

第 5 章

Higher condensation theory

Review of [?].

5.1 Definition of condensation

In this section, we refer to an (n, n) -category as an (∞, n) -category which is “truncated” at morphism degree n . Intuitively, if \mathcal{C} is an (n, n) -category, then, for arbitrary $n-1$ -morphisms $f, g \in \mathcal{C}_{n-1}$, the space of n -morphisms between f and g has discrete topology. Therefore, we can compose n -morphisms strictly (not up to homotopy), associatively, and unitally.

Note that a $(0, 0)$ -category \mathcal{C} is merely a set^{*1}.

定義 5.1: 0-condensation

Let \mathcal{C} be a $(0, 0)$ -category. A **0-condensation** on \mathcal{C} is an equality between elements of a set \mathcal{C}_0 .

n -condensation is defined by induction.

定義 5.2: n -condensation

Fix $n \geq 0$. Let \mathcal{C} be an (n, n) -category, and let $x, y \in \mathcal{C}_0$ be objects of \mathcal{C} . We define n -condensation by induction on n .

n -condensation of x onto y in \mathcal{C} consists of three data:

- A 1-morphism $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- An $(n-1)$ -condensation of $r \circ i$ onto Id_y in $\text{Map}_{\mathcal{C}}(y, y)$ ^a

^a Roughly speaking, a mapping space $\text{Map}_{\mathcal{C}}(y, y)$ itself is an $(n-1, n-1)$ -category. Strickly speaking, we need enriched ∞ -category theory.

^{*1} That is, \mathcal{C} consists of a set \mathcal{C}_0 of objects (0-morphisms) only.

【例 5.1.1】 1-condensation

Let \mathcal{C} be a $(1, 1)$ -category (i.e. an ordinary category). **1-condensation** of x onto y in \mathcal{C} consists of these data:

- A 1-morphism $r \in \text{Hom}_{\mathcal{C}}(x, y)$
- A 1-morphism $i \in \text{Hom}_{\mathcal{C}}(y, x)$
- A 0-condensation of $r \circ i$ onto Id_y in $\text{Hom}_{\mathcal{C}}(y, y)$. i.e. $r \circ i = \text{Id}_y$.

In the context of $(1, 1)$ -categories, such y is called a **retract of x** .

【例 5.1.2】 2-condensation

Let \mathcal{C} be a $(2, 2)$ -category (i.e. a bicategory). **2-condensation** of x onto y in \mathcal{C} consists of these data:

- A 1-morphism $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 1-condensation of $r \circ i$ onto Id_y in $(1, 1)$ -category $\text{Map}_{\mathcal{C}}(y, y)$.

By **【例 5.1.1】** ,

- A 1-morphism $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism (in \mathcal{C}) $(r \circ i \xrightarrow{\rho} \text{Id}_y) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism (in \mathcal{C}) $(\text{Id}_y \xrightarrow{\iota} r \circ i) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- An equality $\rho \circ \iota = \text{Id}_{\text{Id}_y}$

【例 5.1.3】 3-condensation

Let \mathcal{C} be a $(3, 3)$ -category. By **【例 5.1.2】** , **3-condensation** of x onto y in \mathcal{C} consists of these data:

- A 1-morphism $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism $(r \circ i \xrightarrow{\rho} \text{Id}_y) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism $(\text{Id}_y \xrightarrow{\iota} r \circ i) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 3-morphism

$$\left(\begin{array}{ccc} & \xrightarrow{\rho \circ \iota} & \\ \text{Id}_y & \Downarrow \alpha & \text{Id}_y \\ & \xleftarrow{\text{Id}_{\text{Id}_y}} & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

- A 3-morphism

$$\left(\begin{array}{ccc} & \text{Id}_{\text{Id}_y} & \\ \text{Id}_y & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \beta \\ \xrightarrow{\quad} \end{array} & \text{Id}_y \\ & \text{Id}_{\text{Id}_y} & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

- An equality

$$\begin{array}{ccc} & \text{Id}_{\text{Id}_y} & \\ \text{Id}_y & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \beta \\ \xrightarrow{\quad} \end{array} & \text{Id}_y \\ & \text{Id}_{\text{Id}_y} & \end{array} \begin{array}{c} \xrightarrow{\rho \circ \iota} \\ \Downarrow \alpha \\ \xrightarrow{\quad} \end{array} \text{Id}_y = \begin{array}{ccc} & \text{Id}_{\text{Id}_y} & \\ \text{Id}_y & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \text{Id}_{\text{Id}_{\text{Id}_y}} \\ \xrightarrow{\quad} \end{array} & \text{Id}_y \\ & \text{Id}_{\text{Id}_y} & \end{array}$$

【例 5.1.4】 4-condensation

Let \mathcal{C} be a $(4, 4)$ -category. By 【例 5.1.2】 , **3-condensation** of x onto y in \mathcal{C} consists of these data:

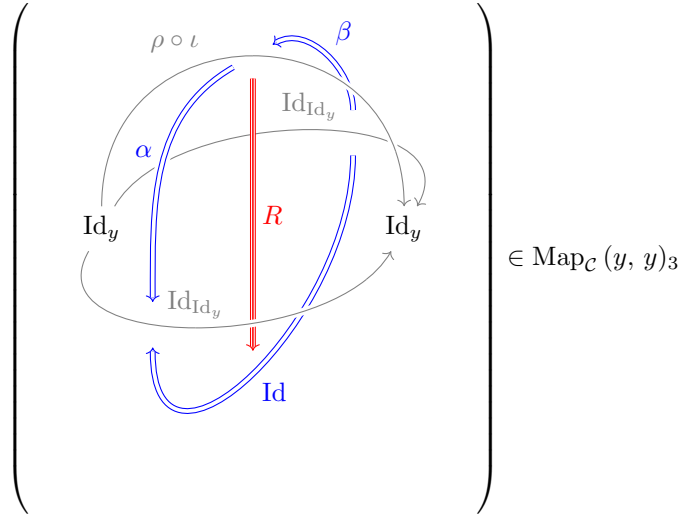
- A 1-morphism $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism $(r \circ i \xrightarrow{\rho} \text{Id}_y) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism $(\text{Id}_y \xrightarrow{\iota} r \circ i) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 3-morphism

$$\left(\begin{array}{ccc} & \rho \circ \iota & \\ \text{Id}_y & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \alpha \\ \xrightarrow{\quad} \end{array} & \text{Id}_y \\ & \text{Id}_{\text{Id}_y} & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

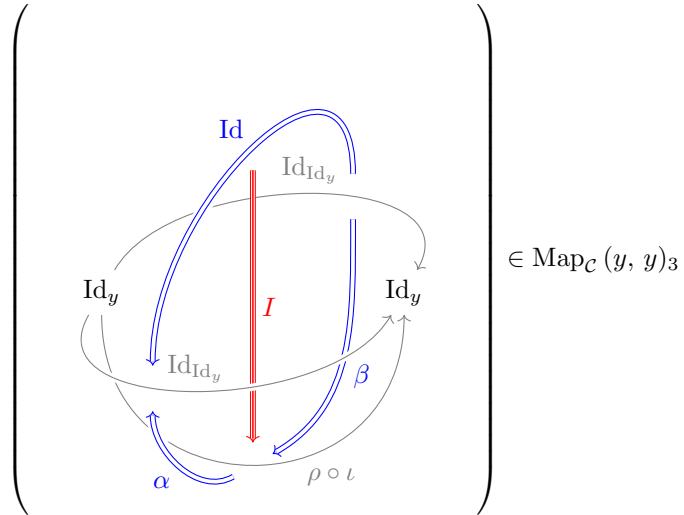
- A 3-morphism

$$\left(\begin{array}{ccc} & \text{Id}_{\text{Id}_y} & \\ \text{Id}_y & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \beta \\ \xrightarrow{\quad} \end{array} & \text{Id}_y \\ & \rho \circ \iota & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

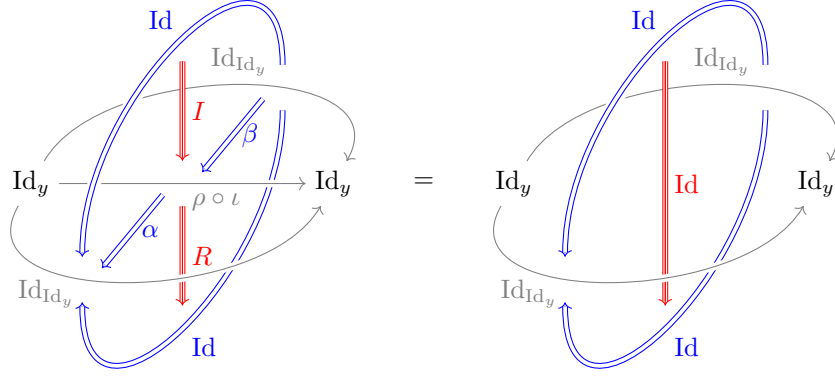
- A 4-morphism



- A 4-morphism



- An equality



5.1.1 Walking condensation

Let \mathcal{C} be an (n, n) -category. Roughly speaking, a **walking n -condensation** is a (strict?) n -category \clubsuit_n which “generates” n -condensation in \mathcal{C} . i.e. the functor category $\mathcal{F}\mathbf{un}(\spadesuit_n, \mathcal{C})$ is equivalent to the category of n -condensations in \mathcal{C} .

5.1.2 Condensation monad

Let $\clubsuit_n \subset \spadesuit_n$ be a full-subcategory which consists of walking n -condensations on only one points.

定義 5.3: condensation monad

5.2 Physical interpretation