ECEn 671: Mathematics of Signals and Systems

Randal W. Beard

Brigham Young University

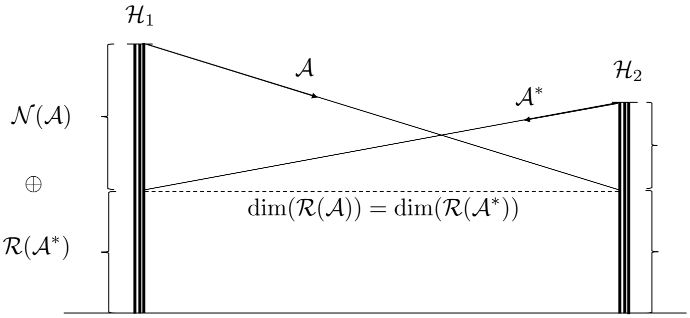
September 1, 2023

Section 1

Fundamental Subspaces

Fundamental Subspaces

Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. Then $\mathcal{A}^* : \mathcal{H}_2 \to \mathcal{H}_1$ and we have the following picture:



Fundamental Subspaces, cont.

Lemma

1. $\mathcal{H}_1 = \mathcal{N}(\mathcal{A}) \oplus \mathcal{R}(\mathcal{A}^*)$
2. $\mathcal{H}_2 = \mathcal{N}(\mathcal{A}^*) \oplus \mathcal{R}(\mathcal{A})$
3. $\dim(\mathcal{H}_1) = \dim(\mathcal{N}(\mathcal{A})) + \dim(\mathcal{R}(\mathcal{A}^*))$
4. $\dim(\mathcal{H}_2) = \dim(\mathcal{N}(\mathcal{A}^*)) + \dim(\mathcal{R}(\mathcal{A}))$
5. $\dim(\mathcal{R}(\mathcal{A})) = \dim(\mathcal{R}(\mathcal{A}^*))$

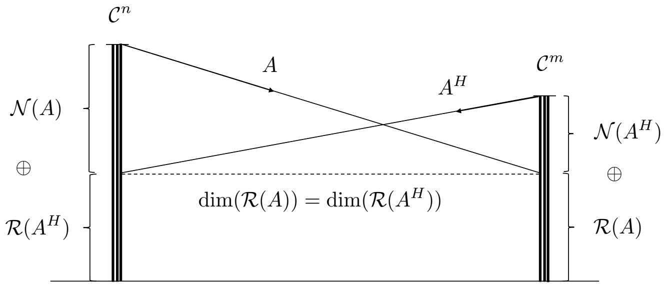
Proofs to follow.

Fundamental Subspaces for Matrices

For matrices, the picture looks as follows:

$$A : \mathbb{C}^n \to \mathbb{C}^m$$

$$A^* = A^H : \mathbb{C}^m \to \mathbb{C}^n$$



$$\dim(\mathcal{R}(A^H)) = \dim(\mathcal{R}(A))$$

Fundamental Subspaces, cont

Theorem (Moon Theorem 4.5)

Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{R}(\mathcal{A})$ and $\mathcal{R}(\mathcal{A}^*)$ be closed, then

1. $[\mathcal{R}(\mathcal{A})]^\perp = \mathcal{N}(\mathcal{A}^*)$
2. $[\mathcal{R}(\mathcal{A}^*)]^\perp = \mathcal{N}(\mathcal{A})$

Theorem 4.5, Proof

(1): To show that $[\mathcal{R}(\mathcal{A})]^{\perp} = \mathcal{N}(\mathcal{A}^*)$ we need to show that $\mathcal{N}(\mathcal{A}^{\perp}) \subseteq [\mathcal{R}(\mathcal{A})]^{\perp}$ and $[\mathcal{R}(\mathcal{A})]^{\perp} \subseteq \mathcal{N}(\mathcal{A}^*)$.

We first show that $\mathcal{N}(\mathcal{A}^*) \subseteq [\mathcal{R}(\mathcal{A})]^{\perp}$:

Select any $y \in \mathcal{N}(\mathcal{A}^*)$ and any $\hat{y} \in \mathcal{R}(\mathcal{A})$. Then $\exists \hat{x} \in \mathcal{H}_1$ such that $\hat{y} = \mathcal{A}\hat{x}$. Therefore

$$
\begin{aligned}
\langle \hat{y}, y \rangle &= \langle \mathcal{A}\hat{x}, y \rangle \\
&= \langle \hat{x}, \mathcal{A}^* y \rangle \\
&= \langle \hat{x}, 0 \rangle = 0 \\
\Rightarrow &\quad y \in [\mathcal{R}(\mathcal{A})]^{\perp} \\
\Rightarrow &\quad \mathcal{N}(\mathcal{A}^*) \subseteq [\mathcal{R}(\mathcal{A})]^{\perp}
\end{aligned}
$$

Theorem 4.5, Proof, cont.

<u>We first show that $[\mathcal{R}(\mathcal{A})]^{\perp} \subseteq \mathcal{N}(\mathcal{A}^*)$:</u>

Select any $y \in [\mathcal{R}(\mathcal{A})]^{\perp}$. For every $\hat{x} \in \mathcal{H}_1$ we have $\hat{y} = \mathcal{A}\hat{x} \in \mathcal{R}(\mathcal{A})$, and therefore

$$\langle \hat{y}, y \rangle = \langle \mathcal{A}\hat{x}, y \rangle = 0$$

By definition of the adjoint, we therefore have that

$$\langle \hat{x}, \mathcal{A}^* y \rangle = 0$$

Since this is true for every $\hat{x} \in \mathcal{H}_1$ it must be that $\mathcal{A}^* y = 0$. Therefore

$$y \in \mathcal{N}(\mathcal{A}^*),$$

which implies that

$$[\mathcal{R}(\mathcal{A})]^{\perp} \subseteq \mathcal{N}(\mathcal{A}^*).$$

Item (2) is shown similarly.

Fundamental Subspaces, cont

Theorem 2.10 states that if $\mathcal{H}$ is a Hilbert space and if $\mathbb{V}$ a closed subspace in $\mathcal{H}$ then

$$\mathcal{H} = \mathbb{V} \oplus \mathbb{V}^{\perp}$$

Therefore Theorem 4.5 implies that

$$\begin{aligned} \mathcal{H}_1 &= \mathcal{R}(\mathcal{A}^*) \oplus \mathcal{N}(\mathcal{A}) \\ \mathcal{H}_2 &= \mathcal{R}(\mathcal{A}) \oplus \mathcal{N}(\mathcal{A}^*) \end{aligned}$$

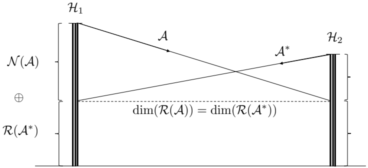
Which also implies that

$$\begin{aligned} \dim(\mathcal{H}_1) &= \dim(\mathcal{R}(\mathcal{A}^*)) + \dim(\mathcal{N}(\mathcal{A})) \\ \dim(\mathcal{H}_2) &= \dim(\mathcal{R}(\mathcal{A})) + \dim(\mathcal{N}(\mathcal{A}^*)) \end{aligned}$$

Fundamental Subspaces, cont

Lemma

- $\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}\mathcal{A}^*)$
- $\mathcal{R}(\mathcal{A}^*) = \mathcal{R}(\mathcal{A}^*\mathcal{A})$



Proof.

We will prove (1) by showing that:

(a) $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{A}\mathcal{A}^*)$

(b) $\mathcal{R}(\mathcal{A}\mathcal{A}^*) \subseteq \mathcal{R}(\mathcal{A})$

Fundamental Subspaces, cont

Proof (cont.)

(a) Let $y \in \mathcal{R}(\mathcal{A}) \Rightarrow \exists x \in \mathcal{H}_1$ such that $y = \mathcal{A}x$
Since $\mathcal{H}_1 = \mathcal{R}(\mathcal{A}^*) \oplus \mathcal{N}(\mathcal{A}), \quad x = x_n + x_r$ where

$$x_n \in \mathcal{N}(\mathcal{A}) \text{ and } x_r \in \mathcal{R}(\mathcal{A}^*)$$

$$\Rightarrow \exists \hat{y} \in \mathcal{H}_2 \text{ such that } x_r = \mathcal{A}^* \hat{y}$$

so

$$y = \mathcal{A}x = \mathcal{A}(x_n + x_r) = \mathcal{A}\mathcal{A}^* \hat{y}$$

$$\Rightarrow y \in \mathcal{R}(\mathcal{A}\mathcal{A}^*)$$

(b) let $y \in \mathcal{R}(\mathcal{A}\mathcal{A}^*) \Rightarrow \exists \hat{y} \in \mathcal{H}_2$ such that

$$y = \mathcal{A}\mathcal{A}^* \hat{y} \Rightarrow y = \mathcal{A}\hat{x} \text{ where } \hat{x} \in \mathcal{H}_1$$

$$\Rightarrow y \in \mathcal{R}(\mathcal{A}).$$

$\square$

Fundamental Subspaces, cont

Theorem

$$\dim(\mathcal{R}(\mathcal{A})) = \dim(\mathcal{R}(\mathcal{A}^*))$$

Proof.

We need to show that

(a) $\dim(\mathcal{R}(\mathcal{A})) \leq \dim(\mathcal{R}(\mathcal{A}^*))$

(b) $\dim(\mathcal{R}(\mathcal{A}^*)) \leq \dim(\mathcal{R}(\mathcal{A}))$

Fundamental Subspaces, cont

Proof (cont.)

(a) Let $P = \{p_1, p_2, \ldots\}$ be a Hamel basis for $\mathcal{R}(\mathcal{A})$ so $\dim(\mathcal{R}(\mathcal{A})) =$ *cardinality* of P.

$$p_i \in \mathcal{R}(\mathcal{A}) \Rightarrow \exists \hat{q}_i \in \mathcal{H}_1 \text{ such that } p_i = \mathcal{A}\hat{q}_i$$

$$\mathcal{H}_1 = \mathcal{R}(\mathcal{A}^*) \oplus \mathcal{N}(\mathcal{A}) \Rightarrow \hat{q}_i = q_{i,n} + q_i$$

$$\text{where } q_{i,n} \in \mathcal{N}(\mathcal{A}) \text{ and } q_i \in \mathcal{R}(\mathcal{A}^*)$$

$$\Rightarrow p_i = \mathcal{A}q_i,$$

let

$$Q = \{q_1, q_2, \ldots\}$$

we will show that Q is linearly independent $\Rightarrow$ any Hamel basis of $\mathcal{R}(A^*)$ contains $Q \Rightarrow \dim(\mathcal{R}(A^*)) \geq \dim(\mathcal{R}(A))$,

Fundamental Subspaces, cont

Proof (cont.)

P is a Hamel basis $\Rightarrow$ all finite subsets of P are linearly independent, i.e.

$$\sum_{i \in I} c_i p_i = 0 \iff c_i = 0, i \in I$$

where I is a finite index set. But,

$$\sum_I c_i p_i = 0 \iff \sum_I c_i \mathcal{A} q_i = 0 \iff \mathcal{A}(\sum_I c_i q_i) = 0$$

but $\sum_I c_i q_i \in \mathcal{R}(\mathcal{A}^*) \perp \mathcal{N}(\mathcal{A})$
so

$$\iff \sum_I c_i q_i = 0 \iff c_i = 0, i \in I$$

$$\Rightarrow Q \text{ is linearly independent}$$

(b) Substitute $\mathcal{A}$ for $\mathcal{A}^*$ and $\mathcal{A}^*$ for $\mathcal{A}$ is above argument.

Solution of Operator Equations

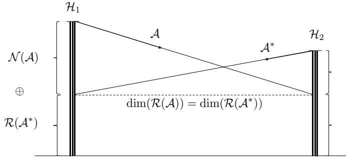
We turn to solutions to the linear operator equation

$$\mathcal{A}x = y$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded, $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert and $\mathcal{R}(\mathcal{A})$ is closed.

Fact 1. $\mathcal{A}x = y$ has a solution $\iff y \in \mathcal{R}(\mathcal{A})$

Fact 2. $\mathcal{A}x = y$ has a solution $\iff y \perp \mathcal{N}(\mathcal{A}^*)$

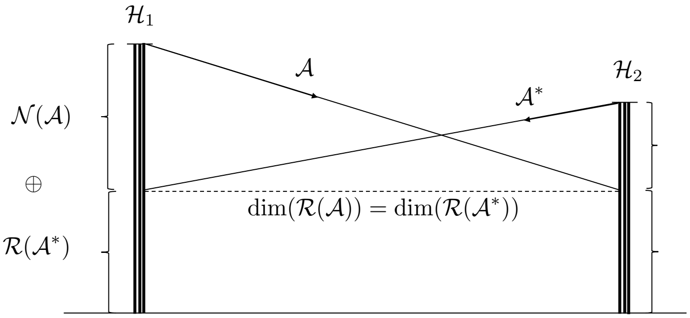


Solution of Operator Equations

Fact 3. If $\mathcal{A}x = y$ has a solution then it is unique $\iff \mathcal{N}(\mathcal{A}) = \{0\}$

Fact 4. If $\mathcal{N}(\mathcal{A}) \neq \{0\}$ and $y \in \mathcal{R}(\mathcal{A})$ then $\mathcal{A}x = y$ has an infinite number of solutions.

Fact 5 . $\mathcal{A}^{-1}$ exists $\Rightarrow \mathcal{N}(\mathcal{A}) = \{0\}$ (otherwise can't get back to all of $\mathcal{H}$.



Matrix Rank

Definition (Row Rank)

The row rank of $A \in \mathbb{C}^{m \times n}$ is the number of linearly independent rows.

Definition (Column Rank)

The column rank of $A \in \mathbb{C}^{m \times n}$ is the number of linearly independent columns.

- Since $\mathcal{R}(A) = span\{\text{columns of } A\}$ we have that $\dim(\mathcal{R}(A)) = $ column rank
- Since $\mathcal{R}(A^H) = span\{\text{rows of } A\}$ we have that $\dim(\mathcal{R}(A^*)) = $ row rank
- Therefore $\dim(\mathcal{R}(A)) = \dim(\mathcal{R}(A^H))$ implies that column rank $=$ row rank

Matrix Rank

Definition

The rank of A is the number of linearly independent rows or columns.

Lemma

$$rank(A) = rank(A^H)$$

Definition

$A : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is full rank if $rank(A) = \min(n, m)$

Sylvester's Inequality

Lemma (Sylvester's Inequality)

Let $A \in \mathbb{C}^{q \times n}$ *and* $B \in \mathbb{C}^{n \times p}$ *then*

$$rank(A) + rank(B) - n \leq rank(AB) \leq \min(rank(A), rank(B)).$$

Example

Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ then

$$rank(xy^\top) = 1$$