ECEn 671: Mathematics of Signals and Systems

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Geometric and Algebraic Multiplicity

Definition

Factor the characteristic polynomial as follows:

$$\chi_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p}$$

m_i is the algebraic multiplicity of eigenvalue λ_i.

Definition

The geometric multiplicity of eigenvalue λ_i is defined as

$$q_i = dim(\mathcal{N}(\lambda_i I - A)).$$

Geometric and Algebraic Multiplicity: Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then

$$\chi_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2.$$

Therefore, the algebraic multiplicity of $\lambda_1 = 1$ is $m_1 = 2$.
What is the geometric multiplicity?

$$q_1 = dim(\mathcal{N}(\lambda_1 I - A)) = dim\left(\mathcal{N}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = dim(\mathbb{R}^2) = 2.$$

Note that the eigenvectors $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$ are linearly independent!

Geometric and Algebraic Multiplicity: Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then

$$\chi_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2$$

so the algebraic multiplicity of $\lambda_1 = 1$ is $m_1 = 2$.
The geometric multiplicity is

$$\begin{aligned} q_1 &= dim(\mathcal{N}\{I - A\}) = dim(\mathcal{N} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) \\ &= dim(\{x \in \mathbb{R}^2 \mid x_2 = 0\}) = 1 \neq m_1 \end{aligned}$$

What are the eigenvectors assocaited with A?

$$(\lambda I - A)x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} x_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_{12} = 0$$

so $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ are the eigenvectors associated with λ_1. There are <u>not</u> two linearly independent eigenvectors.

Linearly Independent Eigenvectors

In general we have,

Lemma

Let $A \in \mathbb{C}^{n\times n}$, then there are n-linearly independent eigenvectors if and only if

$$\text{algebraic multiplicity} = \text{geometric multiplicity}$$

for each eigenvalue of A.

Linearly Independent Eigenvectors: Proof

Proof.

First prove that if $\lambda_i \neq \lambda_j$ then

$$\mathcal{N}(\lambda_i I - A) \cap \mathcal{N}(\lambda_j I - A) = \{0\}.$$

To prove the claim, suppose not, then

$$\exists x \neq 0 \text{ such that } x \in \mathcal{N}(\lambda_i I - A) \text{ and } x \in \mathcal{N}(\lambda_j I - A)$$

$$\begin{aligned}
\iff \quad & Ax = \lambda_i x \text{ and } Ax = \lambda_j x \\
\iff \quad & \lambda_i x = \lambda_j x \\
\iff \quad & (\lambda_i - \lambda_j)x = 0 \\
\iff \quad & \lambda_i = \lambda_j
\end{aligned}$$

which is a contradiction.

Linearly Independent Eigenvectors: Proof

Note that the number of linearly independent eigenvectors associated with λ_i is the geometric multiplicity q_i since we can find q_i linearly independent vectors that span $\mathcal{N}(\lambda_i - A)$.

The previous claim shows that if $x_i \in \mathcal{N}(\lambda_i I - A)$ then $x_i \notin \mathcal{N}(\lambda_j I - A)$ which implies that there are $\sum q_i$ linearly independent eigenvectors of A. Since $\sum m_i = n$, the lemma follows.

Note that if the eigenvalues are all distinct then $m_i = 1$. Also since $1 \leq q_i \leq m_i$, for each i, we must have that the algebraic multiplicity equals the geometric multiplicity.

$\square$

Linearly Independent Eigenvectors

Suppose that there are n-linearly independent eigenvectors (where some of the eigenvalues might be repeated), then we can write

$$
\begin{aligned}
& \begin{pmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{pmatrix} \\
\Longleftrightarrow\ & A \underbrace{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}}_{\Lambda} \\
\Longleftrightarrow\ & AS = S\Lambda
\end{aligned}
$$

Since the eigenvectors are linearly independent, S is invertible. Therefore

$$
\begin{aligned}
& A = S\Lambda S^{-1} \\
\Longleftrightarrow\ & \Lambda = S^{-1}AS
\end{aligned}
$$

Therefore, we say that S diagonalizes A.

Section 1

Jordan Form

Jordan Form

What if the algebraic multiplicity does not equal the geometric multiplicity? (i.e., $q_i \neq m_i$ for some eigenvalue λ_i of A)?

Then we cannot diagonalize A using a similarity transformation. However we can "almost" diagonalize A.

The resulting "almost diagonal" matrix is called the Jordan form of A.

Jordan Form, cont.

Suppose the algebraic multiplicity of λ_1 is $m_1 > 1$ but the geometric multiplicity is $q_1 = 1$.

Then $\exists$ one linearly independent eigenvector x_1 s.t. $Ax_1 = \lambda_1 x_1$.

Now form the following chain:

$$\begin{aligned} A\xi_{11} &= \lambda_1 \xi_{11} + x_1 \\ A\xi_{12} &= \lambda_1 \xi_{12} + \xi_{12} \\ &\vdots \\ A\xi_{1,m_1} &= \lambda_1 \xi_{1,m_1} + \xi_{1,(m_1-1)} \end{aligned}$$

$\xi_{11} \cdots \xi_{1,m_1}$ are called the "generalized eigenvectors" associated with x_1.

Jordan Form, cont.

Note that we can write the generalized eigenvector equations as

$$A\begin{pmatrix} x_1 & \xi_{11} & \cdots & \xi_{1,m_1}\end{pmatrix} = \begin{pmatrix} x_1 & \xi_{11} & \cdots & \xi_{1,m_1}\end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & 1 & \ddots & & \\ & \lambda_1 & 1 & 0 & \\ \ddots & & \lambda_1 & \ddots & \ddots \\ & 0 & & \ddots & 1 \\ & & \ddots & & \lambda_1 \end{pmatrix}}_{\text{This is called a Jordan block}}$$

Lemma

If the geometric multiplicity of λ_i is $q_i = 1$ then the associated $m_1 - 1$ generalized eigenvectors are linearly independent of the other eigenvectors.

Jordan Form, cont.

If $1 < q_i < m_i$ then the problem is slightly more complicated.

There are precisely q_i linearly independent eigenvectors associated with λ_i and there will be q_i Jordan blocks associated with λ_i. What are the sizes of the Jordan blocks? For example, suppose $m_i = 4$ and $q_i = 2$, the possible Jordan blocks are:

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1 \end{pmatrix} \text{ i.e., } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

or

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \text{ i.e., } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

Which option is correct?

Jordan Form, cont.

To decide, generate the generalized eigenvector for each eigenvector and pick the linearly independent ones.
Example: Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $\det(\lambda I - A) = (\lambda - 1)^4$ we have $\lambda_1 = 1$ and $m_1 = 4$.

$$q_1 = dim(\mathcal{N} \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = 2$$

since there are 2 linearly independent rows.

Jordan Form, cont.

So there are two linearly independent eigenvectors:

$$(\lambda_1 I - A)x_1 = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{pmatrix} = \begin{pmatrix} -x_{12} + x_{13} - x_{14} \\ -x_{14} \\ -x_{14} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies x_{14} = 0 \text{ and } -x_{12} + x_{13} - x_{14} = 0$$

so

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ is an eigenvector, and so is } x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Jordan Form, cont.

Find the possible generalized eigenvector associated with eigenvector x_1:

$$A\xi_{11} = \xi_{11} + x_1 \Rightarrow (\lambda_1 I - A)\xi_{11} = -x_1$$

$$\text{i.e.} \quad -\xi_{112} + \xi_{113} - \xi_{114} = 1 \qquad \xi_{114} = 0$$

$$\xi_{112} = \xi_{113} + 1 \qquad \text{so} \qquad \xi_{11} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \text{ is valid}$$

$$(\lambda_1 I - A)\xi_{12} = \xi_{12} \text{ so } \begin{pmatrix} -\xi_{122} + \xi_{123} - \xi_{124} \\ -\xi_{124} \\ -\xi_{124} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \leftarrow \text{ can't use .}$$

Jordan Form, cont.

Note: There are an infinite number of possibilities of generalized eigenvectors from each true eigenvector, but you can only pick ones that are linearly independent. This second eigenvector forms a linearly dependent subset of one of the real eigenvectors.

Therefore, one Jordan block is of size 2.

Also solve $(\lambda_1 I - A)\xi_{21} = x_2$ i.e.

$$\begin{pmatrix} -\xi_{212} + \xi_{213} - \xi_{214} \\ -\xi_{214} \\ -\xi_{214} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \xi_{214} = 1, \xi_{213} = \xi_{212} + 1$$

so $\xi_{21} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$.

Jordan Form, cont.

In summary

$$A \underbrace{\begin{pmatrix} x_1 & \xi_{11} & x_2 & \xi_{21} \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} x_1 & \xi_{11} & x_2 & \xi_{21} \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{J}$$

or

$$A = SJS^{-1}$$

J is called the "Jordan" form of A

If the eigenvalues are distinct or $q_i = m_i$ for each i then $J = \Lambda$ (is diagonal).

Otherwise J is block diagonal with Jordan blocks along the diagonal (q_i Jordan blocks for each eigenvalue).

Jordan Form, cont.

Example: suppose there are 3 eigenvalues with $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, and $m_1 = 1, m_2 = 2, m_3 = 3$, and $q_1 = 1, q_2 = 1, q_3 = 2$. There are two possible Jordan forms:

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & 1 & & 0 & \\ & & \lambda_2 & & & \\ & & & \lambda_3 & 1 & \\ & 0 & & & \lambda_3 & \\ & & & & & \lambda_3 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & 1 & & 0 & \\ & & \lambda_2 & & & \\ & & & \lambda_3 & & \\ & 0 & & & \lambda_3 & 1 \\ & & & & & \lambda_3 \end{pmatrix}$$