ECEn 671: Mathematics of Signals and Systems

Randal W. Beard

Brigham Young University

September 1, 2023

Section 1

Vector Spaces

Vector Spaces

A field is a set of scalars with well defined addition and multiplication operations.

Example of fields:

- $\mathbb{R}$ with normal addition and multiplication operations
- $\mathbb{C}$ with complex addition and complex multiplication
- The set of quaternions, with addition and quaternion multiplication
- Binary numbers $\{0, 1\}$ where addition is the “or” operator and multiplication is the “and” operator.

Vector Spaces

Definition (Linear Vector Space)

A linear vector space is a pair $(\mathbb{X}, \mathbb{F})$, where $\mathbb{X}$ is a set of objects, and $\mathbb{F}$ is a field, this is closed under addition and scalar multiplication. i.e.,

- $x \in \mathbb{X}, \alpha \in \mathbb{F} \Rightarrow \alpha x \in \mathbb{X}$
- $x, y \in \mathbb{X} \Rightarrow x + y \in \mathbb{X}$.

By implication

- $x \in \mathbb{X}, \alpha, \beta \in \mathbb{F} \Rightarrow (\alpha + \beta)x = \alpha x + \beta x \in \mathbb{X}$
- $x, y \in \mathbb{X}, \alpha \in \mathbb{F} \Rightarrow \alpha(x + y) = \alpha x + \alpha y \in \mathbb{X}$
- $x, y \in \mathbb{X}, \alpha, \beta \in \mathbb{F} \Rightarrow \alpha x + \beta y \in \mathbb{X}$.

Vector Spaces: Subspace

Definition (Subspace)

A subspace $V \subset \mathbb{X}$ is a subset of $\mathbb{X}$ that is also a linear vector space, in particular it contains zero.

Important property: A vector space contains a zero element.

Vector Spaces: Examples

The following are vector spaces:

- $(\mathbb{R}^n, \mathbb{R})$, $(\mathbb{C}^n, \mathbb{C})$, $(\mathbb{R}^{m \times n}, \mathbb{R})$, $(C[a,b], \mathbb{R})$, $(\ell^\infty, \mathbb{R})$, $(L^\infty, \mathbb{R})$.

The following are NOT vector spaces:

- The set $\mathbb{X} = \mathbb{R} \times [0, 2\pi]$, (a cylinder) is not a vector space for any field $\mathbb{F}$. This is the state space for an inverted pendulum.
- The set of rotation matrices is not a vector space for any field $\mathbb{F}$. This is in the configuration space for robots and satellites.
- The set of unit quaternions is not a vector space for any field. Quaternions are used extensively in robotics, quantum mechanics, and computer graphics.
- There are many useful spaces that are NOT linear vector spaces.

Vector Spaces: Linear Independence

Let S be a vector space and let $T \subset S$. (T may have uncountable infinite members). T is linearly independent if for each finite nonempty subset of T. i.e., $\{p_1, \cdots, p_n\}$ where $p_i \in T$, we have that

$$c_1 p_1 + \cdots + c_n p_n = 0 \qquad \Longleftrightarrow \qquad c_1 = c_2 = \cdots = c_n = 0.$$

Otherwise T is linearly dependent.

Vector Spaces: Linear Independence

Example

Let $S = \mathbb{R}^3$ then the set $T = \{(1,0,0)^\top, (0,1,0)^\top\} \subset \mathbb{R}^3$ is linearly independent since

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if and only if $c_1 = c_2 = 0$.
However, the set $T = \{(1,1,0)^\top, (2,2,0)^\top\} \subset \mathbb{R}^3$ is linearly dependent since

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ c_1 + 2c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

when $c_1 = -2$ and $c_2 = 1$ (as only on example).

Vector Spaces: Span

Definition (Span)

Let S be a vector space, then span(T) is the set of all linear combinations of $T \subseteq S$.

Example

$$\text{span}\left\{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)\right\} = \left\{\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} |\alpha \in \mathbb{R}\right\}$$

Example

$$\text{span}\left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right\} = \left\{\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R}\right\} = \mathbb{R}^2.$$

Vector Spaces: Basis

Definition (Basis)

T is a <u>basis</u> for the vector space S if T is linearly independent and $\mathrm{span}(T) = S$.

Definition (Dimension)

The dimension of the vector space S is the smallest number of linearly independent vectors needed to span S.

Example

One possible basis for $\mathbb{R}^n$ is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

Therefore $\dim(\mathbb{R}^n) = n$.

Vector Spaces: Basis

Example

One possible basis for ℓ^∞ is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \cdots \right\}$$

Therefore $\dim(\ell^\infty) = \infty$.

Vector Spaces: Basis

Example

The set of all polynomials P is a vector space with basis

$$\{1, t, t^2, \cdots\}$$

Therefore $\dim(P) = \infty$.

Example

The set of all polynomials of degree $\leq q$ P^q is a vector space with basis

$$\{1, t, t^2, \cdots, t^q\}$$

Therefore $\dim(P^q) = q$.