ECEn 671: Mathematics of Signals and Systems

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Section 1

Projections

Projections

- Suppose that V and W are disjoint subspaces of S such that $V + W = S$, i.e.

$$x \in S \Rightarrow x = v + w$$

  where $v \in V$ and $w \in W$ is a unique decomposition.
- Define the linear operator $P : S \to V \subset S$ as

$$Px = P(v + w) = v$$

- Note that $P(Px) = Pv = v$

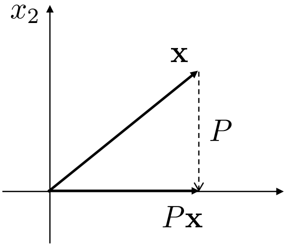
Projections, cont.

Definition (Projection Operator)

Let $P : S \to S$ such that $P^2 = P$, then P is called a <u>projection</u> operator or <u>idempotent</u>.

Example

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$
i.e. P projects elements of P onto the x_1 axis:



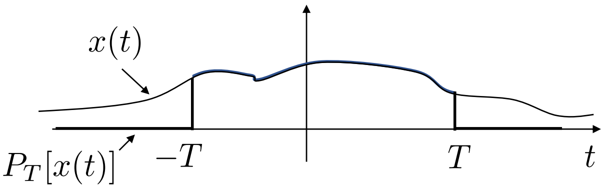
Projections, cont.

Example

Truncation: let

$$(P_T x)(t) = \begin{cases} x(t), & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Then P_T projects $x(t)$ onto its truncated function:



Projections, cont.

Theorem (Moon 2.7)

Let $P : S \to S$ be a projection operator, then

$$S = R(P) + N(P)$$

Proof.

Homework problem. $\square$

Projections, cont.

Theorem

If $P : S \to S$ is a projection operator then so is $(I - P) : S \to S$

Proof.

$$\begin{aligned}(I-P)^2 = (I-P)(I-P) &= \\ &= I - P - P - P^2 \\ &= I - P - P + P \\ &= I - P\end{aligned}$$

$\square$

Projections, cont.

- Note that if $P : S \to V$ and $I - P : S \to W$ then V and W are disjoint and $S = V + W$ since

$$x = \underbrace{Px}_{\in V} + \underbrace{(I-P)x}_{\in W}.$$

- V and W are disjoint. If not, then $\exists x_0 (\neq 0) \in S$ such that

$$\begin{aligned} Px_0 &= (I-P)x_0 = x_0 - Px_0 \\ 2Px_0 &= x_0 \\ \Rightarrow Px_0 &= \frac{1}{2}x_0 \\ \text{and } P^2x_0 &= \frac{1}{4}x_0 = \frac{1}{2}x_0 \Leftrightarrow x_0 = 0 \end{aligned}$$

Projections, cont.

Definition (Orthogonal Projection)

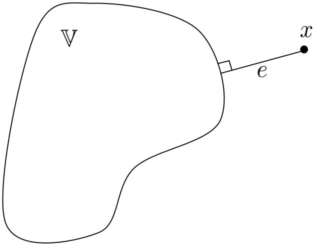
If V and W are orthogonal then P is an orthogonal projection

Theorem

P is an orthogonal projection iff $R(P) \perp N(P)$

Applications to Engineering

Given a point $x \in S$, suppose that we want to approximate x by a point in $\mathbb{V} \subset S$ (assuming $x \notin \mathbb{V}$) then we want to find the point in $\mathbb{V}$ that is closest to x.



This is given by the orthogonal projection of x onto $\mathbb{V}$. i.e.
$\langle e, v \rangle = 0 \qquad \forall v \in \mathbb{V}$

Applications to Engineering

Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$ (i.e., $\|\mathbf{n}\| = 1$), then

$$\Pi_{\mathbf{n}}^{\perp} \triangleq \mathbf{n}\mathbf{n}^{\top}$$

is a projection operation. Geometrically $\Pi_{\mathbf{n}}^{\perp} x = \mathbf{n}\mathbf{n}^{\top} x$ find the projection of x along the unit vector $\mathbf{n}$
Also

$$\Pi_{\mathbf{n}} = I - \mathbf{n}\mathbf{n}^{\top}$$

is a projection operator. Geometrically, $\Pi_{\mathbf{n}} x$ projections x onto the 2D space that is orthogonal to $\mathbf{n}$.

The Projection Theorem

Theorem

Let $\mathbb{S}$ be a Hilbert space and let $\mathbb{V}$ be a closed subspace of $\mathbb{S}$. For any $x \in \mathbb{S}$ there exists a unique $v_0 \in \mathbb{V}$ closest to x; i.e.
$\|x - v_0\| \leq \|x - v\| \quad \forall v \in \mathbb{V}$.

Furthermore v_0 minimizes $\|x - v_0\|$ iff $x - v_0$ is orthogonal to $\mathbb{V}$.

The Projection Theorem, proof

Step 1. Show that v_0 exists.
Assume $x \notin \mathbb{V}$ and let $\delta = \inf_{v \in \mathbb{V}} \|x - v\|$ We need to show that in fact $\exists v_0 \in \mathbb{V}$ such that $\|x - v_0\| = \delta$
Let $\{v_i\}$ be a sequence in $\mathbb{V}$ such that $\|x - v_i\| \to \delta$ and show that $\{v_i\}$ is Cauchy.

Need parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

Consider

$$\begin{aligned} &\|(v_j - x) + (x - v_i)\|^2 + \|(v_j - x) - (x - v_i)\|^2 \\ &\qquad = 2\|(v_j - x)\|^2 + 2\|x - v_i\|^2 \end{aligned}$$

$$\Rightarrow \|v_j - v_i\|^2 = 2\|v_j - x\|^2 + 2\|x - v_i\|^2 - 4\left\|\frac{(v_j + v_i)}{2} - x\right\|^2$$

The Projection Theorem, proof

$$v_i, v_j \in \mathbb{V} \quad \Rightarrow \quad \frac{v_j + v_i}{2} \in \mathbb{V} \quad \Rightarrow \quad \left\| \frac{(v_j - v_i)}{2} - x \right\|^2 \geq \delta^2$$

$$\Rightarrow \|v_j - v_i\|^2 \leq 2\|v_j - x\|^2 + 2\|v_i - x\|^2 - 4\delta^2$$

But $\|v_j - x\| \to \delta$

$$\Rightarrow \|v_j - v_i\| \to 0$$

and is therefore Cauchy.

Since $\mathbb{V}$ is a Hilbert space

$$v_i \to v_0 \in \mathbb{V}.$$

Note that this proof is not constructive, i.e. it doesn't tell you how to construct the sequence $\{v_i\}$.

The Projection Theorem, proof

Step 2. Show that $v_0 = \arg\min_{v \in \mathbb{V}} \|x - v\| \Rightarrow x - v_0 \perp \mathbb{V}$.
Proof by contradiction. Suppose that $x - v_0$ is not perpendicular to $\mathbb{V}$. Then there exists a $v \in \mathbb{V}$ such that

$$\langle x - v_0, v \rangle = \delta \neq 0$$

and w.l.o.g. (why?) let $\|v\| = 1$

Let $z = v_0 + \delta v \in \mathbb{V}$ then

$$\begin{aligned}\|x - z\|^2 = \|x - v_0 - \delta v\|^2 &= \|x - v_0\|^2 - 2Re\,\langle x - v_0, \delta v \rangle + \|\delta v\|^2 \\ &= \|x - v_0\|^2 - 2\delta^2 + \delta^2 < \|x - v_0\|^2\end{aligned}$$

which is a contradiction since v_0 is the minimizer.

The Projection Theorem, proof

Step 3. Suppose $(x - v_0) \perp \mathbb{V}$ then $\forall v \in \mathbb{V}$ such that $v \neq v_0$

$$\begin{aligned} \|x - v\|^2 &= \|x - v_0 + v_0 - v\|^2 \\ &= \|x - v_0\|^2 + 2Re\,\langle x - v_0, v_0 - v\rangle + \|v_0 - v\|^2 \\ &= \|x - v_0\|^2 + \|v_0 - v\|^2 \\ &> \|x - v_0\|^2 \end{aligned}$$

Step 4. **Uniqueness** Same as proof on page 25 of notes.

Closed Subspace

Theorem (Moon Theorem 2.10)

Let $\mathbb{V}$ be a closed subspace of a Hilbert space $\mathbb{S}$, then

$$\mathbb{S} = \mathbb{V} \oplus \mathbb{V}^{\perp}$$
$$\mathbb{V} = \mathbb{V}^{\perp\perp}$$

Proof.

In book. $\square$