ECEn 671: Mathematics of Signals and Systems

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Spline problems:

- Smoothing: Given a set of data $\{(t_i, y_i)\}$, fit a 5th order polynomial that minimizes the sum of the squared error.
- Polynomial fit: Given a set of data $\{(t_i, y_i)\}$ find the lowest order polynomial that fits the data exactly.
- Path planning: Given a start position and velocity, and an end position and velocity, find a trajectory that satisfies the end points.

RWB:Add figures

Path Planning

Given start position $p_s \in \mathbb{R}^2$,
start velocity $p_s^{'} \in \mathbb{R}^2$, end position $p_e \in \mathbb{R}^2$, and end velocity $p_e^{'} \in \mathbb{R}^2$.
Suppose that the trajectory will be given by a 4th order spline:

$$p(t) = \begin{pmatrix} c_{0x} \\ c_{0y} \end{pmatrix} + \begin{pmatrix} c_{1x} \\ c_{1y} \end{pmatrix} t + \begin{pmatrix} c_{2x} \\ c_{2y} \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} c_{3x} \\ c_{3y} \end{pmatrix} \frac{t^3}{3!}.$$

Rewrite in matrix notation as:

$$\begin{aligned} p(t) &= \begin{pmatrix} c_{0x} & c_{1x} & c_{2x} & c_{3x} \\ c_{0y} & c_{1y} & c_{2y} & c_{3y} \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \frac{t^3}{3!} \end{pmatrix} \\ &= C\phi(t), \end{aligned}$$

where

Path Planning (cont.)

Then

$$\begin{aligned} p_s &= p(0) = C\phi(0) \\ p_s^{'} &= p^{'}(0) = C\phi^{'}(0) \\ p_e &= p(1) = C\phi(1) \\ p_e^{'} &= p^{'}(1) = C\phi^{'}(1) \end{aligned}$$

where

$$\phi^{'}(t) = \frac{d}{dt}\phi(t) = \begin{pmatrix} 0 \\ 1 \\ t \\ \frac{t^2}{2!} \end{pmatrix}.$$

Path Planning (cont.)

Collecting all of these equations into a matrix we get

$$\begin{aligned}\begin{pmatrix} p_s & p_s' & p_e & p_e' \end{pmatrix} &= \begin{pmatrix} C\phi(0) & C\phi'(0) & C\phi(1) & C\phi'(1) \end{pmatrix} \\ &= C\begin{pmatrix} \phi(0) & \phi'(0) & \phi(1) & \phi'(1) \end{pmatrix}\end{aligned}$$

or

$$P = C\Phi$$

where

$$\begin{aligned} P &= \begin{pmatrix} p_s & p_s' & p_e & p_e' \end{pmatrix} \\ \Phi &= \begin{pmatrix} \phi(0) & \phi'(0) & \phi(1) & \phi'(1) \end{pmatrix}. \end{aligned}$$

We can find C by inverting Φ as

$$C = P\Phi^{-1}$$

Path Planning (cont.)

But what if we use a 4th order polynomial:

$$
\begin{aligned}
p(t) &= \begin{pmatrix} c_{0x} \\ c_{0y} \end{pmatrix} + \begin{pmatrix} c_{1x} \\ c_{1y} \end{pmatrix} t + \begin{pmatrix} c_{2x} \\ c_{2y} \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} c_{3x} \\ c_{3y} \end{pmatrix} \frac{t^3}{3!} + \begin{pmatrix} c_{4x} \\ c_{4y} \end{pmatrix} \frac{t^4}{4!} \\
&= C\phi(t)
\end{aligned}
$$

where

$$
\begin{aligned}
C &\triangleq \begin{pmatrix} c_{0x} & c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{0y} & c_{1y} & c_{2y} & c_{3y} & c_{4y} \end{pmatrix} \\
\phi(t) &\triangleq \begin{pmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \frac{t^3}{3!} \\ \frac{t^4}{4!} \end{pmatrix}.
\end{aligned}
$$

Path Planning (cont.)

Proceeding as before, and collecting into a matrix we get

$$\begin{aligned}\begin{pmatrix} p_s & p_s^{'} & p_e & p_e^{'} \end{pmatrix} &= \begin{pmatrix} C\phi(0) & C\phi^{'}(0) & C\phi(1) & C\phi^{'}(1) \end{pmatrix} \\ &= C \begin{pmatrix} \phi(0) & \phi^{'}(0) & \phi(1) & \phi^{'}(1) \end{pmatrix}\end{aligned}$$

or

$$P = C\Phi$$

but in this case P is 2×4, C is 2×5 and Φ is 5×4, and so we can't simply invert Φ, because an inverse does not exist.
How should we proceed?