

# 第 5 章

## Higher condensation theory

Review of [?].

### 5.1 Definition of condensation

In this section, we refer to an  **$(n, n)$ -category** as an  $(\infty, n)$ -category which is “truncated” at morphism degree  $n$ . Intuitively, if  $\mathcal{C}$  is an  $(n, n)$ -category, then, for arbitrary  $n-1$ -morphisms  $f, g \in \mathcal{C}_{n-1}$ , the space of  $n$ -morphisms between  $f$  and  $g$  has discrete topology. Therefore, we can compose  $n$ -morphisms strictly (not up to homotopy), associatively, and unitaly.

Note that a  **$(0, 0)$ -category**  $\mathcal{C}$  is merely a set<sup>\*1</sup>.

#### 定義 5.1: 0-condensation

Let  $\mathcal{C}$  be a  $(0, 0)$ -category. A **0-condensation** on  $\mathcal{C}$  is an equality between elements of a set  $\mathcal{C}_0$ .

$n$ -condensation is defined by induction.

#### 定義 5.2: $n$ -condensation

Fix  $n \geq 0$ . Let  $\mathcal{C}$  be an  $(n, n)$ -category, and let  $x, y \in \mathcal{C}_0$  be objects of  $\mathcal{C}$ . We define  $n$ -condensation by induction on  $n$ .

**$n$ -condensation** of  $x$  onto  $y$  in  $\mathcal{C}$  consists of three data:

- A 1-morphism  $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism  $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- An  $(n - 1)$ -condensation of  $r \circ i$  onto  $\text{Id}_y$  in  $\text{Map}_{\mathcal{C}}(y, y)$ <sup>a</sup>

<sup>a</sup> Roughly speaking, a mapping space  $\text{Map}_{\mathcal{C}}(y, y)$  itself is an  $(n - 1, n - 1)$ -category. Strictly speaking, we need enriched  $\infty$ -category theory.

\*1 That is,  $\mathcal{C}$  consists of a set  $\mathcal{C}_0$  of objects ( $0$ -morphisms) only.

### 【例 5.1.1】 1-condensation

Let  $\mathcal{C}$  be a  $(1, 1)$ -category (i.e. an ordinary category). **1-condensation** of  $x$  onto  $y$  in  $\mathcal{C}$  consists of these data:

- A 1-morphism  $r \in \text{Hom}_{\mathcal{C}}(x, y)$
- A 1-morphism  $i \in \text{Hom}_{\mathcal{C}}(y, x)$
- A 0-condensation of  $r \circ i$  onto  $\text{Id}_y$  in  $\text{Hom}_{\mathcal{C}}(y, y)$ . i.e.  $r \circ i = \text{Id}_y$ .

In the context of  $(1, 1)$ -categories, such  $y$  is called a **retract of  $x$** .

### 【例 5.1.2】 2-condensation

Let  $\mathcal{C}$  be a  $(2, 2)$ -category (i.e. a bicategory). **2-condensation** of  $x$  onto  $y$  in  $\mathcal{C}$  consists of these data:

- A 1-morphism  $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism  $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 1-condensation of  $r \circ i$  onto  $\text{Id}_y$  in  $(1, 1)$ -category  $\text{Map}_{\mathcal{C}}(y, y)$ .

By 【例 5.1.1】 ,

- A 1-morphism  $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism  $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism (in  $\mathcal{C}$ )  $(r \circ i \xrightarrow{\rho} \text{Id}_y) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism (in  $\mathcal{C}$ )  $(\text{Id}_y \xrightarrow{\iota} r \circ i) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- An equality  $\rho \circ \iota = \text{Id}_{\text{Id}_y}$

### 【例 5.1.3】 3-condensation

Let  $\mathcal{C}$  be a  $(3, 3)$ -category. By 【例 5.1.2】 , **3-condensation** of  $x$  onto  $y$  in  $\mathcal{C}$  consists of these data:

- A 1-morphism  $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism  $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism  $(r \circ i \xrightarrow{\rho} \text{Id}_y) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism  $(\text{Id}_y \xrightarrow{\iota} r \circ i) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 3-morphism

$$\left( \begin{array}{ccc} & \xrightarrow{\rho \circ \iota} & \\ \text{Id}_y & \Downarrow_{\alpha} & \text{Id}_y \\ & \xrightarrow{\text{Id}_{\text{Id}_y}} & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

- A 3-morphism

$$\left( \begin{array}{ccc} & \xrightarrow{\text{Id}_{\text{Id}_y}} & \\ \text{Id}_y & \Downarrow \beta & \text{Id}_y \\ & \xrightarrow{\rho \circ \iota} & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

- An equality

$$\begin{array}{c} \text{Id}_{\text{Id}_y} \\ \Downarrow \beta \\ \text{Id}_y \xrightarrow{\rho \circ \iota} \text{Id}_y \\ \Downarrow \alpha \\ \text{Id}_{\text{Id}_y} \end{array} = \begin{array}{c} \text{Id}_{\text{Id}_y} \\ \Downarrow \text{Id}_{\text{Id}_{\text{Id}_y}} \text{Id}_y \\ \text{Id}_{\text{Id}_y} \end{array}$$

#### 【例 5.1.4】 4-condensation

Let  $\mathcal{C}$  be a (4, 4)-category. By 【例 5.1.2】 , **3-condensation** of  $x$  onto  $y$  in  $\mathcal{C}$  consists of these data:

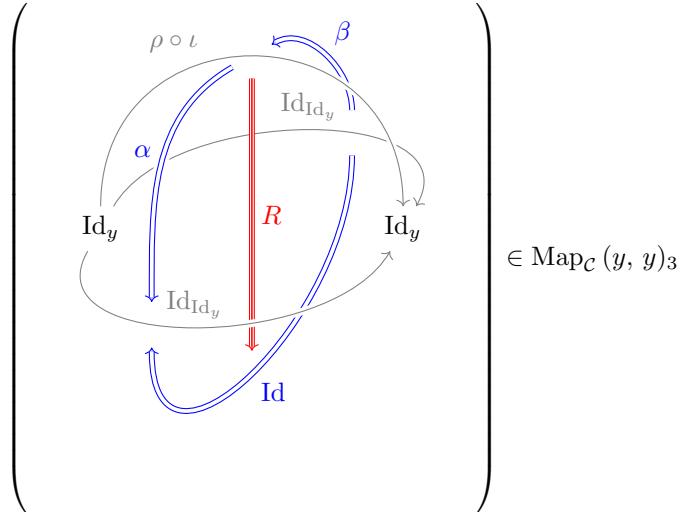
- A 1-morphism  $r \in \text{Map}_{\mathcal{C}}(x, y)_0$
- A 1-morphism  $i \in \text{Map}_{\mathcal{C}}(y, x)_0$
- A 2-morphism  $(r \circ i \xrightarrow{\rho} \text{Id}_y) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 2-morphism  $(\text{Id}_y \xrightarrow{\iota} r \circ i) \in \text{Map}_{\mathcal{C}}(y, y)_1$
- A 3-morphism

$$\left( \begin{array}{ccc} & \xrightarrow{\rho \circ \iota} & \\ \text{Id}_y & \Downarrow \alpha & \text{Id}_y \\ & \xrightarrow{\text{Id}_{\text{Id}_y}} & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

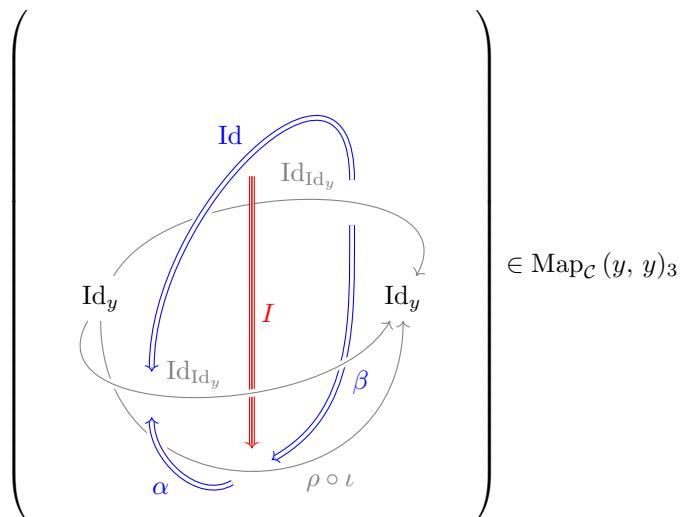
- A 3-morphism

$$\left( \begin{array}{ccc} & \xrightarrow{\text{Id}_{\text{Id}_y}} & \\ \text{Id}_y & \Downarrow \beta & \text{Id}_y \\ & \xrightarrow{\rho \circ \iota} & \end{array} \right) \in \text{Map}_{\mathcal{C}}(y, y)_2$$

- A 4-morphism



- A 4-morphism



- An equality

The diagram illustrates a commutative equality between two configurations of morphisms. On the left, there is a complex network of blue arrows labeled with  $\text{Id}$ ,  $\text{Id}_{\text{Id}_y}$ ,  $\alpha$ ,  $\rho \circ \iota$ ,  $\beta$ ,  $I$ , and  $R$ . On the right, the same objects are shown but with fewer arrows, indicating a simplification or collapse of the network. A red double-headed arrow between the two configurations indicates that they are equal.

### 5.1.1 Walking condensation

Let  $\mathcal{C}$  be an  $(n, n)$ -category. Roughly speaking, a **walking  $n$ -condensation** is a (strict?)  $n$ -category  $\clubsuit_n$  which “generates”  $n$ -condensation in  $\mathcal{C}$ . i.e. the functor category  $\mathcal{F}\text{un}(\spadesuit_n, \mathcal{C})$  is equivalent to the category of  $n$ -condensations in  $\mathcal{C}$ .

### 5.1.2 Condensation monad

Let  $\clubsuit_n \subset \spadesuit_n$  be a full-subcategory which consists of walking  $n$ -condensations on only one points.

**定義 5.3:** condensation monad

## 5.2 Physical interpretation