第 5 章

Higher condensation theory

Review of [?].

5.1 Definition of condensation

In this section, we refer to an (n, n)**-category** as an (∞, n)-category which is "truncated" at morphism degree n. Intuitively, if $\mathcal{C}$ is an (n, n)-category, then, for arbitrary $n-1$-morphisms $f, g \in \mathcal{C}_{n-1}$, the space of n-morphisms between f and g has discrete topology. Therefore, we can compose n-morphisms strictly (not up to homotopy), associatively, and unitally.

Note that a $(0, 0)$**-category** $\mathcal{C}$ is merely a set[*1].

定義 5.1: 0-condensation

Let $\mathcal{C}$ be a $(0, 0)$-category. A **0-condensation** on $\mathcal{C}$ is an equality between elements of a set $\mathcal{C}_0$.

n-condensation is defined by induction.

定義 5.2: n-condensation

Fix $n \geq 0$. Let $\mathcal{C}$ be an (n, n)-category, and let $x, y \in \mathcal{C}_0$ be objects of $\mathcal{C}$. We define n-condensation by induction on n.

$\boldsymbol{n}$**-condensation** of x onto y in $\mathcal{C}$ consists of three data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- An $(n-1)$-condensation of $r \circ i$ onto Id_y in $\mathrm{Map}_{\mathcal{C}}(y, y)$[a]

[a] Roughly speaking, a mapping space $\mathrm{Map}_{\mathcal{C}}(y, y)$ itself is an $(n-1, n-1)$-category. Strickly speaking, we need enriched ∞-category theory.

After [?], we denote an n-condensation of x onto y in $\mathcal{C}$ as $x \twoheadrightarrow y$.

[*1] That is, $\mathcal{C}$ consists of a set $\mathcal{C}_0$ of objects (0-morphisms) only.

【例 5.1.1】1-condensation

Let $\mathcal{C}$ be a (1, 1)-category (i.e. an ordinary category). **1-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Hom}_{\mathcal{C}}(x, y)$
- A 1-morphism $i \in \mathrm{Hom}_{\mathcal{C}}(y, x)$
- A 0-condensation of $r \circ i$ onto Id_y in a (0, 0)-category $\mathrm{Hom}_{\mathcal{C}}(y, y)$. i.e. $r \circ i = \mathrm{Id}_y$.

In the context of (1, 1)-categories, such y is called a **retract of $\boldsymbol{x}$**.

【例 5.1.2】2-condensation

Let $\mathcal{C}$ be a (2, 2)-category (i.e. a bicategory). **2-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- A 1-condensation of $r \circ i$ onto Id_y in a (1, 1)-category $\mathrm{Map}_{\mathcal{C}}(y, y)$.

By【例 5.1.1】,

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism (in $\mathcal{C}$) $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism (in $\mathcal{C}$) $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- An equality $\rho \circ \iota = \mathrm{Id}_{\mathrm{Id}_y}$

【例 5.1.3】3-condensation

Let $\mathcal{C}$ be a (3, 3)-category. By【例 5.1.2】, **3-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism $(r \circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism $(\mathrm{Id}_y \xrightarrow{\iota} r \circ i) \in \mathrm{Map}_{\mathcal{C}}(y, y)_1$
- A 3-morphism

$$\left(\begin{tikzcd} \mathrm{Id}_y \arrow[r, bend left, "\rho\circ\iota"] \arrow[r, bend right, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Id}_y \end{tikzcd} \right) \in \mathrm{Map}_{\mathcal{C}}(y, y)_2$$

- A 3-morphism

$$
\left(
\begin{tikzcd}
\mathrm{Id}_y \arrow[r, bend left, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, bend right, "\rho\circ\iota"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathrm{Id}_y
\end{tikzcd}
\right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2
$$

- An equiality

$$
\begin{tikzcd}
\mathrm{Id}_y \arrow[rr, bend left=50, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[rr, "\rho\circ\iota" description] \arrow[rr, bend right=50, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[rr, phantom, bend left=25, "\Downarrow\beta"] \arrow[rr, phantom, bend right=25, "\Downarrow\alpha"] & & \mathrm{Id}_y
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{Id}_y \arrow[rr, bend left=50, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[rr, bend right=50, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[rr, phantom, "\Downarrow \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_y}}"] & & \mathrm{Id}_y
\end{tikzcd}
$$

【例 5.1.4】4-condensation

Let $\mathcal{C}$ be a $(4,\, 4)$-category. By【例 5.1.2】, **3-condensation** of x onto y in $\mathcal{C}$ consists of these data:

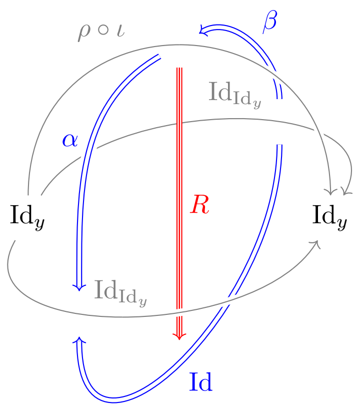
- A 1-morphism $r \in \mathrm{Map}_{\mathcal{C}}(x,\, y)_0$
- A 1-morphism $i \in \mathrm{Map}_{\mathcal{C}}(y,\, x)_0$
- A 2-morphism $(r\circ i \xrightarrow{\rho} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_1$
- A 2-morphism $(\mathrm{Id}_y \xrightarrow{\iota} r\circ i) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_1$
- A 3-morphism

$$
\left(
\begin{tikzcd}
\mathrm{Id}_y \arrow[r, bend left, "\rho\circ\iota"] \arrow[r, bend right, "\mathrm{Id}_{\mathrm{Id}_y}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Id}_y
\end{tikzcd}
\right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2
$$

- A 3-morphism

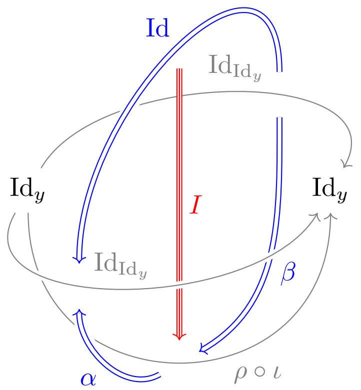
$$
\left(
\begin{tikzcd}
\mathrm{Id}_y \arrow[r, bend left, "\mathrm{Id}_{\mathrm{Id}_y}"] \arrow[r, bend right, "\rho\circ\iota"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathrm{Id}_y
\end{tikzcd}
\right) \in \mathrm{Map}_{\mathcal{C}}(y,\, y)_2
$$

- A 4-morphism



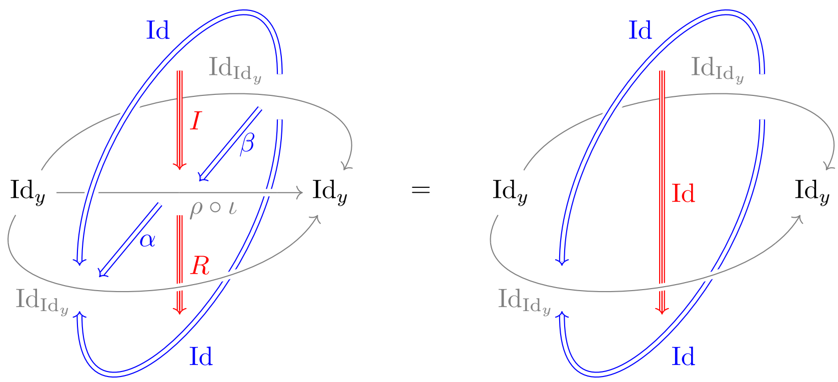
$\in \mathrm{Map}_{\mathcal{C}}(y,\, y)_3$

- A 4-morphism



$\in \mathrm{Map}_{\mathcal{C}}(y,\, y)_3$

- An equality



5.1.1 Walking condensation

Let $\mathcal{C}$ be an (n, n)-category. Roughly speaking, a **walking n-condensation** is a (strict?) n-category $\spadesuit_n$ which "generates" n-condensation in $\mathcal{C}$. i.e. the functor category $\mathcal{F}\mathbf{un}(\spadesuit_n, \mathcal{C})$ is equivalent to the category of n-condensations in $\mathcal{C}$.

5.1.2 Condensation algebra / monad

Fix an (n, n)-category $\mathcal{C}$. Now we will outline the definition of **condensation monad in $\mathcal{C}$**.

Let's pick out an object $x \in \mathcal{C}_0$. Recall that the endmorphism $(n-1, n-1)$-category $\mathrm{End}_{\mathcal{C}}(x) \coloneqq \mathrm{Map}_{\mathcal{C}}(x, x)$ has a natural monoidal structure induced by a composition of 1-morphisms in $\mathcal{C}$. Let $\clubsuit_n \subset \spadesuit_n$ be a subcategory [*2] which consists of walking n-condensations with a single object. A **condensation monad** in $\mathcal{C}$ is a functor $(x \in \mathcal{C}_0;\, e \in \mathrm{End}_{\mathcal{C}}(x)_0,\, e^{\circ 2} \twoheadrightarrow e,\, \dots)\colon \clubsuit_n \longrightarrow \mathcal{C}$. An (n, n)-category $\mathcal{C}$ is said to **have all condensates** if each condensation monad $\clubsuit_n \xrightarrow{A} \mathcal{C}$ extends to an n-condensation $\spadesuit_n \xrightarrow{\bar{A}} \mathcal{C}$:

$$\begin{tikzcd}
\clubsuit_n \arrow[r, "A"] \arrow[d, hook] & \mathcal{C} \\
\spadesuit_n \arrow[ur, dashed, "\bar{A}"'] &
\end{tikzcd}$$

However, there are some technical issues in this definition , so we should work on more concrete definition.

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*2 (18 February, 2026) Full-subness of $\clubsuit_n$ is still a conjecture [?].

定義 5.3: condensation algebra / monad

Let $n \geq 0$ and let $\mathcal{V}^{\otimes}$ be a <u>monoidal</u> (n, n)-category [a]. A **condensation algebra** in $\mathcal{V}^{\otimes}$ is a sequence of "commuting condensation cubes":

$$
A \in (\mathcal{V}^{\otimes})_0, \quad A \otimes A \twoheadrightarrow A, \quad
\begin{tikzcd}
A^{\otimes 3} \arrow[r, two heads] \arrow[d, two heads] & A^{\otimes 2} \arrow[d, two heads] \\
A^{\otimes 2} \arrow[r, two heads] & A
\end{tikzcd}, \quad
\begin{tikzcd}
A^{\otimes 4} \arrow[rr, two heads] \arrow[dr, two heads] \arrow[dd, two heads] & & A^{\otimes 3} \arrow[dr, two heads] \arrow[dd, two heads] & \\
& A^{\otimes 3} \arrow[rr, two heads] \arrow[dd, two heads] & & A^{\otimes 2} \arrow[dd, two heads] \\
A^{\otimes 3} \arrow[rr, two heads] \arrow[dr, two heads] & & A^{\otimes 2} \arrow[dr, two heads] & \\
& A^{\otimes 2} \arrow[rr, two heads] & & A
\end{tikzcd}, \quad \cdots
$$

in which the faces of the k-dimensional cube are either whiskerings of lower-dimensional cubes or are identities [**?**, Definition 2.2.1.].

[a] **Monoidal (0, 0)-category** is a set $\mathcal{V}_0$ with a map $\otimes \colon \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ and a distinguished element $1 \in \mathcal{V}_0$, which makes up of a monoid in $\mathbf{Sets} = \mathbf{Cat}_{(0,\,0)}$.

Let $\mathcal{C}$ be an (n, n)-category[a]. A **condensation monad** in $\mathcal{C}$ is a pair of an object $x \in \mathcal{C}_0$ and a condensation algebra in the monoidal $(n-1, n-1)$-category $\mathrm{End}_{\mathcal{C}}(x)$.

[a] Not necessarily monoidal.

We would like to construct a **higher Morita category of** condensation algebras in $\mathcal{V}^{\otimes}$.

定義 5.4: condensation bimodule

Let $n \geq 0$ and let $\mathcal{V}^{\otimes}$ be a <u>monoidal</u> (n, n)-category. Pick two condensation algebras $(A, A^{\otimes 2} \twoheadrightarrow A, \dots)$, $(B, B^{\otimes 2} \twoheadrightarrow B, \dots)$ in $\mathcal{V}^{\otimes}$.
A **condensation (A, B)-bimodule** in $\mathcal{V}^{\otimes}$ is a sequence of "commuting condensation cubes" [**?**, Definition 2.3.3.]:

$$
M \in (\mathcal{V}^{\otimes})_0, \qquad A \otimes M \twoheadrightarrow M, \qquad M \otimes B \twoheadrightarrow M,
$$

$$
\begin{tikzcd}
A^{\otimes 2} \otimes M \arrow[r, two heads] \arrow[d, two heads] & A \otimes M \arrow[d, two heads] \\
A \otimes M \arrow[r, two heads] & M
\end{tikzcd}, \quad
\begin{tikzcd}
A \otimes M \otimes B \arrow[r, two heads] \arrow[d, two heads] & M \otimes B \arrow[d, two heads] \\
A \otimes M \arrow[r, two heads] & M
\end{tikzcd}, \quad
\begin{tikzcd}
M \otimes B^{\otimes 2} \arrow[r, two heads] \arrow[d, two heads] & M \otimes B \arrow[d, two heads] \\
M \otimes B \arrow[r, two heads] & M
\end{tikzcd}, \quad \cdots
$$

in which $\forall i, j \in \mathbb{Z}_{\geq 0}$, there is a $i+j$-dimensional cube which has $A^{\otimes i} \otimes M \otimes B^{\otimes j} \in (\mathcal{V}^{\otimes})_0$ on its "top" vertex and $M \in (\mathcal{V}^{\otimes})_0$ on its "bottom" vertex.

Let $\mathcal{C}$ be an (n, n)-category[a] and pick two condensation monads:

- A condensation monad $(x \in \mathcal{C}_0;\, e_1,\, e_1^{\circ 2} \twoheadrightarrow e_1,\, \dots)$ in $\mathrm{End}_{\mathcal{C}}(x)$
- A condensation monad $(y \in \mathcal{C}_0;\, e_2,\, e_2^{\circ 2} \twoheadrightarrow e_2,\, \dots)$ in $\mathrm{End}_{\mathcal{C}}(y)$

A **condensation** (e_1, e_2)**-bimodule** is a sequence of "commuting condensation cubes" in the $(n-1, n-1)$-category $\mathrm{Map}_{\mathcal{C}}(y, x)$ [**?**, Definition 2.3.3.]:

$$m \in \mathrm{Map}_{\mathcal{C}}(y, x)_0, \qquad e_1 \circ m \twoheadrightarrow m, \qquad m \circ e_2 \twoheadrightarrow m,$$

$$\begin{tikzcd} e_1^{\circ 2} \circ m \arrow[r, two heads] \arrow[d, two heads] & e_1 \circ m \arrow[d, two heads] \\ e_1 \circ m \arrow[r, two heads] & m \end{tikzcd}, \quad \begin{tikzcd} e_1 \circ m \circ e_2 \arrow[r, two heads] \arrow[d, two heads] & m \circ e_2 \arrow[d, two heads] \\ e_1 \circ m \arrow[r, two heads] & m \end{tikzcd}, \quad \begin{tikzcd} m \circ e_2^{\circ 2} \arrow[r, two heads] \arrow[d, two heads] & m \circ e_2 \arrow[d, two heads] \\ m \circ e_2 \arrow[r, two heads] & m \end{tikzcd}, \quad \cdots$$

in which $\forall i, j \in \mathbb{Z}_{\geq 0}$, there is a $i + j$-dimensional cube which has $e_1^{\circ i} \circ m \circ e_2^{\circ j} \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$ on its "top" vertex and $m \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$ on its "bottom" vertex.

[a] Not necessarily monoidal.

5.2 Physical interpretation

5.3 Tannaka-Krein reconstruction

Fix an algebraically closed field $\mathbb{K}$. From now on, we denote the $(1, 1)$-category of finite dimensional $\mathbb{K}$-vector spaces as $\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$[*3].

5.3.1 A bicategory of 2-vactor spaces $\mathbf{2Vec}_{\mathbb{K}}$

Let $\mathbf{Pr}^{\mathrm{L}}$ be the symmetric monoidal $(\infty, 2)$-category of presentable $(\infty, 1)$-categories, colimit-preserving functors, and natural transformations [**?**, Definition 5.5.3.1.]. Note that $\mathbf{Vec}_{\mathbb{C}}^{\mathrm{fin}}$ is an $\mathbb{E}_2$-algebra in $\mathbf{Pr}^{\mathrm{L}}$. We define $\mathbf{Pr}_{\mathbb{C}} \coloneqq \mathbf{RMod}_{\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}}(\mathbf{Pr}^{\mathrm{L}})$ [**?**, Definition 4.2.1.13.]. More concretely, $\mathbf{Pr}_{\mathbb{C}}$ is the symmetric monoidal $(\infty, 2)$-category of presentable $\mathbf{Vec}_{\mathbb{C}}$-enriched categories, colimit-preserving $\mathbf{Vec}_{\mathbb{C}}^{\mathrm{fin}}$-enriched functors, and natural transformations.

After [**?**], we define $\mathbf{2Vec}_{\mathbb{K}}$ as a $(2, 2)$-category (bicategory) as follows:

- Objects are finite semisimple $\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$-enriched categories
- 1-morphisms are $\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$-enriched functors.
- 2-morphisms are natural transformations.

[*3] This is the fully-dualizable part of $\mathbf{Vec}_{\mathbb{K}}$, which is a $(1, 1)$-category of $\mathbb{K}$-vector spaces.

Note that $\mathbf{2Vec}_{\mathbb{K}}$ is fully-dualizable. In fact, $\mathbf{2Vec}_{\mathbb{K}} \subset \mathbf{Pr}_{\mathbb{C}}$ is the full-subcategory of $\mathbf{Pr}_{\mathbb{C}}$ which consists of fully dualizable objects of $\mathbf{Pr}_{\mathbb{C}}$. For more details, see [?, APPENDIX A.].

5.3.2 Morita bicategory of algebras

Note that $\mathbf{Vec}_{\mathbb{K}}$ ($\mathbf{Vec}_{\mathbb{K}}^{\mathrm{fin}}$?) is a symmetric monoidal $((\infty, 1)$-)category. Then, we obtain a notion of $\mathbb{E}_1$-algebra objects in $\mathbf{Vec}_{\mathbb{K}}$, according to [?]. More concretely, an $\mathbb{E}_1$-algebra object in $\mathbf{Vec}_{\mathbb{K}}$ is none other than an ordinary associative algebra (i.e. $\mathbb{K}$-vector spaces with associative multiplication)[*4].

Morita category of $\mathbb{E}_1$-algebras in $\mathbf{Vec}_{\mathbb{K}}$ is a bicategory $\mathbf{Mor}_{\mathbb{E}_1}(\mathbf{Vec}_{\mathbb{K}})$ consists of the following data:

- Objects are $\mathbb{E}_1$-algebras in $\mathbf{Vec}_{\mathbb{K}}$.
- 1-morphisms between two $\mathbb{E}_1$-algebras $A, B \in \mathrm{Ob}(\mathbf{Mor}_{\mathbb{E}_1}(\mathbf{Vec}_{\mathbb{K}}))$ are (A, B)-bimodules in $\mathbf{Vec}_{\mathbb{K}}$.
- 2-morphisms between two (A, B)-bimodules $M, N \in \mathrm{Ob}(\mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(A, B))$ are (A, B)-bimodule homomorphisms.

Composition of 1-morphisms (or horizontal composition) in $\mathbf{Mor}_{\mathbb{E}_1}(\mathbf{Vec}_{\mathbb{K}})$ is the relative tensor product functor:

$$\otimes_B : \mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(B, C) \boxtimes \mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(A, B) \longrightarrow \mathbf{Bimod}_{\mathbf{Vec}_{\mathbb{K}}}(A, C)$$

5.3.3 Tannaka-Krein reconstruction

5.4 Dualizability in higher Morita categories

[*4] With respect to the open embedding $\emptyset \hookrightarrow \mathbb{R}$, $\mathbb{E}_1$-algebra in $\mathbf{Vec}_{\mathbb{K}}$ has units. However, there appears no physical motivation to introduce unit. Therefore it's better to work on nonunital associative algebras.