefore

$$\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1\Lambda + a_0 I = 0.$$

Cayley-Hamilton Theorem, implications

Recall polynomial division:

$$\frac{f(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)}$$

$$\implies \underbrace{f(x)}_{\text{degree } m} = \underbrace{a(x)}_{\text{degree } (m-n)} \overbrace{q(x)}^{\text{degree } n} + \underbrace{r(x)}_{\text{degree } < n}$$

Application to infinite series like e^x gives

$$\underbrace{e^x}_{\text{degree } \infty} = \underbrace{a(x)}_{\text{degree } \infty} \overbrace{\chi_A(x)}^{\text{degree } n} + \underbrace{r(x)}_{\text{degree } < n}$$

Since $\chi_A(A) = 0$,

$$e^A = \underbrace{r(A)}_{\text{degree } < n} = \cdots b_{n-1}A^{n-1} + \cdots + b_1 A + b_0 I$$

Since $e^{\lambda_i} = r(\lambda_i)$ the coefficients can be found from

$$e^{\lambda_i} = \cdots b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0.$$

Cayley-Hamilton Theorem, example

Find e^A where $A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$.

$$\det(\lambda I - A) = \lambda^2 + 2\lambda + 2 \Rightarrow \lambda_{1,2} = -1 \pm j$$

$$\Rightarrow e^A = b_1 A + b_0 I = \begin{pmatrix} 0 & b_1 \\ -2b_1 & -2b_1 \end{pmatrix} + \begin{pmatrix} b_0 & 0 \\ 0 & b_0 \end{pmatrix} = \begin{pmatrix} b_0 & b_1 \\ -2b_1 & -2b_1 - b_0 \end{pmatrix}$$

where b_0 and b_1 satisfy

$$e^{-1+j} = b_1(-1+j) + b_0$$

$$e^{-1-j} = b_1(-1-j) + b_0$$

$$\Rightarrow \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 & -1+j \\ 1 & -1-j \end{pmatrix}^{-1} \begin{pmatrix} e^{-1+j} \\ e^{-1-j} \end{pmatrix} = \begin{pmatrix} 0.5083 \\ 0.3096 \end{pmatrix}$$

$$\Rightarrow e^A = \begin{pmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{pmatrix}$$