第 5 章

Higher condensation theory

Review of [?].

5.1 Definition of condensation

In this section, we refer to an **(n, n)-category** as an (∞, n)-category which is "truncated" at morphism degree n. Intuitively, if $\mathcal{C}$ is an (n, n)-category, then, for arbitrary $n-1$-morphisms $f, g \in \mathcal{C}_{n-1}$, the space of n-morphisms between f and g has discrete topology. Therefore, we can compose n-morphisms strictly (not up to homotopy), associatively, and unitally.

Note that a **(0, 0)-category** $\mathcal{C}$ is merely a set[*1].

定義 5.1: 0-condensation

Let $\mathcal{C}$ be a $(0, 0)$-category. A **0-condensation** on $\mathcal{C}$ is an equality between elements of a set $\mathcal{C}_0$.

n-condensation is defined by induction.

定義 5.2: n-condensation

Fix $n \geq 0$. Let $\mathcal{C}$ be an (n, n)-category, and let $x, y \in \mathcal{C}_0$ be objects of $\mathcal{C}$. We define n-condensation by induction on n.

n-condensation of x onto y in $\mathcal{C}$ consists of three data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- An $(n-1)$-condensation of $r \circ i$ onto Id_y in $\mathrm{Map}_{\mathcal{C}}(y, y)$[a]

[a] Roughly speaking, a mapping space $\mathrm{Map}_{\mathcal{C}}(y, y)$ itself is an $(n-1, n-1)$-category. Strickly speaking, we need enriched ∞-category theory.

[*1] That is, $\mathcal{C}$ consists of a set $\mathcal{C}_0$ of objects (0-morphisms) only.

【例 5.1.1】1-condensation

Let $\mathcal{C}$ be a $(1,\,1)$-category (i.e. an ordinary category). **1-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Hom}_{\mathcal{C}}(x,\,y)$
- A 1-morphism $i \in \mathrm{Hom}_{\mathcal{C}}(y,\,x)$
- A 0-condensation of $r \circ i$ onto Id_y in $\mathrm{Hom}_{\mathcal{C}}(y,\,y)$. i.e. $r \circ i = \mathrm{Id}_y$.

In the context of $(1,\,1)$-categories, such y is called a **retract of $\boldsymbol{x}$**.

【例 5.1.2】2-condensation

Let $\mathcal{C}$ be a $(2,\,2)$-category (i.e. a bicategory). **2-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x,\,y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y,\,x)_0$
- A 1-condensation of $r \circ i$ onto Id_y in $(1,\,1)$-category $\mathrm{Map}_{\mathcal{C}}(y,\,y)$.

By **【例 5.1.1】**,

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x,\,y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y,\,x)_0$
- A 2-morphism (in $\mathcal{C}$) $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y,\,y)_1$
- A 2-morphism (in $\mathcal{C}$) $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y,\,y)_1$
- An equality $\rho \circ \iota = \mathrm{Id}_{\mathrm{Id}_y}$

【例 5.1.3】3-condensation

Let $\mathcal{C}$ be a $(3,\,3)$-category. By **【例 5.1.2】**, **3-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x,\,y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y,\,x)_0$
- A 2-morphism $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y,\,y)_1$
- A 2-morphism $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y,\,y)_1$
- A 3-morphism

$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\rho\circ\iota"] \arrow[r, bend right, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Id}_y \end{tikzcd}\right) \in \mathrm{Map}_{\mathcal{C}}(y,\,y)_2$$

- A 3-morphism

$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, bend right, "\rho\circ\iota"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2$$

- An equiality

$$\begin{tikzcd}[column sep=large] \mathrm{Id}_y \arrow[r, bend left=60, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, "\rho\circ\iota" description] \arrow[r, bend right=60, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\alpha"] & \mathrm{Id}_y \end{tikzcd} = \begin{tikzcd}[column sep=large] \mathrm{Id}_y \arrow[r, bend left=60, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, "\Downarrow \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_y}}"] & \mathrm{Id}_y \end{tikzcd}$$

【例 5.1.4】4-condensation

Let $\mathcal{C}$ be a $(4,\, 4)$-category. By【例 5.1.2】, **3-condensation** of x onto y in $\mathcal{C}$ consists of these data:

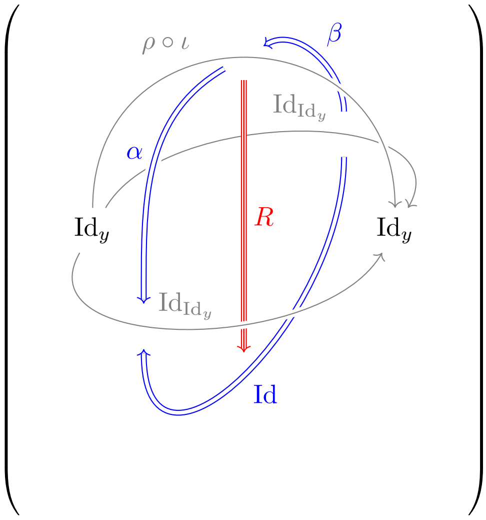
- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x,\, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y,\, x)_0$
- A 2-morphism $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_1$
- A 2-morphism $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_1$
- A 3-morphism

$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\rho\circ\iota"] \arrow[r, bend right, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2$$

- A 3-morphism

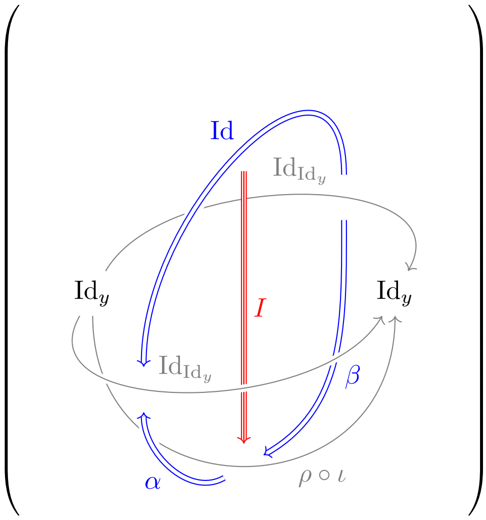
$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, bend right, "\rho\circ\iota"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2$$

- A 4-morphism



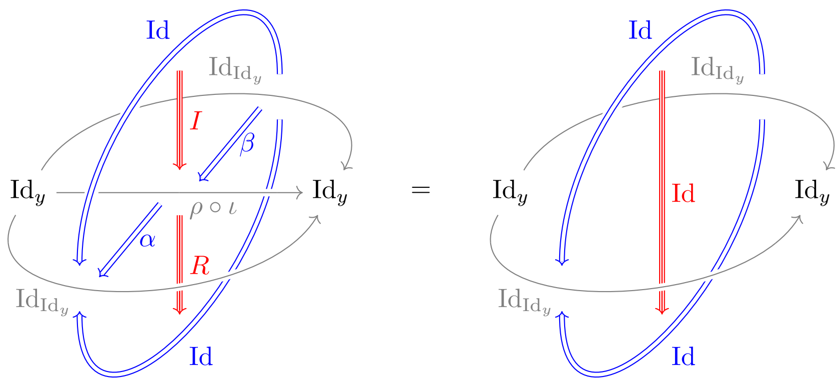
$\in \mathrm{Map}_{\mathcal{C}}(y,\, y)_3$

- A 4-morphism



$\in \mathrm{Map}_{\mathcal{C}}(y,\, y)_3$

- An equality



5.1.1 Walking condensation

Let $\mathcal{C}$ be an (n, n)-category. Roughly speaking, a **walking n-condensation** is a (strict?) n-category $\clubsuit_n$ which "generates" n-condensation in $\mathcal{C}$. i.e. the functor category $\mathcal{F}\mathbf{un}(\spadesuit_n, \mathcal{C})$ is equivalent to the category of n-condensations in $\mathcal{C}$.

5.1.2 Condensation monad

定義 5.3: condensation monad

5.2 Physical interpretation

5.3 Reutter-Theo

Let $\mathbf{Pr}^{\mathrm{L}}$ be the symmetric monoidal $(\infty, 2)$-category of presentable $(\infty, 1)$-categories, colimit-preserving functors, and natural transformations [?, Definition 5.5.3.1.]. Note that $\mathbf{Vec}_{\mathbb{C}}$ is an $\mathbb{E}_2$-algebra in $\mathbf{Pr}^{\mathrm{L}}$. We define $\mathbf{Pr}_{\mathbb{C}} \coloneqq \mathbf{RMod}_{\mathbf{Vec}_{\mathbb{C}}}(\mathbf{Pr}^{\mathrm{L}})$ [?, Definition 4.2.1.13.]. More concretely, $\mathbf{Pr}_{\mathbb{C}}$ is the symmetric monoidal $(\infty, 2)$-category of presentable $\mathbf{Vec}_{\mathbb{C}}$-enriched categories, colimit-preserving $\mathbf{Vec}_{\mathbb{C}}$-enriched functors, and natural transformations.