Topological Quantum: Exercises

S. Simon

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Instructions: Problems labeled with a † are "required" and will be marked for official course credit. This subset of problems are meant to be the easiest of the problems Note: In some problems only certain oarts of the problem are marked as required. You are *highly* encouraged to try all of the problems and hand in as many as you can. This is the only way you will actually learn the subject! All problems will be discussed with the TA. However, only the † problems count for course credit. The completion requirement is that you must obtain at least C-grade on 50% of the required problems.

Problem 1 *Kauffman (Jones) Invariant* †

Using the Kauffman rules, calculate the Kauffman Invariant of the Right and Left handed Trefoil knots. Conclude these two knots are topologically inequivalent. While this statement appears obvious on sight, it was not proved mathematically until the 1900s. (It is trivial using this technique!).

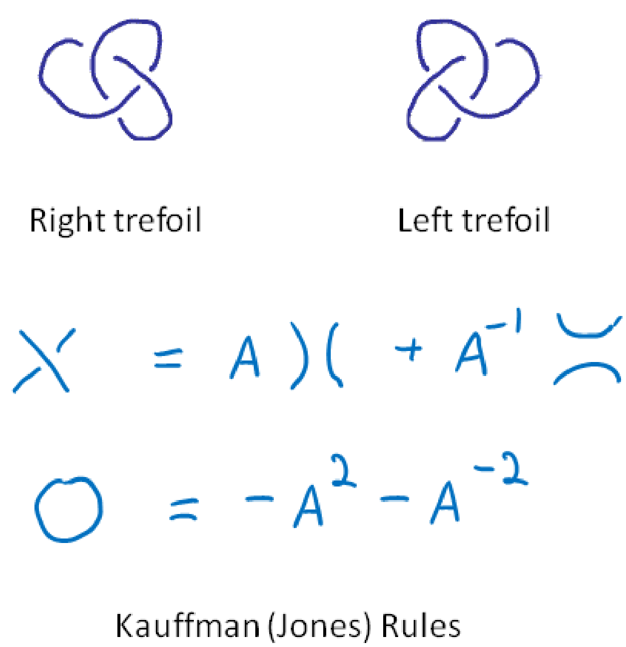

Figure 1: Top: Right and Left Handed Trefoil Knot. Bottom: Kauffman Rules.

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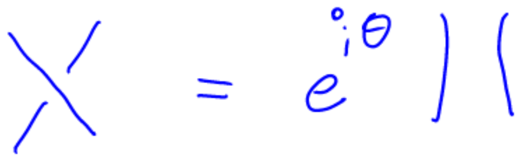
Problem 2 *Abelian Kauffman Anyons*

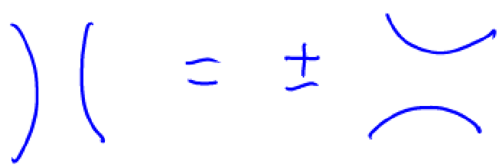
Anyons described by the Kauffman invariant with certain special values of the constant A are abelian anyons – meaning that an exchange introduces only a simple phase.



(a) For $A = \pm e^{i\pi/3}$ (and the complex conjugates of these values), show that the anyons are bosons or fermions respectively (i.e., $e^{i\theta} = \pm 1$).

(b) For $A = \pm e^{i\pi/6}$ (and the complex conjugates of these values) show the anyons are semions (i.e., $e^{i\theta} = \pm i$). In fact these are precisely the anyons that arise for the $\nu = 1/2$ fractional quantum Hall effect of bosons (We will discuss this later in the term. While it is still controversial whether this particular phase of quantum Hall matter has been produced experimentally as of now, it is almost certain that it will be produced experimentally and convinceingly within a few years.)

HINT: Aim to show first in the two respective cases that



If you can't figure it out, try evaluating the Kauffman invariant for a few knots with these values of A and see how the result arises.

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Problem 3 *Quantum Hall Interferometer*

This problem has a long introduction with a lot of words. At least parts (a-c) should not be hard.

Anyons described by the Kauffman invariant with $A = i^{3/4}$ are realized in the $\nu = 5/2$ fractional quantum Hall state. (We will discuss quantum Hall effect later in the term). We will here discuss a real experiment with anyons[1]. A fact we need to know is that currents of anyons run along the edge of the quantum hall sample (in a counter-clockwise direction in this example). The idea of the experiment is to build an interferometer where the anyons can be back-reflected at either of two points labeled (1) and (2). See Fig. 2

The outgoing wave is a superposition of the two paths S for short, and L for long. So we have

$$|\text{out}\rangle = t_1 |S > + t_2 e^{i\phi} |L\rangle$$

where t_1 is the tunneling amplitude at point 1 (i.e, the amplitude that the anyon will jump across the narrow neck at point 1) and where t_2 is the tunneling amplitude at point 2. Here $e^{i\phi}$ is an

[1]The experiment was proposed by Fradkin et in the 1990s, then in more detail Stern and Halperin; Shtengel, Bonderson, and Kitaev in the 2000s. The experiment has been attempted by several groups [Willett et al, Marcus et al, Kang et al] and unfortunately results remain murky. But discussion of the issues in the experiment requires more detailed knowledge of the experimental setup. No reasonable scientist doubts that the anyons exist in the experiment, but various extraneous effects have prevented these anyons from being measured cleanly!

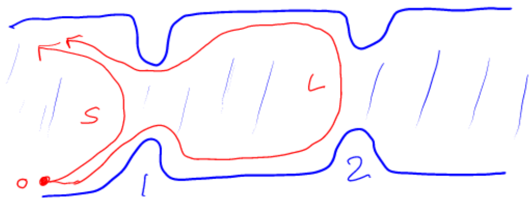

Figure 2: An interferometer where particles can be backrefelceted at points (1) and (2)

additonal phase associated with the additional distance the wave has to travel in order to take the long path. This phase can be varied in experiment.

In the lower left of the figure there is drawn a particle (closed dot) and its antiparticle (open dot). This is supposed to indicate that the particle is created with its antiparticle (pair creation) and the particle is sent through the interferometer while the antiparticle stays still.

The measured backscattering probability is

$$P = \langle \text{out}|\text{out}\rangle = |t_1|^2 + |t_2|^2 + 2\text{Re}[t_1^* t_2 e^{i\phi}\langle S|L\rangle]$$

It is the inner product $\langle S|L\rangle$ which this experiment seeks to measure by looking at amplitude of oscillation in P as ϕ is varied.

First, we consider the case where there are no particles within the interferometer cavity of the interferometer (i.e., in the bulk [not the edge] to the left of 2 and to the right of 1). The space-time diagram for the short path looks like that shown in Fig. 3 here the particle-hole creation event occurs

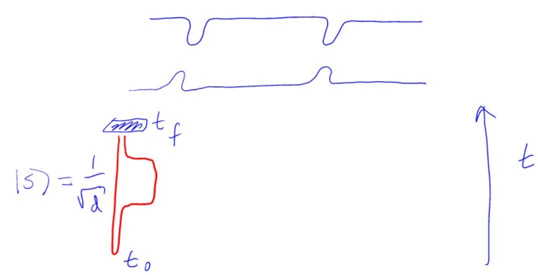

Figure 3: short path $|S\rangle$

at some time t_0, then the particle makes takes the short path (shown as the excursion of the red line) and we have the final ket at time t_f. The prefactor $1/\sqrt{d}$ is included in the definition of $|S\rangle$ so that $\langle S|S\rangle = 1$. As shown in fig. 4

Similarly the space-time diagram for the long path looks like Fig. 5

It should be easy to see that in this case $\langle S|L\rangle = 1$ as well.

(a) Now, consider changing the experiment. Long before the experiment starts, imagine creating a particle-antiparticle pair on far right outside of the interferometer cavity (to the right of point 2). Leave the antiparticle alone, but move the particle into the middle of the interferometer cavity (to the left of 2, to the right of 1). Then try doing the interference experiment. Show that the amplitude of interference is given by the diagram shown in Fig. 6 where the loop on the right is the particle being put in the center of the experiment, and the loop on the left is the probe particle that gets sent through the interferometer.

(b)† For the experimentally relevant case of $A = i^{3/4}$ show that the diagram in figure 6 gives zero and hence the interference vanishes.

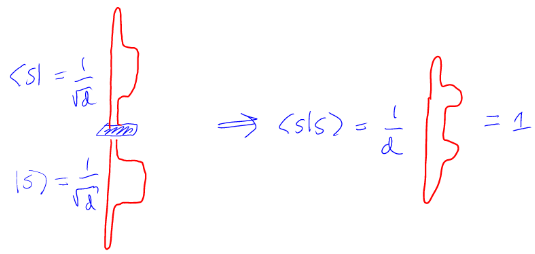

Figure 4: Normalization of ket $|S\rangle$

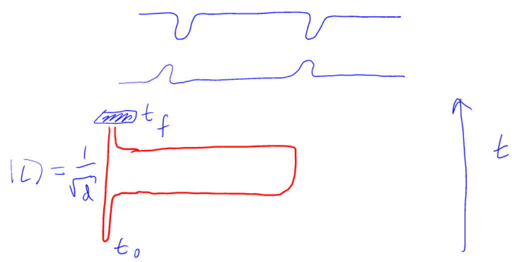

Figure 5: long path $|L\rangle$

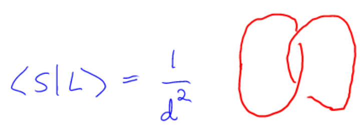

Figure 6: interference term

(c) We now consider putting two particles inside the interferometer cavity. One way to do this is to create a particle-antiparticle pair entirely within the interferometer cavity and then perform the interferometry experiment. Show that this gives the same result as when there are no particles within the interferomter cavity (this is essentially a result of locality).

(d) [Harder]. Now consider creating a particle-antiparticle pair within the interferometer cavity, and another pair outside of the interferometer cavity. Then braid the two pairs with each other as shown in the figure 7 ending up in a state again with two particles inside and two outside as shown in Figure 7.

Then once this setup is prepared, do the interference experiment. (Why do we choose this particular setup procedure? Look at answer to part b!)

(d.i) What knot diagram has to be evaluated?

(d.ii) [Hard] Evaluate this knot diagram (again for $A = i^{3/4}$) and show that $\langle S|L\rangle = -1$ showing that phase of interference is now opposite what it was in part (c).

HINTS to d.ii:

One useful lemma is that the diagram shown in figure 8 is always zero for this value of A. Prove this lemma first. Another useful lemma is that a self-twist in a string give a factor of $-A^{\pm 3}$ depending on the direction of the twist (we showed this in lecture).

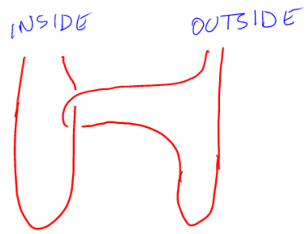

Figure 7: Preparing two quasiparticles inside the interferometer in a nontrivial way

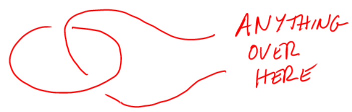

Figure 8: A useful lemma for the case of $A = i^{3/4}$ is that diagrams of this form have zero value.

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Problem 4 *About the Braid Group*

To define the braid group B_M for M particles, line up the M strands from left to right. The generator σ_m for $m = 1 \ldots, M-1$ is a counter-clockwise exchange of particles m and $m+1$ as shown in Fig. 9 The braid generators may be composed in products and inverted as shown in Fig. 10. Each braid

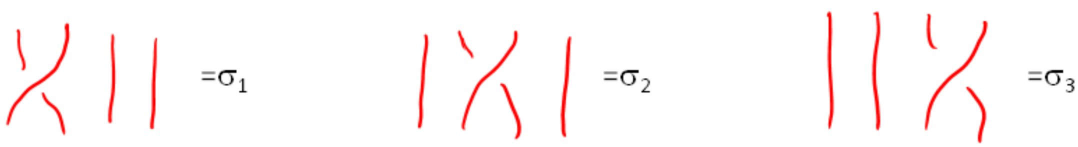

Figure 9: The three generators of the braid group on four strands, B_4

(arbitrary product of the braid generators and their inverses) can be thought of as representing a class of topologically equivalent paths of M particles in a plane moving through time.

(a) Convince yourself geometrically that the defining relations of the braid group are:

$$\sigma_i\,\sigma_{i+1}\,\sigma_i \;=\; \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} \qquad\qquad 1 \le i \le M-2 \tag{1}$$

$$\sigma_i\,\sigma_j \;=\; \sigma_j\,\sigma_i \quad \text{for} \quad |i-j| > 1, \qquad 1 \le i,j \le M-1 \tag{2}$$

(b) Instead of thinking about particles on a plane, let us think about particles on the surface of a sphere. In this case, the braid group of M strands on the sphere is written as $B_M(S^2)$. To think about braids on a sphere, it is useful to think of time as being the radial direction of the sphere, so that braids are drawn as in Fig. 11. The braid generators on the sphere still obey Eqns. 1 and 2, but they also obey one additional identity

$$\sigma_1\sigma_2\ldots\sigma_{M-2}\sigma_{M-1}\sigma_{M-1}\sigma_{M-2}\ldots\sigma_2\sigma_1 = I \tag{3}$$

where I is the identity (or trivial) braid. What does this additional identity mean geometrically?

[In fact, for understanding the properties of anyons on a sphere, Eq. 3 is not quite enough. We will try to figure out below why this is so by using Ising Anyons as an example.]

Figure 10: Multiplying generators of the braid group, B_4

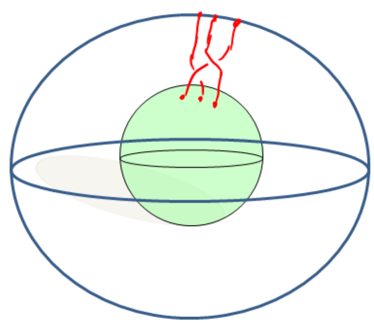

Figure 11: An element of the braid group $B_3(S^2)$. The braid shown here is $\sigma_1\sigma_2^{-1}$

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Problem 5 *Ising Anyons and Majorana Fermions*

The most commonly discussed type of non-Abelian anyon is the Ising anyon (we will discuss this in more depth later). Ising anyons occurs in the Moore-Read quantum Hall state ($\nu = 5/2$), as well as in any chiral p-wave superconductor and in recently experimentally relevant so called "Majorana" systems.

The non-Abelian statistics of these anyons may be described in terms of Majorana fermions by attaching a majorana operator to each anyon. The Hamiltonian for these majoranas is zero – they are completely noninteracting.

In case you haven't seen them before, Majorana Fermions γ_j satisfy the anticommutation relation

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij} \tag{4}$$

as well as being self conjugate $\gamma_i^\dagger = \gamma_i$.

(a) Show that the ground state degeneracy of a system with $2N$ Majoranas is 2^N if the Hamiltonian is zero. Thus conclude that each *pair* of Ising anyons is a two-state system. Hint: Construct a regular (Dirac) fermion operator from two Majorana fermion operators. For example,

$$c^\dagger = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

will then satisfy the usual fermion anti-commutation $\{c, c^\dagger\} = cc^\dagger + c^\dagger c = 1$. (If you haven't run into fermion creation operators yet, you might want to read up on this first!) There is more discussion of this transformation in a later problem on "Ising F-matrix".

(b)$\dagger$ When anyon i is exchanged clockwise with anyon j, the unitary transformation that occurs on the ground state is

$$U_{ij} = \frac{e^{i\alpha}}{\sqrt{2}}\left[1 + \gamma_i\gamma_j\right] \qquad i < j. \tag{5}$$

for some real value of α. Show that these unitary operators form a representation of the braid group. (Refer back to the previous problem, "About the Braid Group"). In other words we must show that replacing σ_i with $U_{i,i+1}$ in Eqns. 1 and 2 yields equalities. This representation is 2^N dimensional since the ground state degeneracy is 2^N.

(c) Consider the operator

$$\gamma^{\text{FIVE}} = (i)^N \gamma_1\gamma_2 \ldots \gamma_{2N} \tag{6}$$

(the notation FIVE is in analogy with the γ^5 of the dirac gamma matrices). Show that the eigenvalues of γ^{FIVE} are ± 1. Further show that this eigenvalue remains unchanged under any braid operation. Conclude that we actually have two 2^{N-1} dimensional representations of the braid group. We will assume that any particular system of Ising anyons is in one of these two representations.

(d) Thus, 4 Ising anyons on a sphere comprise a single 2-state system, or a qubit. Show that by only braiding these four Ising anyons one cannot obtain all possible unitary operation on this qubit. Indeed, braiding Ising anyons is not sufficient to build a quantum computer. [Part d not required to solve part e,f]

(e) [bit harder] Now consider $2N$ Ising anyons on a sphere (See above problem "About the braid group" for information about the braid group on a sphere). Show that in order for either one of the 2^{N-1} dimensional representations of the braid group to satisfy the sphere relation, Eqn. 3, one must choose the right abelian phase α in Eq. 5. Determine this phase.

(f) [a bit harder] The value you just determined is not quite right. It should look a bit unnatural as the abelian phase associated with a braid depends on the number of anyons in the system. Go back to Eqn. 3 and insert an additional abelian phase on the right hand side which will make the final result of part (e) independent of the number of anyons in the system. In fact, there should be such an additional factor — to figure out where it comes from, go back and look again at the geometric "proof" of Eqn. 3. Note that the proof involves a self-twist of one of the anyon world lines. The additional phase you added is associated with one particle twisting around itself. The relation between self-rotation of a single particle and exchange of two particles is a generalized spin-statistics theorem.

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Problem 6 *Small Numbers of Anyons on a Sphere*

On the plane, the braid group of two particles is an infinite group (The free group with one generator). However, this is not true on a sphere

First review the problem "About the Braid Group" about braiding on a sphere.

(a) Now consider the case of two particles on a sphere. Determine the full structure of the braid group. Show it is a well known finite discrete group. What group is it?

(b) [Harder] Now consider three particles on a sphere. Determine the full structure of the braid group. Show that it is a finite discrete group. [Even Harder] What group is it? It is "well known" only to people who know a lot of group theory. But you can google to find information about it on the web with some work. It may be useful to list all the subgroups of the group and the multiplication table of the group elements.

(c) Suppose we have two (or three) anyons on a sphere. Suppose the ground state is two-fold degenerate. If the braid group is discrete, conclude that no possible type of anyon statistics will allow us to do arbitrary $SU(2)$ rotations on this degenerate ground state by braiding.

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Problem 7 *Surgery*

(a) † Beginning with the three-sphere S^3, consider the so-called "unknot" (a simple unknotted circle S^1 with no twists) embedded in this S^3. Thicken the circle into a solid torus ($S^1 \times D^2$) which has boundary $S^1 \times S^1$. Now perform surgery on this torus by excising the solid torus from the manifold S^3 and replacing it with another solid torus that has the longitude and meridian switched. I.e., replace $S^1 \times D^2$ with $D^2 \times S^1$. Note that both of the two soli tori have the same boundary $S^1 \times S^1$ so that the new torus can be smoothly sewed back in where the old one was removed. What is the new manifold you obtain? (This should be easy because we did this in lecture!)

(b) † [Not hard if you think about it right!] Consider two linked rings, know as the Hopf link:

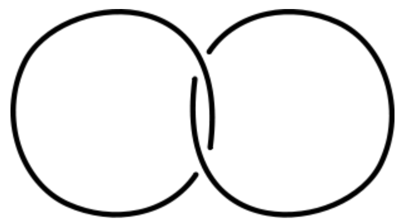

Figure 12: Hopf Link

Consider starting with S^3 and embedding the Hopf link within the S^3 with "blackboard framing" (i.e., don't introduce any additional twists when you embed it). Thicken both strands into solid tori and perform surgery on each of the two links exactly as we did above. Argue that the resulting manifold is S^3.

(c) [Hard] Consider the link shown here known as the Borromean rings. This is an interesting

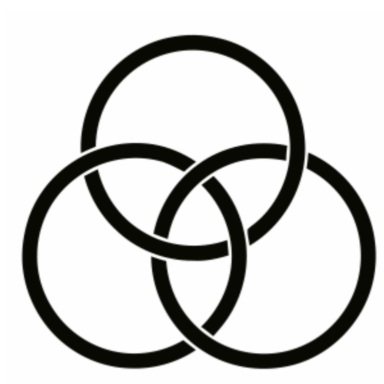

Figure 13: Borromean Rings

link because no two strands are actually linked with each other, but the three links are still tied together. If you remove any one strand the remaining two come apart.

Consider starting with S^3 and embedding the Borromean rings within the S^3 with "blackboard framing. Thicken all three strands into solid tori and perform surgery on each of the three links exactly as we did above. What manifold do you obtain? Hint 1: Think about the group of topologically different loops through the manifold starting and ending at the same point, the so-called "fundamental group" or first homotopy group. For example, the fundamental group of a circle is $\mathbb{Z}$, the group of integers – since closed paths can go any number of times around the circle; the fundamental group of a torus is $\mathbb{Z} \times \mathbb{Z}$ since one can go around either handle any number of times, and it doesn't matter in which order you do this (since it is topologically equivalent to go around the first handle then the second as it is to go around the second then the first — convince yourself of this!). Hint 2: If we say a path around the meridian of one of the three Borromean rings (i.e., threading though the loop) is called a and the path around the meridian of the second ring is called b, then notice that the third ring is topologically equivalent to $aba^{-1}b^{-1}$.

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Problem 8 *Fibonacci Pentagon* †

In a TQFT (indeed, in any tensor category), a change of basis is described by the F-matrix as shown in the figure[2].

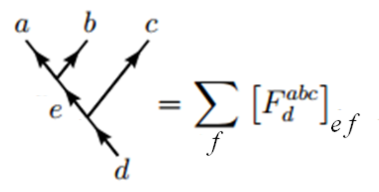

Figure 14: F-matrix

Consistency of F-matrices is enforced by the so-called pentagon equation, which can be described diagramatically by the figure:

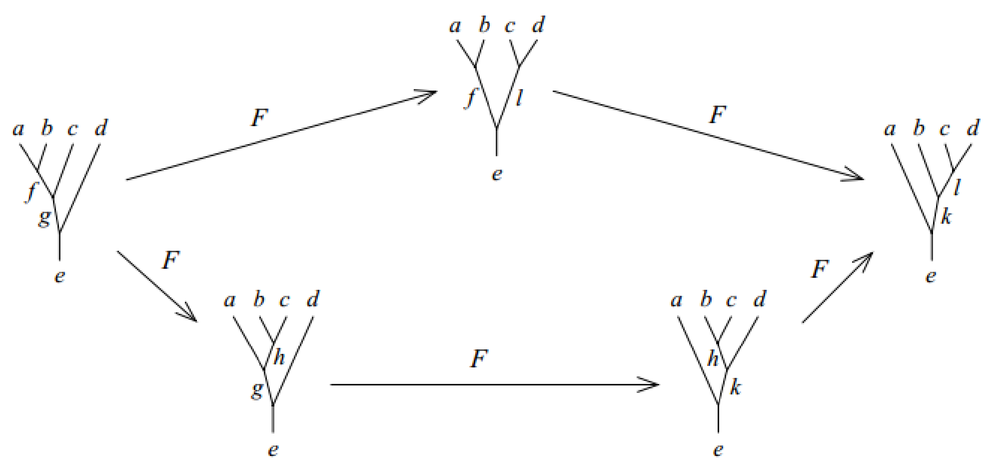

Figure 15: Pentagon Identity (credit for fancy figure, Parsa Bonderson's thesis – thanks Parsa!)

In the Fibonacci anyon model, there are two particle types which are usually called 1 and τ. The fusion rules are (first two lines are obvious)

$$\begin{aligned} 1 \times 1 &= 1 \\ 1 \times \tau &= \tau \\ \tau \times \tau &= 1 + \tau \end{aligned}$$

[2]Strictly speaking when there are fusion multiplicities, $N^a_{bc} > 1$, then one also needs an additional index at each vertex.

With these fusion rules, the F matrix is completely fixed up to a gauge freedom (corresponding to adding a phase to some of the kets). If we choose all elements of the F matrix to be real, then the F matrix is completely determined by the pentagon. Using the pentagon equation determine the F-matrix. (To get you started, note that the variables a,b,c,d,e,f, g,h can only take values 1 and τ. You only need to consider the cases where a,b,c,d are all τ).

Answer: The two by two matrix $F = F_\tau^{\tau\tau\tau}$ is given by

$$\begin{pmatrix} F_{11} & F_{1\tau} \\ F_{\tau 1} & F_{\tau\tau} \end{pmatrix} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$$

with $\phi^{-1} = (\sqrt{5}-1)/2$.

If you are stuck as to how to start, part of the calculation is given in the Nayak, et al, Rev Mod Phys article (see the reference list)

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Problem 9 *Fibonacci Hexagon*

In any TQFT or "braided" (including modular) tensor category (think of all of these as just anyon theories! don't worry about the fine distinctions), a braiding is defined[3] by an R-matrix as shown in the figure 16

Figure 16: R-matrix

Once F matrices are defined for a TQFT, consistency of the R-matrix is enforced by the so-called hexagon equations as shown in the figure diagramatically by the figure:

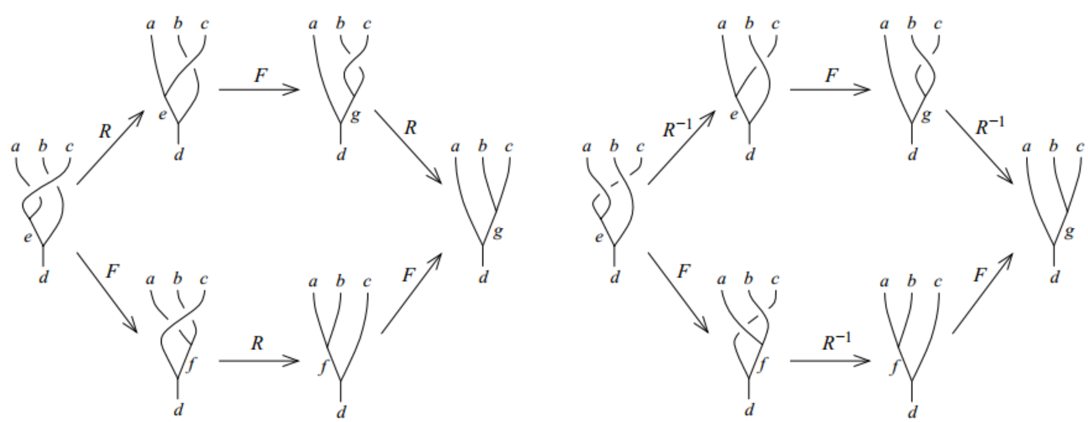

Figure 17: Hexagon Identities (credit for fancy figure, Parsa Bonderson's thesis)

For the Fibonacci anyon theory, once the F matrix is fixed as in the above problem, the R matrices are defined up to complex conjugation (i.e., there is a right and left handed Fibonacci anyon theory — both are consistent). Find these R matrices.

[3]Same comment applies as in the pentagon problem.

Answer: The nontrivial elements of the R matrix are

$$\begin{aligned} R_1^{\tau\tau} &= e^{\pm 6\pi i/5} \\ R_\tau^{\tau\tau} &= e^{\pm 3\pi i/5} \end{aligned}$$

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Problem 10 *Ising F-matrix*

[Hard] As discussed in the earlier problem, "Ising Anyons and Majorana Fermions", one can express Ising anyons in terms of Majorana fermions. As discussed there we can choose any two majoranas and construct a fermion operator

$$c_{12}^\dagger = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

then the corresponding fermion orbital can be either filled or empty. We might write this as $|0_{12}\rangle = c_{12}|1_{12}\rangle$ and $|1_{12}\rangle = c_{12}^\dagger|0_{12}\rangle$. The subscript 12 here meaning that we have made the orbital out of majoranas number 1 and 2. Note however, that we have to be careful that $|0_{12}\rangle = e^{i\phi}|1_{21}\rangle$ where ϕ is a gauge choice which is arbitrary (think about this if it is not obvious already).

Let us consider a system of 4 majoranas, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Consider the basis of states

$$\begin{aligned} |a\rangle &= |0_{12}0_{34}\rangle \\ |b\rangle &= |0_{12}1_{34}\rangle \\ |c\rangle &= |1_{12}0_{34}\rangle \\ |d\rangle &= |1_{12}1_{34}\rangle \end{aligned}$$

rewrite these states in terms of basis of states

$$\begin{aligned} |a'\rangle &= |0_{41}0_{23}\rangle \\ |b'\rangle &= |0_{41}1_{23}\rangle \\ |c'\rangle &= |1_{41}0_{23}\rangle \\ |d'\rangle &= |1_{41}1_{23}\rangle \end{aligned}$$

Hence determine the F-matrix for Ising anyons. Be cautious about fermionic anticommutations: $c_x^\dagger c_y^\dagger = -c_y^\dagger c_x^\dagger$ so if we define $|1_x 1_y\rangle = c_x^\dagger c_y^\dagger |0_x 0_y\rangle$ with the convention that $|0_x 0_y\rangle = |0_y 0_x\rangle$ then we will have $|1_x 1_y\rangle = -|1_y 1_x\rangle$. Note also that you have to make a gauge choice of some phases (analogous to the mentioned gauge choice above). You can choose F to be always real.

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Problem 11 *Quantum Dimension*

Let N_{ab}^c be the fusion multiplicity matrices of a TQFT

$$a \times b = \sum_c N_{ab}^c\, c$$

meaning that N_{ab}^c is the number of distinct ways that a and b can fuse to c. (In many, or even most, theories of interest all N's are either 0 or 1).

The quantum dimension d_a of a particle a is defined as the largest eigenvalue of the matrix $[N_a]_b^c$ where this is now thought of as a two dimensional matrix with a fixed and b, c the indices.

Show that

$$d_a d_b = \sum_c N_{ab}^c\, d_c$$

Try to prove this without invoking the Verlinde formula (which would be assuming the result!).

Hints: Use the fact that $[N_a, N_b] = 0$. Show that all N matrices can be written as $N_a = S\lambda_a S^{-1}$ where λ_a is diagonal with first diagonal element d_a. (In fact S is the modular S matrix, but we do not need this information in order to do the proof!)

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Problem 12 *Fusion and Ground State Degeneracy*

To determine the ground state degeracy of a 2-manifold in a 2+1 dimensional TQFT one can cut the manifold into pieces and sew back together. One can think of the open "edges" or connecting tube-ends as each having a label given by one of the particle types (i.e., one of the anyons) of the theory. Really we are labeling each edge with a basis element of a possible Hilbert space. The labels on two tubes that have been connected together must match (label a on one tube fits into label $\bar{a}$ on another tube.) To calculate the ground state degeneracy we must keep track of all possible ways that these assembled tubes could have been labeled. For example, when we assemble a torus as in Fig. 18, we must match the quantum number on one open end to the (opposite) quantum number on the opposite end. The ground state degeneracy is then just the number of different possible labels,

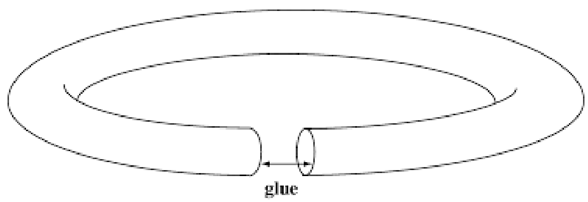

Figure 18: Gluing together a torus

or equivalently the number of different particle types.

For more complicated 2-d manifolds, we can decompose the manifold into so-called pants diagrams that look like Fig. 19. when we sew together pants diagrams, we should include a factor of

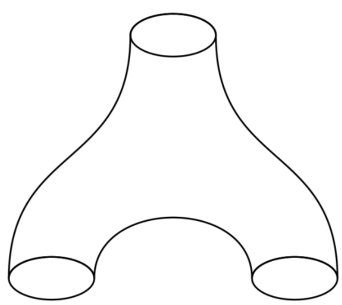

Figure 19: A pants diagram

the fusions multiplicity N_{ab}^{c} for each pants which has its three tube edges labeled with a, b and $\bar{c}$.

(a) Write a general formula for the ground state degeneracy of an M-handled torus in terms of the N matrices.

(b) For the fibonacci anyon model, find the ground state degeneracy of a 4-handled torus.

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Problem 13 *Kirby Color*

From any anyon theory (i.e., TQFT or modular tensor category) we can construct a type of string (a sum of particle types) called an Ω (sometimes ω) string, or sometimes called a Kirby-color string, given by

$$|\Omega\rangle = \mathcal{D}^{-1} \sum_j d_j |j\rangle$$

where d_i here is the quantum dimension of anyon type i and $\mathcal{D}^2 = \sum_i d_i^2$. Here we mean that the Ω string is the sum of all possible string particle labels, each weighted with the respective quantum dimension.

(a) † Evaluating a knot diagram with the evaluation rules of the TQFT gives

$$Z(S^3 \text{with link})/Z(S^3)$$

So the empty diagram is give value 1.

Consider a simple ring (an "unknot" or unknotted loop of string), blackboard framed (meaning no twists) of Kirby color string. Evaluate this diagram.

(b) A knot (or link) of Kirby-colored string is meant to be equivalent to doing surgury on a the knot thickened into a torus. Considering the result of part (a) above as well as part (a) of the above problem on Surgery. Are these results consistent?

(c) Show that the Ω string made into a loop has the so-called "killing property" shown in Fig. 20. In other words, any diagram gives zero unless the particle type going through it is the trivial or vacuum particle (there is a delta function δ_{x0} and $\mathcal{D}$ on the right if you can't read my writing!). Hint:

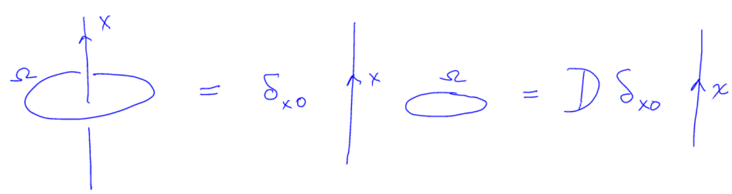

Figure 20: The killing property

Use the fact that the quantum dimension is part of the modular S matrix, and various properties of the S matrix to prove this identity.

(d) Evaluate a Hopf Link of Kirby color string (See Fig. 12). Does this match the result of part (b) of the surgery problem above?

(e) [Harder] Evaluate the Borromean rings of Kirby color string (See Fig. 13). Compare your result to that of part (c) of the Surgery problem above, and also the discussion in the problem on "Ground State Degeneracy" above.

Hints: Consider the F-move shown in figure 21. By closing up the top and bottom, show that $F^{aa0}_{aa0} = 1/d_a$ with d_a the quantum dimension of the particle a. You will need the locality law (also known as the "no-tadpole" law), which says that diagrams of the type shown in Fig. 22 must be zero unless the incoming particle is the vacuum, $p = 0$.

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Problem 14 *Handle Slide*

One can describe a 3-manifold by giving a knot (or link) diagram which should be thickened into a tube and surgered. A handle-slide of a link diagram (which corresponds to sliding a handle of the manifold over another handle, but leaving the manifold topologically unchanged) involves splitting

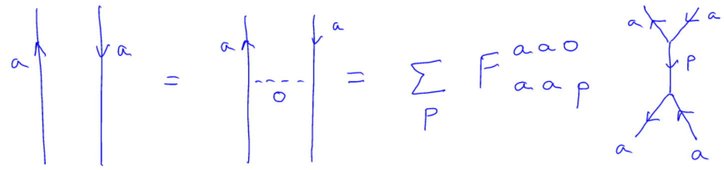

Figure 21: An F-move. The far left diagram can be thought of as having a vacuum particle go between the two strings (middle). Then we can use an F-move to obtain the diagram on the right

Figure 22: A tadpole diagram must be zero unless $p = 0$, by locality

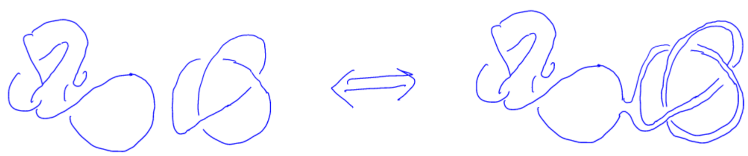

Figure 23: A handle-slide

one strand, having it trace the path of a second strand and then reconnecting. An example is shown in the figure

In TQFT, one uses a string of Kirby color to represent the knot or link to be surgered. In fact, the evaluation of the link in the diagramatic calculus is unchanged by handle-slides. While it takes a bit more diagramatic calculus rules to derive the handle-slide invariance in general, a simple case of the handle-slide is fairly easy to derive. Consider instead a handle-slide over an untwisted loop as shown in the figure 24. Use the killing property. You will have to think about fusion, but you should not need to do any detailed calculations with F matrices.

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Problem 15 *Jones-Wenzl projectors*

The Temperly-Lieb algebra is the algebra of strings in two dimension that are not allowed to cross. As with the Kauffman invariant, a closed loop gets a value d. As in the lecture, we can "cable" together two strings to make two types of "particles". Projectors onto these two particles types are given by the so-called Jones-Wenzl projectors

(a) Show that the projectors satisfy $P^2 = P$, so their eigenvalues are 0 and 1. Further show that the two projectors are orthogonal $P_0P_2 = P_2P_0 = 0$. (should be easy, we did this in lecture)

(b) Show that for $d = \pm 1$ we have $P_2 = 0$ in the evaluation of any diagram. The result means that in these models there is no new particle which can be described as the fusion of two elementary

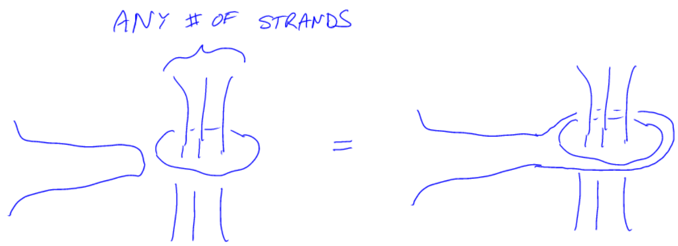

Figure 24: A handle-slide over an untwisted loop. In this figure all strings are meant to be Ω strings or Kirby color (a weighted sum of all particle types).

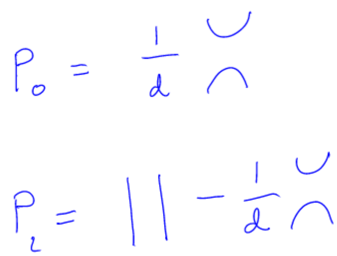

Figure 25: Jones-Wentzl projectors for two strands

anyons. Why should this be obvious? Hint: Look back at the the problem called "Abelian Kauffman Anyons".

(c) Using the Jones-Wenzl projectors with the Kauffman braiding rules, and choosing $A = i^{3/4}$ (which corresponds to the anyons of the $\nu = 5/2$ fractional quantum Hall state), show that the P_2 particle is a fermion, (The P_0 particle is a boson, but this should be trivial!). Hint: Show this by calculating the twist factor of the P_2 particle — it is easier, but equivalent to, calculating exchange of two particles.

(d) The three strand Jones-Wenzl projector must be of the form shown in the figure 26.

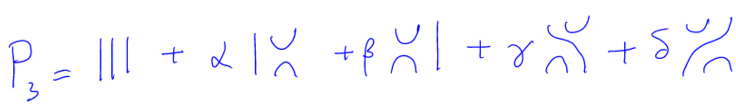

Figure 26: Jones-Wentzl projector for three strands

The coefficients $\alpha, \beta, \gamma, \delta$ are defined by the projector condition $P_3^2 = P_3$ and also by the condition that P_3 is orthogonal to P_0 which is shown in the figure 27.

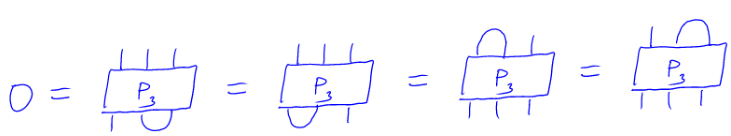

Figure 27: Condition on Jones-Wentzl projector for three strands

Calculate the coefficients $\alpha, \beta, \gamma, \delta$ in P_3.

(e) Choosing $A = i^{3/4}$ again, show that $P_3 = 0$ in the evaluation of any diagram. We can then conclude that in this model there is no new particle that is the fusion of three elementary strands. Hint: think about how the "Abelian Kauffman Anyon" problem was solved.

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Problem 16 *Quasiparticles in Toric Code Loop Gas*

As discussed in lecture, the toric code ground state can be considered to be a loop gas with the rules given in Fig. 28

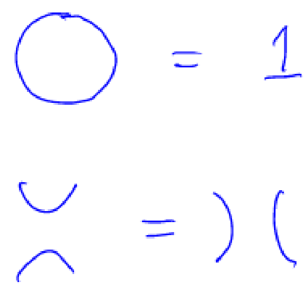

Figure 28: Loop gas rules for the toric code

Certain quasiparticle excitations can be indicated as ends of strings in the loop gas.

(a) † Show that the linear combinations of string ends shown in the figure 29 are eigenstates of the rotation operator – with the boson accumulating no phase under rotation and the fermion accumulating a minus sign. (We did this in lecture so it should be easy).

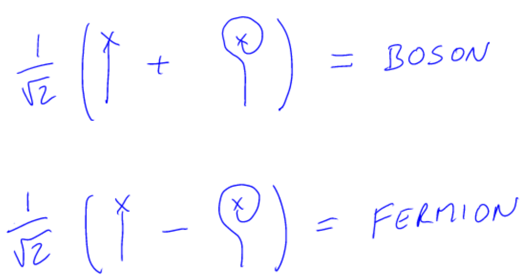

Figure 29: Boson and Fermion quasiparticles as string ends in the toric code loop gas

(b) Consider exchanging two such quasiparticles. To get a general idea of how the calculation goes, you will have to evaluate diagrams of the form of Fig. 30. Show that one obtains bosonic or fermionic exchange statistics respectively for the two linear combinations shown above.

(c) [Harder] Consider fusing the boson (the electric particle e) and the fermion together. Show that this creates a magnetic defect which does not have a trailing string. You will have to recall that the operator that creates a magnetic particle is sum of the identity operator and minus an operator that draws a loop all the way around the region. (This operator is a projector that forces a magnetic defect into a region; the orthogonal projector assures that there is no magnetic defect within the region).

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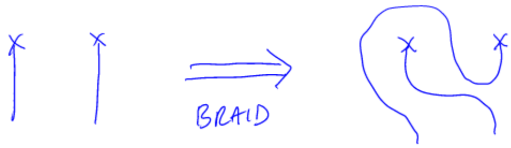

Figure 30: Braiding defects

Problem 17 *Quasiparticles in Double Semion Loop Gas*

As discussed in lecture, the doubled semion model ground state can be considered to be a loop gas with the rules given in Fig. 31. Note that these rules are the same as the semion rules from the problem "Abelian Kauffman Anyons" which we considered earlier (although in that model there is only one chirality of semion particle!)

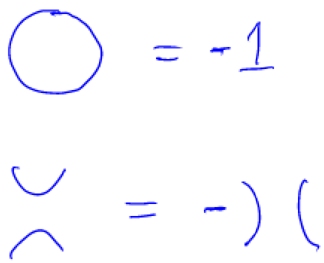

Figure 31: Loop gas rules for the doubled semion model

Again certain quasiparticle excitaitons can be indicated as ends of strings in the loop gas.

(a)† Show that the linear combinations of string ends shown in the figure 32 are eigenstates of the rotation operator – with the two particles accumulating a factor of i or $-i$ under rotation (We also did this in lecture so it should be easy).

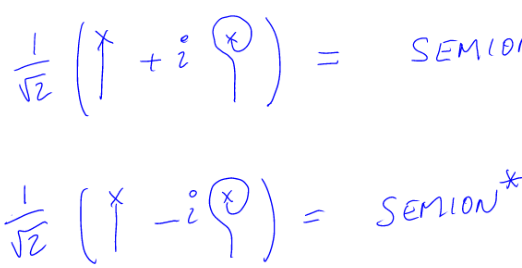

Figure 32: Semion and anti-semion string ends in the doubled semion loop gas

(b) Consider exchanging two such quasiparticles. Show that under exchange one obtains factor of i or $-i$ as expected for semions and anti-semions. Note: The anti-semion is not the antiparticle of the semion (I know it is bad nomenclature!) – The antisemion is the opposite handed particle. The semion is its own antiparticle.

(c) [Harder] Consider fusing the semion and anti-semion together. Show that this creates a "magnetic defect." What is the projector that produces a magnetic defect in a region?

. . . .

Problem 18 *Double Fibonacci String Net*

(a) As discussed in lecture, the double Fibonacci model ground state can be viewed as a branching string net with graphical rules given by Fig. 33 (Compare to the problem on Fibonacci pentagon relation) where $\phi^{-1} = (\sqrt{5}-1)/2$. In the ground state no endpoints of strings are allowed, but branching is allowed.

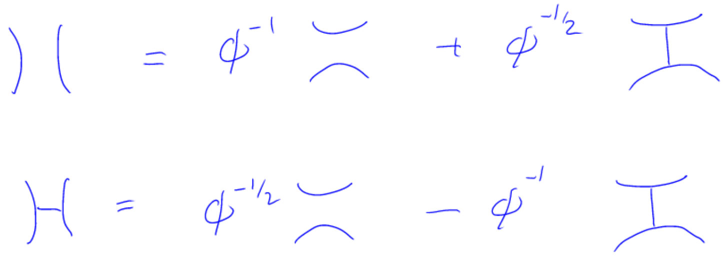

Figure 33: String net rules for the doubled Fibonacci model

To complete the graphical rules we must also use the rules shown in Fig. 34 for some values of the variables, d, X and T.

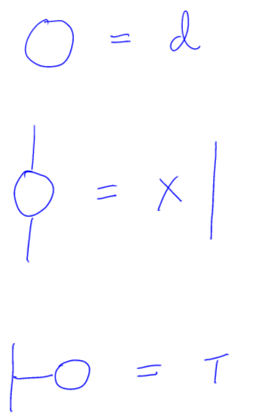

Figure 34: Additionnal string net rules for the doubled Fibonacci model

(a) Show that the consistent solutions is $d = \phi$ with $X = \phi^{1/2}$ and $T = 0$. We did much of this in lecture. What was left out is proving that any $T \neq 0$ solution is not self-consistent. Hint: Try evaluating a circle with three legs coming out of it. That should enable you to derive a useful identity. Then see if you can use this identity to derive a contradiction when $T \neq 0$.

(b) Consider quasiparticles which are the ends of strings. The general form of a quasiparticle is as shown in Fig 35 with coefficients a, b, c that need to be determined. Find the eigenvalues/eigenvectors of the rotation operator to determine the quasiparticle types and their spins. (We did most of this in lecture except the explicit evaluation of the eigenvalue problem!) Compare your result to the result of the problem "Fibonacci Hexagon Equation".

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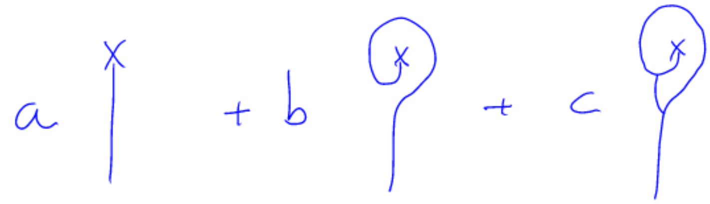

Figure 35: Combination of defect types for the doubled Fibonacci model

Problem 19 Quantum Hall Conductivity vs Conductance †

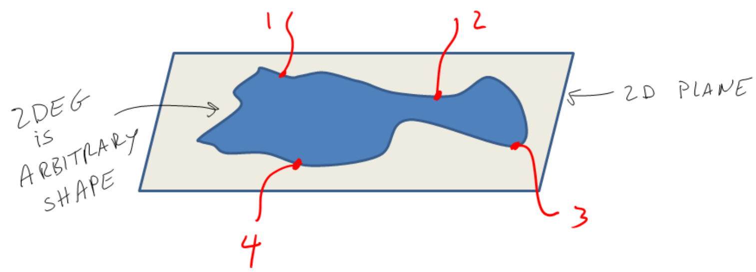

Figure 36: A 2D electron gas of arbitrary shape with contacts 1,2,3,4 attached on its perimeter in clockwise order

Consider a two dimensional electron gas (2DEG) of arbitrary shape in the plane with four contacts (1,2,3,4) attached at its perimeter in a clockwise order as shown in Fig. 36. The conductivity tensor σ_{ij} relates the electric field to the current via

$$j_i = \sigma_{ij} E_j \tag{7}$$

where indices i and j take values $\hat{x}$ and $\hat{y}$ (and sum over j is implied). Assume that this is a quantized hall system with quantized hall conductance s. In other words, assume that

$$\sigma = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} \tag{8}$$

Show that the following two statements are true independent of the shape of the sample.

(a) Suppose current I is run from contact 1 to contact 2, show that the voltage measured between contact 3 and 4 is zero.

(b) Suppose current I is run from contact 1 to contact 3, show that the voltage measured between contact 2 and 4 is $V = I/s$.

Note: The physical measurements proposed here measure the *conductance* of the sample, the microscopic quantity σ is the *conductivity*.

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Problem 20 *About the Lowest Landau Level*

If you have never before actually solved the problem of an electron in two dimensions in a magnetic field, it is worth doing. Even if you have done it before, it is worth doing again.

Consider a two dimensional plane with a perpendicular magnetic field $\vec{B}$. Work in symmetric gauge $\vec{A} = \frac{1}{2}\vec{r} \times \vec{B}$.

(a)† (This is the hard part, see below for hints if you need them.) Show that the single electron Hamiltonian can be rewritten as

$$H = \hbar\omega_c(a^\dagger a \ + \frac{1}{2}) \tag{9}$$

where $\omega_c = eB/m$ and

$$a = \sqrt{2}\ell\left(\bar{\partial} + \frac{1}{4\ell^2}z\right) \tag{10}$$

with $z = x+iy$ and $\bar{\partial} = \partial/\partial\bar{z}$ with the overbar meaning complex conjugation. Here ℓ is the magnetic length $\ell = \sqrt{\hbar/eB}$.

(b) † Confirm that

$$[a, a^\dagger] = 1 \tag{11}$$

and therefore that the energy spectrum is that of the harmonic oscillator

$$E_n = \hbar\omega_c(n + \frac{1}{2}) \tag{12}$$

(c) † Once you obtain Eq. 9, show that any wavefunction

$$\psi = f(z)e^{-|z|^2/4\ell^2} \tag{13}$$

with f any analytic function is an eigenstate with energy $E_0 = \frac{1}{2}\hbar\omega_c$. Show that an orthogonal basis of wavefunctions in the lowest Landau level (i.e., with eigenenergy E_0) is given by

$$\psi_m = N_m z^m e^{-|z|^2/4\ell^2} \tag{14}$$

where N_m is a normalization constant. Show that the maximum amplitude of the wavefunction ψ_m is a ring of radius $|z| = \ell\sqrt{2m}$ and calculate roughly how the amplitude of the wavefunction decays as the radius is changed away from this value.

(d) Defining further

$$b = \sqrt{2}\ell\left(\partial + \frac{1}{4\ell^2}\bar{z}\right) \tag{15}$$

with $\partial = \partial/\partial z$, Show that the operator b also has canonical commutations

$$[b, b^\dagger] = 1 \tag{16}$$

but both b and $b^\dagger$ commute with a and $a^\dagger$. Conclude that applying b or $b^\dagger$ to a wavefunction does not change the energy of the wavefunction.

(e) show that the $\hat{z}$ component of angular momentum (angular momentum perpendicular to the plane) is given by

$$L = \hat{z}\cdot(\vec{r}\times\vec{p}) = \hbar(b^\dagger b \ - a^\dagger a \ ) \tag{17}$$

Conclude that applying b or $b^\dagger$ to a wavefunction changes its angular momentum, but not its energy.

(f) [Harder] Let us write an arbitrary wavefunction (not necessarily lowest Landau level) as a polynomial in z and $\bar{z}$, times the usual gaussian factor. Show that projection of this wavefunction to the lowest Landau level can be performed by moving all of the $\bar{z}$ factors all the way to the left and replacing each $\bar{z}$ with $2\ell^2\partial_z$.

Hints to part a: First, define the antisymmetric tensor ϵ_{ij}, so that the vector potential may be written as $A_i = \frac{1}{2}B\epsilon_{ij}r_j$. We have variables p_i and r_i that have canonical commutations (four scalar variables total). It is useful to work with a new basis of variables. Consider the coordinates

$$\begin{aligned} \pi_i^{(\alpha)} &= p_i + \alpha\frac{\hbar}{2\ell^2}\epsilon_{ij}r_i && (18)\\ &= \frac{\hbar}{\ell^2}\epsilon_{ij}\xi_j && (19)\end{aligned}$$

defined for $\alpha = \pm 1$. Here $\alpha = +1$ gives the canonical momentum. Show that

$$\left[\pi_i^{(\alpha)}, \pi_j^{(\beta)}\right] = i\alpha\epsilon_{ij}\delta_{\alpha\beta}\frac{\hbar^2}{\ell^2} \tag{20}$$

The Hamiltonian

$$H = \frac{1}{2m}(p_i + eA_i)(p_i + eA_i) \tag{21}$$

can then be rewritten as

$$H = \frac{1}{2m}\pi_i^{(+1)}\pi_i^{(+1)} \tag{22}$$

with a sum on $i = \hat{x}, \hat{y}$ implied. Finally use

$$a = (-\pi_y^{(+1)} + i\pi_x^{(+1)})\frac{\ell}{\sqrt{2}\hbar} \tag{23}$$

$$b = (\pi_y^{(-1)} + i\pi_x^{(-1)})\frac{\ell}{\sqrt{2}\hbar} \tag{24}$$

to confirm that a and b are given by Eqs. 10 and 15 respectively. Finally confirm Eq. 9 by rewriting Eq. 22 using Eqs. 23 and 24.

A typical Place to get confused is the definition of ∂. Note that

$$\partial z = \bar{\partial}\bar{z} = 1 \tag{25}$$

$$\bar{\partial} z = \partial\bar{z} = 0 \tag{26}$$

Hints to part f: Rewrite the operators $a, a^\dagger, b, b^\dagger$ such that they operate on polynomials, but not on the Gaussian factor. Construct $\bar{z}$ in terms of these operators. Then project.

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Problem 21 *Filled Lowest Landau Level*

Show that the filled Lowest Landau level of non-interacting electrons (a single slater determinant) can be written as

$$\Psi_m^0 = \mathcal{N} \prod_{1 \le i < j \le N} (z_i - z_j)^1 \prod_{1 \le i \le N} e^{-|z|^2/4\ell^2} \tag{27}$$

with $\mathcal{N}$ some normalization constant. I.e, this is the Laughlin wavefunction with exponent $m = 1$.

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Problem 22 *Laughlin Plasma Analogy*

Consider the Laughlin wavefunction for N electrons at positions z_i

$$\Psi_m^0 = \mathcal{N} \prod_{1 \le i < j \le N} (z_i - z_j)^m \prod_{1 \le i \le N} e^{-|z|^2/4\ell^2} \tag{28}$$

with $\mathcal{N}$ a normalization constant. The probability of finding particles at positions $\{z_1, \ldots, z_N\}$ is given by $|\Psi_m(z_1, \ldots z_N)|^2$.

Consider now N classical particles at temperature $\beta = \frac{1}{k_b T}$ in a plane interacting with logarithmic interactions $v(\vec{r}_i - \vec{r}_j)$ such that

$$\beta v(\vec{r}_i - \vec{r}_j) = -2m \log(|\vec{r}_i - \vec{r}_j|) \tag{29}$$

in the presence of a background potential u such that

$$\beta u(|\vec{r}|) = |\vec{r}|^2/(2\ell^2) \tag{30}$$

Note that this log interaction is "Coulombic" in 2d (i.e., $\nabla^2 v(\vec{r}) \propto \delta(\vec{r})$).

(a) Show that the probability that these classical particles will take positions $\{\vec{r}_1, \ldots, \vec{r}_N\}$ is given by $|\Psi^0_m(z_1, \ldots z_N)|^2$ where $z_j = x_j + iy_j$ is the complex representation of position $\vec{r}_i$. Argue that the mean particle density is constant up to a radius of roughly $\ell\sqrt{Nm}$. (Hint: Note that u is a neutralizing background. What configuration of charge would fully screen this background?)

(b) Now consider the same Laughlin wavefunction, but now with M quasiholes inserted at positions $w_1, \ldots, w_M$.

$$\Psi_m = \mathcal{N}(w_1, \ldots, w_M) \left[\prod_{1 \leq i \leq N} \prod_{1 \leq \alpha \leq M} (z_i - w_\alpha) \right] \Psi^0_m \tag{31}$$

where $\mathcal{N}$ is a normalization constant which may now depend on the positions of the quasiholes. Using the plasma analogy, show that the $w-z$ factor may be obtained by adding additional logarithmically interacting charges at positions w_i,with $1/m$ of the charge of each of the z particles

(c) Note that in this wavefunction the z's are physical parameters (and the wavefunction must be single-valued in z's), but the w's are just parameters of the wavefunction – and so the function $\mathcal{N}$ could be arbitrary — and is only fixed by normalization. Argue using the plasma analogy that in order for the wavefunction to remain normalized (with respect to integration over the z's) as the w's are varied, we must have

$$|\mathcal{N}(w_1, \ldots, w_M)| = \mathcal{K} \prod_{1 \leq \alpha < \gamma \leq M} |w_\alpha - w_\gamma|^{1/m} \prod_{1 \leq \alpha \leq M} e^{-|w_\alpha|^2/(4m\ell^2)} \tag{32}$$

with $\mathcal{K}$ a constant so long as the $w's$ are not too close to each other. (Hint: a plasma will screen a charge).

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Problem 23 *Bose Vertex Operators*

In lecture we needed the following identity

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \ldots V_{\alpha_N}(z_N) \rangle = \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j} \tag{33}$$

where

$$\sum_i \alpha_i = 0 \tag{34}$$

where the vertex operators are defined by

$$V_\alpha(z) =: e^{i\alpha\phi(z)} : \tag{35}$$

with ϕ a chiral bose field and colons meaning normal ordering.

(a) To get to this result, let us first show that for a bose operator a, such that $[a, a^\dagger] = 1$, we have

$$e^{\alpha a} e^{\beta a^\dagger} = e^{\beta a^\dagger} e^{\alpha a} e^{\alpha \beta} \tag{36}$$

(b) Thus derive

$$\langle V_{A_1} V_{A_2} \ldots V_{A_N} \rangle = e^{\sum_{i<j} \langle A_i A_j \rangle} \tag{37}$$

where

$$A_i = u_i a^\dagger + v_i a \tag{38}$$

and

$$V_{A_i} =: e^{A_i} := e^{u_i a^\dagger} e^{v_i a} \tag{39}$$

with the colons meaning normal ordering (all daggers moved to the left).

(c) Show that Eq. 37 remains true for any operators A_i that are sums of different bose modes a_k, i.e., if

$$A_i = \sum_k [u_i(k) a_k^\dagger + v_i(k) a_k] \tag{40}$$

Set $A_i = i\alpha_i \phi(z_i)$ such that $V_{A_i} = V_\alpha(z_i)$. If ϕ is a free massless chiral bose filed which can be written as the sum of fourier modes of bose operators such that

$$\langle \phi(z)\phi(w)\rangle = -\ln(z-w) \tag{41}$$

conclude that Eq. 33 holds.

Note: This result is not quite correct, as it fails to find the constraint Eq. 34 properly. The reason it fails is a subtlety which involves how one separates a bose field into two chiral components. (More detailed calculations that get this part right are given in the Big Yellow CFT book (P. Di Francesco, P. Mathieu, and D. Senechal) and in a different language in A. Tsvelik's book.)

There is, however, a quick way to see that the constraint must be true. Note that the lagrangian of a massless chiral bose field is

$$\mathcal{L} = \frac{1}{2\pi} \partial_x \phi (\partial_x + v \partial_t) \phi \tag{42}$$

which clearly must be invariant under the global transformation $\phi \to \phi + b$.

(d)Show that the correlator Eq. 33 (with Eq. 35) cannot be invariant under this transformation unless Eq. 34 is satisfied, or unless the value of the correlator is zero.

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Problem 24 $\mathbb{Z}_4$ Quantum Hall State

In this problem we intend to construct a quantum hall state from the the $\mathbb{Z}_4$ parafermion conformal field theory (Details of the CFT can be found in A. B. Zamolodchikov and V. A. Fateev, Soviet Physics JETP 62, 216 (1985), but we will not need too many of the details here).

The wavefunction we construct is known as the $\mathbb{Z}_4$ Read-Rezayi wavefunction (N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999)).

The $\mathbb{Z}_4$ parafermion conformal field theory has 10 fields with corresponding conformal weights

(scaling dimension)

field	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
weight h	0	$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{9}{16}$	$\frac{9}{16}$

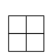
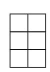
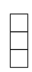
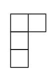
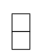
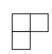
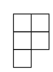
and the fusion table is given by

$\times$	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
$\mathbf{1}$	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
ψ_1	ψ_1	ψ_2	ψ_3	$\mathbf{1}$	χ_-	σ_+	ρ	ϵ	σ_-	χ_+
ψ_2	ψ_2	ψ_3	$\mathbf{1}$	ψ_1	χ_+	χ_-	ϵ	ρ	σ_+	σ_-
ψ_3	ψ_3	$\mathbf{1}$	ψ_1	ψ_2	σ_-	χ_+	ρ	ϵ	χ_-	σ_+
σ_+	σ_+	χ_-	χ_+	σ_-	$\psi_1+\rho$	$\mathbf{1}+\epsilon$	$\sigma_++\chi_+$	$\sigma_-+\chi_-$	$\psi_3+\rho$	$\psi_2+\epsilon$
σ_-	σ_-	σ_+	χ_-	χ_+	$\mathbf{1}+\epsilon$	$\psi_3+\rho$	$\sigma_-+\chi_-$	$\sigma_++\chi_+$	$\psi_2+\epsilon$	$\psi_1+\rho$
ϵ	ϵ	ρ	ϵ	ρ	$\sigma_++\chi_+$	$\sigma_-+\chi_-$	$\mathbf{1}+\psi_2+\epsilon$	$\psi_1+\psi_3+\rho$	$\sigma_++\chi_+$	$\sigma_-+\chi_-$
ρ	ρ	ϵ	ρ	ϵ	$\sigma_-+\chi_-$	$\sigma_++\chi_+$	$\psi_1+\psi_3+\rho$	$\mathbf{1}+\psi_2+\epsilon$	$\sigma_-+\chi_-$	$\sigma_++\chi_+$
χ_+	χ_+	σ_-	σ_+	χ_-	$\psi_3+\rho$	$\psi_2+\epsilon$	$\sigma_++\chi_+$	$\sigma_-+\chi_-$	$\psi_1+\rho$	$\mathbf{1}+\epsilon$
χ_-	χ_-	χ_+	σ_-	σ_+	$\psi_2+\epsilon$	$\psi_1+\rho$	$\sigma_-+\chi_-$	$\sigma_++\chi_+$	$\mathbf{1}+\epsilon$	$\psi_3+\rho$

If I have not made any mistake in typing this table, the fusion rules should be associative

$$(a \times b) \times c = a \times (b \times c) \tag{43}$$

Note of interest: These fusion rules may look mysterious, but in fact they are very closely related to the fusion rules of $SU(2)$ appropriately truncated (i.e., this is the $SU(2)_4$ WZW model). We can write each field as a young tableau with no more than 2 (for $SU(2)$) columns and no more than $4-1=3$ rows

field	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
tableau	empty									

The fusion rules are just a *slight* modification of the usual young tableau manipulations for $SU(2)$ where columns are removed if they have 4 boxes. (See the big yellow book for details).

Using the techniques discussed in lecture:

(a) Use the operator product expansion (dimension counting) to find the singularity as two ψ_1 fields come close together. I.e, find the exponent α in the relation

$$\lim_{z' \to z} \psi_1(z')\psi_1(z) \sim (z'-z)^{\alpha}\, \psi_2(z) \tag{44}$$

(b) Construct all possible "electron" fields by making a product of the ψ_1 field and a chiral bose vertex operator of the form

$$\psi_e(z) = \psi_1(z) e^{i\beta\phi(z)} \tag{45}$$

that give a single-valued and nonsingular wavefunction for the electron. (See Eq. 33, but ignore the sum condition Eq. 34) I.e., find all acceptable values of β. Consider both the case where the "electron" is a boson or a fermion. What filling fractions do these correspond to? (There are multiple allowable solutions for both bosons and fermions). Consider among the bosonic solution, the one solution of the highest density. The ground state wavefunction in this case is the highest density zero energy state of a 5-point delta function interaction. Show that the wavefunction does not vanish when 4 particles come to the same point, but does indeed vanish as 5 particles come to the same point.

(c) Given a choice of the electron field, construct all possible quasihole operators from all fields φ in the above table

$$\phi_{qh}(w) = \varphi(w) e^{i\kappa\phi(w)} \tag{46}$$

For each case, fix the values of κ by insisting that the wavefunction remain single-valued in the electron coordinates. Determine the quasihole with the lowest possible (nonzero) electric charge. What is this charge?

(d) Two such quasiholes can fuse together in two possible fusion channels. What is the monodromy in each of these channels. I.e, what phase is accumulated when the two quasiholes are transported around each other (assuming the Berry matrix is zero – which is a statement about wavefunctions being properly orthonormal – which we usually assume is true).

(e) Draw a Bratteli diagram (a tree) describing the possible fusion channels for many of these elementary particles. Label the number of paths in the diagram for up to 10 quasiholes. If there are 8 quasiparticles and the number of electrons is divisible by 4, what is the degeneracy of the ground state? If there are 4 quasiparticles and the number of electrons is $4m+2$ what is the degeneracy of the ground state?

(f) Construct a 5 by 5 transfer matrix and show how to calculate the ground state degeneracy in the presence of any number of quasiholes. Finding the largest eigenvalue of this matrix allows you to calculate the "quantum dimension" d which is the scaling

$$\text{Degeneracy} \sim d^{[\text{Number of Quasiholes}]} \tag{47}$$

in the limit of large number of quasiholes. While diagonalizing a 5 by 5 matrix seems horrid, this one can be solved in several easy ways (look for a trick or a nice factorization of the characteristic polynomial).

(g) Consider instead constructing a wavefunction from the ψ_2 field

$$\psi_e(z) = \psi_2(z) e^{i\beta\phi(z)} \tag{48}$$

What filling fraction does this correspond to (for bosons or fermions). In the highest density case, what are the properties of this wavefunction (how does it vanish as how many many electrons come to the same point).

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