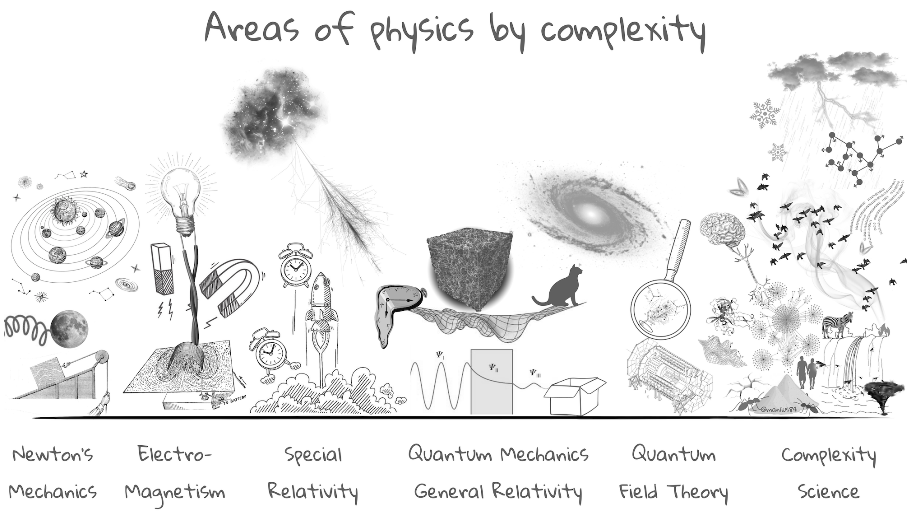
Final Report

Physics of Complex Networks: Structure and Dynamics

Università degli Studi di Padova



Physics of Complex Networks:

Bains, Arman Singh

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1 | Axelrod's Model

Task leader(s): *Bains, Arman Singh*

1.1 | Introduction

The goal of Axelrod's model [1] hereby discussed is to explain why and how distinct cultural differences may persist, despite the interactions among individuals which should bring homogeneity since its expected that interacting individuals will eventually become more alike. The model proposes an agent-based adaptive model that should show how local convergence may indeed lead to global polarization or homogeneity.
In this report, Axelrod's model will be firstly implemented on a lattice, in which nodes interact with their neighbors assimilating into their culture, possibly leading to a giant connected component in which all nodes have the same cultural characteristics. Then, the model will be also applied to different network topologies to see if there are similarities in how such a giant connected component emerges. Finally, in appendix A.1 a slightly different model will be presented, showing how an external force might affect the results which were previously obtained only for a decentralized model.

1.2 | Dynamics

The basic model is implemented via a 10-by-10 lattice, in which each node is connected to its immediate neighbors (though free boundary conditions have been applied at the boundaries). To each node is assigned a culture, represented abstractly by a set of integer numbers. The size of the set is equal to the number of cultural features being examined (e.g. language, religion, etc.), while each integer number represents the specific cultural trait of the respective feature (e.g. italian or hindi for language). For the models described in this report, the initial cultural traits are assigned to each node by sampling from a uniform distribution. Cultural similarity among two neighbors is computed as the percentage of the features that have identical trait.
The process of social influence start by picking at random a node to be active, and then one of its neighbors. These two nodes interact with probability equal to their cultural similarity, with the active site copying one of the other's features.
With only five distinct cultural traits and features, it seems that a giant connected component arises in a relatively small number of iterations ($\sim 10^5$) as can be seen in figure 1.1.

Such convergence is not guaranteed though, as changing parameters such as the numbers of traits or features may make it significantly less likely to have a single giant connected component. Using table 1.1 as reference, one sees that if the simulation is allowed to run indefinitely until there are no more active sites (i.e. until all sites

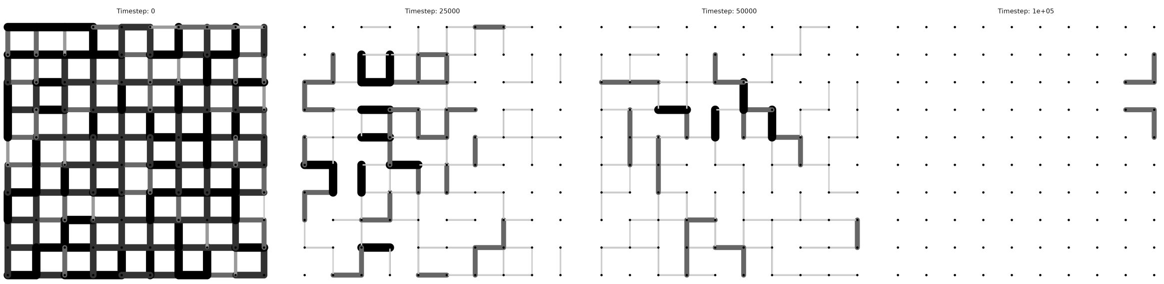

Figure 1.1: Cultural diffusion at several iteration steps. Edges' thickness and darkness are proportional to the dissimilarity among the respective nodes. Timestep is specified on top of the respective figure.

have either the same culture as their neighbors, or one so different they won't ever interact), certain combination of parameters will results in several distinct cultural regions, thus showing that local convergence may indeed lead to global polarization and to an absence of giant connected component.

Average Number of Stable Regions

Number of Traits	**Number of Features**		
	5	10	15
5	1	1	1
10	2	1	1
15	14	2	1

Table 1.1: Average number of stable regions per pair of number of traits and features.

1.3 | Additional topologies

As noted in the original paper [1], changing neighborhood definitions will induce different convergence tendencies, thus it's likely that changing network topology altogether should also offer alternative insights. The plots below represent the number of cultural regions and active sites (the reason why they are over 100 is because they are recounted depending on how many differing neighbors they have) for the following additional networks:

- Barabasi-Albert network (ba): this network models preferential attachment, in order to show how hubs may accelerate cultural homogenization. The parameters chosen are 1 for the power and 2 for the number of edges to add at each time step.
- Watts-Strogatz Small-World Network (ws): this network allows to understand the role of shortcuts in breaking down cultural clusters. Each node is connected to 4 neighbors, with a probability of rewiring of 0.1.
- Stochastic Block Model (sbm): this network that allows to compare for explicit community structure. The graph was partitioned into two equal-sized groups,

where nodes of the same community connect with proabability of 90%, while nodes from different communities connect with probability of 10%.

Plot 1.4 compares how these models fare against the standard lattice described previously, when setting 5 features and 15 traits. Note that the plot actually takes the average across 10 replications, with the shades around plot lines representing the standard deviation for each model at any specific timestep.

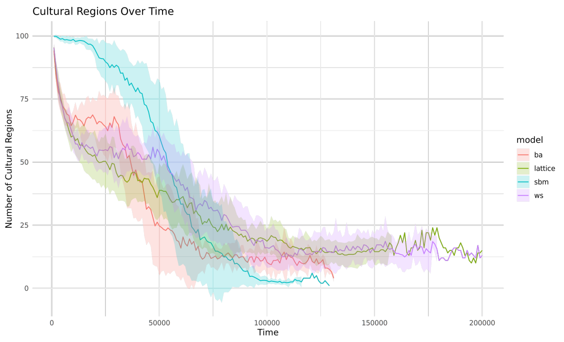

Figure 1.2: Number of cultural regions over time.

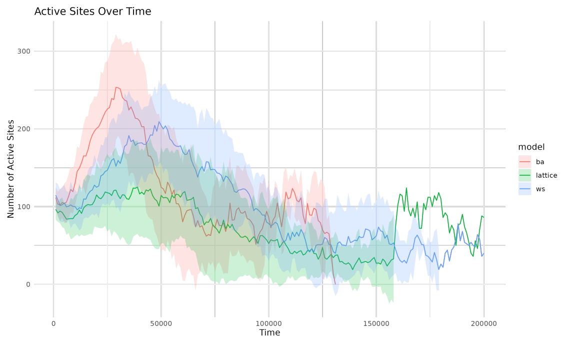

Figure 1.3: Number of active pairs over time. 'sbm' has not been plotted as its number of active sites was too much higher then the rest through the first iterations.

Figure 1.4

As it's clear from the plot, the persistence of multiple cultural regions depends on network structure, which may or may not make it easier for individuals to interact and transmit cultural traits. Interestingly, the SBM tends to be more resistant to homogeneity at first, but then it converges much more quickly.

2 | Schelling Segregation Model

Task leader(s): *Bains, Arman Singh*

2.1 | Introduction

The stated goal of this project is to simulate a well known model of Sociophysics, possibly expanding on it through the lens of statistical mechanics and network theory. In particular, this paper focused on a segregation model developed first by Thomas C. Schelling [6], where individuals of two types on a network are allowed to move to vacant nodes to surround themselves with others of the same type, leading to de facto segregation. A second model, inspired by this, will also be discussed here, with the goal of showing that simulating the behavior of agents moving on top of the network is not necessary, rather it is sufficient to allow nodes to rewire some connections to expose that even a slight bias against dissimilar types will lead to segregation even in more abstract networks. In appendix A.2 Schelling's original model will be discussed again after simulating how various migration patterns might affect its dynamics.

2.2 | Dynamics

The one dimensional Schelling's model is implemented by employing a one dimensional lattice, with free boundary conditions. A fraction r_0 of the nodes will be labeled 0, representing vacancies, while the remaining nodes will be labeled ± 1, with 1 representing agents of the majority type, -1 representing agents of the minority type and m being the difference between the fraction of majority and minority agents. An individual is deemed to have utility 0 (i.e. unhappy) if more than half of his neighbors are of the opposite type (otherwise he will have utility 1).
The dynamics proceed by picking at random an unhappy agent, and checking if moving him in place of a random vacancy will increase its utility. As it can be seen in figure 2.3, displaying the mean total utility for the nodes of the minority type after 20 trials (as a function of the fraction of vacancies), a phase transition seems to occur. Indeed, a small number of vacancies may lead nodes of the minority type to be unhappy (because it would be unlikely for them to find a neighborhood with others of their type), whereas increasing vacancies will make it easier for them to relocate. It should be noted that the length of the lattice was initially set to $L = 1,000$ to make the computations easier, but, as it can be seen from the plot, increasing L to $10,000$ makes the curve smoother. That is because in $L = 1,000$ there are too few vacancies, so it's quite hard for unhappy agents to find better nodes. Similarly, if the fraction of agents of the minority type is too little (i.e. m is bigger), it's harder for them to find neighborhoods of similar agents where to relocate, increasing their total unhappiness.

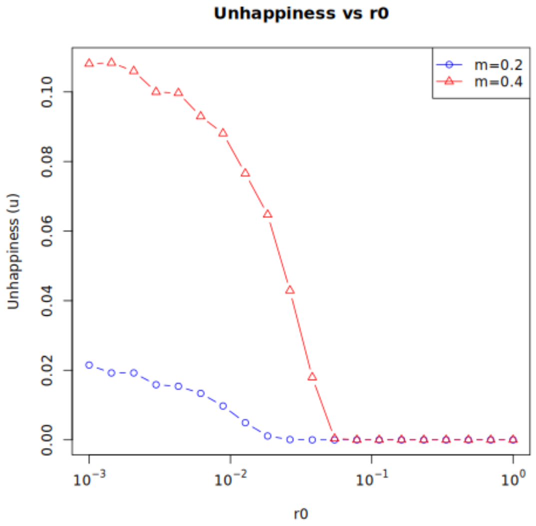

Figure 2.1: L=1,000.

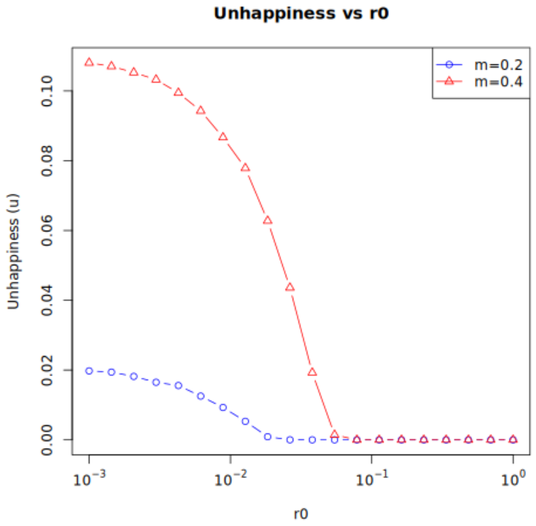

Figure 2.2: L=10,000

Figure 2.3

2.3 Abstract generalization

In [2], it was shown that Schelling's initial proposal was unnecessarily restrictive if the goal was to prove that mild aversion to dissimilar types could result in segregation. Indeed, instead of simulating the dynamics of agents moving over a network, one could analyze the dynamics of the network itself, allowing its nodes to rewire themselves with a small probability. Therefore, what has been done for this project is to run several simulations where each node has been assigned to one of two types (with 70%/30% ratios), with the subsequent dynamics consists of picking an edge at random connecting dissimilar nodes, than allowing one to rewire such edge to another of similar type. Figure 2.8 shows the homophily (i.e. the fractions of edges connecting nodes of the same type) and minority stress (i.e. the average fraction of dissimilar neighbors across all nodes of minority type), for networks of 400 nodes. The networks proposed have the same structure as those cited in chapter 1, with the exception of the Erdős–Rényi network which was implemented with probability of edge drawing of 0.02.

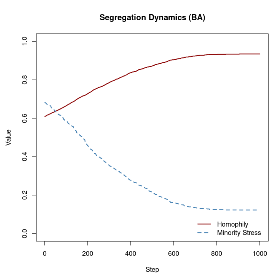

Figure 2.4: Barabási–Albert

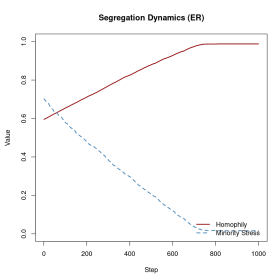

Figure 2.5: Erdős–Rényi

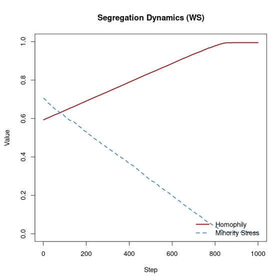

Figure 2.6: Watts–Strogatz

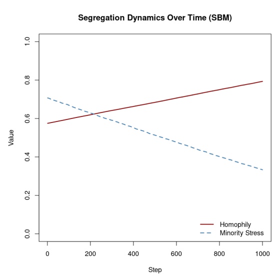

Figure 2.7: Stochastic Block Model

Figure 2.8: Comparison of network models under the Henry framework.

3 | Social Connectedness Index

Task leader(s): *Bains, Arman Singh*

3.1 | Introduction

The goal of this project is to extract a network for each country from the Facebook Scaled Social Connectedness Index data [4], which offers information on active Facebook users and their friendship networks. The *Social Cennectedness Index* $SCI_{i,j}$ between two locations i and j is defined as:

$$SCI_{i,j} = \frac{FB\_Connections_{i,j}}{FB\_Users_i \cdot FB\_Users_j}$$

where FB_Users_i, FB_Users_j are the number of Facebook users in locations i, j and $FB\_Connections_{i,j}$ is the total number of Facebook friendship connections between individuals in the two locations. The *Social Connectedness Index* available is actually rescaled so to have a maximum value of $1 \cdot 10^6$ and minimum of 1.
The main output consists of two files for each country, one concerning the nodes and other the edges of the networks (it's implied that such edges have weight equal to one, i.e. the networks are unweighted). An additional file has been produced, containing each edge's SCI, which was used to compute a few analytics, presented at the end of this chapter,

3.2 | Networks extraction

The data downloaded [4] consisted in a TSV file built on the Database of Global Administrative Areas (GADM, version 2.8) and the European Nomenclature of Territorial Units for Statistics (NUTS 2016) areas. To ease the computation process, only the top 100 countries (encoded in *gadm1*, *gadm2* or *nuts3* encoding) by total SCI were considered (excluding USA).
The desired output consisted specifically in two files:

- Edges: a file containing both nodes' IDs for each edge, alongside the name and ISO3 encoding for the country of the node from which the edge was originated
- Nodes: a file containing the ID and label of each node, alongside its latitude and longitude

Whereas for *gadm* encoded countries it was possible to convert the encoding to ISO3 just by string manipulation, *nuts3* encoded nodes required the use of the `eurostat R` package, in particular of its function `get_eurostat_geospatial`, which allows one to store a shapefile that can be used to convert encodings. The encodings

`EL` and `UK` were manually mapped to `GR` and `GB` respectively.
To add the coordinates to each node, for *nuts3* encoded ares, the `get_eurostat_geospatial` function was again used, with a resolution of 60. In the other cases, it was possible to use a shapefile that was provided on the website alongside the rest of the data [4].
After obtaining all the required data, for each country two separate CSV files were produced, with the characteristics described above. An extra file was produced for each country, in which in addition to the edges' information there is also their SCI.

A few graphs have been also proposed to gain a visual understanding of the networks. Figure 3.1 shows the relationship between mean degree and mean betweenness centrality, using as weight the SCI of each edge. It's interesting to see that countries such as India and Germany seem to have very high mean degree but low mean betwenness, which is actually expected because they are subdivided in many interconnected administrative units, thus increasing their degree while lowering the average centrality. Figure 3.2 instead shows instead the mean eigenvector centrality over the total SCI per node.

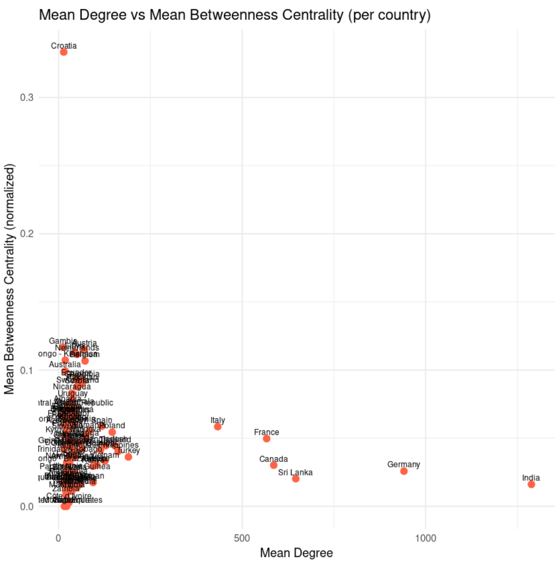

Figure 3.1

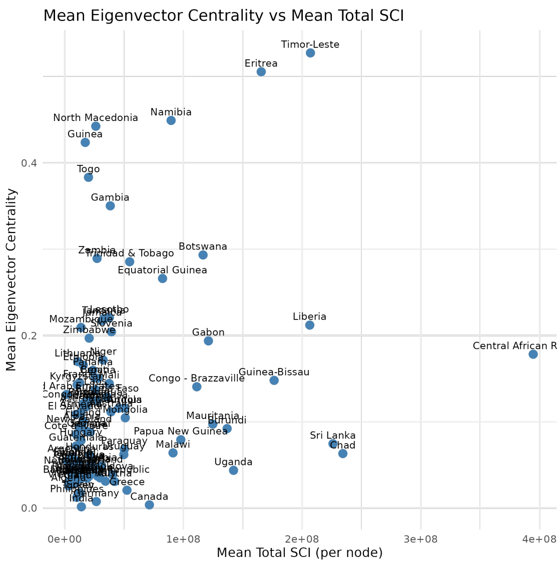

Figure 3.2

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A | Appendix

Task leader(s): *Bains, Arman Singh*

A.1 | Axelrod's Model

A feature of Axelrod's model is that it's decentralized. It could be interesting though to see how the results might change if an external agent is introduced. I tested two cases, differing in how the external agent behaves.
In the first case, the external agent is referred to as "dictator", as it's supposed to represent an autocratic figure which is able to impose his culture on the others. In this variant, according to a certain probability p_d, at each iteration a node at random is picked and the whole culture of the dictator is coercively copied onto him.
In the second case, the external agent is referred to as the suasor, because it should be representing the case in which the agent is able to non-coercively influence the other nodes (for instance like cultural institutions are supposed to do).
The simulations were done using five different cultural features and fifteen traits. The average of five replications over 10^4 iterations can be seen in tables A.1 and A.2. The network used was the same ones defined previously in chapter 1 for the basic implementation of Axelrod's model.

Despite the number of iterations being kept short, they are informative. In addition to them, a simulation of the behavior of the dictator has been run for 10^5 iterations and is present in table A.3, though it was done only for a reduced number of graphs. Still, it highlights that even at subsequent timestamps the trends seen before persist.

In particular, What seems to happen is that an external force may temporarily increase the number of cultural regions, but only because it's forcing random nodes to have a culture different from that of their neighbors, whom are possibly connected to a large connected component. Indeed, if such a force intervenes quite frequently, we see much less fragmentation.
An additional experiment has been run, with the same conditions as in table A.3 but with a slight change: this time, three dictators have been used. It was decided to make sure that their cultural traits were different from each other. Of course, this was inspired by Locke's and Montesquieu's concept of separation of powers [3][5], according to which separating the state in at least three bodies and having multiple powers competing with each other might alleviate the risks of having a single dictator overtaking everything and imposing his will. Table A.4 summarizes the results, showing that, especially at high probability of interaction, having multiple coercive agents will prevent the formation of a single giant connected component, and thus a uniform and homogeneous culture.

Table A.1: Axelrod's model with external dictator

model	p_d	avg regions	sd regions
lattice	0.000	35.6	6.43
lattice	1e-04	44.4	4.67
lattice	0.001	43.8	11.37
lattice	0.010	5.4	4.93
lattice	0.100	2.6	1.67
ba	0.000	21.0	8.86
ba	1e-04	25.0	15.22
ba	0.001	47.6	17.17
ba	0.010	4.4	2.41
ba	0.100	1.2	0.45
ws	0.000	50.8	11.30
ws	1e-04	50.4	4.98
ws	0.001	51.0	8.69
ws	0.010	2.4	0.89
ws	0.100	1.2	0.45
sbm	0.000	48.4	22.14
sbm	1e-04	57.8	21.25
sbm	0.001	54.0	25.62
sbm	0.010	2.8	0.84
sbm	0.100	2.6	1.82

Table A.2: Axlrod's model with external suasor.

model	p_d	avg regions	sd regions
lattice	0.000	28.8	11.92
lattice	1e-04	50.4	5.86
lattice	0.001	43.4	10.48
lattice	0.010	46.0	8.09
lattice	0.100	22.0	5.61
ba	0.000	20.8	16.12
ba	1e-04	22.0	10.72
ba	0.001	27.4	5.41
ba	0.010	22.4	16.70
ba	0.100	18.2	4.44
ws	0.000	36.4	10.06
ws	1e-04	49.0	7.52
ws	0.001	44.0	10.51
ws	0.010	54.6	8.62
ws	0.100	22.0	8.89
sbm	0.000	48.0	33.37
sbm	1e-04	45.2	25.62
sbm	0.001	71.2	7.40
sbm	0.010	64.4	20.56
sbm	0.100	4.6	1.82

Table A.3: Axelrod's model for external dictator and 10^5 iterations

model	p_d	avg regions	sd regions
lattice	0.000	16.4	5.55
lattice	1e-04	23.0	5.79
lattice	0.001	21.0	8.89
lattice	0.010	2.2	1.30
lattice	0.100	2.4	1.52
sbm	0.000	2.2	1.10
sbm	1e-04	3.0	2.92
sbm	0.001	4.0	3.32
sbm	0.010	1.8	0.84
sbm	0.100	1.0	0.00

Table A.4: Axelrod's model with three dictators

model	p_d	avg regions	sd regions
lattice	0.000	29.0	3.61
lattice	1e-04	21.2	4.97
lattice	0.001	30.2	15.37
lattice	0.010	46.0	11.34
lattice	0.100	80.4	3.58
sbm	0.000	1.4	0.55
sbm	1e-04	1.2	0.45
sbm	0.001	4.2	2.39
sbm	0.010	28.8	13.22
sbm	0.100	64.0	3.46

A.2 | Schelling Segregation Model

Schelling's original work sought to answer questions regarding how segregation could arise in an environment like his, which was the USA in the 1960s, where segregation occurred for a long time even after the civil rights movement tried to contrast it [7]. Segregation doesn't happen only in static environments tough, indeed it can emerge in more dynamic situations, where for instance migration and assimilation play a role in shaping communities [8].
Plot A.3 displays the results of adding a migration dynamic to Schelling's 1D model described in chapter 2. In particular, two probabilities were defined, named enter and leave. The first one determines the probability that at the end of each cycle agents of majority type will emigrate, while the second one determines the probability vacancies will be filled with agents of minority types.

Due to limits of computational resources, the plots have been done for only a few

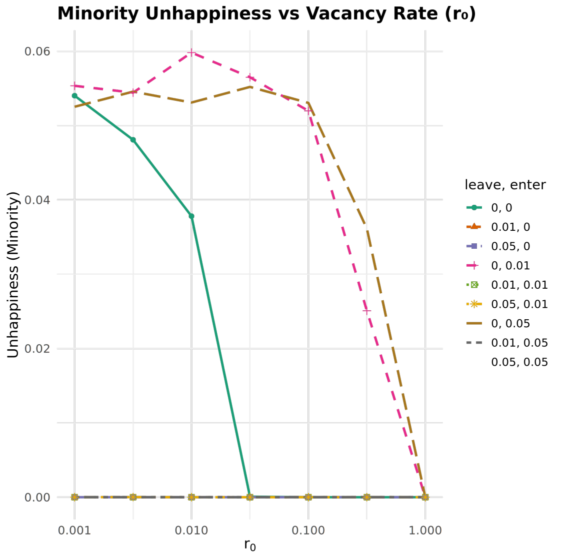

Figure A.1: Unhappiness of agents of the minority type.

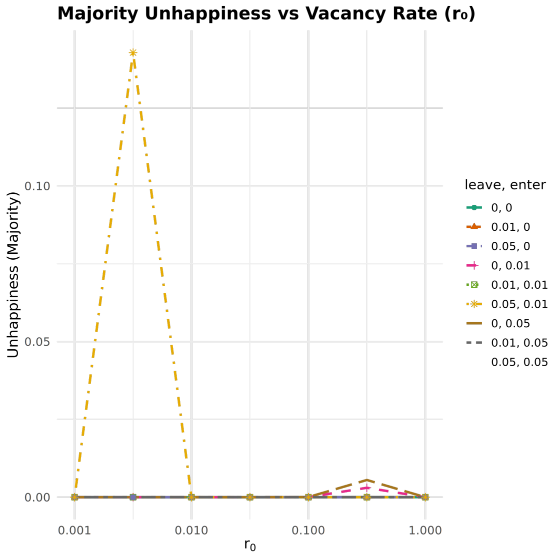

Figure A.2: Unhappiness of agents of the majority type.

Figure A.3

values of r_0 (seven actually), few replications (again, seven) and few iteration steps. Nevertheless, it's interesting to note that it does seem the case that high immigration rate will reduce the unhappiness of agents of minority type, though it might increase the unhappiness of the agents of originally majority type, especially if paired with high emigration rates on their part.

Another overlooked aspect of segregation is how agents interacting with each other might become more alike, as shown in Axelrod's model of social influence [1]. Since incorporating both models could be too complicated for the scope of this project, what will be presented here is a simpler implementation, where on top of the dynamics of the basic 1D Schelling model, at each iteration an unhappy agent, with a certain probability, might become of the same type as the agents surrounding him. The result

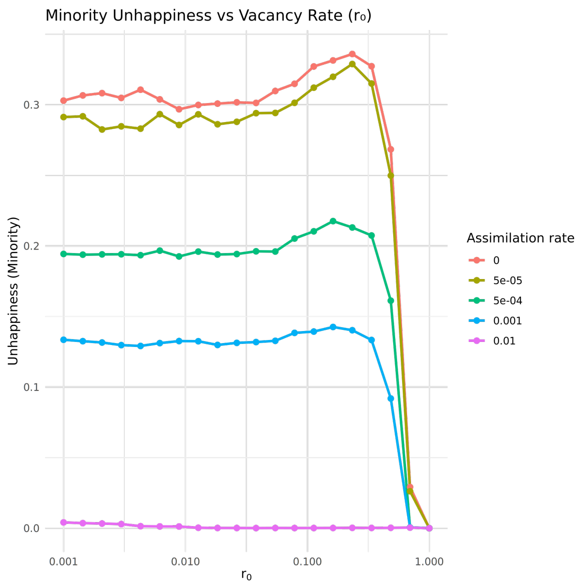

Figure A.4: Unhappiness of minority type agents if assimilation may happen.

of such dynamics is shown in plot A.4, where the average of 20 replications of the dynamics (with few iterations only within each replication though) shows that adding an assimilation rate lowers unhappiness. The implications of this suggest that cultures that are already similar and compatible, and thus more likely to assimilate each other, might not give rise to significant segregation.

At this point, it could be interesting combining both concepts, in order to get a realistic intuition of how immigration might lead to segregation. An attempt to do this is presented in figure A.7, where one can see the average unhappiness across seven replications of several combinations of both assimilation and migration dynamics. Again, high immigration with little-to-no assimilation causes the majority to be surprisingly unhappy, whereas a more balanced net-migration and a good assimilation rate allow both minority and majority agents to have low unhappiness.

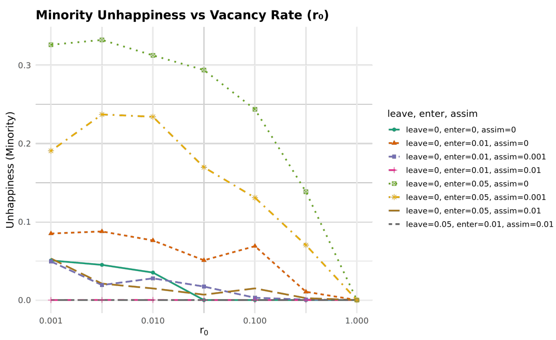

Figure A.5

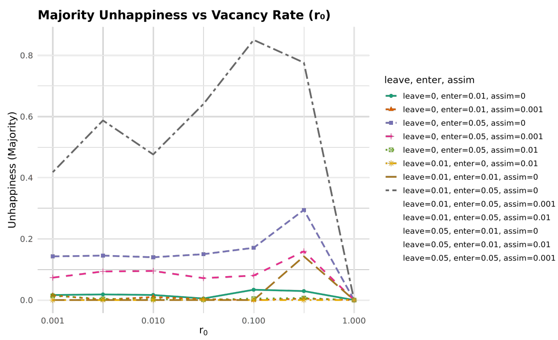

Figure A.6

Figure A.7: Segregation with both migration and assimilation dynamics. Only most relevant combinations have been plotted.