第 5 章

Higher condensation theory

Review of [**?**].

▎ 5.1 Definition of condensation

In this section, we refer to an **(n, n)-category** as an (∞, n)-category which is "truncated" at morphism degree n. Intuitively, if $\mathcal{C}$ is an (n, n)-category, then, for arbitrary $n-1$-morphisms f, $g \in \mathcal{C}_{n-1}$, the space of n-morphisms between f and g has discrete topology. Therefore, we can compose n-morphisms strictly (not up to homotopy), associatively, and unitally.

Note that a **$(0, 0)$-category** $\mathcal{C}$ is merely a set[*1].

定義 5.1: 0-condensation

Let $\mathcal{C}$ be a $(0, 0)$-category. A **0-condensation** on $\mathcal{C}$ is an equality between elements of a set $\mathcal{C}_0$.

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n-condensation is defined by induction.

定義 5.2: n-condensation

Fix $n \geq 0$. Let $\mathcal{C}$ be an (n, n)-category, and let x, $y \in \mathcal{C}_0$ be objects of $\mathcal{C}$. We define n-condensation by induction on n.

n-condensation of x onto y in $\mathcal{C}$ consists of three data:

- A 1-morphism $f \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $g \in \mathrm{Map}_{\mathcal{C}}(y, x)_0$
- An $(n-1)$-condensation of $f \circ g$ onto Id_y in $\mathrm{Map}_{\mathcal{C}}(y, y)$[a]

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[a] Roughly speaking, a mapping space $\mathrm{Map}_{\mathcal{C}}(y, y)$ itself is an $(n-1, n-1)$-category. Strickly speaking, we need enriched ∞-category theory.

[*1] That is, $\mathcal{C}$ consists of a set $\mathcal{C}_0$ of objects (0-morphisms) only.

【例 5.1.1】 1-condensation

Let $\mathcal{C}$ be a $(1, 1)$-category (i.e. an ordinary category). **1-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $f \in \mathrm{Hom}_{\mathcal{C}}\,(x,\, y)$
- A 1-morphism $g \in \mathrm{Hom}_{\mathcal{C}}\,(y,\, x)$
- A 0-condensation of $f \circ g$ onto Id_y in $\mathrm{Hom}_{\mathcal{C}}\,(y,\, y)$. i.e. $f \circ g = \mathrm{Id}_y$

【例 5.1.2】 2-condensation

Let $\mathcal{C}$ be a $(2, 2)$-category (i.e. a bicategory). **2-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $f \in \mathrm{Map}_{\mathcal{C}}\,(x,\, y)_0$
- A 1-morphism $g \in \mathrm{Map}_{\mathcal{C}}\,(y,\, x)_0$
- A 1-condensation of $f \circ g$ onto Id_y in $(1, 1)$-category $\mathrm{Map}_{\mathcal{C}}\,(y,\, y)$.

By 【例 5.1.1】 ,

- A 1-morphism $f \in \mathrm{Map}_{\mathcal{C}}\,(x,\, y)_0$
- A 1-morphism $g \in \mathrm{Map}_{\mathcal{C}}\,(y,\, x)_0$
- A 2-morphism (in $\mathcal{C}$) $(f \circ g \xrightarrow{\varphi} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}\,(y,\, y)_1$
- A 2-morphism (in $\mathcal{C}$) $(\mathrm{Id}_y \xrightarrow{\psi} f \circ g) \in \mathrm{Map}_{\mathcal{C}}\,(y,\, y)_1$
- An equality $\varphi \circ \psi = \mathrm{Id}_{\mathrm{Id}_y}$

【例 5.1.3】 3-condensation

Let $\mathcal{C}$ be a $(3, 3)$-category. By 【例 5.1.2】 , **3-condensation** of x onto y in $\mathcal{C}$ consists of these data:

- A 1-morphism $f \in \mathrm{Map}_{\mathcal{C}}\,(x,\, y)_0$
- A 1-morphism $g \in \mathrm{Map}_{\mathcal{C}}\,(y,\, x)_0$
- A 2-morphism $(f \circ g \xrightarrow{\varphi} \mathrm{Id}_y) \in \mathrm{Map}_{\mathcal{C}}\,(y,\, y)_1$
- A 2-morphism $(\mathrm{Id}_y \xrightarrow{\psi} f \circ g) \in \mathrm{Map}_{\mathcal{C}}\,(y,\, y)_1$
- A 3-morphism

$$
\left(
\mathrm{Id}_y
\begin{array}{c}
\overset{\varphi \circ \psi}{\overbrace{\Big\Downarrow \alpha}} \\
\underset{\mathrm{Id}_{\mathrm{Id}_y}}{\underbrace{}}
\end{array}
\mathrm{Id}_y
\right) \in \mathrm{Map}_{\mathcal{C}}\,(y,\, y)_2
$$

- A 3-morphism

$$\left(\mathrm{Id}_y \; \overset{\mathrm{Id}_{\mathrm{Id}_y}}{\underset{\varphi \circ \psi}{\Downarrow \beta}} \; \mathrm{Id}_y \right) \in \mathrm{Map}_{\mathcal{C}}\,(y,\,y)_2$$

- An equiality

$$\mathrm{Id}_y \; \overset{\mathrm{Id}_{\mathrm{Id}_y}}{\underset{\mathrm{Id}_{\mathrm{Id}_y}}{\overset{\Downarrow \beta}{\underset{\Downarrow \alpha}{\varphi \circ \psi}}}} \; \mathrm{Id}_y \;\; = \;\; \mathrm{Id}_y \; \overset{\mathrm{Id}_{\mathrm{Id}_y}}{\underset{\mathrm{Id}_{\mathrm{Id}_y}}{\Big\Vert \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_y}}}} \; \mathrm{Id}_y$$

5.1.1 Walking condensation

Let $\mathcal{C}$ be an $(n,\,n)$-category. Roughly speaking, a **walking n-condensation** is a (strict?) n-category $\clubsuit_n$ which "generates" n-condensation in $\mathcal{C}$. i.e. the functor category $\mathcal{F}\mathbf{un}(\spadesuit_n, \mathcal{C})$ is equivalent to the category of n-condensations in $\mathcal{C}$.

5.1.2 Condensation monad

Let $\clubsuit_n \subset \spadesuit_n$ be a full-subcategory which consists of walking n-condensations on only one points.

> **定義 5.3: condensation monad**
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5.2 Physical interpretation