ECEn 671: Mathematics of Signals and Systems

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Section 1

Gram Schmidt Orthogonalization

Application: Gram Schmidt Orthogonalization

Given a set $T = \{p_1, \ldots, p_n\}$
Find a set $T' = \{q_1, \ldots, q'_n\}$ $\quad n' \leq n$ such that

$$span(T') = span(T) \text{ and } < q_i, q_j >= \delta_{ij}$$

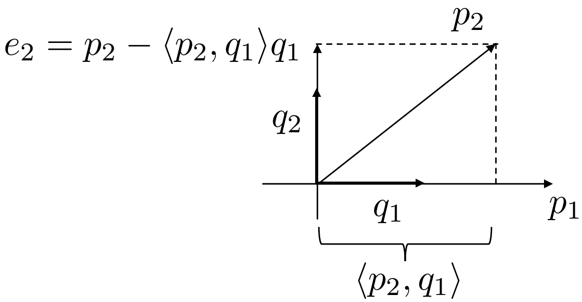
Step 1. Normalize the First Vector

$$q_1 = \frac{p_1}{\|p_1\|} \qquad (\text{i.e. } \langle q_1, q_1 \rangle = 1 \ )$$

Application: Gram Schmidt Orthogonalization, cont

Step 2. Let e_2 be p_2 minus the projection of p_2 on q_1 i.e.

$$e_2 = p_2 - \langle p_2, q_1 \rangle \, q_1$$



Then normalize e_2:

$$q_2 = \frac{e_2}{\|e_2\|}$$

Application: Gram Schmidt Orthogonalization, cont

Step 3. Let e_k be p_k minus the projection of p_k on $q_1, \ldots, q_{k-1}$:

$$e_k = p_k - \sum_{j=1}^{k-1} \langle p_k, q_j \rangle \, q_j \Rightarrow q_k = \frac{e_k}{\|e_k\|}$$

Example: Gram Schmidt Orthogonalization

Given the set

$$T = \{p_1, p_2, p_3\} \triangleq \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

find a set $T' = \{q_1, q_2, q_3\}$ where the vectors in T' are orthonormal and $\text{span}(T) = \text{span}(T')$.

$$q_1 = \frac{p_1}{\|p_2\|} = \frac{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Example: Gram Schmidt Orthogonalization, cont.

$$
\begin{aligned}
e_2 &= p_2 - \langle p_2, q_1 \rangle \, q_1 \\
&= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}^{\top} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}
\end{aligned}
$$

Therefore $q_2 = \frac{e_2}{\|e_2\|} = \frac{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Example: Gram Schmidt Orthogonalization, cont.

$$
\begin{aligned}
e_3 &= p_3 - \langle p_3, q_1 \rangle\, q_1 - \langle p_3, q_2 \rangle\, q_2 \\
&= \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix}^\top \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix}^\top \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \\
&= \begin{pmatrix}1\\2\\3\end{pmatrix} - 1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} - 2 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} \\
&= \begin{pmatrix}0\\0\\3\end{pmatrix}
\end{aligned}
$$

Therefore $q_3 = \frac{e_3}{\|e_3\|} = \frac{\begin{pmatrix}0\\0\\3\end{pmatrix}}{3} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Example: Gram Schmidt Orthogonalization, cont.

Therefore, the Gram Schmidt orthonormalization of

$$T = \{p_1, p_2, p_3\} \triangleq \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

is

$$T' = \{q_1, q_2, q_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Note that $\mathrm{span}(T) = \mathrm{span}(T')$.