ECEn 671: Mathematics of Signals and Systems

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Section 1

Orthogonality

Orthogonality

Let $x, y \in \mathbb{X}$ where $\mathbb{X}$ is an inner product space. Then the angle between x and y is

$$\theta = cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\,\|y\|}\right).$$

i.e.

$$\langle x, y\rangle = \|x\|\,\|y\|\,cos\theta$$

Orthogonality, cont.

Definition (Colinear)

Two vectors $x, y \in \mathbb{X}$ are said to be colinear if

$$\theta = 180 * n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Definition (Orthogonal)

Two vectors $x, y \in \mathbb{X}$ are said to be orthogonal if

$$\theta = 90 * n \qquad n = \pm 1, \pm 3, \pm 5, \ldots$$

i.e., $\langle x, y \rangle = 0$.

If $\langle x, y \rangle = 0$ we write $x \perp y$.

Orthogonality, cont.

Example (Vectors in $L_2[0, 2\pi)$)

The functions $x = sin(t)$ and $y = cos(t)$ are orthogonal since

$$\langle x, y \rangle = \int_0^{2\pi} sin(t)cos(t)dt = 0.$$

Example (Vectors in ℓ)

The sequences

$$x = (1, 1, 1, 1, 0, 0, \dots)$$
$$y = (1, -1, 1, -1, 1, \dots)$$

are orthogonal since

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i = 0.$$

Other useful inner products and norms: Weighting

Definition (Positive Definite Matrix)

A matrix $W : \mathbb{R}^k \to \mathbb{R}^k$ is positive definite (PD) if
$\forall x \in \mathbb{R}^k \qquad x^T W x > 0$

- W is positive semi-definite (PSD) if $x^T W x \geq 0$
- W is negative definite (ND) if $x^T W x < 0 \qquad \forall x \in \mathbb{R}^k$
- W is negative semi-definite (NSD) if $x^T W x \leq 0 \qquad \forall x \in \mathbb{R}^k$
- Otherwise it is indefinite

Examples of positive definiteness

- $W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is PD since

$$x^T W x = x_1^2 + x_2^2 > 0 \qquad \forall x \in \mathbb{R}^2$$

- $W = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is PSD since

$$x^T W x = x_1^2 = 0 \qquad \forall x = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \neq 0$$

- $W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is indefinite since

$$x^T W x = -x_1^2 + x_2^2$$

which can be positive or negative depending on x.

Examples of Inner Products

Weighted Inner Products and Norms
If $W > 0$ then $\langle x, y \rangle_W = x^H W y$ is a valid inner product which induces the weighted norm $\|x\|_W = (x^H W x)^{\frac{1}{2}}$
We can define weighted inner products for functions:

$$\langle f, g \rangle_W = \int f(t) g(t) w(t) dt$$

where $w(t) > 0$ except on a set of measure zero.

Examples of Inner Products

Definition (Expectation)

Expectation is a weighted inner product with weight $f_{\mathbb{X}\mathbb{Y}}(x,y)$

$$\langle \mathbb{X}, \mathbb{Y} \rangle = \int xyf_{\mathbb{X}\mathbb{Y}}(x,y)dxdy \qquad = E[\mathbb{X}\mathbb{Y}]$$

if $\mathbb{X}$ is a zero mean then

$$\langle x, x \rangle = var(x)$$

is the norm induced by $E[\cdot\cdot]$

Examples of Inner Products

- Let $\mathbb{I}(m, n)$ be the set of grayscale images with $m \times n$ pixels, each valued between $[0, 255]$.
- A valid inner on $\mathbb{I}(m, n)$ is given by

$$\langle I, J \rangle = \sum_{u=1}^{m} \sum_{v=1}^{n} I(u, v) J(u, v), \qquad \forall I, J \in \mathbb{I}(m, n).$$

Orthogonal Subspaces

Definition (Orthogonal Subspaces)

Let V, W be subspaces of S. $V \perp W$ if

$$\forall v \in V \text{ and } \forall w \in W, \qquad \langle v, w \rangle = 0$$

Definition (Orthogonal Complement)

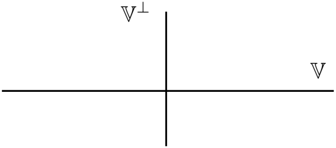
$V^\perp$, called the orthogonal complement of V, is the set

$$V^\perp = \{x \in S : \forall v \in V, \langle x, v \rangle = 0\}$$

Orthogonal Subspaces, cont.

Example

Let $S = \mathbb{R}^2$ and $V = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$ then $V^{\perp} = \left\{ \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\}$



Orthogonal Subspaces, cont.

Theorem

Let V and W be subspaces of an inner product space S (not necessarily Hilbert). Then

1. $V^{\perp}$ *is a closed subspace of* S
2. $V \subset V^{\perp\perp}$ ($V = V^{\perp\perp}$ *if* S *is complete)*
3. *If* $V \subset W$ *then* $W^{\perp} \subset V^{\perp}$
4. $V^{\perp\perp\perp} = V^{\perp}$
5. *If* $x \in V \cap V^{\perp}$ *then* $x = 0$
6. $\{0\}^{\perp} = S$ *and* $S^{\perp} = \{0\}$

Prove one in class.

Inner Sum and Direct Sum

Definition (Inner Sum)

If V and W are linear subspaces then

$$V + W = \{x : x = v + w, v \in V \text{ and } w \in W\}$$

is the inner sum.

Definition (Orthogonal Sum)

If V and W are orthogonal subspaces then the sum

$$V \oplus W = \{x : x = v + w, v \in V \text{ and } w \in W\}$$

is called the orthogonal sum.

Definition (Disjoint Subspaces)

Two subspaces are said to be disjoint if

$$V \cap W = \{0\}$$

Inner Sum and Direct Sum, cont.

Lemma

Let $V + W$ be subspaces of S and let $x \in V + W$ then the representation $x = v + w$ is unique iff $V + W$ are disjoint.

Proof.

($\Leftarrow$) Assume V, W are disjoint but $x = v + w$ is not unique i.e. $x = v_1 + w_1 = v_2 + w_2$ where $v_1 \neq v_2$ and $w_1 \neq w_2$. This implies that $v_1 - v_2 = w_2 - w_1$ but $v_1 - v_2 \in V$ and $w_2 - w_1 \in W$ since V, W are subspaces. Since $V \cap W = \{0\}$ we must have that $v_1 - v_2 = w_2 - w_1 = 0$ or $v_1 = v_2$ and $w_1 = w_2$ which is a contradiction. $\square$

Inner Sum and Direct Sum, cont.

Lemma

If V and W are orthogonal subspaces then the representation of $x \in V \oplus W$ is unique (i.e. $x = v + w$, where $v \in V$ and $w \in W$).

Example

Let $S = \mathbb{R}^2$, let $V = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$, let $W = \left\{ \begin{pmatrix} 0 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$ Then

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

is a <u>unique</u> decomposition.

Difference between a Hamel basis and a Complete basis.

Definition

An orthonormal set of basis vectors $T = \{p_i, p_2, \ldots\}$ is said to be a compelte basisfor a Hilbert space S if every $x \in S$ can be represented as

$$x = \sum_{j=1}^{\infty} c_j p_j$$

<u>Examples of complete bases:</u> Fourier functions: $e^{j\omega t}$
Legendre & Chebyshev polynomials

<u>Difference:</u> A Hamel basis $\Rightarrow$ every x can be represented by a <u>finite</u> representation. A complete basis allows functions through a limiting process.