ECEn 671: Mathematics of Signals and Systems

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Section 1

Inequality Constraints: Kuhn-Tucker Conditions

Inequality Constraints

Lets first consider the problem with just inequality constraints, i.e.

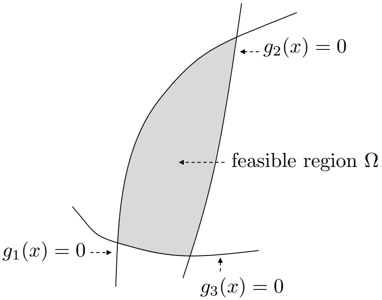
$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \mathbf{g}(x) \leq 0 \end{aligned}$$

where $\mathbf{g}(x) \leq 0$ means that

$$\begin{pmatrix} g_1(x) \\ \vdots \\ g_q(x) \end{pmatrix} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

i.e., element-wise.

For example, let $x \in \mathbb{R}^2$ and let $q = 3$.



Inequality Constraints

Case I. If the local min is in the interior of Ω, then clearly

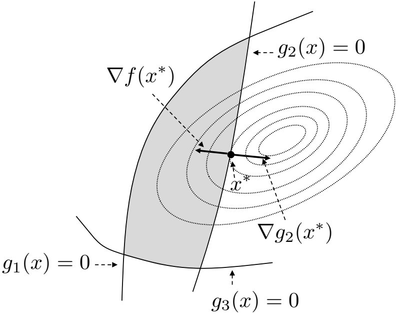
$$\nabla f(x^*) = 0$$

or

$$\nabla f(x^*) + 0 \cdot \nabla g_1(x^*) + 0 \cdot \nabla g_2(x^*) + 0 \cdot g_3(x^*) = 0.$$

Inequality Constraints

Case II. The local minimum is on the boundary but not at a corner

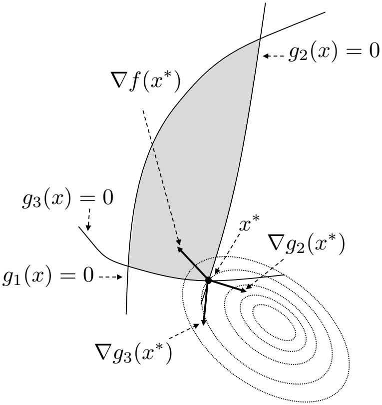


Since in this case g_1 is an equality constraint, we must have that $\nabla f(x^*) \parallel \nabla g_1(x^*)$. In fact, in this case the two vectors point in opposite directions! Therefore

$$\nabla f(x^*) + \mu_1 \nabla g_1(x^*) + 0 \cdot \nabla g_2(x^*) + 0 \cdot g_3(x^*) = 0.$$

Inequality Constraints

Case III.



In this case, $\nabla f(x^*)$ is in the linear span of $\nabla g_1(x^*)$ and $\nabla g_2(x^*)$ where the coefficients are negative. Therefore

$$\nabla f(x^*) + \mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) + 0 \cdot g_3(x^*) = 0$$

where $\mu_1 > 0$ and $\mu_2 > 0$.

Inequality Constraints

In general, for inequality constraints at a local minimum x^* we have that

1. $\nabla f(x^*) + \nabla \mathbf{g}(x^*)\mu = 0$
2. $\mathbf{g}(x^*)^\top \mu = 0$
3. $\mu \geq 0$

Conditions (1) and (3) together mean that $\nabla f(x^*)$ is contained in the (negative) linear span of $\{\nabla g_1(x^*), \ldots, \nabla g_q(x^*)\}$.

Condition (2): Note that if the constraint is active, i.e. $g_i(x^*) = 0$ then μ_i can be nonzero, but if g_i is inactive, i.e. $g_i(x^*) < 0$ then μ_i must be zero to satisfy (2).

Inequality Constraints

Now lets go back to the general constrained optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \mathbf{h}(x) = 0, \\ & \mathbf{g}(x) \leq 0 \end{aligned}$$

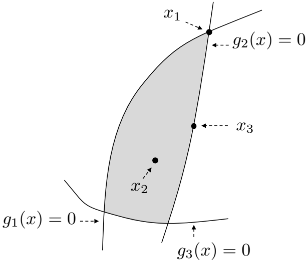
where $f : \mathbb{R}^n \to \mathbb{R}$, $h(x) : \mathbb{R}^n \to \mathbb{R}^p$, $g(x) : \mathbb{R}^n \to \mathbb{R}^q$.

Definition

x^* is a <u>regular point</u> if $\nabla h_i(x^*), i = 1, \ldots, p$ and $\nabla g_j(x^*)$ are linearly independent for all $j = 1, \ldots, q$ such that $g_j(x^*)$ is active.

Inequality Constraints

For example, suppose that $\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, and $\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$.



Then x^* is a regular point at:

- x_1 if $\{\nabla h_1(x_1), \nabla h_2(x_1), \nabla g_1(x_1), \nabla g_2(x_1)\}$ are linearly independent.
- x_2 if $\{\nabla h_1(x_2), \nabla h_2(x_2)\}$ are linearly independent.
- x_3 if $\{\nabla h_1(x_3), \nabla h_2(x_3), \nabla g_1(x_3)\}$ are linearly independent.

Kuhn Tucker Conditions: Necessary Conditions

Theorem (Moon Theorem 18.6)

Let x^* be a regular local minimum, then $\exists \lambda \in \mathbb{R}^p$ (regular Lagrange multipliers), and $\exists \mu \in \mathbb{R}^q$, such that

1. $\mu \geq 0$ *(element wise)*
2. $\mathbf{g}^\top(x^*)\mu = 0$
3. $\nabla f(x^*) + \nabla \mathbf{h}^\top(x^*)\lambda + \nabla \mathbf{g}^\top(x^*)\mu = 0.$

Kuhn Tucker Conditions: Sufficient Conditions

Theorem (Moon 18.7)

Suppose f, g, h are in C_2. If there exist $\lambda \in \mathbb{R}^p, \mu \in \mathbb{R}^q$ such that at x^*

1. $\mu \geq 0$
2. $\mathbf{g}^\top(x^*)\mu = 0$
3. $\nabla f(x^*) + \nabla \mathbf{h}^\top(x^*)\lambda + \nabla \mathbf{g}^\top(x^*)\mu = 0$
4. $p^\top(\nabla^2 f(x^*) + \sum_{k=1}^{p} \nabla^2 h_k(x^*)\lambda_k + \sum_{k=1}^{q} \nabla g_k(x^*)\mu_k)p > 0$

for all p in the tangent plane of the active constraints, then x^* is a local constrained minimum.

Kuhn Tucker Conditions: Example 18.9.1

$$
\begin{aligned}
\min \quad & 3x_1^2 + 4x_2^2 + 6x_1x_2 - 8x_2 - 6x_1 \\
\text{s.t.} \quad & x_1^2 + x_2^2 - 9 \leq 0, \\
& 2x_1 - x_2 - 4 \leq 0
\end{aligned}
$$

The necessary conditions are:

$$
\begin{aligned}
6x_1 + 6x_2 - 6 + \mu_1(2x_1) + \mu_2(2) &= 0 \\
8x_2 + 6x_1 - 8 + \mu_1(2x_2) + \mu_2(-1) &= 0 \\
\mu_1(x_1^2 + x_2^2 - 9) + \mu_2(2x_1 - x_2 - 4) &= 0 \\
\mu_1 \geq 0, \mu_2 &\geq 0
\end{aligned}
$$

Kuhn Tucker Conditions: Example 18.9.1

Lets try various combinations of active constraints:

Case I (Both inactive) i.e.

$\mu_1 = \mu_2\ 0$

Therefore, must solve

$$6x_1 + 6x_2 - 6 = 0$$
$$8x_2 + 6x_1 - 8 = 0$$

i.e.,

$$\begin{pmatrix} 6 & 6 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$
$$\implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Check inequality constraints:

$$g_1(x) = 1 - 9 = -8 \leq 0$$
$$g_2(x) = -1 - 4 \leq 0$$

Therefore

$$x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mu^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

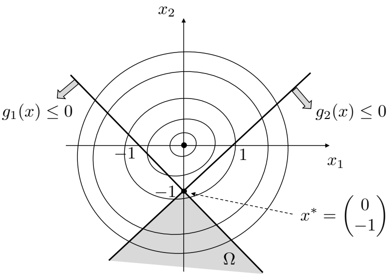
satisfies necessary conditions.
Sufficient condition:

$$\nabla^2 f = \begin{pmatrix} 6 & 6 \\ 6 & 8 \end{pmatrix} > 0$$

implies local minimum.

Kuhn Tucker Conditions: Example

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 + 1 \leq 0, \\ & -x_1 + x_2 + 1 \leq 0 \end{aligned}$$



Kuhn Tucker Conditions: Example

The necessary conditions are:

$$
\begin{aligned}
2x_1 + \mu_1 - \mu_2 &= 0 \\
2x_2 + \mu_1 + \mu_2 &= 0 \\
\mu_1(x_1 + x_2 + 1) + \mu_2(-x_1 + x_2 + 1) &= 0 \\
\mu_1 \geq 0, \mu_2 \geq 0
\end{aligned}
$$

Try various combinations of active constraints Case 1: (Both inactive)

$$
\begin{aligned}
2x_1 &= 0 \\
2x_2 &= 0
\end{aligned}
$$

$$\implies x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

However, both constraints are violated since

$$
\begin{aligned}
g_1^*(x^*) &= 1 \geq 0 \\
g_2(x^*) &= 1 \geq 0.
\end{aligned}
$$

Kuhn Tucker Conditions: Example

Case 2: g_1-active, g_2-inactive

$$
\begin{aligned}
2x_1 + \mu_1 &= 0 \qquad \Longrightarrow \; x_1 = -\frac{1}{2}\mu_1 \\
2x_2 + \mu_1 &= 0 \qquad \Longrightarrow \; x_2 = -\frac{1}{2}\mu_1 \\
\mu_1(x_1 + x_2 + 1) &= 0 \\
\mu_1 &> 0
\end{aligned}
$$

Last two equations imply that

$$\mu_1(-\frac{1}{2}\mu_1 - \frac{1}{2}\mu_1 + 1) = -\mu_1^2 + \mu_1 = \mu_1(1 - \mu_1) = 0.$$

Solving for μ_1 gives $\mu_1 = 0$ or $\boxed{\mu_1 = 1}$. Therefore

$$x^* = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Kuhn Tucker Conditions: Example

Checking constraints:

$$g_1(x^*) = -\frac{1}{2} - \frac{1}{2} + 1 = 0 \leq 0 \qquad \text{ok}$$
$$g_2(x^*) = \frac{1}{2} - \frac{1}{2} + 1 = 1 \geq 0 \qquad \text{no}$$

Case 3: g_1-inactive, g_2-active Similar results to Case 2.

Case 4: Both active

$$\begin{aligned} &\mu_1(\frac{1}{2}\mu_2 - \frac{1}{2}\mu_1 - \frac{1}{2}\mu_2 - \frac{1}{2}\mu_1 + 1) \\ &\quad + \mu_2(-\frac{1}{2}\mu_2 + \frac{1}{2}\mu_1 - \frac{1}{2}\mu_1 - \frac{1}{2}\mu_2 + 1) = 0 \\ \Longrightarrow\ &\mu_1(1-\mu_1) + \mu_2(1-\mu_2) = 0 \end{aligned}$$

Kuhn Tucker Conditions: Example

A positive solution is

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0$$

which gives

$$x^* = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Constraints can be verified to be satisfied.

Sufficient condition:

$$\nabla^2 f + \nabla^2 g \mu = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} 1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} 1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0$$

Therefore x^* is a local minimum.