ECEn 671: Mathematics of Signals and Systems

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Section 1

Neumann Expansion

Geometric Series

One of the most important series in analysis is the geometric series

$$S = 1 + x + x^2 + \ldots = \sum_{i=0}^{\infty} x^i$$

Noting that

$$\begin{aligned} & 1 + xS = 1 + x + x^2 + \ldots = S \\ \Rightarrow \quad & S(1 - x) = 1 \end{aligned}$$

Therefore

$$S = \sum_{i=0}^{\infty} x^i = \frac{1}{1 - x} = (1 - x)^{-1}$$

The series converges if $|x| < 1$.

Geometric Series for Operators (Neumann Expansion)

For operators we have a similar expression:

Theorem (Moon Theorem 4.3)

Suppose $\|\cdot\|$ is a norm satisfying the submultiplicative property and $\|\mathcal{A}\| < 1$. Then $(I - \mathcal{A})^{-1}$ exists and

$$(I - \mathcal{A})^{-1} = \sum_{i=0}^{\infty} \mathcal{A}^i = I + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \dots$$

where

$$\begin{aligned} \mathcal{A}^2 &= \mathcal{A}\mathcal{A} \\ \mathcal{A}^3 &= \mathcal{A}\mathcal{A}^2 \\ \mathcal{A}^k &= \mathcal{A}\mathcal{A}^{k-1}. \end{aligned}$$

Neumann Expansion, Proof

Suppose that $(I - \mathcal{A})^{-1}$ does not exist. Then $\mathcal{N}(I - A)$ is non-trivial.
Therefore, $\exists x \neq 0$ such that

$$\begin{aligned}(I - \mathcal{A})x = 0 \quad &\Longleftrightarrow \quad x = \mathcal{A}x \\ &\Longleftrightarrow \quad \|x\| = \|\mathcal{A}x\| \leq \|\mathcal{A}\| \, \|x\| < \|x\| \,,\end{aligned}$$

which is a contradiction.

Therefore $(I - \mathcal{A})^{-1}$ exists.

Neumann Expansion, cont.

Note that $\left\|\mathcal{A}^k\right\| \le \|\mathcal{A}\|^k$ since $\|\cdot\|$ satisfies the submultiplication property.

Since $\|\mathcal{A}\| < 1$

$$\lim_{k\to\infty} \left\|\mathcal{A}^k\right\| = 0 \quad \iff \quad \lim_{k\to\infty} \mathcal{A}^k = 0$$

Note that

$$(I - \mathcal{A})(I + \mathcal{A} + \mathcal{A}^2 + \cdots + \mathcal{A}^{k-1}) = I - \mathcal{A}^k$$

$k \to \infty$ gives

$$(I - \mathcal{A})\left(\sum_{i=0}^{\infty} \mathcal{A}^i\right) = I$$

Therefore

$$\sum_{i=0}^{\infty} \mathcal{A}^i = (I - \mathcal{A})^{-1}.$$