ECEn 671: Mathematics of Signals and Systems

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Section 1

Matrix Inverses

Matrix Inverses

Definition

$A \in \mathbb{C}^{m \times n}$ has a left inverse if $\exists B \in \mathbb{C}^{n \times m}$ such that

$$\underset{n \times m}{B} \quad \underset{m \times n}{A} = \underset{n \times n}{I}$$

Definition

$A \in \mathbb{C}^{m \times n}$ has a right inverse if $\exists D \in \mathbb{C}^{n \times m}$ such that

$$\underset{m \times n}{A} \quad \underset{n \times m}{C} = \underset{m \times m}{I}$$

Matrix Inverses, cont

Example

The matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \end{pmatrix}.$$

has an infinite number of right inverses, namely

$$C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{7} \\ c_1 & c_2 \end{pmatrix} \qquad \forall c_1, c_2 \in \mathbb{R}$$

since

$$AC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Matrix Inverses, cont

- Suppose A has a left inverse, then

$$Ax = b \iff BAx = Bb \iff x = Bb$$

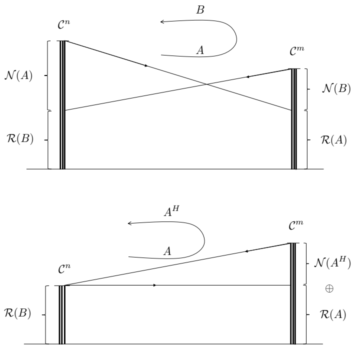
- Suppose A has a right inverse, then let

$$x = Cb \Rightarrow Ax = ACb = b$$

so $x = Cb$ is a solution.

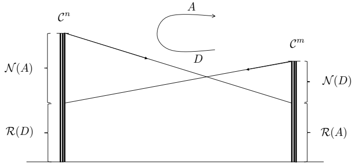
Left Inverse

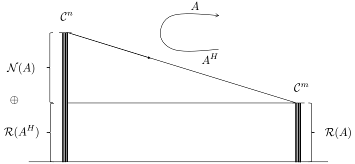
- Let B be a left inverse of A.
- Then $BA = I : \mathbb{C}^n \to \mathbb{C}^n$.
- Of necessity we must have that $\mathcal{N}(A) = \{0\}$, otherwise there are vectors $x \in \mathcal{N}(\mathcal{A}) \subseteq \mathbb{C}^n$ such that $BAx = B0 = 0 \neq x$, i.e., $BA \neq I$.
- Therefore $Ax = b$ has <u>at most</u> one solution
  (since b may not be in $\mathcal{R}(A)$).



Right Inverse

- Let D be a right inverse of A.
- Then $AD = I : \mathbb{C}^m \to \mathbb{C}^m$.
- Of necessity we must have that $\mathcal{N}(A^H) = \{0\}$, otherwise $D^H A^H = I$ is impossible.
- $\mathcal{N}(A)$ may be nontrivial therefore if $\hat{x}$ is a solutions so is $\hat{x} + x_n$ where $x_n \in \mathcal{N}(A)$ since $A(\hat{x} + x_n) = A\hat{x} = b$. Therefore, there is at least one solution.





Right and Left Inverses

Lemma

1. *If A has a left inverse then $Ax = b$ has at most one solution.*
2. *If A has a right inverse then $Ax = b$ has at least one solution.*

Regular Inverse

If $A \in \mathbb{C}^{n \times n}$ when the following statements are equivalent:

1. A^{-1} exists
2. $\mathcal{N}(A) = \{0\}$ and $\mathcal{N}(A^H) = \{0\}$.
3. $rank(A) = n$
4. $det(A) \neq 0$
5. (right inverse of A) = (left inverse of A) = A^{-1}
6. there are no zero eigenvalues of A
7. $A^H A$ is positive definite
8. A is nonsingular

Regular Inverse, cont.

If A^{-1} exists then

$$\boxed{A^{-1} = \frac{adj(A)}{det(A)}}$$

where $adj(A)$ is the adjugate of A where $adj(A) = [B_{ij}]^\top$ and $B_{ij} = (-1)^{i+j} det(M_{ij})$ and M_{ij} is the $(i,j)^{th}$ minor of A.

Example

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$adj(A) = \begin{pmatrix} (-1)^2|d| & (-1)^3|c| \\ (-1)^3|b| & (-1)^4|a| \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

so $A^{-1} = \dfrac{\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}}{det(A)} = \dfrac{\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}}{ad-cb}$

Matrix Rank

Lemma

Let $A : \mathbb{C}^n \to \mathbb{C}^m$ *then*

$$rank(\underset{m\times n}{A}) = rank(\underset{n\times m}{A^H}) = rank(\underset{n\times n}{A^H A}) = rank(\underset{m\times m}{AA^H})$$

Proof.

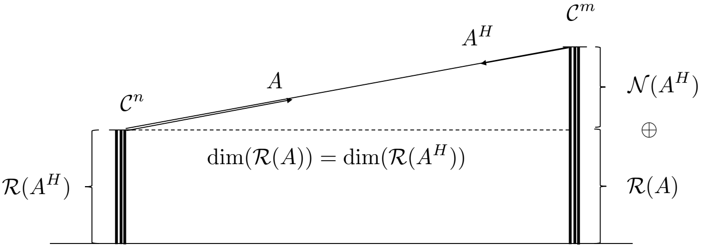
$$\begin{aligned} rank(B) &= \ \# \text{ of linearly independent columns} = \dim(\mathcal{R}(B)) \\ &= \ \# \text{ of linearly independent rows } \ = \dim(\mathcal{R}(B^H)). \end{aligned}$$

Therefore

$$\begin{aligned} rank(A) &= \dim(\mathcal{R}(A)) = \dim(\mathcal{R}(A^H)) = rank(A^H) \\ &= \dim(\mathcal{R}(AA^H)) = rank(AA^H) \text{ Since } \mathcal{R}(A^*) = \mathcal{R}(AA^*) \\ &= \dim(\mathcal{R}(A^H A)) = rank(A^H A) \text{ Since } \mathcal{R}(A) = \mathcal{R}(A^* A) \end{aligned}$$

Left Inverse: Least Squares

- Consider the solution of $Ax = b$ where $m > n$, i.e., A is tall.
- Assume A is full rank, i.e., $rank(A) = n$.
- Assume $b \in \mathcal{R}(A)$



- Map b to $\mathcal{R}(A^*)$: $A^H b = A^H A x$
- Since $rank(A) = n \iff rank(A^H A) = n$ so $(A^H A)^{-1}$ exists

$$\Rightarrow \boxed{x = (A^H A)^{-1} A^H b}$$

Left Inverse: Least Squares, cont.

What if $b \notin \mathcal{R}(A)$? This is the least squares problem, e.g.

$$\underbrace{\underbrace{\begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{b}}_{\text{linear regression}}$$

Since there is no solution, it is reasonable to find x that minimizes $\|e\|_2$ where

$$e = Ax - b$$

.

Left Inverse: Least Squares, cont.

- Note that $b = b_r + b_n$ where $b_r \in \mathcal{R}(A)$ and $b_n \in \mathcal{N}(A^H)$ so $e = Ax - b_r - b_n$.
- Since $Ax - b_r \in \mathcal{R}(A) \perp \mathcal{N}(A^H)$ the best we can do is make $Ax = b_r \Rightarrow e = b_n$.
- Since $b_n \in \mathcal{N}(A^H)$ we have

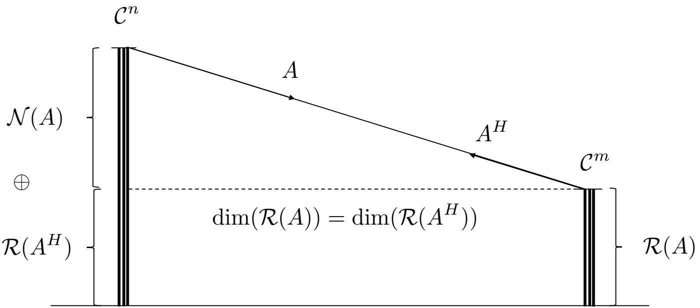
$$
\begin{aligned}
& 0 = A^H Ax - A^H b_r \\
\Rightarrow \quad & \underbrace{A^H Ax}_{\text{projection of } x \text{ onto } \mathcal{R}(A^H)} = A^H b_r = \underbrace{A^H b}_{\text{projection of } b \text{ onto } \mathcal{R}(A^H)}
\end{aligned}
$$

- Since $rank(A^H A) = rank(A) = n$ we have

$$
\underbrace{x = (A^H A)^{-1} A^H b}_{\text{least square solution}}
$$

Right Inverse: Min-Norm Solution

- Consider the solution of $Ax = b$ where $m < n$, i.e., A is fat.
- Assume A is full rank, i.e., $rank(A) = m$.



We would like to solve $Ax = b$ note that since $x = x_r + x_n$ where $x_r \in \mathcal{R}(A^H)$ and $x_n \in \mathcal{N}(A)$ and $\mathcal{N}(A) \neq \{0\}$ there are an infinite number of solutions (i.e. add any thing in $\mathcal{N}(A)$ to a solution). The minimum norm solution will be the element of $\mathcal{R}(A^H)$ that satisfies $Ax_r = b$.

Right Inverse: Min-norm Solution, cont.

$$x_r \in \mathcal{R}(A^H) \Rightarrow x_r = A^H y \text{ where } y \in \mathbb{C}^m$$

so we need to solve

$$(\underset{m\times n}{A}\underset{n\times m}{A^H})\underset{m\times 1}{y} = \underset{m\times 1}{b}$$

Since $rank(A) = rank(AA^H) = m$, $(AA^H)^{-1}$ exists.

$$\Rightarrow y = (AA^H)^{-1} b$$

$$\Rightarrow \boxed{x_r = A^H(AA^H)^{-1} b}$$

Note that this is the same solution as

$$\begin{aligned} \min \quad & \|x\|_2 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Right and Left Inverses

Lemma

If $A \in \mathbb{C}^{m\times n}$ where $m > n$ and A is full rank, then $(A^H A)^{-1} A^H$ is a left inverse of A.

Proof.

$$(A^H A)^{-1} A^H A = I_n \qquad \square$$

Lemma

If $A \in \mathbb{C}^{m\times n}$ where $m < n$ and A is full rank, then $A^H (AA^H)^{-1} b$ is a right inverse of A.

Proof.

$$AA^H (AA^H)^{-1} = I_m \qquad \square$$

- Both are examples of pseudo-inverses.
- $A^H (AA^H)^{-1}$ is called the Moore-Penrose pseudo-inverse.
- In Matlab type `pinv(A)`.