ECEn 671: Mathematics of Signals and Systems

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Section 1

Normed Spaces

Norms and Normed Spaces

Definition (Norm)

Let S be a vector space, $\|x\|$ is a norm if:

$$
\begin{array}{ll}
(N1) & \|x\| \geq 0 \quad \forall x \in S \\
(N2) & \|x\| = 0 \quad \Leftrightarrow x = 0 \\
(N3) & \|\alpha x\| = \alpha \|x\| \\
(N4) & \|x + y\| \leq \|x\| + \|y\| \quad \text{(triangle inequality)}
\end{array}
$$

Differences between norms and metrics:

- Norms only have one argument (the length of a vector), where metrics are distances between elements of a set.
- Norms are only defined for vector spaces!
  (i.e. there is no norm for rotation matrices, but there are metrics!)
- Norms scale with the vector (N3)
  (there are metrics that don't scale), e.g.
  $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$
- Every norm is also a metric

$$\|x - y\| = d(x, y)$$

$$\|x\| = d(x, 0)$$

Definition: Normed Space

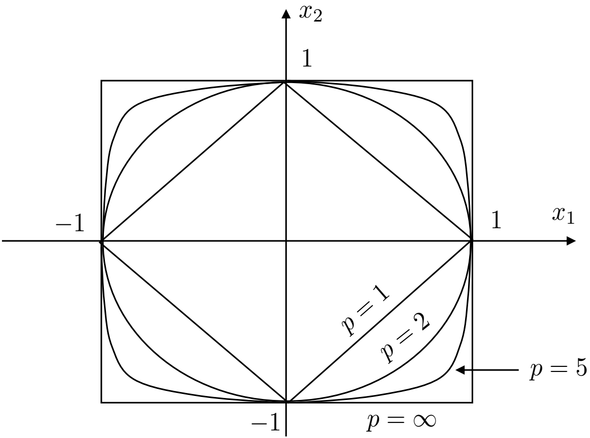
A normed space is a pair $(\mathbb{X}, \|\cdot\|)$ where $\mathbb{X}$ is a vector space and $\|\cdot\|$ is a norm.

Example (Normed Spaces)

$\mathbb{R}^n$ is a vector space. All of the following norms are valid:

- one-norm $\quad \|x\|_1 = \sum_{i=1}^{n} |x_i| \quad$ (power vectors)
- two-norm $\quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \quad$ (energy vectors)
- infinity-norm $\quad \|x\|_\infty = \max_{i=1,\cdots,n} |x_i| \quad$ (bounded vectors)
- p-norm $\quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$

Unit Circle in $\mathbb{R}^2$



Normed Space Example: Sequence Spaces

Let $\boldsymbol{\ell}$ be the set of sequences: $x = (x_1, x_2, x_3, \cdots)$. The following normed vector spaces can be defined:

- $\boldsymbol{\ell}_1$: (power sequences) If $\|x\|_{\ell_1} = \sum_{i=1}^{\infty} |x_i|$ then $\boldsymbol{\ell}_1 \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_1} < \infty\}$
- $\boldsymbol{\ell}_2$: (energy sequences) If $\|x\|_{\ell_2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$ then $\boldsymbol{\ell}_2 \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_2} < \infty\}$
- $\boldsymbol{\ell}_\infty$: (bounded sequences) If $\|x\|_{\ell_\infty} = \sup_{j\in\mathbb{N}} |x_j|$ then $\boldsymbol{\ell}_\infty \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_\infty} < \infty\}$
- $\boldsymbol{\ell}_p$: If $\|x\|_{\ell_p} = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ then $\boldsymbol{\ell}_p \triangleq \{x \in \boldsymbol{\ell} : \|x\|_{\ell_p} < \infty\}$ for $1 \le p \le \infty$

Normed Space Examples

Example

Consider the sequence $x = (1, 1, 1, \ldots)$:

- $x \in \ell_\infty$, but
- $x \notin \ell_p$ for $1 \leq p < \infty$.

Example

Consider the sequence $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

- $x \notin \ell_1$ (prove this), but
- $x \in \ell_p \quad p > 1$ (prove this)

Normed Space Example: Function Spaces

Let $L^n(\Omega)$ be the set of functions on Ω. $x \in L^n(\Omega)$ is an equivalent classes of functions, i.e. equal except on a set of measure zero. (picture) The following norms are valid:

- $L_1^n(\Omega)$ (power signals). If $\|x\|_{L_1^n(\Omega)} = \int_\Omega \|x(t)\|\, dt$, then $L_1^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_1^n(\Omega)} < \infty\}$.
- $L_2^n(\Omega)$ (energy signals). If $\|x\|_{L_2^n(\Omega)} = \left(\int_\Omega \|x(t)\|^2\, dt\right)^{1/2}$, then $L_2^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_2^n(\Omega)} < \infty\}$.
- $L_p^n(\Omega)$. If $\|x\|_{L_p^n(\Omega)} = \left(\int_\Omega \|x(t)\|^p\, dt\right)^{1/p}$, then $L_p^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_p^n(\Omega)} < \infty\}$, $1 \leq p \leq \infty$.
- $L_\infty^n(\Omega)$ (bounded signals). If $\|x\|_{L_\infty^n(\Omega)} = \sup_{t\in\Omega} \|x(t)\|$, then $L_\infty^n(\Omega) = \{x \in L^n(\Omega) |\, \|x\|_{L_\infty^n(\Omega)} < \infty\}$.