ECEn 671: Mathematics of Signals and Systems

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Section 1

Eigenvalues and Eigenvectors

Eigenpair

Let $A \in \mathbb{C}^{n \times n}$.

Definition

- (λ, x) is a right eigen-pair if $Ax = \lambda x$ and $x \neq 0$.
- (λ, x) is a left eigen-pair if $x^H A = \lambda x^H$ and $x \neq 0$.

Note that $Ax = \lambda x$ can be written as

$$(\lambda I - A)x = 0.$$

Therefore for x to be an eigenvector (associated with λ) then $x \in \mathcal{N}(\lambda I - A)$, and

$$x \neq 0 \Rightarrow \mathcal{N}(\lambda I - A) \neq \{0\} \Rightarrow det(\lambda I - A) = 0$$

This formula can be used to find the eigenvalues and eigenvectors of a matrix.

Eigenpair: Example

Let $A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$. Find the eigenvalues and eigenvectors.

<u>Eigenvalues:</u>

$$det(\lambda I - A) = det \begin{pmatrix} \lambda & -1 \\ 2 & \lambda + 2 \end{pmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

implies that

$$\lambda = -1 \pm \sqrt{1 - 2} = -1 \pm j$$

so that

$$\lambda_1 = -1 + j \qquad \lambda_2 = -1 - j.$$

Which one is larger? Note, there is no possible ordering among complex numbers.

Eigenpair: Example

Eigenvectors: The eigenvectors can be found from the formula $(\lambda I - A)x = 0$.

$$\lambda_1 : \begin{pmatrix} -1+j & -1 \\ 2 & 1+j \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that the rows are linearly dependent since

$$\begin{aligned} &(-1+j)\begin{pmatrix} -1+j & -1 \end{pmatrix} + \begin{pmatrix} 2 & 1+j \end{pmatrix} \\ &= \begin{pmatrix} -2 & -(1+j) \end{pmatrix} + \begin{pmatrix} 2 & 1+j \end{pmatrix} \\ &= 0. \end{aligned}$$

Therefore, solving $(-1+j)x_{11} - x_{12} = 0$ gives

$$x_{12} = (-1+j)x_{11}$$

Let $x_{11} = 1$ then $x_{12} = -1+j$.
So

$$x_1 = \begin{pmatrix} 1 \\ -1+j \end{pmatrix}$$

is an eigenvector.

Eigenpair: Example

$$\lambda_2 : \begin{pmatrix} -1-j & -1 \\ 2 & 1-j \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Again the rows are linearly dependent so solve to get $(-1-j)x_{21} = x_{22}$ Let $x_{21} = 1$ then $x_{22} = -1-j$.
So

$$x_2 = \begin{pmatrix} 1 \\ -1-j \end{pmatrix}$$

is an eigenvector.

Characteristic Polynomial

Definition

The polynomial

$$\chi_A(\lambda) = det(\lambda I - A)$$

is called the characteristic polynomial of A. The eigenvalues of A are the roots of $\chi_A(\lambda) = 0$. The set of roots of $\chi_A(\lambda) = 0$ is called the spectrum of A, denoted $\lambda(A)$.

Relationship between transfer function and state space models

Given a state space system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, what is the transfer function?
Take the Laplace transform to get

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s)$$

From the first equation we get

$$X(s) = (sI - A)^{-1}BU(s)$$

From the second equation we get

$$Y(s) = \underbrace{C(sI - A)^{-1}B}_{(p \times m) \text{ transfer matrix}} U(s)$$

Relationship between transfer function and state space models

What are the poles of the system?

$$\begin{aligned} Y(s) &= C(sI - A)^{-1} B U(s) \\ &= \frac{C\,\mathrm{adj}(sI - A) B}{\det(sI - A)} U(s) \end{aligned}$$

Therefore, the poles are when

$$det(sI - A) = 0,$$

i.e. when s is an eigenvalue of A.

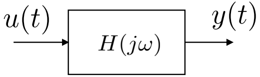
The poles of an LTI system and the eigenvalues of A are equivalent!

Generalized Eigenvalues

Eigenvalues and eigenvectors can be defined for more general operators than just matrices.

Example

Let $h(t)$ be the impulse response of an LTI system with Fourier transform $H(j\omega)$.



Recall that if $u(t) = e^{j\omega_0 t}$ then

$$\begin{aligned} y(t) &= |H(j\omega_0)| e^{j(\omega_0 t + \angle H(j\omega_0))} \\ &= |H(j\omega_0)| e^{j\angle H(j\omega_0)} e^{j\omega_0 t} \end{aligned}$$

i.e. if a sinusoid goes in then the output will be a sinusoid of the same frequency but different magnitude and phase.

Generalized Eigenvalues

Lemma

Let $\mathcal{A}[u] = \int_0^{\top} h(t-\tau)u(\tau)d\tau$ *then*

$$(\lambda, x) = \left(H(j\omega)e^{j\angle H(j\omega)}, e^{j\omega t} \right)$$

is an eigenpair of $\mathcal{A}$*.*

Proof.

$$\mathcal{A}[e^{j\omega t}] = \left(|H(j\omega)|e^{j\angle H(j\omega)} \right) e^{j\omega t}.$$

Note that for $\mathcal{A}$ there are an uncountable infinite number of eigenpairs. $\square$

Geometric and Algebraic Multiplicity

Definition

Factor the characteristic polynomial as follows:

$$\chi_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p}$$

m_i is the algebraic multiplicity of eigenvalue λ_i.

Definition

The geometric multiplicity of eigenvalue λ_i is defined as

$$q_i = dim(\mathcal{N}(\lambda_i I - A)).$$

Geometric and Algebraic Multiplicity: Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then

$$\chi_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2.$$

Therefore, the algebraic multiplicity of $\lambda_1 = 1$ is $m_1 = 2$.
What is the geometric multiplicity?

$$q_1 = dim(\mathcal{N}(\lambda_1 I - A)) = dim\left(\mathcal{N}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = dim(\mathbb{R}^2) = 2.$$

Note that the eigenvectors $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$ are linearly independent!

Geometric and Algebraic Multiplicity: Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then

$$\chi_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2$$

so the algebraic multiplicity of $\lambda_1 = 1$ is $m_1 = 2$.
The geometric multiplicity is

$$\begin{aligned} q_1 &= dim(\mathcal{N}\{I - A\}) = dim(\mathcal{N} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) \\ &= dim(\{x \in \mathbb{R}^2 \mid x_2 = 0\}) = 1 \neq m_1 \end{aligned}$$

What are the eigenvectors assocaited with A?

$$(\lambda I - A)x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} x_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_{12} = 0$$

so $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ are the eigenvectors associated with λ_1. There are <u>not</u> two linearly independent eigenvectors.

Linearly Independent Eigenvectors

In general we have,

Lemma

Let $A \in \mathbb{C}^{n \times n}$, then there are n-linearly independent eigenvectors if and only if

$$algebraic\ multiplicity = geometric\ multiplicity$$

for each eigenvalue of A.

Linearly Independent Eigenvectors: Proof

Proof.

First prove that if $\lambda_i \neq \lambda_j$ then

$$\mathcal{N}(\lambda_i I - A) \cap \mathcal{N}(\lambda_j I - A) = \{0\}.$$

To prove the claim, suppose not, then

$$\exists x \neq 0 \text{ such that } x \in \mathcal{N}(\lambda_i I - A) \text{ and } x \in \mathcal{N}(\lambda_j I - A)$$

$$\begin{aligned}
\iff \quad & Ax = \lambda_i x \text{ and } Ax = \lambda_j x \\
\iff \quad & \lambda_i x = \lambda_j x \\
\iff \quad & (\lambda_i - \lambda_j)x = 0 \\
\iff \quad & \lambda_i = \lambda_j
\end{aligned}$$

which is a contradiction.

Linearly Independent Eigenvectors: Proof

Note that the number of linearly independent eigenvectors associated with λ_i is the geometric multiplicity q_i since we can find q_i linearly independent vectors that span $\mathcal{N}(\lambda_i - A)$.

The previous claim shows that if $x_i \in \mathcal{N}(\lambda_i I - A)$ then $x_i \notin \mathcal{N}(\lambda_j I - A)$ which implies that there are $\sum q_i$ linearly independent eigenvectors of A. Since $\sum m_i = n$, the lemma follows.

Note that if the eigenvalues are all distinct then $m_i = 1$. Also since $1 \leq q_i \leq m_i$, for each i, we must have that the algebraic multiplicity equals the geometric multiplicity.

$\square$

Linearly Independent Eigenvectors

Suppose that there are n-linearly independent eigenvectors (where some of the eigenvalues might be repeated), then we can write

$$\begin{pmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{pmatrix}$$

$$\iff A \underbrace{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}}_{\Lambda}$$

$$\iff AS = S\Lambda$$

Since the eigenvectors are linearly independent, S is invertible. Therefore

$$A = S\Lambda S^{-1}$$
$$\iff \Lambda = S^{-1}AS$$

Therefore, we say that S diagonalizes A.

Similarity Transformation

Definition

Two matrices, A and B are said to be similar if $\exists$ an invertible T such that

$$A = TBT^{-1}.$$

Lemma

Similar matrices have the same eigenvalues.

Proof.

Let (λ, x) be an eigenpair of A, then

$$\begin{aligned}
& Ax = \lambda x \\
\Longleftrightarrow\ & TBT^{-1}x = \lambda x \\
\Longleftrightarrow\ & BT^{-1}x = \lambda T^{-1}x \\
\Longleftrightarrow\ & By = \lambda y \text{ where } y = T^{-1}x \\
\Longleftrightarrow\ & (\lambda, y) \text{ is an eigenpair of } B
\end{aligned}$$