ECEn 671: Mathematics of Signals and Systems

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Section 1

Linear Operators

Operators and Transformations

Definition (Linear Operator)

Let $\mathcal{L} : \mathbb{X} \to \mathbb{Y}$ be an operator from $\mathbb{X}$ to $\mathbb{Y}$. $\mathcal{L}$ is a linear operator if

1. $\mathcal{L}[\alpha x] = \alpha \mathcal{L}[x] \qquad \forall x \in \mathbb{X} \qquad \forall \alpha \in \mathbb{F}$
2. $\mathcal{L}[x_1 + x_2] = \mathcal{L}[x_1] + \mathcal{L}[x_2], \qquad \forall x_1, x_2 \in \mathbb{X}$

Examples of Linear Operators

Example (Matrices)

Operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ are $m \times n$ matrices.

$$A(\alpha x + \beta y) = \alpha A x + \beta A y$$

A is a linear operator.

Example (Differential Equations with no input)

The differential equation $\dot{x} = Ax; \quad x(0) = x_0$ defines a linear operator from $\mathbb{R}^n$ to $L_2[0, T]$

$$y(t) = \mathcal{L}[x_0] \text{ where } \mathcal{L}[x_0] = e^{At} x_0$$

$\mathcal{L}$ is linear since

$$e^{At}(\alpha x_{01} + \beta x_{01}) = \alpha e^{At} x_{01} + \beta e^{At} x_{02}$$

Examples of Linear Operators

Example (Convolution)

Convolution is a linear operator from L_∞ to L_∞ if $h(t) \in L_1[-\infty, \infty]$, i.e.

$$y(t) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

(Recall: for a system to be BIBO stable required that $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$ i.e. $h(t) \in L_1[-\infty, \infty]$)

Examples of Linear Operators

Example (Fourier Transform)

(E4) The Fourier transform defines a linear operator from $L_2[-\infty,\infty]$ to $L_2[-\infty,\infty]$.

$$X(j\omega) = \mathcal{L}[x(t)] \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

<u>There are many examples of linear operators!</u>

Range and Null Space of an Operator

Definition (Range Space)

Let $\mathcal{L} : \mathbb{X} \to \mathbb{Y}$ be a linear operator. The range space (or image) of $\mathcal{L}$ is

$$\mathcal{R}(\mathcal{L}) = \{y \in \mathbb{Y} : y = \mathcal{L}[x] \text{ and } x \in \mathbb{X}\} \subseteq \mathbb{Y}$$

Definition (Null Space)

The Null space or kernel of $\mathcal{L}$ is

$$\mathcal{N}(\mathcal{L}) = \{x \in \mathbb{X} : \mathcal{L}[x] = 0\} \subseteq \mathbb{X}$$

Example of Range and Null Space

- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which defines a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^2$.
- Note that $y = Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$.
- Therefore, the range space is

$$\mathcal{R}(A) = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\} \subset \mathbb{R}^2.$$

- Similarly, the null space is

$$\mathcal{N}(A) = \left\{ \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$