第 5 章

Higher condensation theory

Review of [?].

5.1 Definition of condensation

In this section, we refer to an **(n, n)-category** as an (∞, n)-category which is "truncated" at morphism degree n. Intuitively, if $\mathcal{C}$ is an (n, n)-category, then, for arbitrary $n-1$-morphisms $f, g \in \mathcal{C}_{n-1}$, the space of n-morphisms between f and g has discrete topology. Therefore, we can compose n-morphisms strictly (not up to homotopy), associatively, and unitally.

Note that a **(0, 0)-category** $\mathcal{C}$ is merely a set[*1].

定義 5.1: 0-condensation

Let $\mathcal{C}$ be a $(0, 0)$-category. A **0-condensation** on $\mathcal{C}$ is an equality between elements of a set $\mathcal{C}_0$.

n-condensation is defined by induction.

定義 5.2: n-condensation

Fix $n \geq 0$. Let $\mathcal{C}$ be an (n, n)-category, and let $x, y \in \mathcal{C}_0$ be objects of $\mathcal{C}$. We define n-condensation by induction on n.

***n*-condensation** of x onto y in $\mathcal{C}$ consists of three data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- An $(n-1)$-condensation of $r \circ i$ onto Id_y in $\mathrm{Map}_{\mathcal{C}}(y, y)$[a]

[a] Roughly speaking, a mapping space $\mathrm{Map}_{\mathcal{C}}(y, y)$ itself is an $(n-1, n-1)$-category. Strickly speaking, we need enriched ∞-category theory.

[*1] That is, $\mathcal{C}$ consists of a set $\mathcal{C}_0$ of objects (0-morphisms) only.

【例 5.1.1】1-condensation

Let $\mathcal{C}$ be a $(1, 1)$-category (i.e. an ordinary category). **1-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Hom}_{\mathcal{C}}(x, y)$
- A 1-morphism $i \in \mathrm{Hom}_{\mathcal{C}}(y, x)$
- A 0-condensation of $r \circ i$ onto Id_y in $\mathrm{Hom}_{\mathcal{C}}(y, y)$. i.e. $r \circ i = \mathrm{Id}_y$.

In the context of $(1, 1)$-categories, such y is called a **retract of $\boldsymbol{x}$**.

【例 5.1.2】2-condensation

Let $\mathcal{C}$ be a $(2, 2)$-category (i.e. a bicategory). **2-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- A 1-condensation of $r \circ i$ onto Id_y in $(1, 1)$-category $\mathrm{Map}_{\mathcal{C}}(y, y)$.

By【例 5.1.1】,

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism (in $\mathcal{C}$) $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism (in $\mathcal{C}$) $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- An equality $\rho \circ \iota = \mathrm{Id}_{\mathrm{Id}_y}$

【例 5.1.3】3-condensation

Let $\mathcal{C}$ be a $(3, 3)$-category. By【例 5.1.2】, **3-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- A 3-morphism

$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\rho\circ\iota"] \arrow[r, bend right, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y, y)_2$$

- A 3-morphism

$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, bend right, "\rho\circ\iota"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2$$

- An equiality

$$\begin{tikzcd} \mathrm{Id}_y \arrow[rr, bend left=60, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[rr, "\rho\circ\iota" description] \arrow[rr, bend right=60, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[rr, phantom, shift left=4, "\Downarrow\beta"] \arrow[rr, phantom, shift right=4, "\Downarrow\alpha"] & & \mathrm{Id}_y \end{tikzcd} = \begin{tikzcd} \mathrm{Id}_y \arrow[rr, bend left=60, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[rr, bend right=60, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[rr, phantom, "\Downarrow \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_y}}"] & & \mathrm{Id}_y \end{tikzcd}$$

【例 5.1.4】4-condensation

Let $\mathcal{C}$ be a (4, 4)-category. By【例 5.1.2】, **3-condensation** of x onto y in $\mathcal{C}$ consists of these data:

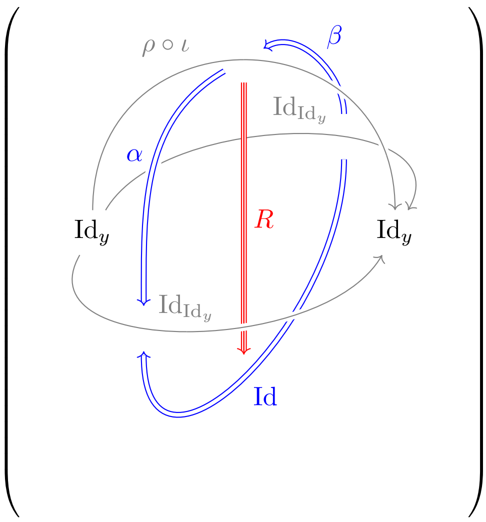
- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x,\, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y,\, x)_0$
- A 2-morphism $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_1$
- A 2-morphism $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_1$
- A 3-morphism

$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\rho\circ\iota"] \arrow[r, bend right, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2$$

- A 3-morphism

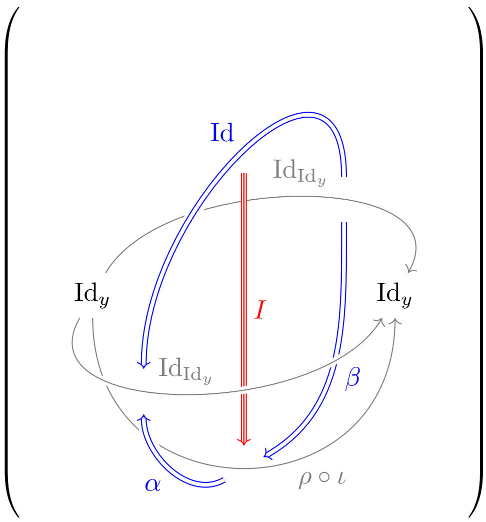
$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, bend right, "\rho\circ\iota"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2$$

- A 4-morphism



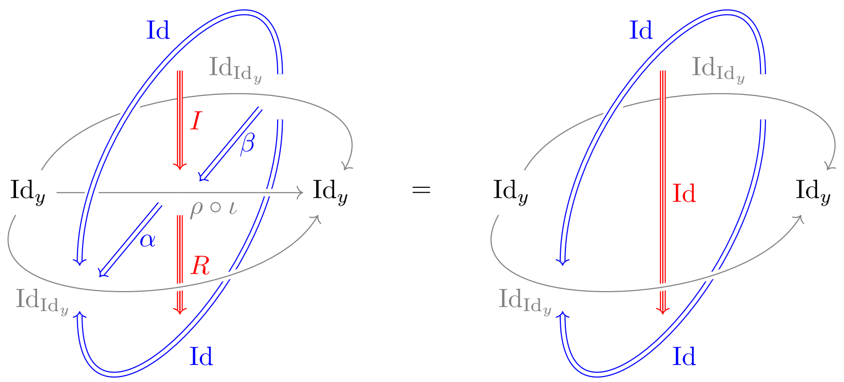
$\in \mathrm{Map}_{\mathcal{C}}(y,\, y)_3$

- A 4-morphism



$\in \mathrm{Map}_{\mathcal{C}}(y,\, y)_3$

- An equality



5.1.1 Walking condensation

Let $\mathcal{C}$ be an (n, n)-category. Roughly speaking, a **walking n-condensation** is a (strict?) n-category $\spadesuit_n$ which "generates" n-condensation in $\mathcal{C}$. i.e. the functor category $\mathcal{F}\mathbf{un}(\spadesuit_n, \mathcal{C})$ is equivalent to the category of n-condensations in $\mathcal{C}$.

5.1.2 Condensation monad

Fix an (n, n)-category $\mathcal{C}$. Now we will outline the definition of **condensation monad in $\mathcal{C}$**.

Let $\clubsuit_n \subset \spadesuit_n$ be a subcategory [*2] which consists of walking n-condensations with a single object. A **condensation monad** in $\mathcal{C}$ is a functor $A\colon \clubsuit_n \longrightarrow \mathcal{C}$. An (n, n)-category $\mathcal{C}$ is said to **have all condensates** if each condensation monad $\clubsuit_n \xrightarrow{A} \mathcal{C}$ extends to an n-condensation $\spadesuit_n \xrightarrow{\bar{A}} \mathcal{C}$:

$$\begin{tikzcd} \clubsuit_n \arrow[r, "A"] \arrow[d, hook] & \mathcal{C} \\ \spadesuit_n \arrow[ur, dashed, "\bar{A}"'] & \end{tikzcd}$$

However, there are some technical issues in this definition , so we should work on more concrete definition.

[*2] (18 February, 2026) Full-subness of $\clubsuit_n$ is still conjecture [?].

定義 5.3: condensation monad

Let $\mathcal{C}$ be an (n, n)-category. A **condensation monad** in $\mathcal{C}$ is a sequence of "commuting condensation squares":

$$
\begin{aligned}
\spadesuit^{\times 0} &\xrightarrow{e} \mathcal{C}, \\
\spadesuit^{\times 1} &\longrightarrow \mathcal{C}_{/e}, \\
\spadesuit^{\times 2} &\longrightarrow \mathcal{C}_{/e}, \\
&\vdots
\end{aligned}
$$

More explicitly, condensation monad consists of the following diagrams:

$$
\begin{aligned}
e &\in \mathcal{C}_0, \\
e \circ e &\twoheadrightarrow e,
\end{aligned}
$$

5.2 Physical interpretation

5.3 Tannaka-Krein reconstruction

Fix an algebraically closed field $\mathbb{K}$. From now on, we denote the $(1, 1)$-category of finite dimensional $\mathbb{K}$-vector spaces as $\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$[*3].

5.3.1 A bicategory of 2-vactor spaces $\mathbf{2Vec}_{\mathbb{K}}$

Let $\mathbf{Pr}^{\mathrm{L}}$ be the symmetric monoidal $(\infty, 2)$-category of presentable $(\infty, 1)$-categories, colimit-preserving functors, and natural transformations [?, Definition 5.5.3.1.]. Note that $\mathbf{Vec}_{\mathbb{C}}^{\mathrm{fin}}$ is an $\mathbb{E}_2$-algebra in $\mathbf{Pr}^{\mathrm{L}}$. We define $\mathbf{Pr}_{\mathbb{C}} := \mathbf{RMod}_{\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}}(\mathbf{Pr}^{\mathrm{L}})$ [?, Definition 4.2.1.13.]. More concretely, $\mathbf{Pr}_{\mathbb{C}}$ is the symmetric monoidal $(\infty, 2)$-category of presentable $\mathbf{Vec}_{\mathbb{C}}$-enriched categories, colimit-preserving $\mathbf{Vec}_{\mathbb{C}}^{\mathrm{fin}}$-enriched functors, and natural transformations.

After [?], we define $\mathbf{2Vec}_{\mathbb{K}}$ as a $(2, 2)$-category (bicategory) as follows:

- Objects are finite semisimple $\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$-enriched categories
- 1-morphisms are $\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$-enriched functors.
- 2-morphisms are natural transformations.

Note that $\mathbf{2Vec}_{\mathbb{K}}$ is fully-dualizable. In fact, $\mathbf{2Vec}_{\mathbb{K}} \subset \mathbf{Pr}_{\mathbb{C}}$ is the full-subcategory of $\mathbf{Pr}_{\mathbb{C}}$ which consists of fully dualizable objects of $\mathbf{Pr}_{\mathbb{C}}$. For more details, see [?, APPENDIX A.].

[*3] This is the fully-dualizable part of $\mathbf{Vec}_{\mathbb{K}}$, which is a $(1, 1)$-category of $\mathbb{K}$-vector spaces.

5.3.2 Morita bicategory of algebras

Note that $\mathbf{Vec}_{\mathbb{K}}$ ($\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$?) is a symmetric monoidal ((∞, 1)-)category. Then, we obtain a notion of $\mathbb{E}_1$-algebra objects in $\mathbf{Vec}_{\mathbb{K}}$, according to [?]. More concretely, an $\mathbb{E}_1$-algebra object in $\mathbf{Vec}_{\mathbb{K}}$ is none other than an ordinary associative algebra (i.e. $\mathbb{K}$-vector spaces with associative multiplication)[*4].

Morita category of $\mathbb{E}_1$-algebras in $\mathbf{Vec}_{\mathbb{K}}$ is a bicategory $\mathbf{Mor}_{\mathbb{E}_1}(\mathbf{Vec}_{\mathbb{K}})$ consists of the following data:

- Objects are $\mathbb{E}_1$-algebras in $\mathbf{Vec}_{\mathbb{K}}$.
- 1-morphisms between two $\mathbb{E}_1$-algebras $A, B \in \mathrm{Ob}(\mathbf{Mor}_{\mathbb{E}_1}(\mathbf{Vec}_{\mathbb{K}}))$ are (A, B)-bimodules in $\mathbf{Vec}_{\mathbb{K}}$.
- 2-morphisms between two (A, B)-bimodules $M, N \in \mathrm{Ob}(\mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(A, B))$ are (A, B)-bimodule homomorphisms.

Composition of 1-morphisms (or horizontal composition) in $\mathbf{Mor}_{\mathbb{E}_1}(\mathbf{Vec}_{\mathbb{K}})$ is the relative tensor product functor:

$$\otimes_B : \mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(B, C) \boxtimes \mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(A, B) \longrightarrow \mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(A, C)$$

5.3.3 Tannaka-Krein reconstruction

[*4] With respect to the open embedding $\emptyset \hookrightarrow \mathbb{R}$, $\mathbb{E}_1$-algebra in $\mathbf{Vec}_{\mathbb{K}}$ has units. However, there appears no physical motivation to introduce unit. Therefore it's better to work on nonunital associative algebras.