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ON HOW TO COUNT GOLDSTONE BOSONS

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A generalization of Goldstone's theorem is presented which is valid for theories either with or without relativistic invariance. The central suggestion is that, under certain specified assumptions, all Goldstone bosons can be divided into two classes, termed type I and type II, in accordance with the behaviour of their dispersion laws. A Goldstone boson is a member of either the first or the second class according as its energy, in the limit of long wavelengths, is proportional to an odd or an even power of its momentum, respectively. The major result then is that, if each Goldstone boson of type I is counted once and that of type II is counted twice, the total number of "bosons" so obtained is always equal to or greater than the number of symmetry generators that are spontaneously broken. An immediate corollary is the familiar result that for relativistically invariant theories the number of Goldstone bosons can never be less than the number of spontaneously broken generators. Throughout the proof of the above result particular emphasis is placed on theories which are not Lorentz invariant.

1. Introduction

The usual form of Goldstone's theorem states that if the Hamiltonian of a system has a certain continuous symmetry which is *not* a symmetry of the ground state of the system, then there exists an excitation of the system whose energy tends to zero in the long wavelength limit [1,2]. In the case of systems which are not relativistically invariant it is well-known that certain further assumptions are needed to guarantee the validity of this theorem. We shall briefly elaborate upon these assumptions later.

Suppose that the Lagrangian L of a system is invariant under an n-parameter symmetry group of continuous transformations and that m of the group generators are spontaneously broken. Of such a situation one might enquire: how many distinct Goldstone excitations does the system possess [3]? Let this number be denoted by l. Certainly, in general l can be less than m, as is evidenced in the case of the Heisenberg ferromagnet. The ferromagnet in its ground state, with all spins aligned along the z-axis, say, has two spontaneously broken rotation generators but only one magnon.

Nevertheless, for many purposes, it is convenient to count the Goldstone bosons in a manner which ensures that their number is always equal to or greater than the

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number of broken generators. The specification of the requisite counting prescription involves an investigation of the dispersion laws of the various bosons, and constitutes the main purpose of this paper. We present our result in the form of a theorem.

2. Statement of the theorem

Let G denote the n(real) parameter symmetry group of a Lagrangian L, and let Q_a (a = 1 ..., n) be the consequent conserved charges. Assume further that:

(i) m of the symmetry generators are spontaneously broken. A more precise formulation of this condition is that there exist m fields Φ_i and a vacuum state $|0\rangle$ such that

$$\det \langle 0|\, [\Phi_i, Q_a]\, |0\rangle \neq 0 \;, \quad i, a = 1, \dots, m \;, \tag{1}$$

(ii) the theory obeys a commutativity condition of the form that if $A(x)$ and $B(0)$ are any two local operators then

$$|\boldsymbol{x}| \to \infty : \quad |\langle 0|\, [A(\boldsymbol{x}, t), B(0)]\, |0\rangle\, | \to \mathrm{e}^{-\tau |\boldsymbol{x}|} \;, \quad \tau > 0 \;, \tag{2}$$

(iii) the translational invariance of the theory is not entirely broken spontaneously.

In such a system there occur two types of Goldstone excitations: one for which the energy is proportional to an odd power of the momentum and the other for which the energy is proportional to an even power of the momentum in the limit of long wavelengths. If every excitation of the former type is counted singly and of the latter type doubly then there are at least m Goldstone "bosons".

3. Brief discussion of the assumptions

It is not our intention here to repeat the frequently quoted proof of Goldstone's theorem. Nevertheless, we feel that, at least in the case of readers who might be unaware of the early controversy [4–6] surrounding the proof of Goldstone's theorem for theories which are not relativistically invariant, it may be helpful to elucidate the meaning of the somewhat obscure assumptions (ii) and (iii) above. For this purpose we follow a method of proof due to Gilbert [6]. It should be emphasized at the outset, and, indeed, will be abundantly clear in the sequel, that these assumptions are necessary to guarantee any theorem in the first place.

Suppose, then, that we have a theory which is not necessarily Lorentz invariant, but involves spatio-temporal concepts. We may artificially embed this theory into a "relativistic" framework provided we take into account the presence of an additional associated time-like vector which, for convenience, may be chosen to be $n^\mu = (1, \mathbf{0})$. The usual proofs now proceed by a consideration of the Fourier transform of the quantity $\langle 0|\, [\Phi_i, j_a^\mu]\, |0\rangle$, written simply as $(\mathrm{FT})^\mu$, where

$$Q_a = \int (\mathrm{d}\boldsymbol{x})\, j_a^0(\boldsymbol{x}, t) \;, \tag{3}$$

and the particular choice of i and a ensures that

$$\langle 0|\,[\Phi_i, Q_a]\,|0\rangle = \frac{1}{2\pi}\int (\mathrm{d}k)\,\mathrm{e}^{-ik^0x^0}\,\delta(k)\,(\mathrm{FT})^0 \neq 0\,. \tag{4}$$

The most general form of the FT may now be exhibited as

$$(\mathrm{FT})^\mu = k^\mu\delta(k^2)\,\chi(nk) + [k^2n^\mu - k^\mu(nk)]\,\rho(k^2, nk)$$
$$+\, n^\mu\,\delta(nk)\,\Delta(k^2) + Cn^\mu\,\delta^4(k)\,, \tag{5}$$

where χ, ρ and Δ are arbitrary functions of their indicated arguments and C is a constant.

Let us now consider briefly the contribution that each of the terms in $(\mathrm{FT})^\mu$ makes to the quantity $M_{ia} \equiv \langle 0|\,[\Phi_i, Q_a]\,|0\rangle$. If $C \neq 0$, there would be a contribution to M_{ia} from the last term $Cn^\mu\delta^4(k)$ which would come from the isolated energy momentum eigenstate $k^\mu = 0$. This corresponds to an intermediate spurion state, one that we would certainly wish to exclude. Since

$$C = (2\pi)^4\,\langle 0|\,[\Phi_i, j_a^0(x,t)]\,|0\rangle \neq 0\,,$$

it is now obvious that one purpose of assumption (ii) above is to preclude precisely the possibility of such an intermediate state.

Trouble also ensues if the term $n^\mu\delta(nk)\,\Delta(k^2)$ makes a non-vanishing contribution to M_{ia}. For in such a case the dispersion law of the intermediate state associated with such a term would be of the form $k^0 = 0$ for all $\boldsymbol{k}$. If such an intermediate state existed it would be possible to generate from any given vacuum other vacua which have zero energy but non-zero momenta. Such vacua would then imply a complete spontaneous breakdown of the translational invariance of the theory. Assumption (iii) guarantees that such a situation will not occur and vitiate Goldstone's theorem.

We shall not discuss any further the first two terms in $(\mathrm{FT})^\mu$. When either of these two terms contributes to M_{ia} there occurs the presence of bonafide Goldstone excitations in the system. These excitations may have the dispersion law $E_p \sim |p|$ (corresponding to the first term) or might encompass more general possibilities (arising from the second term). The details are too familiar to merit repetition.

4. Proof of the theorem

After having made the above preliminary remarks let us proceed to the proof of the theorem enunciated in sect. 2. This may be conveniently divided, for the sake of clarity, into three distinct steps.

(i) We shall first show that the minimum number of Goldstone bosons is always greater than or equal to half the number of symmetry generators that are spontaneously broken. As we have noted above the quantity of real interest is $\langle 0|\,[\Phi_i, Q_a]\,|0\rangle$

(eq. (4)). The customary manipulation of introducing a complete set of intermediate states into $\langle 0|[\Phi_i, j_a^0(x,t)]|0\rangle$ enables us to present it as

$$\langle 0|[\Phi_i, Q_a]|0\rangle = \sum_{n=1}^{l} [\mathrm{e}^{-iE_k t} \langle 0|\Phi_i|n_k\rangle \langle n_k|j_a^0|0\rangle$$

$$-\mathrm{e}^{iE_{-k}t} \langle 0|j_a^0|n_{-k}\rangle \langle n_{-k}|\Phi_i|0\rangle]|_{k=0} \,, \tag{6}$$

where we have employed a non-relativistic normalization for the states. Now, we know that as $k \to 0$, non-vanishing contributions to M_{ia} can come only if $E \to 0$ simultaneously, i.e., only from those intermediate states which are Goldstone states. Assume that there are l such states in the theory. The notation $|n_k\rangle$ then denotes a momentum eigenstate of the nth particle species. Defining further the $m \times m$ matrix $\underset{\sim}{v}$, whose elements $\underset{\sim}{v}_{ia}$ are also expressible as the ith element of its ath column, $(\mathfrak{v}_a)_i$,

$$\underset{\sim}{v}_{ia} \equiv (\mathfrak{v}_a)_i = \sum_{n=1}^{l} \langle 0|\Phi_i|n_0\rangle \langle n_0|j_a^0|0\rangle \,, \tag{7}$$

we may write

$$\langle 0|\,[\Phi_i, Q_a]|0\rangle = 2i \,\mathrm{Im}\, \underset{\sim}{v}_{ia} \,, \tag{8}$$

whence, from assumption (i) (the broken symmetry condition),

$$\det (\mathrm{Im}\, \underset{\sim}{v}) \neq 0 \,, \tag{9}$$

which, in turn, implies that

$$\mathrm{rank}\, (\mathrm{Im}\, \underset{\sim}{v}) = m \,. \tag{10}$$

On the other hand, the columns $\mathfrak{v}_a$ are, by definition, linear combinations of the l column vectors

$$A_n = \begin{bmatrix} \langle 0|\Phi_1|n_0\rangle \\ \langle 0|\Phi_2|n_0\rangle \\ \vdots \\ \langle 0|\Phi_m|n_0\rangle \end{bmatrix}, \qquad n = 1, \ldots, l \,, \tag{11}$$

which may in fact have further linear dependences among them. We thus conclude that

$$\mathrm{rank}\, \underset{\sim}{v} \leqslant l \,, \tag{12}$$

and write

$$\mathfrak{v}_a = \sum_{n=1}^{l} \gamma_{an} A_n \,, \quad a = 1, \ldots, m \,, \tag{13}$$

where the $\gamma_{an} \equiv \langle n_0 | j_a^0 | 0\rangle$ are complex coefficients. Then it follows that

$$\mathrm{Im}\, \mathfrak{v}_a = \sum_{n=1}^{l} \mathrm{Re}\, \gamma_{an}\, \mathrm{Im}\, A_n + \sum_{n=1}^{l} \mathrm{Im}\, \gamma_{an}\, \mathrm{Re}\, A_n \;, \tag{14}$$

which states that every column of the matrix Im $\underset{\sim}{\mathfrak{v}}$ is expressible as a linear combination of the $2l$ columns Im A_n and Re A_n. To avoid contradiction with eq. (10) we must then have $2l \geqslant m$, which proves the assertion.

(ii) We next wish to show that for $\frac{1}{2}m \leqslant l < m$ there must exist at least one Goldstone boson whose energy, in the limit of long wavelengths, is proportional to an even power of its momentum. Note, first, that, since $p \equiv \mathrm{rank}\, \underset{\sim}{\mathfrak{v}} \leqslant l$, there must exist $m-p$ independent linear relations amongst the m vectors $\mathfrak{v}_a$:

$$\sum_{a=1}^{m} C_a^\alpha \mathfrak{v}_a = 0 \;, \quad \alpha = 1, \dots, m-p \,. \tag{15}$$

For any given α the complex coefficients C_a^α are not all zero. But, then, it must also be true that

$$\sum_{a=1}^{m} C_a^{\alpha *}\, \mathfrak{v}_a \neq 0 \;, \tag{16}$$

for, if this were not so, we would have

$$\sum_{a=1}^{m} (\mathrm{Re}\, C_a^\alpha)\, \mathrm{Im}\, \mathfrak{v}_a = 0 \;, \tag{17}$$

in violation of the broken symmetry condition, eq. (10). Consider now the following quantity,

$$\langle 0 | \, [\Phi_i, \sum_{a=1}^{m} C_a^\alpha j_a^0(x)] \, | 0 \rangle$$

$$= \sum_{a=1}^{m} C_a^\alpha \sum_f \int \frac{(\mathrm{d}p)}{(2\pi)^3} \, [\mathrm{e}^{ipx} \langle 0 | \Phi_i | f_p \rangle \langle f_p | j_a^0 | 0 \rangle - \mathrm{e}^{-ipx} \langle 0 | j_a^0 | f_p \rangle \langle f_p | \Phi_i | 0 \rangle] \tag{18}$$

(f denotes all the intermediate states and the summation includes integrations over the internal variables of the multiparticle states), of which the four dimensional Fourier transform may be written as

$$\mathrm{FT} = 2\pi \sum_f \, [\delta(k^0 - E_k) \sum_{a=1}^{m} C_a^\alpha \langle 0 | \Phi_i | f_k \rangle \langle f_k | j_a^0 | 0 \rangle$$

$$-\delta(k^0 + E_{-k})\{\sum_{a=1}^{m} C_a^{\alpha *}\langle 0|\,\Phi_i|f_{-k}\rangle\langle f_{-k}|j_a^0|0\rangle\}^*] \,. \tag{19}$$

It is imperative to observe that for every value of α there must be at least one value of i for which the object

$$\langle 0|[\Phi_i, \sum_{a=1}^{m} C_a^{\alpha} Q_a]|0\rangle$$

is non-vanishing so as to accord with the broken symmetry condition. We know, however, that only those intermediate states in eq. (19) which have a Goldstone like dispersion law can contribute to this object. There must therefore be at least one Goldstone state that couples to the combination $\Sigma_a C_a^{\alpha} j_a^0$, and it is on the dispersion law of such a state that we focus our attention.

Let us now confine our attention to the immediate neighbourhood of the point $k = 0$ (and, hence, also of $k^0 = 0$). This is the only region of interest to us. Assumption (ii) assures us that the FT is an analytic function of k whence, in virtue of the previously derived linear relations eqs. (15) and (16), we may omit the first term in eq. (19) for small k, and write

$$\mathrm{FT}\,(k \simeq 0) = -2\pi \sum_{n=1}^{l} [\delta(k^0 + E_{-k}) \sum_{a=1}^{m} C_a^{\alpha}\langle 0|\,j_a^0\,|n_{-k}\rangle\langle n_{-k}|\Phi_i\,|0\rangle]_{k\simeq 0} \,, \tag{20}$$

where we have replaced the summation over f by a summation over only the Goldstone states n *. Since $E \geqslant 0$, the right hand side of eq. (20) is non-zero only for $k^0 < 0$. Essentially this property, as well as the assumed differentiability of the FT as a function of k, enables us to conclude that the FT $(k \simeq 0)$ is different from zero only on a hypersurface in the $k^0 - k$ space which is tangential to the hypersurface $k^0 = 0$ from below. Implicit in such behaviour of the FT $(k \simeq 0)$ is the implication that the energy of the Goldstone boson under consideration is proportional to an even power of its momentum in the limit of long wavelengths. We employ the term "Goldstone boson of type II" to describe a particle with a dispersion law of this type, and the foregoing discussion demonstrates the existence of at least one such particle whenever $l < m$.

For visual clarity, a pictorial representation of the above remarks is depicted in figs. 1 and 2 for the simple case of a world involving only one spatial dimension. These should be contrasted with fig. 3 which exhibits, again in the one dimensional case, a typical dispersion law of a "Goldstone boson of type I". For such bosons the

* Emphasis should be laid on the fact that, in the region under consideration in eq. (20), phase-space considerations allow one to consider the states $|n\rangle$ as effectively single particle states, whence it follows that the summation over n is devoid of any implicit integrations.

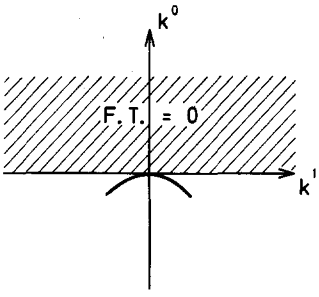

Fig. 1. The support of the function $\mathrm{FT}(k^1 \simeq 0)$ in the special instance of only one spatial dimension. The main features of $\mathrm{FT}(k^1 \simeq 0)$ are its vanishing nature for $k^0 > 0$ and its assumed differentiability with respect to k^1. From these then follow the differentiable character of the support curve as a function of k^1, and the fact that k^0 can only be proportional to an even power of k^1 in the immediate vicinity of $k^1 = 0$.

energy, in the long wavelength limit, is proportional to some odd power of their momentum, and throughout our discussion we typically embody this characteristic situation in the dispersion law $E_p \sim |p|$.

(iii) It only remains to show that there are $(m-p)$ Goldstone bosons of type II. This follows almost immediately by asserting the impossibility of the existence of a set of non-trivial constants β_α such that

$$\sum_{\alpha=1}^{m-p} \beta_\alpha \sum_{a=1}^{m} C_a^{\alpha *} v_a = 0 . \tag{22}$$

For the truth of eq. (22) would imply

$$\sum_{a=1}^{m} \Big[\sum_{\alpha=1}^{m-p} (\beta_\alpha C_a^{\alpha *} + \beta_\alpha^* C_a^\alpha) \Big] \mathrm{Im}\, v_a = 0 , \tag{23}$$

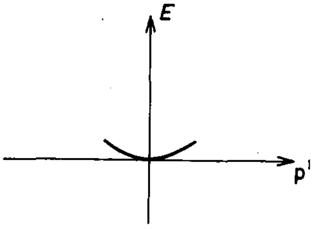

Fig. 2. The dispersion law of a type II Goldstone boson, in the limit of long wavelengths, assuming only one spatial dimension. The distinctive feature of such bosons is that their energy is proportional to an even power of their momentum. In the subsequent enumeration of Goldstone states every particle of this type is to be counted twice.

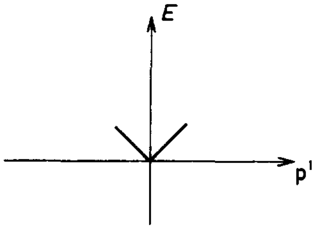

Fig. 3. For purposes of comparison with fig. 2, this diagram exhibits (for one dimensional space) the long wavelength limit of the dispersion law of a Goldstone boson of type I, here typified by the form $E \sim |p^1|$. In general, the members of this category have energies which are proportional to some odd power of their momentum. However, in contradistinction to those of type II, such particles are to be counted only singly. Evidently, every Goldstone boson occurring within the context of a relativistically invariant theory must belong to this class.

in contradiction to eq. (10). Hence the $(m-p)$ vectors $\boldsymbol{\rho}^\alpha$,

$$(\boldsymbol{\rho}^\alpha)_i = \sum_{a=1}^{m} C_a^{\alpha *} (\mathbf{v}_a)_i = \sum_{n=1}^{l} \langle 0|\Phi_i|n_0\rangle \langle n_0| \sum_{a=1}^{m} C_a^{\alpha *} j_a^0 |0\rangle \,, \tag{24}$$

are linearly independent. It is now clear that at least $m-p$ distinct Goldstone bosons must couple to the $m-p$ combinations

$$\sum_{a=1}^{m} C_a^\alpha j_a^0, \qquad (\alpha = 1, ..., m-p)\,.$$

Otherwise it would be possible to express all the $\boldsymbol{\rho}^\alpha$'s as linear combinations of certain vectors from the set $\{A_n, n = 1, ... l\}$, which are less than $m-p$ in number. This would contradict the linear independence of the vectors $\boldsymbol{\rho}^\alpha$.

If now every Goldstone boson of type II is counted as two and of type I as one, the total number of them, adopting this mode of enumeration, is

$$\kappa \geqslant [l - (m-p)] + 2(m-p) \geqslant m\,. \tag{25}$$

This completes the proof of the theorem. An immediate consequence is the result that, since every Goldstone boson of a relativistically invariant theory must necessarily be of type I, such theories must exhibit at least as many Goldstone bosons as the number of symmetry generators that are spontaneously broken *.

5. An example

In this concluding section we mention somewhat cursorily an amusing example which illustrates some of the foregoing ideas. Consider a Heisenberg ferromagnet.

* This result is implicit in the work of Kibble [7].

As is well-known [8], the ground state of such a system consists of all the spins aligned along the same direction which we may conveniently label as the z direction. This ground state is, of course, infinitely degenerate with respect to rotations even though the Hamiltonian of the system itself is rotationally invariant, and, indeed, it is simple to verify that two of the generators of the rotation group are spontaneously broken. Excitations of the ground state then cause the occurrence of spin waves [9] or, when quantized, magnons which reduce the value of the z component of the total spin of the system. But there is only one ferromagnetic magnon; this is the relevant Goldstone boson in this case. The theorem then decrees that this must be a Goldstone boson of type II. Indeed, it is found that the ferromagnetic magnon has a dispersion law of the form $E_{\mathrm{p}} \sim p^2$, in the limit of long wavelengths.

Now think of an antiferromagnet, a system of two coupled sublattices of which the ground state consists of all the spins of one sublattice aligned along some direction and all the spins of the other sublattice aligned along the opposite direction. Unlike the case of a ferromagnet, however, such a system has two distinct magnons. Since, again, only two of the generators of the rotation group are spontaneously broken, the theorem this time allows the possibility of having a linear dispersion law for each of the antiferromagnetic magnons: $E_p \sim |p|$. This, in fact, is the case in reality. The change in the dispersion law from a ferromagnetic to an antiferromagnetic magnon is not without some important consequences (for instance, the temperature dependence of the spontaneous magnetization at low temperatures in an antiferromagnet is different from that in a ferromagnet), but that is not the subject of our present discussion.

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References

[1] J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127 (1962) 965.
[2] S.A. Bludman and A. Klein, Phys. Rev. 131 (1963) 2364.
[3] T.W.B. Kibble, Proc. 1967 Int. Conf. on particles and fields, ed. C.R. Hagen, G. Guralnik and V.S. Mathur (Wiley, 1967), see the remarks made in the discussion period after the lecture especially p. 303.
[4] A. Klein and B.W. Lee, Phys. Rev. Letters, 12 (1964) 266.
[5] R.V. Lange, Phys. Rev. Letters 14 (1965) 3.
[6] W. Gilbert, Phys. Rev. Letters 12 (1964) 713.
[7] T.W.B. Kibble, Phys. Rev. 155 (1967) 1554.
[8] C. Kittel, Quantum theory of solids (Wiley, 1963) p. 49.
[9] J. Keffer, H. Kaplan and Y. Yafet, Am. J. Phys. 21 (1953) 250.