ECEn 671: Mathematics of Signals and Systems
Moon: Chapter 6.

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Section 1

Eigenvalues and Eigenvectors

Eigenpair

Let $A \in \mathbb{C}^{n \times n}$.

Definition

- (λ, x) is a right eigen-pair if $Ax = \lambda x$ and $x \neq 0$.
- (λ, x) is a left eigen-pair if $x^H A = \lambda x^H$ and $x \neq 0$.

Note that $Ax = \lambda x$ can be written as

$$(\lambda I - A)x = 0.$$

Therefore for x to be an eigenvector (associated with λ) then $x \in \mathcal{N}(\lambda I - A)$, and

$$x \neq 0 \Rightarrow \mathcal{N}(\lambda I - A) \neq \{0\} \Rightarrow det(\lambda I - A) = 0$$

This formula can be used to find the eigenvalues and eigenvectors of a matrix.

Eigenpair: Example

Let $A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$. Find the eigenvalues and eigenvectors.

Eigenvalues:

$$det(\lambda I - A) = det \begin{pmatrix} \lambda & -1 \\ 2 & \lambda + 2 \end{pmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

implies that

$$\lambda = -1 \pm \sqrt{1 - 2} = -1 \pm j$$

so that

$$\lambda_1 = -1 + j \qquad\qquad \lambda_2 = -1 - j.$$

Which one is larger? Note, there is no possible ordering among complex numbers.

Eigenpair: Example

Eigenvectors: The eigenvectors can be found from the formula $(\lambda I - A)x = 0$.

$$\lambda_1 : \begin{pmatrix} -1+j & -1 \\ 2 & 1+j \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that the rows are linearly dependent since

$$\begin{aligned} &(-1+j)\begin{pmatrix} -1+j & -1 \end{pmatrix} + \begin{pmatrix} 2 & 1+j \end{pmatrix} \\ &= \begin{pmatrix} -2 & -(1+j) \end{pmatrix} + \begin{pmatrix} 2 & 1+j \end{pmatrix} \\ &= 0. \end{aligned}$$

Therefore, solving $(-1+j)x_{11} - x_{12} = 0$ gives

$$x_{12} = (-1+j)x_{11}$$

Let $x_{11} = 1$ then $x_{12} = -1+j$.
So

$$x_1 = \begin{pmatrix} 1 \\ -1+j \end{pmatrix}$$

is an eigenvector.

Eigenpair: Example

$$\lambda_2 : \begin{pmatrix} -1-j & -1 \\ 2 & 1-j \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Again the rows are linearly dependent so solve to get $(-1-j)x_{21} = x_{22}$ Let $x_{21} = 1$ then $x_{22} = -1-j$. So

$$x_2 = \begin{pmatrix} 1 \\ -1-j \end{pmatrix}$$

is an eigenvector.

Characteristic Polynomial

Definition

The polynomial

$$\chi_A(\lambda) = det(\lambda I - A)$$

is called the characteristic polynomial of A. The eigenvalues of A are the roots of $\chi_A(\lambda) = 0$. The set of roots of $\chi_A(\lambda) = 0$ is called the spectrum of A, denoted $\lambda(A)$.

Relationship between transfer function and state space models

Given a state space system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, what is the transfer function?
Take the Laplace transform to get

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s)$$

From the first equation we get

$$X(s) = (sI - A)^{-1}BU(s)$$

From the second equation we get

$$Y(s) = \underbrace{C(sI - A)^{-1}B}_{(p \times m)\text{ transfer matrix}} U(s)$$

Relationship between transfer function and state space models

What are the poles of the system?

$$\begin{aligned} Y(s) &= C(sI - A)^{-1} BU(s) \\ &= \frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)} U(s) \end{aligned}$$

Therefore, the poles are when

$$det(sI - A) = 0,$$

i.e. when s is an eigenvalue of A.

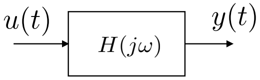
The poles of an LTI system and the eigenvalues of A are equivalent!

Generalized Eigenvalues

Eigenvalues and eigenvectors can be defined for more general operators than just matrices.

Example

Let $h(t)$ be the impulse response of an LTI system with Fourier transform $H(j\omega)$.



Recall that if $u(t) = e^{j\omega_0 t}$ then

$$
\begin{aligned}
y(t) &= |H(j\omega_0)| e^{j(\omega_0 t + \angle H(j\omega_0))} \\
&= |H(j\omega_0)| e^{j\angle H(j\omega_0)} e^{j\omega_0 t}
\end{aligned}
$$

i.e. if a sinusoid goes in then the output will be a sinusoid of the same frequency but different magnitude and phase.

Generalized Eigenvalues

Lemma

Let $\mathcal{A}[u] = \int_0^{T} h(t-\tau)u(\tau)d\tau$ *then*

$$(\lambda, x) = \left(H(j\omega)e^{j\angle H(j\omega)}, e^{j\omega t}\right)$$

is an eigenpair of $\mathcal{A}$.

Proof.

$$\mathcal{A}[e^{j\omega t}] = \left(|H(j\omega)|e^{j\angle H(j\omega)}\right) e^{j\omega t}.$$

Note that for $\mathcal{A}$ there are an uncountable infinite number of eigenpairs. $\square$

Geometric and Algebraic Multiplicity

Definition

Factor the characteristic polynomial as follows:

$$\chi_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p}$$

m_i is the algebraic multiplicity of eigenvalue λ_i.

Definition

The geometric multiplicity of eigenvalue λ_i is defined as

$$q_i = dim(\mathcal{N}(\lambda_i I - A)).$$

Geometric and Algebraic Multiplicity: Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then

$$\chi_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2.$$

Therefore, the algebraic multiplicity of $\lambda_1 = 1$ is $m_1 = 2$.
What is the geometric multiplicity?

$$q_1 = dim(\mathcal{N}(\lambda_1 I - A)) = dim\left(\mathcal{N}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = dim(\mathbb{R}^2) = 2.$$

Note that the eigenvectors $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$ are linearly independent!

Geometric and Algebraic Multiplicity: Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then

$$\chi_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2$$

so the algebraic multiplicity of $\lambda_1 = 1$ is $m_1 = 2$.
The geometric multiplicity is

$$\begin{aligned} q_1 &= dim(\mathcal{N}\{I - A\}) = dim(\mathcal{N} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) \\ &= dim(\{x \in \mathbb{R}^2 \mid x_2 = 0\}) = 1 \neq m_1 \end{aligned}$$

What are the eigenvectors assocaited with A?

$$(\lambda I - A)x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} x_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_{12} = 0$$

so $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ are the eigenvectors associated with λ_1. There are <u>not</u> two linearly independent eigenvectors.

Linearly Independent Eigenvectors

In general we have,

Lemma

Let $A \in \mathbb{C}^{n \times n}$, then there are n-linearly independent eigenvectors if and only if

$$\textit{algebraic multiplicity} = \textit{geometric multiplicity}$$

for each eigenvalue of A.

Linearly Independent Eigenvectors: Proof

Proof.

First prove that if $\lambda_i \neq \lambda_j$ then

$$\mathcal{N}(\lambda_i I - A) \cap \mathcal{N}(\lambda_j I - A) = \{0\}.$$

To prove the claim, suppose not, then

$$\exists x \neq 0 \text{ such that } x \in \mathcal{N}(\lambda_i I - A) \text{ and } x \in \mathcal{N}(\lambda_j I - A)$$

$$\begin{aligned}
\iff &\qquad Ax = \lambda_i x \text{ and } Ax = \lambda_j x \\
\iff &\qquad \lambda_i x = \lambda_j x \\
\iff &\qquad (\lambda_i - \lambda_j)x = 0 \\
\iff &\qquad \lambda_i = \lambda_j
\end{aligned}$$

which is a contradiction.

Linearly Independent Eigenvectors: Proof

Note that the number of linearly independent eigenvectors associated with λ_i is the geometric multiplicity q_i since we can find q_i linearly independent vectors that span $\mathcal{N}(\lambda_i - A)$.

The previous claim shows that if $x_i \in \mathcal{N}(\lambda_i I - A)$ then $x_i \notin \mathcal{N}(\lambda_j I - A)$ which implies that there are $\sum q_i$ linearly independent eigenvectors of A. Since $\sum m_i = n$, the lemma follows.

Note that if the eigenvalues are all distinct then $m_i = 1$. Also since $1 \leq q_i \leq m_i$, for each i, we must have that the algebraic multiplicity equals the geometric multiplicity.

$\square$

Linearly Independent Eigenvectors

Suppose that there are n-linearly independent eigenvectors (where some of the eigenvalues might be repeated), then we can write

$$
\begin{aligned}
&\begin{pmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{pmatrix} \\
\iff & A \underbrace{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}}_{\Lambda} \\
\iff & AS = S\Lambda
\end{aligned}
$$

Since the eigenvectors are linearly independent, S is invertible. Therefore

$$
\begin{aligned}
A &= S\Lambda S^{-1} \\
\iff \Lambda &= S^{-1}AS
\end{aligned}
$$

Therefore, we say that S diagonalizes A.

Similarity Transformation

Definition

Two matrices, A and B are said to be similar if $\exists$ an invertible T such that

$$A = TBT^{-1}.$$

Lemma

Similar matrices have the same eigenvalues.

Proof.

Let (λ, x) be an eigenpair of A, then

$$\begin{aligned} & Ax = \lambda x \\ \iff & TBT^{-1}x = \lambda x \\ \iff & BT^{-1}x = \lambda T^{-1}x \\ \iff & By = \lambda y \text{ where } y = T^{-1}x \\ \iff & (\lambda, y) \text{ is an eigenpair of } B \end{aligned}$$

Section 2

Jordan Form

Jordan Form

What if the algebraic multiplicity does not equal the geometric multiplicity? (i.e., $q_i \neq m_i$ for some eigenvalue λ_i of A)?

Then we cannot diagonalize A using a similarity transformation. However we can "almost" diagonalize A.

The resulting "almost diagonal" matrix is called the Jordan form of A.

Jordan Form, cont.

Suppose the algebraic multiplicity of λ_1 is $m_1 > 1$ but the geometric multiplicity is $q_1 = 1$.

Then $\exists$ one linearly independent eigenvector x_1 s.t. $Ax_1 = \lambda_1 x_1$.

Now form the following chain:

$$\begin{aligned}
A\xi_{11} &= \lambda_1 \xi_{11} + x_1 \\
A\xi_{12} &= \lambda_1 \xi_{12} + \xi_{12} \\
&\vdots \\
A\xi_{1,m_1} &= \lambda_1 \xi_{1,m_1} + \xi_{1,(m_1-1)}
\end{aligned}$$

$\xi_{11} \cdots \xi_{1,m_1}$ are called the "generalized eigenvectors" associated with x_1.

Jordan Form, cont.

Note that we can write the generalized eigenvector equations as

$$A \begin{pmatrix} x_1 & \xi_{11} & \cdots & \xi_{1,m_1} \end{pmatrix} = \begin{pmatrix} x_1 & \xi_{11} & \cdots & \xi_{1,m_1} \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & 1 & \ddots & & \\ & \lambda_1 & 1 & 0 & \\ \ddots & & \lambda_1 & \ddots & \ddots \\ & 0 & & \ddots & 1 \\ & & \ddots & & \lambda_1 \end{pmatrix}}_{\text{This is called a Jordan block}}$$

Lemma

If the geometric multiplicity of λ_i is $q_i = 1$ then the associated $m_1 - 1$ generalized eigenvectors are linearly independent of the other eigenvectors.

Jordan Form, cont.

If $1 < q_i < m_i$ then the problem is slightly more complicated.

There are precisely q_i linearly independent eigenvectors associated with λ_i and there will be q_i Jordan blocks associated with λ_i. What are the sizes of the Jordan blocks? For example, suppose $m_i = 4$ and $q_i = 2$, the possible Jordan blocks are:

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1 \end{pmatrix} \text{ i.e., } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

or

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \text{ i.e., } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

Which option is correct?

Jordan Form, cont.

To decide, generate the generalized eigenvector for each eigenvector and pick the linearly independent ones.
Example: Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $\det(\lambda I - A) = (\lambda - 1)^4$ we have $\lambda_1 = 1$ and $m_1 = 4$.

$$q_1 = dim(\mathcal{N} \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = 2$$

since there are 2 linearly independent rows.

Jordan Form, cont.

So there are two linearly independent eigenvectors:

$$(\lambda_1 I - A)x_1 = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{pmatrix} = \begin{pmatrix} -x_{12} + x_{13} - x_{14} \\ -x_{14} \\ -x_{14} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies x_{14} = 0 \text{ and } -x_{12} + x_{13} - x_{14} = 0$$

so

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ is an eigenvector, and so is } x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Jordan Form, cont.

Find the possible generalized eigenvector associated with eigenvector x_1:

$$A\xi_{11} = \xi_{11} + x_1 \Rightarrow (\lambda_1 I - A)\xi_{11} = -x_1$$

$$\text{i.e.} \quad -\xi_{112} + \xi_{113} - \xi_{114} = 1 \qquad\qquad \xi_{114} = 0$$

$$\xi_{112} = \xi_{113} + 1 \qquad \text{so} \qquad \xi_{11} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \text{ is valid}$$

$$(\lambda_1 I - A)\xi_{12} = \xi_{12} \text{ so } \begin{pmatrix} -\xi_{122} + \xi_{123} - \xi_{124} \\ -\xi_{124} \\ -\xi_{124} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \leftarrow \text{ can't use .}$$

Jordan Form, cont.

Note: There are an infinite number of possibilities of generalized eigenvectors from each true eigenvector, but you can only pick ones that are linearly independent. This second eigenvector forms a linearly dependent subset of one of the real eigenvectors.

Therefore, one Jordan block is of size 2.

Also solve $(\lambda_1 I - A)\xi_{21} = x_2$ i.e.

$$\begin{pmatrix} -\xi_{212} + \xi_{213} - \xi_{214} \\ -\xi_{214} \\ -\xi_{214} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \xi_{214} = 1, \xi_{213} = \xi_{212} + 1$$

so $\xi_{21} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$.

Jordan Form, cont.

In summary

$$A\underbrace{\begin{pmatrix} x_1 & \xi_{11} & x_2 & \xi_{21} \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} x_1 & \xi_{11} & x_2 & \xi_{21} \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{J}$$

or

$$A = SJS^{-1}$$

J is called the "Jordan" form of A

If the eigenvalues are distinct or $q_i = m_i$ for each i then $J = \Lambda$ (is diagonal).

Otherwise J is block diagonal with Jordan blocks along the diagonal (q_i Jordan blocks for each eigenvalue).

Jordan Form, cont.

Example: suppose there are 3 eigenvalues with $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, and $m_1 = 1, m_2 = 2, m_3 = 3$, and $q_1 = 1, q_2 = 1, q_3 = 2$. There are two possible Jordan forms:

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & 1 & & 0 & \\ & & \lambda_2 & & & \\ & & & \lambda_3 & 1 & \\ & 0 & & & \lambda_3 & \\ & & & & & \lambda_3 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & 1 & & 0 & \\ & & \lambda_2 & & & \\ & & & \lambda_3 & & \\ & 0 & & & \lambda_3 & 1 \\ & & & & & \lambda_3 \end{pmatrix}$$

Section 3

Cayley-Hamilton Theorem

Functions of Matrices

Lemma

A square matrix can always be put in Jordan form.

This implies that we can always write

$$A = SJS^{-1}$$

This implies that

$$\begin{aligned} A^k &= \underbrace{AA\cdots A}_{k \text{ times}} \\ &= SJS^{-1}SJS^{-1}\cdots SJS^{-1} \\ &= SJ^kS^{-1} \end{aligned}$$

This is particularly simple if $J = \Lambda$ since

$$A^k = S\Lambda^k S^{-1} \text{ where } \Lambda^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

Functions of Matrices, cont.

For square matrices we can define analytic functions of matrices. Analytic functions are functions that can be expanded as a Taylor seires, e.g.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The corresponding Matrix function is defined in terms of its Taylor series, e.g.,

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$
$$\cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \cdots$$
$$\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots$$

Cayley-Hamilton Theorem

Computing infinite series of matrices is a pain. Fortunately we have the following theorem:

Theorem (Cayley-Hamilton Theorem)

Every matrix satisfies its own characteristic polynomial, i.e.

$$\chi_A(A) = 0.$$

Cayley-Hamilton Theorem, proof

The proof holds for all A but we will only prove the case when $q_i = m_i$ for each λ_i. In this case $A = S\Lambda S^{-1}$. Note that for each eigenvalue $\chi_A(\lambda_i) = 0$ since $\chi_A(\lambda_i) = \det(\lambda_i I - A)$
Let

$$\chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

then

$$\begin{aligned}
\chi_A(A) &= A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I \\
&= S\Lambda^n S^{-1} + a_{n-1}S\Lambda^{n-1}S^{-1} + \cdots + a_1 S\Lambda S^{-1} + a_0 SS^{-1} \\
&= S(\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1\Lambda + a_0 I)S^{-1}
\end{aligned}$$

Note that the matrix

$$\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1\Lambda + a_0 I$$

is diagonal with each element on the diagonal equal to $\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \cdots + a_1\lambda_i + a_0 = 0.$
Therefore

$$\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1\Lambda + a_0 I = 0.$$

Cayley-Hamilton Theorem, implications

Recall polynomial division:

$$\frac{f(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)}$$

$$\implies \underbrace{f(x)}_{\text{degree } m} = \underbrace{a(x)}_{\text{degree } (m-n)} \overbrace{q(x)}^{\text{degree } n} + \underbrace{r(x)}_{\text{degree } < n}$$

Application to infinite series like e^x gives

$$\underbrace{e^x}_{\text{degree } \infty} = \underbrace{a(x)}_{\text{degree } \infty} \overbrace{\chi_A(x)}^{\text{degree } n} + \underbrace{r(x)}_{\text{degree } < n}$$

Since $\chi_A(A) = 0$,

$$e^A = \underbrace{r(A)}_{\text{degree } < n} = \cdots b_{n-1}A^{n-1} + \cdots + b_1 A + b_0 I$$

Since $e^{\lambda_i} = r(\lambda_i)$ the coefficients can be found from

$$e^{\lambda_i} = \cdots b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0.$$

Cayley-Hamilton Theorem, example

Find e^A where $A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$.

$$\det(\lambda I - A) = \lambda^2 + 2\lambda + 2 \Rightarrow \lambda_{1,2} = -1 \pm j$$

$$\Rightarrow e^A = b_1 A + b_0 I = \begin{pmatrix} 0 & b_1 \\ -2b_1 & -2b_1 \end{pmatrix} + \begin{pmatrix} b_0 & 0 \\ 0 & b_0 \end{pmatrix} = \begin{pmatrix} b_0 & b_1 \\ -2b_1 & -2b_1 - b_0 \end{pmatrix}$$

where b_0 and b_1 satisfy

$$e^{-1+j} = b_1(-1+j) + b_0$$

$$e^{-1-j} = b_1(-1-j) + b_0$$

$$\Rightarrow \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 & -1+j \\ 1 & -1-j \end{pmatrix}^{-1} \begin{pmatrix} e^{-1+j} \\ e^{-1-j} \end{pmatrix} = \begin{pmatrix} 0.5083 \\ 0.3096 \end{pmatrix}$$

$$\Rightarrow e^A = \begin{pmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{pmatrix}$$

Section 4

Self Adjoint Matrices

Self Adjoint Matrices

Definition

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be <u>self adjoint</u> (also called <u>Hermitian</u>) if

$$A = A^H.$$

Lemma (Moon 6.2)

If $A = A^H$ then the eigenvalues of A are real.

Proof.

Let (λ, x) be a right eigen-pair, then

$$Ax = \lambda x, \text{ and } x^H A^H = \bar{\lambda} x^H.$$

Therefore

$$\begin{aligned} & x^H A x = \lambda x^H x, \text{ and } x^H A^H x = \bar{\lambda} x^H x \\ \implies & \lambda x^H x = \bar{\lambda} x^H x \implies \bar{\lambda} = \lambda, \\ \implies & \lambda \text{ is real.} \end{aligned}$$

Self Adjoint Matrices

Lemma (Moon 6.3)

If $A = A^H$ and the eigenvalues are distinct, then the eigenvectors are orthogonal.

Proof.

Let (λ_1, x_1) and (λ_2, x_2) be distinct eigenpairs, i.e. $\lambda_1 \neq \lambda_2$, then

$$x_2^H A x_1 = \lambda_1 x_2^H x_1$$
$$\text{and } x_2^H A^H x_1 = \lambda_2 x_2^H x_1$$

Therefore $(\lambda_1 - \lambda_2) x_2^H x_1 = 0$. Because $\lambda_1 \neq \lambda_2$ we must have that

$$x_2^H x_1 = 0$$

which implies that x_1 and x_2 are orthogonal. $\square$

Note the eigenvectors can always be chosen to be orthonormal.

Self Adjoint Matrices

Theorem (Moon Theorem 6.2 (Special Theorem))

If $A \in \mathbb{C}^{n\times n}$ is Hermitian, then $q_i = m_i$ for each eigenvalue λ_i.

Corollary

If $A = A^H$, then $\exists$ a unitary U and real diagonal Λ such that

$$A = U\Lambda U^H.$$

Eigenvalues and Rank

Lemma (Moon Lemma 6.5)

Let $A \in \mathbb{C}^{m \times m}$ be of rank $r < m$. Then at least $m - r$ of the eigenvalues of A are equal to zero

Section 5

Invariant Subspaces

Invariant Subspaces

Definition

Let A be a square matrix. If $\mathbb{S} \subset \mathcal{R}(A)$ is such that $x \in \mathbb{S} \implies Ax \in \mathbb{S}$ then $\mathbb{S}$ is an invariant subspace of A.

Example

An eigenvector forms an invariant subspace i.e.

$$\mathbb{S} = \{\alpha x \mid x \text{ is an eigenvector}\}$$

is invariant since $\hat{x} \in \mathbb{S} \Rightarrow A\hat{x} = \lambda\hat{x} \in \mathbb{S}$.

Invariant Subspaces

Example

The span of any subset of eigenvectors is invariant: Let $x_1 \dots x_p$ be eigenvectors with associated eigenvalues $\lambda_1 \dots \lambda_p$.
Let

$$\mathbb{S} = \text{span}\{x_1 \dots x_p\}$$

then

$$\begin{aligned}
&\hat{x} \in \mathbb{S} \\
\implies &\hat{x} = \alpha_1 x_1 + \cdots + \alpha_p x_p \\
\implies &A\hat{x} = \alpha_1 A x_1 + \cdots + \alpha_p A x_p \\
\implies &A\hat{x} = \alpha_1 \lambda_1 x_1 + \cdots + \lambda_p \alpha_p x_p \\
\implies &A\hat{x} \in \mathbb{S}.
\end{aligned}$$

Applications to Differential Equations

Consider the differential equation $\dot{x} = Ax$ with initial condition $x(0) = x_0$.

Lemma

The solution is given by $x(t) = e^{At}x_0$ *where* $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$.

Proof.

Plug into equation

$$\begin{aligned}
\frac{dx(t)}{dt} &= \frac{d}{dt}(e^{At})x_0 + e^{At}\frac{d}{dt}(x_0) = \frac{d}{dt}e^{At}x_0 \\
&= \frac{d}{dt}(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \cdots x_0 \\
&= (A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots)x_0 \\
&= A(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots)x_0 \\
&= Ae^{At}x_0 = Ax(t)
\end{aligned}$$

so $x(t) = e^{At}x_0$ satisfies $\dot{x} = Ax$ with initial condition x_0. $\square$

Applications to Differential Equations

Lemma

If $\mathbb{S}$ is an invariant subspace of A then $\mathbb{S}$ is an invariant subspace of e^{At}

Proof.

Let $x_0 \in \mathbb{S}$ then

$$
\begin{aligned}
Ax_0 \in \mathbb{S} &\implies tAx_0 \in \mathbb{S} \\
&\implies A(Ax_0) \in \mathbb{S} \\
&\implies \frac{A^2t^2}{2!}x_0 \in \mathbb{S} \\
&\cdots
\end{aligned}
$$

Therefore

$$
x(t) = Ix_0 + Atx_0 + \frac{A^2t^2}{2!}x_0 + \frac{A^3t^3}{3!}x_0 + \cdots \in \mathbb{S}.
$$

Applications to Differential Equations: Example

Consider the differential equation

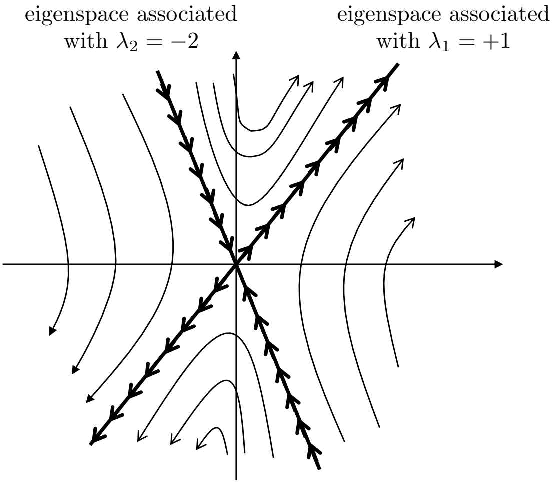
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} x.$$

The eigenvalues of A are given by $\det(\lambda I - \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}) = (\lambda - 1)(\lambda + 2) = 0$ and so $\lambda_1 = 1$ and $\lambda_2 = -2$.

The associated eigenvector are

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Applications to Differential Equations: Example



- If the initial condition is on $span\{x_1\}$, then the solution remains on $span\{x_1\}$.
- If the initial condition is on $span\{x_2\}$, then the solution remains on $span\{x_2\}$.

Applications to Difference Equations: Example

RWB:Change system to eigenvalues in unit circle. Show that eigenspaces are invariant. Provide example.

Consider the differential equation

$$x[k+1] = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} x[k].$$

Again the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -2$, and the eigenvectors are

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Section 6

Quadratic Forms

Quadratic Forms

Definition
A real square matrix is <u>symmetric</u> if $A^\top = A$

Definition
A real square matrix is <u>skew-symmetric</u> if $A^\top = -A$

Quadratic Forms

Lemma

Any real square matrix $B \in \mathbb{R}^{n \times n}$ can be written as

$$B = B_s + B_{ss}$$

where B_s is symmetric and B_{ss} is skew-symmetric.

Proof.

$$B = \frac{B + B^\top}{2} + \frac{B - B^\top}{2} \triangleq B_s + B_{ss}$$

where

$$B_s^\top = \left(\frac{B + B^\top}{2}\right)^\top = \frac{B^\top - B}{2} = \frac{B + B^\top}{2} = B_s$$

$$B_{ss}^\top = \left(\frac{B - B^\top}{2}\right)^\top = \frac{B^\top - B}{2} = -\left(\frac{B - B^\top}{2}\right) = -B_{ss}$$

Quadratic Forms

Lemma

For any real square matrix A and for all y

$$y^\top A y = y^\top A_s y$$

where A_s is the symmetric part of A.

Proof.

$$y^\top A y = y^\top A_s y + y^\top A_{ss} y$$

but

$$y^\top A_{ss} y = y^\top \left(\frac{A - A^\top}{2} \right) y = \frac{1}{2} y^\top A y - \frac{1}{2} y^\top A^\top y.$$

But since

$$y^\top A^\top y = (y^\top A^\top y)^\top = y^\top A y \implies y^\top A_{ss} y = 0.$$

Quadratic Forms

Definition

A quadratic form of a real square matrix A is $Q_A(y) = \mathbf{y}^\top A\mathbf{y}$.

w.l.o.g. A can be assumed to be symmetric. If not, we can always limit our attention to the symmetric part of A since

$$\mathbf{y}^\top A\mathbf{y} = \mathbf{y}^\top A_s\mathbf{y}.$$

Quadratic forms show up in numerous places. For example, the pdf for a Gaussian random variable is

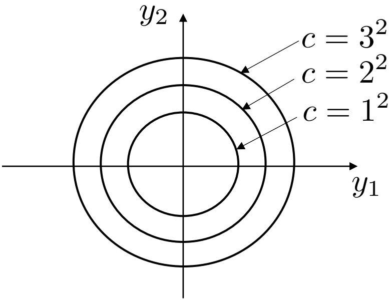
$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right).$$

Quadratic Forms

Example

Let

$$Q_A(\mathbf{y}) = \mathbf{y}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} = y_1^2 + y_2^2 = c$$



The level curves of $Q_A(\mathbf{y})$ are circles of radius $\sqrt{c}$.

Quadratic Forms

Example

Consider the quadratic equation

$$f(x) = 2y_1^2 + 3y_1 y_2 + 4y_2^2,$$

and note that

$$\begin{aligned} f(x) &= 2y_1^2 + 3y_1 y_2 + 4y_2^2 \\ &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 2y_1 + \frac{3}{2}y_2 \\ 4y_2 + \frac{3}{2}y_1 \end{pmatrix} \\ &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \mathbf{y}^\top A\mathbf{y} \end{aligned}$$

Any quadratic equation in n variables can be written in the form $\mathbf{y}^\top A\mathbf{y}$.

Quadratic Forms

By the spectral theorem, A is diagonalizable. In other words, there exists an invertible U so that $A = U\Lambda U^\top$.

From Moon Lemma 6.2 the eigenvalues are real so we can order them as

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

with

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

Quadratic Forms

Lemma

Level curves of the quadratic form

$$Q_A(x - x_0) = (x - x_0)^\top A(x - x_0) = c$$

are hyper-ellipsoids with the length of the axes given by $\frac{1}{\sqrt{\lambda_i}}$.

Proof.

Let $z = U^\top y$ then

$$\begin{aligned} Q_A(y) &= y^\top A y = y^\top U \Lambda U^\top y = z^\top \Lambda z \\ &= \begin{pmatrix} z_1 & \cdots & z_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \\ &= \lambda_1 z_1^2 + \cdots + \lambda_n z_n^2 \end{aligned}$$

Quadratic Forms

Note that in the variable z, the quadratic form is an ellipsoid:

$$Q_A(\mathbf{y}) = (\sqrt{\lambda_1})^2 z_1^2 + (\sqrt{\lambda_2})^2 z_2^2 + \cdots + (\sqrt{\lambda_n})^2 z_n^2 = 1$$

or

$$Q_A(\mathbf{y}) = \frac{z_1^2}{\left(\frac{1}{\sqrt{\lambda_1}}\right)^2} + \frac{z_2^2}{\left(\frac{1}{\sqrt{\lambda_2}}\right)^2} + \cdots + \frac{z_n^2}{\left(\frac{1}{\sqrt{\lambda_n}}\right)^2} = 1$$

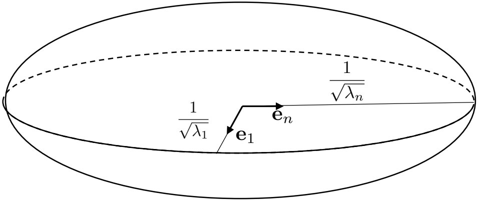
Either of these are the general equation for an ellipsoid with minor axis $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^\top$ and major axis $\mathbf{e}_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\top$

Quadratic Forms

In the original space, what is $\mathbf{e}_1$?

Note that along $\mathbf{e}_1$, the stretching is

$$(\sqrt{\lambda_1})^2 z_1^2 = 1 \Rightarrow z_1 = \frac{1}{\sqrt{\lambda_1}}$$



$$\mathbf{e}_1 = U^\top \mathbf{y} = \begin{pmatrix} \mathbf{u}_1^\top \\ \vdots \\ \mathbf{u}_n^\top \end{pmatrix} \mathbf{y}$$

$$= \begin{pmatrix} \mathbf{u}_1^\top \mathbf{y} \\ \vdots \\ \mathbf{u}_n^\top \mathbf{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore $\mathbf{y} = \mathbf{u}_1$ since U is orthogonal.
i.e. $\mathbf{u}_i = U\mathbf{e}_i$.

Quadratic Forms

Therefore the major axis is given by the eigenvector associated with the smallest eigenvalue, and the minor axis is given by the eigenvector associated with the largest eigenvalue.

Question: What is the geometric picture associated with

$$(x - x_0)^\top A(x - x_0) = c$$

where c is a constant and A is symmetric and positive definite?

Answer: An ellipsoid of radius $\sqrt{c}$ centered at x_0 with axes along the eigenvectors of A and stretching along each axis given by $\dfrac{1}{\sqrt{\lambda_i}}$.

Quadratic Forms

Question: What if we would like to maximize

$$Q_A(y) = y^\top A y \text{ where } \|y\| = 1.$$

Which axis provides the most bang-for-the-buck?

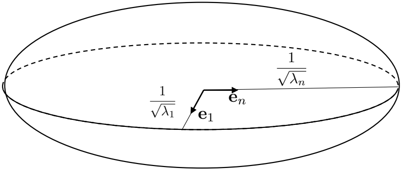
Answer: The <u>major</u> axis! i.e. the axis associated with the largest eigenvalue.

Quadratic Forms

Rather than drawing

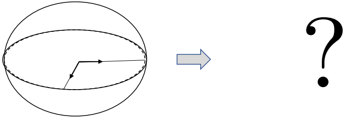
$$\lambda_1 z_1^2 + \lambda_2 z_2^2 + \cdots + \lambda_n z_n^2 = 1$$

which is



lets draw the mapping of the unit circle through $\mathbf{y}^\top A\mathbf{y}$ i.e.

$$\{\|\mathbf{y}\| = 1\} \xrightarrow{Q_A(\mathbf{y})} \{\mathbf{y}^\top A\mathbf{y}\}$$



Quadratic Forms

If $A = A^\top$ then $A = U\Lambda U^\top$ where U is orthogonal, i.e., $UU^\top = U^\top U = I$. Then

$$\max_{\|\mathbf{y}\|=1} \mathbf{y}^\top A\mathbf{y} = \max_{\|\mathbf{y}\|=1} \mathbf{y}^\top U\Lambda U^\top \mathbf{y}.$$

Let $\mathbf{z} = U^\top \mathbf{y}$ and note that $\|\mathbf{z}\| = \left\|U^\top \mathbf{y}\right\| = \|\mathbf{y}\|$ since U is orthogonal. Then

$$\begin{aligned}\max_{\|\mathbf{y}\|=1} \mathbf{y}^\top U\Lambda U^\top \mathbf{y} &= \max_{\|\mathbf{z}\|=1} \mathbf{z}^\top \Lambda\mathbf{z} \\ &= \max_{\|\mathbf{z}\|=1} \left(\lambda_1 z_1^1 + \lambda_2 z_2^2 + \cdots + \lambda_n z_n^2\right)\end{aligned}$$

where Λ is arranged such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Quadratic Forms

The maximum is therefore

$$\mathbf{z}^* = \begin{pmatrix} 1 & 0 & \vdots & 0 \end{pmatrix}^\top$$

where it is clear that $\|\mathbf{z}\| = 1$.
Furthermore

$$\max_{\|\mathbf{z}\|=1} \mathbf{z}^\top \Lambda \mathbf{z} = \lambda_1,$$

which implies that

$$\mathbf{y}^* = U\mathbf{z}^* = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{u}_1.$$

Quadratic Forms

This mapping also forms an ellipsoid but with a different effect.
Let $\mathbf{y} = \mathbf{u}_1 \implies \|\mathbf{y}\| = 1$ to get

$$Q_A(\mathbf{y}) = \lambda_1$$

Question: Is it possible to pick a $\hat{\mathbf{y}}$ where $\|\hat{\mathbf{y}}\| = 1$ such that

$$Q_A(\hat{\mathbf{y}}) > Q_A(\mathbf{u}_1)?$$

Answer: No.

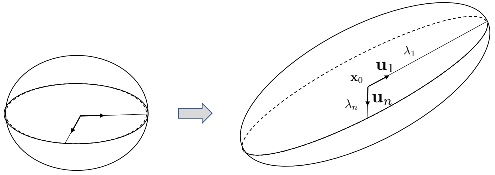
Quadratic Forms

Explanation: Recall that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\|\hat{\mathbf{y}}\| = y_1^2 + \cdots + y_n^2 = 1$.

Therefore

$$\begin{aligned} Q_A(\hat{\mathbf{y}}) &= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \\ &\leq \lambda_1 y_1^2 + \lambda_1 y_2^2 + \cdots + \lambda_1 y_n^2 \\ &= \lambda_1 \|\hat{\mathbf{y}}\|^2 \\ &= Q_A(\mathbf{u}_1) \end{aligned}$$

So the mapping of the unit circle looks like



Quadratic Forms

We have essentially proved the following theorem:

Theorem (Moon Theorem 6.5)

For a positive semi-definite Hermitian matrix A, the maximum

$$\lambda_1 = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^H A \mathbf{x}$$

where λ_1 is the largest eigenvalue of A, and the maximizing $\mathbf{x}$ is $\mathbf{x} = \mathbf{u}_1$, the associated eigenvector.
Furthermore if we maximize $\mathbf{x}^H A \mathbf{x}$ subject to the constraints

$$\begin{aligned} \langle \mathbf{x}, \mathbf{u}_i \rangle &= 0 \qquad i = 1, \ldots, k-1, \\ \|\mathbf{x}\|_2 &= 1 \end{aligned}$$

then the maximum is λ_k and $\mathbf{x}_{max} = \mathbf{u}_k$.

Quadratic Forms

Note that if A is positive semi-definite Hermitian then

$$\|A\|_2 = \sup_{\|\mathbf{x}\|_2 \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \max_{\|\mathbf{x}\|_2 = 1} \sqrt{\mathbf{x}^H A^H A \mathbf{x}} = \sqrt{\lambda_1 \mathbf{u}_1^H \mathbf{u}_1} = \sqrt{\lambda_1}$$

where λ_1 is the largest eigenvalue of $A^H A$.

More generally,

$$R(x) = \frac{\mathbf{x}^\top A \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$$

is called a Rayleigh quotient and

$$\max_{\|\mathbf{x}\| \neq 0} R(\mathbf{x}) = \lambda_1$$

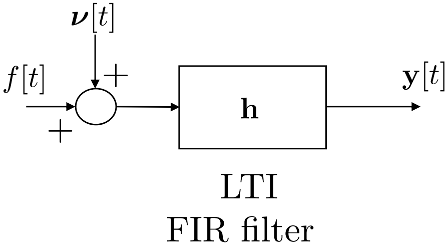
$$\min_{\|\mathbf{x}\| \neq 0} R(\mathbf{x}) = \lambda_n.$$

Section 7

Eigenfilters

Eigenfilters for Random Signals

Problem: Given



where ν is white noise with variance σ^2, and f is a stationary, zero-mean random process.

Find **h** to maximize the signal-to-noise ratio.

Eigenfilters for Random Signals

Let

$$\mathbf{f}(t) = \begin{pmatrix} f(t) \\ f(t-1) \\ \vdots \\ f(t-(m-1)) \end{pmatrix}$$

then

$$y(t) = \mathbf{h}^H \mathbf{f}(t).$$

The output power due to the signal $\mathbf{f}$ is

$$\begin{aligned} P_0 &= E|y(t)|^2 = E|\mathbf{h}^H \mathbf{f}(t)|^2 = E\{\mathbf{h}^H \mathbf{f}(t) \mathbf{h}^H \mathbf{f}(t)\} \\ &= E\{\mathbf{h}^H \mathbf{f}(t) \mathbf{f}^H(t) \mathbf{h}\} = \mathbf{h}^H E\{\mathbf{f}(t) \mathbf{f}^H(t)\} \mathbf{h} \\ &= \mathbf{h}^H R \mathbf{h} \end{aligned}$$

where $R = E\{\mathbf{f}(t)\mathbf{f}^H(t)\}$

Eigenfilters for Random Signals

Let

$$\boldsymbol{\nu}(t) = \begin{pmatrix} v(t) \\ v(t-1) \\ \vdots \\ v(t-m+1) \end{pmatrix}$$

Then the output due to the noise is

$$\mathbf{h}\boldsymbol{\nu}(t)$$

and the average noise power is

$$N_0 = E\{\mathbf{h}^H \boldsymbol{\nu}(t) \boldsymbol{\nu}^H(t) \mathbf{h}\} = \sigma^2 \mathbf{h}^H \mathbf{h}$$

Eigenfilters for Random Signals

The signal-to-noise ratio is

$$SNR = \frac{P_0}{N_0} = \frac{1}{\sigma^2} \cdot \underbrace{\frac{\mathbf{h}^H R \mathbf{h}}{\sigma^2 \mathbf{h}^H \mathbf{h}}}_{\text{Rayleigh quotient}}$$

Therefore

$$SNR_{max} = \frac{\lambda_1}{\sigma^2}$$

where λ_1 is the largest eigenvalue of R and $\mathbf{h} = q_1$ the largest eigenvector of R.