ECEn 671: Mathematics of Signals and Systems

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Section 1

Linear Operators

Linear Operators

Recall from Chapter 3 the definition of a Linear operator:

Definition

Let $\mathbb{X}$ and $\mathbb{Y}$ be vector spaces, then $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is a linear operator if

$$\mathcal{A}[\alpha_1 x_1 + \alpha_2 x_2] = \alpha_1 \mathcal{A}[x_1] + \alpha_2 \mathcal{A}[x_2]$$

$\forall x_1, x_2 \in \mathbb{X}$ and $\forall \alpha_1, \alpha_2 \in \mathbb{F}$

See chapter 2 notes (slides 79–83) for examples of linear operators.

Norm of a Linear Operator

An important concept is the norm of an operator. There are several ways to define norms for operators. The most important is the "induced" or "subordinate" norm.

Definition

Let $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ then

$$\begin{aligned} \|\mathcal{A}\| &= \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{\mathbb{Y}}}{\|x\|_{\mathbb{X}}} \\ &= \sup_{\|x\|_{\mathbb{X}}=1} \|\mathcal{A}[x]\|_{\mathbb{Y}} \end{aligned}$$

Different norms on $\mathcal{A}$ are defined by taking different norms in $\mathbb{X}$ and $\mathbb{Y}$.

Norm of a Linear Operator, Examples

Example

Let $\mathcal{A} : L_2 \to L_2$ then

$$\|\mathcal{A}\|_2 = \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{L_2}}{\|x\|_{L_2}}$$
$$= sup_{\|x\|_{L_2}=1} \|\mathcal{A}[x]\|_{L_2}$$

Example

Let $\mathcal{A} : L_\infty \to L_\infty$ then

$$\|\mathcal{A}\|_\infty = \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{L_\infty}}{\|x\|_{L_\infty}}$$
$$= \sup_{\|x\|_{L_\infty}=1} \|\mathcal{A}[x]\|_{L_\infty}$$

Norm of a Linear Operator, Examples

Example

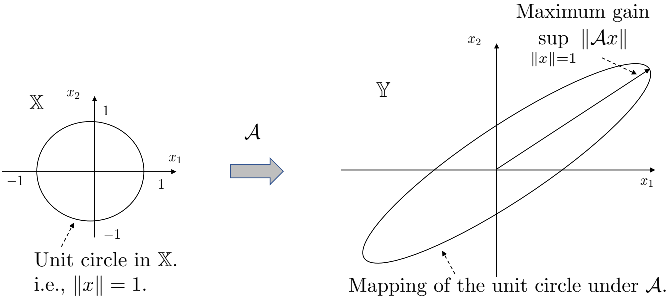
Let $\mathcal{A} : L_p \to L_p$ then

$$\|\mathcal{A}\|_p = \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{L_p}}{\|x\|_{L_p}} = \sup_{\|x\|_{L_p}=1} \|\mathcal{A}[x]\|_{L_p}$$

Why is it called the induced or subordinate norm? The norm on the operator is induced by the vector norm.

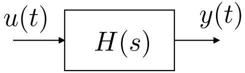
Norm of a Linear Operator, Geometric Interpretation

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$



Norm of a Linear Operator, System Interpretation

Given a linear system



The norm of the system $H(s)$ is the maximum gain of the system.

Norm of BIBO System

Let $\mathcal{A} : L_\infty \to L_\infty$ be an LTI system that is BIBO stable with impulse response $h(t)$, then

$$\begin{aligned} y(t) &= \int_0^t h(t-\tau)u(\tau)d\tau \\ &\triangleq \mathcal{A}[u] \end{aligned}$$

Find $\|\mathcal{A}\|_\infty$.

Norm of BIBO System, cont

Lemma

$$\begin{aligned}\|\mathcal{A}\|_\infty &= \|h\|_{L_1[0,\infty]} \\ &\triangleq \int_0^\infty |h(t)|dt\end{aligned}$$

Proof.

We need to prove two things

1. $\|\mathcal{A}\|_\infty \leq \int_0^\infty |h(t)|dt$
2. $\int_0^\infty |h(t)|dt \leq \|\mathcal{A}\|_\infty$

$\square$

Norm of BIBO System, Proof

Proof of 1.

$$
\begin{aligned}
\sup_{\|x\|_\infty=1} \|\mathcal{A}[u]\|_\infty &= \sup_{\|u\|_\infty=1} \left\| \int_0^t h(t-\tau)u(\tau)d\tau \right\|_\infty \\
&= \sup_{\|u\|_\infty=1} \left[\sup_{t>0} \left| \int_0^t h(t-\tau)u(\tau)d\tau \right| \right] \\
&\leq \sup_{\|u\|_\infty=1} \left[\sup_{t>0} \int_0^t |h(t-\tau)u(\tau)| \, d\tau \right] \\
&\leq \sup_{\|u\|_\infty=1} \left[\|u\|_\infty \sup_{t>0} \int_0^t |h(t-\tau)| \, d\tau \right] \\
&\leq \int_0^\infty |h(\tau)| \, d\tau = \|h\|_{L_1[0,\infty]}
\end{aligned}
$$

Norm of BIBO System, Proof

Proof of 2.

Let $\hat{u}_t(\tau) = \begin{cases} 1 & \text{if } h(t-\tau) \geq 0 \\ -1 & \text{otherwise} \end{cases}$.

Note that $\|\hat{u}_t\|_\infty = 1 \; \forall t > 0$, we have that

$$\int_0^t h(t-\tau)\hat{u}_t(\tau)d\tau = \int_0^t |h(t-\tau)|\, d\tau.$$

Therefore for this particular choice of $\hat{u}_t$ we have that

$$\sup_{t>0}\left[\int_0^t |h(t-\tau)|\, d\tau\right] = \|A\hat{u}_\infty\|_\infty = \int_0^\infty |h(\tau)|\, d\tau.$$

By definition of sup

$$\int_0^\infty |h(\tau)|\, d\tau = \|A\hat{u}_\infty\|_\infty \leq \sup_{\|u\|=1} \|Au\|_\infty.$$

Operator Norm: Proof Technique

The proof technique shown here is the general approach to show that the norm of an operator is some value.

Suppose that you would like to prove that

$$\|\mathcal{A}\| = M.$$

You need to show two things

1. $\|\mathcal{A}\| \leq M$
2. $M \leq \|\mathcal{A}\|$.

Operator Norm: Proof Technique

To show (1) use triangle and other inequalities to show that

$$\|\mathcal{A}x\| \leq M\,\|x\|$$

which implies that

$$\sup_{\|x\|=1} \|\mathcal{A}x\| \leq \sup_{\|x\|=1} M\,\|x\| = M$$

To show (2), construct a specific $\hat{x}$ such that

$$\|\hat{x}\| = 1 \text{ and } \|\mathcal{A}\hat{x}\| = M.$$

This implies that

$$M \leq \sup_{\|x\|=1} \|\mathcal{A}x\| = \|\mathcal{A}\|\,.$$

Properties of Linear Operators

Lemma

For any induced operator norm,

$$\|\mathcal{A}x\| \leq \|\mathcal{A}\| \, \|x\| \, .$$

Proof.

$$\|\mathcal{A}\| = \sup_{x \neq 0} \frac{\|\mathcal{A}x\|}{\|x\|}.$$

Therefore for any $x \neq 0$ we must have that

$$\|\mathcal{A}\| \geq \frac{\|\mathcal{A}x\|}{\|x\|}$$
$$\Rightarrow \|\mathcal{A}x\| \leq \|\mathcal{A}\| \, \|x\| \, .$$

$\square$

Properties of Linear Operators, cont

Lemma

All induced operator norms satisfy the "submultiplicative property," i.e.,

$$\|\mathcal{A}\mathcal{B}\| \leq \|\mathcal{A}\| \, \|\mathcal{B}\|$$

Proof.

$$\begin{aligned}
\|\mathcal{A}\mathcal{B}\| &= \sup_{\|x\|=1} \|\mathcal{A}\mathcal{B}x\| \\
&\leq \sup_{\|x\|=1} \|\mathcal{A}\| \, \|\mathcal{B}x\| \\
&\leq \sup_{\|x\|=1} \|\mathcal{A}\| \, \|\mathcal{B}\| \, \|x\| \\
&= \|\mathcal{A}\| \, \|\mathcal{B}\|
\end{aligned}$$

□

Properties of Linear Operators, cont

Definition

An operator $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is <u>bounded</u> if $\|\mathcal{A}\| < \infty$

Definition

The following three statements are equivalent

1. $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is <u>continuous</u>
2. $x_n \to x^* \Rightarrow \mathcal{A}[x_n] \to \mathcal{A}[x^*]$ for all convergent sequences in $\mathbb{X}$
3. $\forall \epsilon > 0, \quad \exists \delta > 0$ such that

$$\|x - y\| \le \delta \quad \Rightarrow \quad \|\mathcal{A}[x] - \mathcal{A}[y]\| < \epsilon \qquad \forall x, y \in \mathbb{X}$$

Properties of Linear Operators, cont

Theorem (Moon Theorem 4.1)

A linear operator is bounded iff it is continuous.

Proof.

□

$(\Rightarrow)$ Suppose $\|\mathcal{A}\| = M < \infty$,let $\{x_n\}$ be any convergent sequence with limit $x^* \in \mathbb{X}$, then

$$\begin{aligned} \|\mathcal{A}x_n - \mathcal{A}x^*\| &= \|\mathcal{A}(x_n - x^*)\| \leq \|\mathcal{A}\| \|x_n - x^*\| \\ &= M \|x_n - x^*\| \rightarrow 0 \Rightarrow \|\mathcal{A}x_n - \mathcal{A}x^*\| \rightarrow 0. \end{aligned}$$

Therefore $\mathcal{A}$ is continuous.

Proof, cont

($\Leftarrow$) Assume $\mathcal{A}$ is continuous and let $\epsilon = 1$ and $y = 0$ then $\exists\delta$ such that $\|x\| \leq \delta \Rightarrow \|\mathcal{A}x\| < 1$

Now let $0 \neq x \in \mathbb{X}$ be arbitrary, then

$$\left\| \frac{\delta x}{\|x\|} \right\| = \frac{\delta}{\|x\|} \|x\| = \delta \leq \delta$$

implies that

$$\left\| \mathcal{A}\left(\frac{\delta x}{\|x\|} \right) \right\| = \frac{\delta}{\|x\|} \|\mathcal{A}x\| < 1$$

which implies that

$$\|\mathcal{A}x\| \leq \frac{1}{\delta} \|x\|$$

Therefore $\mathcal{A}$ is bounded.

Properties of Linear Operators, cont

Theorem (Moon Theorem 4.2)

Let $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a linear operator. If $\mathbb{X}$ is a finite dimensional Hilbert space, then $\mathcal{A}$ is bounded.

Proof.

$\square$

Let $\dim(\mathbb{X}) = n$ and let $\{p_1, \cdots p_n\}$ be an orthonormal basis for $\mathbb{X}$, then

$$x = \sum_{k=1}^{n} \langle x, p_k \rangle \, p_k$$

Proof, cont.

Define $D = \max\{\|\mathcal{A}p_1\|, \|\mathcal{A}p_2\|, \ldots, \|\mathcal{A}p_n\|\}$ then

$$\begin{aligned}
\|\mathcal{A}x\| &= \left\|\mathcal{A}\left(\sum_{k=1}^{n} \langle x, p_k\rangle p_k\right)\right\| \\
&\leq \sum_{k=1}^{n} |\langle x, p_k\rangle| \, \|\mathcal{A}p_k\| \\
&\leq D \sum_{k=1}^{n} |\langle x, p_k\rangle| \\
&\leq D \sum_{k=1}^{n} \|x\| \, \|p_k\| \qquad (Caucy - Schwartz) \\
&= Dn \|x\|
\end{aligned}$$

Therefore $\mathcal{A}$ is bounded.