ECEn 671: Mathematics of Signals and Systems

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Section 1

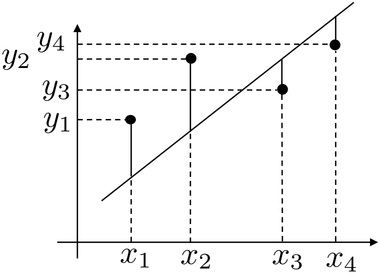
Applications

Application: Total least squares

If we are trying to fit a line to

$$y_i = ax_i$$

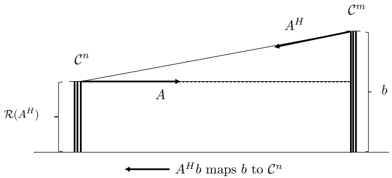
where (y_i, x_i) are measured. The least squares solution minimizes $e_i = y_i - ax_i$. Therefore $y_i - e_i = ax_i$.



In other words: fix the x_i's and play with a to minimize the error.

Application: Total least squares

For the general problem $\min \|Ax - b\|$ we assume A is perfect and that the imperfection is completely in b



Recall $A^H Ax = A^H b$. When we premultiply by A^H we zero everything in b that was in the null space of A^H (i.e. we get rid of the bad parts of b).

Application: Total least squares

However A often comes from noisy data as well (like when fitting a line to data) e.g. if $\mathbf{u}_i = ax_i + b$, then

$$\begin{pmatrix} y_1 \\ \vdots \\ \underbrace{y_n}_{\text{noisy}} \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ \underbrace{x_n}_{\text{noisy}} & \underbrace{1}_{\text{perfect}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Application: Total least squares

Another interpretation of least squares is to find the smallest perturbation of b, i.e., δb such that

$$Ax = b + \delta b$$

where $b + \delta b \in \mathcal{R}(A)$.

The total lest squares problem is to find the smallest perturbation of b and A, denoted δb, δA such that

$$(A + \delta A)x = (b + \delta b)$$

supposing that $\begin{pmatrix} A & b \end{pmatrix}$ is full rank.

Application: Total least squares

This can be written as

$$\begin{pmatrix} A & b \end{pmatrix} \begin{pmatrix} x \\ -1 \end{pmatrix} + \begin{pmatrix} \delta A & \delta b \end{pmatrix} \begin{pmatrix} x \\ -1 \end{pmatrix} = 0$$

or

$$\left[\begin{pmatrix} A & b \end{pmatrix} + \begin{pmatrix} \delta A & \delta b \end{pmatrix} \right] \begin{pmatrix} x \\ -1 \end{pmatrix} = 0.$$

Define

$$C \triangleq \begin{pmatrix} A & b \end{pmatrix} \text{ and } \Delta = \begin{pmatrix} \delta A & \delta b \end{pmatrix}$$

then

$$(C + \Delta) \begin{pmatrix} x \\ -1 \end{pmatrix} = 0.$$

Application: Total least squares

So $\begin{pmatrix} x \\ -1 \end{pmatrix} \in \mathcal{N}(C + \Delta)$ which implies that $C + \Delta$ is not full rank.

The problem is then to find the smallest perturbation Δ such that $C + \Delta$ looses rank.

Note that since $C = \begin{pmatrix} A & b \end{pmatrix} \in \mathbb{C}^{m \times (n+1)}$, for C to be full rank, we must have that $m > n$. Therefore we can write

$$C = \sum_{j=1}^{n+1} \sigma_j \mathbf{u}_j \mathbf{v}_j^H.$$

Application: Total least squares

Hence, the smallest Δ that reduces the rank of C is

$$\Delta = -\sigma_{n+1}\mathbf{u}_{n+1}\mathbf{v}_{n+1}^H.$$

Note that $\mathbf{v}_{n+1} \in \mathcal{N}(C + \Delta)$ since

$$(C + \Delta)\mathbf{v}_{n+1} = \sum_{j=1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^H \mathbf{v}_{n+1} = 0$$

since $\mathbf{v}_i \mathbf{v}_j = \delta_{ij}$.

Application: Total least squares

Therefore

$$\begin{pmatrix} x \\ -1 \end{pmatrix} = \alpha \mathbf{v}_{n+1} = \alpha \begin{pmatrix} \mathbf{v}_{n+1}(n:1) \\ \mathbf{v}_{n+1}(n+1) \end{pmatrix}$$

Letting $\alpha = -\frac{1}{\mathbf{v}_{n+1}(n+1)}$ gives

$$x = \alpha \mathbf{v}_{n+1}(n:1)$$

This is valid if $\mathbf{v}_{n+1}(n+1) \neq 0$. Note that if σ_{n+1} is not a unique minimum singular value, i.e. $\sigma_{n+1} = \sigma_n = \cdots = \sigma_{k+1}$ then we want to find the smallest norm x such that

$$\begin{pmatrix} x \\ -1 \end{pmatrix} \in span\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{n+1}\}$$

Application: Homography Matrix

Application: MIMO Communication

Consider the MIMO communication system modeled by

$$\underbrace{Y(j\omega)}_{p\times 1} = \underbrace{H(j\omega)}_{1\times m}\underbrace{X(j\omega)}_{m\times 1}$$

What is the maximum gain of the system?

$$\|Y(j\omega)\| = \|H(j\omega)X(j\omega)\| \leq \|H(j\omega)\| \, \|X(j\omega)\|$$

Therefore, the maximum gain is given by

$$\begin{aligned}
\gamma_{\mathrm{max}}(j\omega) &= \max_{X(j\omega)\neq 0} \frac{\|H(j\omega)X(j\omega)\|}{\|X(j\omega)\|} \\
&= \|H(j\omega)\| \\
&= \bar{\sigma}(H(j\omega)),
\end{aligned}$$

where $\bar{\sigma}(H(j\omega))$ is the maximum singular value of $H(j\omega)$.

Application: MIMO Communication

How do you achieve this gain? Since

$$H(j\omega) = \Sigma \sigma_k(j\omega)\mathbf{u}_k(j\omega)\mathbf{v}_k^H(j\omega),$$

letting

$$X(j\omega) = \mathbf{v}_1(j\omega)$$

maximizes the gain in the system over the set $\|X(j\omega)\| = 1$.