Formal Power Series in Maxima

An Informal presentation
Bertrand Teguia Tabuguia
February 2021

1 Introduction

What if I tell you that to find
the power series formula of f(z):=
exp(z)+cos(z) one has to compute a
differential equation for f,
convert that differential equation
into a recurrence equation and
solve that recurrence equation for
the power series coefficients? Of
course, some other steps should
also be considered, but I will not
bother you with many details.
Let's present how this works for
f(z):=exp(z)+cos(z).

First, I load my package FPS
(Formal Power Series) which
contains all the necessary
functions.

(%i1) batchload(FPS)$

Version 1.0
Copyright (c) May 2020, Bertrand Teguia Tabuguia
https://www.bertrandteguia.com
bteguia@mathematik.uni-kassel.de
University of Kassel, Germany

Yeah! I completed this package
last year with my Ph.D. thesis
which is the main reference for
the mathematical background behind
the algorithms implemented.
Nevertheless, the code gets better
and better since June 2020 when I
started the Maple version of the
package. By the way, I made a
demonstration int the Maple
Conference 2020. Now, I would say
it is 'almost-surely' (I don't
know the measure) 'perfect'. I
recommend mathematician to read my
Ph.D. thesis or the papers
      (1) Teguia Tabuguia, Bertrand.
A variant of van Hoeij's algorithm
to compute hypergeometric term
solutions of holonomic recurrence
equations.
      (2) Teguia Tabuguia, Bertrand
and Koepf, Wolfram. Symbolic
computation of hypergeometric type
power series
That will appear soon...
Let us come back to our example.
First, we compute a differential
equation.

(%i2) `f:exp(z)+cos(z)$`

(%i3) `DE:HolonomicDE(f,F(z));`

(%o3) $\dfrac{d^3}{d z^3} F(z) - \dfrac{d^2}{d z^2} F(z) + \dfrac{d}{d z} F(z) - F(z) = 0$

Then we convert it into a
recurrence equation.

(%i4) `DEtoRE(DE,F(z),a[n]);`

(%o4) $(n+1)(n+2)(n+3) a_{n+3} - (n+1)(n+2) a_{n+2} + (n+1) a_{n+1} - a_n = 0$

This can also be done directly as
follow.

(%i5) RE:FindRE(f,z,a[n]);

(%o5) $(n+1)(n+2)(n+3) a_{n+3} - (n+1)(n+2) a_{n+2} + (n+1)$
$a_{n+1} - a_n = 0$

The next step is to solve that
recurrence equation. Here we use a
new algorithm from my Ph.D.:
mfoldHyper, to compute what we
call m-fold hypergeometric term
solutions.

(%i6) mfoldHyper(RE,a[n]);

(%o6) $[[1,\{\frac{1}{n!}\}],[2,\{\frac{(-1)^n}{(2n)!}\}]]$

Talking about mfoldHyper may need
another presentation... In this
presentation, I only use it.
You may have recognized the
coefficients of the series of
exp(z) and cos(z). Finally we get
the representation

(%i7) FPS(f,z,n);

(%o7) $\left(\sum_{n=0}^{inf}\left(\frac{(-1)^n z^{2n}}{(2n)!}\right)\right) + \sum_{n=0}^{inf}\left(\frac{z^n}{n!}\right)$

Isn't this amazing?! And by this
process, FPS is able to do more!
Even Laurent-Puiseux series are
handled, I mean series with
positive, negative and fractional
powers.

I will only present examples for
expansions in the positive
neighborhood of 0 since the
general case is deduced easily
from it.

(%i8) FPS(log(1+sqrt(z)+z+z^(3/2)),z,n);

(%o8) $\left(\sum_{n=0}^{inf}\left(\frac{(-1)^n z^{n+1}}{n+1}\right)\right) + \sum_{n=0}^{inf}\left((-1)^n \frac{z^{\frac{n+1}{2}}}{n+1}\right)$

(%i9) FPS(sin(z)/z^5+sqrt(1+z)+1/sqrt(1+z),z,n);

$$(\%o9)\quad \left(\sum_{n=0}^{inf} \left(- \frac{(-1)^n\ (2\ n-1)!\ ((2\ n+1)!-3\ (2\ n)!)\ z^n}{4^n\ n!^2\ ((2\ n-1)!-(2\ n)!)} \right) \right) +$$

$$\sum_{n=0}^{inf} \left(\frac{(-1)^n\ z^{2\ (n-2)}}{(2\ n+1)!} \right)$$

Could this be done before? This (symbolic) approach was introduced in 1992 by Wolfram Koepf, but since m-fold hypergeometric terms could not be computed before, the algorithm was reduced to some particular cases, and the result was still amazing at that time. With the results of Marko Petkovsek and Mark van Hoeij, some improvements could be done around the 2000s. Currently the only Computer Algebra System (CAS) that uses a cutting-edge implementation of Koepf's approach on ist release version is Maple; Mathematica also has an implementation but van Hoej's algorithm seems to not be used. But gess what? None of the above computations can be obtained in Maple (see my pdf file: Formal Power Series in Maple). This is normal because the coefficients cannot be detected directly by the Maple implementation. Regarding Maxima, all these computations are new and I am happy to contribute in making it among the top CASs for computing power series. Therefore there is no need to compare my implementation with the built-in Maxima command "powerseries". Let's move on and present more examples.

2 Hypergeometric type power series

You can find a definition of this subtitle in paper (2) mentioned above.

(%i10) `FPS(asin(z)^2,z,n);`

(%o10) $\displaystyle\sum_{n=0}^{inf}\left(\frac{4^n\,n!^2\,z^{2\,(n+1)}}{(n+1)\,(2\,n+1)!}\right)$

(%i11) `FPS(atan(z^2),z,n);`

(%o11) $\displaystyle\sum_{n=0}^{inf}\left(\frac{(-1)^n\,z^{2\,(2\,n+1)}}{2\,n+1}\right)$

(%i12) `FPS(exp(asinh(z)),z,n);`

(%o12) $\displaystyle\frac{\sum_{n=0}^{inf}\left(\frac{(-1)^n\,(2\,n)!\,z^{2\,(n+1)}}{(n+1)\,4^n\,n!^2}\right)}{2}+z+1$

(%i13) `FPS(integrate(erf(t),t,0,z),z,n);`

(%o13) $-\dfrac{\displaystyle\sum_{n=0}^{inf}\left(\frac{(-1)^n\,z^{2\,n}}{(2\,n-1)\,n!}\right)}{\sqrt{\%pi}}$

(%i14) `FPS(exp(z)-2·exp(-z/2)·cos(sqrt(3)·z/2 + %pi/3),z,n);`

(%o14) $3\displaystyle\sum_{n=0}^{inf}\left(\frac{z^{3\,n+1}}{\left(\frac{2}{3}\right)_n\left(\frac{4}{3}\right)_n\,27^n\,n!}\right)$

Note that (x)_{n+1} (I hope you know LaTeX, index notation) = x(x+1)...(x+n). This is what we call raised factorial or Pochhammer symbol.

(%i15) `FPS(z·atan(z)-log(1+z^2)/2,z,n);`

(%o15) $\dfrac{\displaystyle\sum_{n=0}^{inf}\left(\frac{(-1)^n\,z^{2\,(n+1)}}{(n+1)\,(2\,n+1)}\right)}{2}$

(%i16) `declare(k,constant);`

(%o16) *done*

I may need to find how to avoid
such specification for the user in
Maxima...

(%i17) `FPS(z^2/(1-z)^k,z,n);`

(%o17) $\sum\limits_{n=0}^{inf}\left(\dfrac{(k)_n\, z^{n+2}}{n!}\right)$

(%i18) `FPS(z^3·(cos(k·z)+sin(k·z)),z,n);`

(%o18) $\left(\sum\limits_{n=0}^{inf}\left(\dfrac{|k|^{2n}\,(-1)^n\,z^{2n+3}}{(2n)!}\right)\right)+k\sum\limits_{n=0}^{inf}\left(\dfrac{|k|^{2n}\,(-1)^n\,z^{2(n+2)}}{(2n+1)!}\right)$

(%i19) `FPS(exp(z)·cos(z),z,n);`

(%o19) $\left(\sum\limits_{n=0}^{inf}\left(-\dfrac{(-1)^n\,4^n\,z^{4n+3}}{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n(2n+1)(4n+1)(4n+3)\,64^n\,(2n)!}\right)\right)$

$+\left(\sum\limits_{n=0}^{inf}\left(\dfrac{(-1)^n\,4^n\,z^{4n+1}}{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n(4n+1)\,64^n\,(2n)!}\right)\right)+$

$\sum\limits_{n=0}^{inf}\left(\dfrac{(-1)^n\,4^n\,z^{4n}}{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n\,64^n\,(2n)!}\right)$

(%i20) `FPS(sin(z)+z·cos(z),z,n);`

(%o20) $\sum\limits_{n=0}^{inf}\left(\dfrac{2(n+1)(-1)^n\,z^{2n+1}}{(2n+1)(2n)!}\right)$

(%i21) `FPS((-1/2·x + 1/6·x^3)·atan(x) + (-1/4·x^2 + 1/12)·log(x^2 + 1)`
`       + 5/12·x^2 + 1/4,x,n);`

(%o21) $\left(\sum\limits_{n=0}^{inf}\left(\dfrac{(-1)^n\,x^{2(n+2)}}{4(n+1)(n+2)(2n+1)(2n+3)}\right)\right)+\dfrac{1}{4}$

(%i22) `declare(q1,constant,q2,constant)$`

(%i23) `FPS(1/(q1-z^2)/(q2-z^3),z,n);`

(%o23)
$$\left(\sum_{n=0}^{inf}\left(-\frac{q1\,q2^{-n-1}\,z^{3n+2}}{q2^2-q1^3}\right)\right)+\left(\sum_{n=0}^{inf}\left(-\frac{z^{3n+1}}{(q2^2-q1^3)\,q2^n}\right)\right)+$$

$$\left(\sum_{n=0}^{inf}\left(\frac{z^{2n+1}}{(q2^2-q1^3)\,q1^n}\right)\right)+\left(\sum_{n=0}^{inf}\left(-\frac{q1^2\,q2^{-n-1}\,z^{3n}}{q2^2-q1^3}\right)\right)+$$

$$\sum_{n=0}^{inf}\left(\frac{q2\,q1^{-n-1}\,z^{2n}}{q2^2-q1^3}\right)$$

(%i24) `FPS(tan(3·atan(z^2)),z,n);`

(%o24)
$$\left(\sum_{n=0}^{inf}(8\,3^{n-1}\,z^{4n+2})\right)+\frac{z^2}{3}$$

(%i25) `FPS((sin(z)+cos(z))^3,z,n);`

(%o25)
$$\left(\sum_{n=0}^{inf}\left(\frac{3\left((-1)^n+(-9)^n\right)z^{2n+1}}{2\,(2n+1)\,(2n)!}\right)\right)+$$

$$\sum_{n=0}^{inf}\left(\frac{\left(3\,(-1)^n-(-9)^n\right)z^{2n}}{2\,(2n)!}\right)$$

(%i26) `FPS(atan(z)+asin(z),z,n);`

(%o26)
$$\sum_{n=0}^{inf}\left(\frac{\left((2n)!+(-1)^n\,4^n\,n!^2\right)z^{2n+1}}{(2n+1)\,4^n\,n!^2}\right)$$

(%i27) `FPS(cos(z)^3+log(1+z^2),z,n);`

(%o27)
$$\Bigg(\sum_{n=0}^{inf}\left(\frac{\left(4\,(-1)^n\,(2\,(n+1))!+n\,(-9)^{n+1}+(-9)^{n+1}-3\,n\,(-1)^n-3\,(-1)^n\right)z^{2\,(n+1)}}{4\,(n+1)\,(2\,(n+1))!}\right)\Bigg)+1$$

(%i28) `FPS(log(1+z+z^2+z^3)+sqrt(1+z^3),z,n);`

(%o28)
$$\left(\sum_{n=0}^{inf} \left(\frac{(-1)^n \, z^{2\,(n+1)}}{n+1} \right) \right) + \left(\sum_{n=0}^{inf} \left(\frac{(-1)^n \, z^{n+1}}{n+1} \right) \right) +$$
$$\sum_{n=0}^{inf} \left(-\frac{(-1)^n \, (2\,n)! \, z^{3\,n}}{(2\,n-1) \, 4^n \, n!^2} \right)$$

(%i29) `FPS(exp(z^2)+sinh(z)+atan(z^2),z,n);`

(%o29)
$$\left(\sum_{n=0}^{inf} \left(\frac{(-1)^n \, z^{4\,n+2}}{2\,n+1} \right) \right) + \left(\sum_{n=0}^{inf} \left(\frac{z^{2\,n+1}}{(2\,n+1) \, (2\,n)!} \right) \right) +$$
$$\sum_{n=0}^{inf} \left(\frac{z^{2\,n}}{n!} \right)$$

(%i30) `FPS(atan(z/(1+z^2)),z,n);`

(%o30)
$$\sum_{n=0}^{inf} \left(\frac{ \left(\frac{\sqrt{5} \, (\sqrt{5}+3)^n \, (-1)^n}{2^n} + \frac{(\sqrt{5}+3)^n \, (-1)^n}{2^n} - \frac{(\sqrt{5}-3)^n \, \sqrt{5}}{2^n} + \frac{(\sqrt{5}-3)^n}{2^n} \right) z^{2\,n+1}}{2\,(2\,n+1)} \right)$$

(%i31) `Nmax:11$`

This variable is used to change
the maximal order of the
differential equation to be
computed.
In the following case, a
differential equation of order 11
is needed.

(%i32) `FPS((exp(z)-sin(z))^3,z,n);`

(%o32)
$$\left(\sum_{n=0}^{inf} \left(\frac{3 \left((-9)^{n+1} + (-1)^n \right) z^{2\,(n+1)+1}}{8\,(n+1)\,(2\,n+1)\,(2\,n+3)\,(2\,n)!} \right) \right) +$$
$$\sum_{n=0}^{inf} \left(-\frac{\left(3\,5^{n/2}\,\cos(atan(2)\,n) + 6\,5^{n/2}\,\sin\left(atan\left(\frac{1}{2}\right)n\right) - 2\,3^n - 3 \right) z^n}{2\,n!} \right)$$

I set the default value of Nmax to
10. Note that such example may
increase the CPU time use in the
computations.

2.1 Detecting identities

One of the main reason to have a
direct approach is because it can
simplify inputs to polynomials and
constants in particular. Therefore
it helps to discover identities.

(%i33) `FPS(asin(z)+acos(z),z,n);`

(%o33) $\dfrac{\%pi}{2}$

Note that no prior knowledge is
used, the result is obtained by
means of symbolic computations
only.

(%i34) `FPS(cos(4·acos(z)),z,n);`

(%o34) $8\,z^4 - 8\,z^2 + 1$

(%i35) `FPS(sin(2·asin(z))+cos(3·acos(z)),z,n);`

(%o35) $\left(\displaystyle\sum_{n=0}^{inf}\left(-\dfrac{2\,(2\,n)!\,z^{2\,n+1}}{(2\,n-1)\,4^n\,n!^2}\right)\right) + 4\,z^3 - 3\,z$

(%i36) `FPS(cos(2·z)-cos(z)^2+sin(z)^2,z,n);`

(%o36) 0

(%i37) `f:atan((z+z^2)/(1-z^3));`

(%o37) $atan\left(\dfrac{z^2+z}{1-z^3}\right)$

(%i38) `FPS(f,z,n);`

(%o38) $\left(\displaystyle\sum_{n=0}^{inf}\left(\dfrac{sin\left(\dfrac{\%pi\,(n+1)}{2}\right)z^{2\,(n+1)}}{n+1}\right)\right) +$

$\displaystyle\sum_{n=0}^{inf}\left(\dfrac{sin\left(\dfrac{\%pi\,(n+1)}{2}\right)z^{n+1}}{n+1}\right)$

In this previous example, the solution of the underlying recurrence equation contains complex numbers which reduce to real numbers when they are written in their algebraic representations.

(%i39) g:atan(z)+atan(z^2);

(%o39) $atan(z^2)+atan(z)$

(%i40) FPS(g,z,n);

(%o40) $\left(\sum_{n=0}^{inf}\left(\frac{sin\left(\frac{\%pi\,(n+1)}{2}\right)\,z^{2\,(n+1)}}{n+1}\right)\right)+$

$\sum_{n=0}^{inf}\left(\frac{sin\left(\frac{\%pi\,(n+1)}{2}\right)\,z^{n+1}}{n+1}\right)$

(%i41) FPS(f-g,z,n);

(%o41) 0

(%i42) declare(y,constant);

(%o42) $done$

(%i43) FPS(atan((z+y)/(1-z·y)),z,n);

(%o43) $\left(\sum_{n=0}^{inf}\left(\frac{(-1)^n\,z^{2\,n+1}}{2\,n+1}\right)\right)+atan(y)$

(%i44) f:sqrt((1-sqrt(1-z))/z);

(%o44) $\frac{\sqrt{1-\sqrt{1-z}}}{\sqrt{z}}$

(%i45) FPS(f,z,n);

(%o45) $\frac{\sum_{n=0}^{inf}\left(\frac{\left(\frac{1}{4}\right)_n\,\left(\frac{3}{4}\right)_n\,4^n\,z^n}{(2\,n+1)!}\right)}{\sqrt{2}}$

(%i46) g:(sqrt(1+sqrt(z))-sqrt(1-sqrt(z)))/(sqrt(2·z));

(%o46) $\frac{\sqrt{\sqrt{z}+1}-\sqrt{1-\sqrt{z}}}{\sqrt{2}\,\sqrt{z}}$

(%i47) FPS(g,z,n);

$$(\%o47) \quad \frac{\sum_{n=0}^{inf} \left(\frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n 4^n z^n}{(2n+1)!} \right)}{\sqrt{2}}$$

(%i48) FPS(f-g,z,n);

$(\%o48) \quad 0$

Isn't this nice?!

2.2 Some Puiseux series (series with fractional powers)

(%i49) FPS(asin(z^(1/3)),z,n);

$$(\%o49) \quad \sum_{n=0}^{inf} \left(\frac{(2n)! \, z^{\frac{2n+1}{3}}}{(2n+1) \, 4^n \, n!^2} \right)$$

(%i50) FPS(log(1+z^(1/5)),z,n);

$$(\%o50) \quad \sum_{n=0}^{inf} \left(\frac{(-1)^n \, z^{\frac{n+1}{5}}}{n+1} \right)$$

(%i51) FPS(1/(1-z^(1/7)),z,n);

$$(\%o51) \quad \sum_{n=0}^{inf} \left(z^{n/7} \right)$$

(%i52) FPS(cos(sqrt(z))+sin(z^(3/4)),z,n);

$$(\%o52) \quad \left(\sum_{n=0}^{inf} \left(\frac{(-1)^n \, z^{\frac{6n+3}{4}}}{(2n+1) \, (2n)!} \right) \right) + \sum_{n=0}^{inf} \left(\frac{(-1)^n \, z^n}{(2n)!} \right)$$

(%i53) FPS(atan(sqrt(z))+asinh(z^(1/3)),z,n);

$$(\%o53) \quad \left(\sum_{n=0}^{inf} \left(\frac{(-1)^n \, z^{\frac{6n+3}{6}}}{2n+1} \right) \right) + \sum_{n=0}^{inf} \left(\frac{(-1)^n \, (2n)! \, z^{\frac{4n+2}{6}}}{(2n+1) \, 4^n \, n!^2} \right)$$

(%i54) FPS(sqrt(sqrt(8·z^3+1)-1)+sqrt(13·z^4+7),z,n);

$$(\%o54) \quad \left(\sum_{n=0}^{inf} \left(\frac{2 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n (-8)^n 4^n z^{\frac{6n+3}{2}}}{(2n+1)(2n)!} \right) \right) +$$

$$\sum_{n=0}^{inf} \left(- \frac{7^{1/2-n} (-13)^n (2n)! z^{4n}}{(2n-1) 4^n n!^2} \right)$$

The sum of these two latter series
can be written as a sum of four
series as shown below.

(%i55) FPS(asin(z^(1/3))+log(1+z^(1/5)),z,n);

$$(\%o55) \quad \left(\sum_{n=0}^{inf} \left(\frac{(2n)! z^{\frac{10n+5}{15}}}{(2n+1) 4^n n!^2} \right) \right) + \left(\sum_{n=0}^{inf} \left(\frac{3 (-1)^n z^{\frac{3n+2}{15}}}{3n+2} \right) \right) +$$

$$\left(\sum_{n=0}^{inf} \left(- \frac{3 (-1)^n z^{\frac{3n+1}{15}}}{3n+1} \right) \right) + \sum_{n=0}^{inf} \left(\frac{3 (-1)^n z^{\frac{n+1}{15}}}{n+1} \right)$$

One can also recover the basic
representation by using
map(lambda([f],FPS(f,z,n)),.). The
above result is related to a
linear system solved in the
algorithm. When there are many
possible representations, the
algorithm selects a particular one.

(%i56) map(lambda([f],FPS(f,z,n)),asin(z^(1/3))+log(1+z^(1/5)));

$$(\%o56) \quad \left(\sum_{n=0}^{inf} \left(\frac{(2n)! z^{\frac{2n+1}{3}}}{(2n+1) 4^n n!^2} \right) \right) + \sum_{n=0}^{inf} \left(\frac{(-1)^n z^{\frac{n+1}{5}}}{n+1} \right)$$

I can provide many more examples,
but let's stop here and see other
kind of representations.

2.3 Series related to hypergeometric type series

Some formulas do not fit into the
category of hypergeometric type
series. However we can still
connect their representations with
hypergeometric type series. This
is the situation for products and
reciprocals of hypergeometric type
series. The outputs are given as a
recursive formula, by application
of the Cauchy product rule or
both.
Our first example is the
generating function of the
Bernoulli numbers.

(%i57) FPS(1/(exp(z)-1),z,n);

(%o57) $\left[\sum_{n=0}^{inf} \left\langle A_n \, z^n \right\rangle, A_n = \sum_{k=1}^{n} \left(- \frac{A_{n-k}}{k!} \right), A_0 = 1 \right]$

(%i58) FPS(exp(z)·log(1+z),z,n);

(%o58) $\sum_{n=0}^{inf} \left(\left(\sum_{k=0}^{n} \left(\frac{(-1)^{n-k}}{k!\,(n-k+1)} \right) \right) z^{n+1} \right)$

(%i59) FPS(log(1+z)/exp(z),z,n);

(%o59) $\sum_{n=0}^{inf} \left((-1)^n \left(\sum_{k=0}^{n} \left(\frac{1}{k!\,(n-k+1)} \right) \right) z^{n+1} \right)$

(%i60) FPS(tan(z),z,n);

(%o60) $\left[\sum_{n=0}^{inf} \left(\left(\sum_{k=0}^{n} \left(\frac{A_k \, (-1)^{n-k}}{(2n-2k+1)!} \right) \right) z^{2n+1} \right), A_k = \right.$

$\left. \sum_{j=1}^{k} \left(- \frac{A_{k-j} \, (-1)^j}{(2j)!} \right), A_0 = 1 \right]$

(%i61) FPS(1/atan(z),z,n);

(%o61) $\left[\sum_{n=0}^{inf} \left\langle A_n \, z^{2n-1} \right\rangle, A_n = \sum_{k=1}^{n} \left(- \frac{(-1)^k \, A_{n-k}}{2k+1} \right), A_0 = 1 \right]$

(%i62) `FPS(atan(z)·sin(z),z,n);`

(%o62) $\sum\limits_{n=0}^{inf}\left((-1)^n\left(\sum\limits_{k=0}^{n}\left(\dfrac{1}{(2\,k+1)\,(2\,n-2\,k+1)\,!}\right)\right)z^{2\,n+2}\right)$

(%i63) `FPS(atan(z)^2,z,n);`

(%o63) $\sum\limits_{n=0}^{inf}\left((-1)^n\left(\sum\limits_{k=0}^{n}\left(\dfrac{1}{(2\,k+1)\,(2\,n-2\,k+1)}\right)\right)z^{2\,n+2}\right)$

2.4 **Recursive representations from recurrence equations**

In certain cases the algorithm finds a recurrence equation but does not find a representation related to hypergeometric type series. In this case it uses the recurrence equation obtained to write a recursive formula for the coefficients.

(%i64) `FPS(exp(z+z^2),z,n);`

(%o64) $\left[\sum\limits_{n=0}^{inf}\left\langle A_n\,z^n\right\rangle, A_{n+2}=\dfrac{A_{n+1}+2\,A_n}{n+2}, n>=0, [A_0=1, A_1=1]\right]$

(%i65) `FPS(sin(1+z+z^2),z,n);`

(%o65) $\left[\sum\limits_{n=0}^{inf}\left\langle A_n\,z^n\right\rangle, A_{n+5}=\right.$

$\dfrac{-2\,(n+2)\,(n+4)\,A_{n+4}-A_{n+3}-6\,A_{n+2}-12\,A_{n+1}-8\,A_n}{(n+4)\,(n+5)}, n>=0, [A_0=$

$sin(1), A_1=cos(1), A_2=-\dfrac{sin(1)-2\,cos(1)}{2}, A_3=-$

$\left.\dfrac{6\,sin(1)+cos(1)}{6}, A_4=-\dfrac{11\,sin(1)+12\,cos(1)}{24}\right]\right]$

(%i66) `FPS(atan(z)·exp(z),z,n);`

(%o66) $\left[\sum_{n=0}^{inf} \left\langle A_n \, z^n \right\rangle \right), A_{n+4} =$

$$\frac{2\,(n+3)\,A_{n+3} - \left\langle (n+2)^2 + n + 3 \right) A_{n+2} + 2\,(n+2)\,A_{n+1} - A_n}{(n+3)\,(n+4)}, n >= 0, [A_0$$

$$= 0, A_1 = 1, A_2 = 1, A_3 = \frac{1}{6}]]$$

Observe that this can be used to compute truncated series. My package contains the function 'Taylor' with the same syntax as the internal Maxima 'taylor' to compute Taylor polynomials.

(%i67) `Taylor(sin(z),z,0,7);`

(%o67) $-\frac{z^7}{5040} + \frac{z^5}{120} - \frac{z^3}{6} + z$

It is easy to see that for larger order expansions this approach has a better efficiency than what is internally implemented. Let us consider truncated series of arctan(z)^2.

(%i68) `Taylor(atan(z)^2,z,0,10);`

(%o68) $\frac{563\,z^{10}}{1575} - \frac{44\,z^8}{105} + \frac{23\,z^6}{45} - \frac{2\,z^4}{3} + z^2$

(%i69) `taylor(atan(z)^2,z,0,10);`

(%o69)/T/ $z^2 - \frac{2\,z^4}{3} + \frac{23\,z^6}{45} - \frac{44\,z^8}{105} + \frac{563\,z^{10}}{1575} + \ldots$

(%i70) `showtime:true$`

Evaluation took 0.0000 seconds (0.0000 elapsed) using 0 bytes.

(%i71) `Taylor(atan(z)^2,z,0,1000)$`

Evaluation took 0.8900 seconds (0.8880 elapsed) using 385.493 MB.

(%i72) `taylor(atan(z)^2,z,0,1000)$`

Evaluation took 9.5630 seconds (9.5980 elapsed) using 1248.239 MB.

(%i73) `showtime:false$`

Really interesting, right?

3 Non-holonomic power series

I will only present three examples for this section. This are series that do not satisfy linear differential equations. The outputs are given as recursive formulas from computed algebraic ordinary differential equations. The most important aspect of this approach is that it can prove difficult identities.

(%i74) FPS(log(1+sin(z)),z,n);

(%o74) $\left[\sum_{n=0}^{inf} \left\langle A_n \, z^n \right\rangle, A_{n+4} = \right.$

$$\left(-\left(\sum_{k=1}^{n} ((k+1)\,(k+2)\,A_{k+2}\,(n-k+2)\,A_{n-k+2}) \right) - (n+2)\,(n+3)\,A_{n+3} + (n+2)\,A_{n} \right)$$

$/((n+2)\,(n+3)\,(n+4)), n >= 0, [A_0 = 0, A_1 = 1, A_2 = -\dfrac{1}{2}, A_3 = \dfrac{1}{6}$

$] \,]$

(%i75) f:(1+tan(z))/(1-tan(z));

(%o75) $\dfrac{tan(z)+1}{1-tan(z)}$

(%i76) FPS(f,z,n);

(%o76) $\left[\sum_{n=0}^{inf} \left\langle A_n \, z^n \right\rangle, A_{n+3} = \right.$

$$\dfrac{2 \left(\sum_{k=1}^{n} ((k+1)\,A_{k+1}\,A_{n-k+1}) \right) + 2\,(n+2)\,A_{n+2} + 4\,A_{n+1}}{(n+2)\,(n+3)}, n >= 0, [A_0 = 1$$

$, A_1 = 2, A_2 = 2 \,] \,]$

(%i77) g:exp(2·atanh(sin(2·z)/(1+cos(2·z))));

(%o77) $\%e^{2\,atanh\left(\frac{sin(2\,z)}{cos(2\,z)+1} \right)}$

(%i78) `FPS(g,z,n);`

(%o78) $\left[\sum_{n=0}^{inf} \left\langle A_n z^n \right), A_{n+4} = \left(3 \right.\right.$

$$\left(\sum_{k=1}^{n} ((k+1)\ (k+2)\ A_{k+2}\ (n-k+2)\ A_{n-k+2}) \right) -$$

$$\left(\sum_{k=1}^{n} ((k+1)\ (k+2)\ (k+3)\ A_{k+3}\ A_{n-k+1}) \right) - 4$$

$$\left(\sum_{k=1}^{n} ((k+1)\ A_{k+1}\ A_{n-k+1}) \right) + 6\ (n+2)\ (n+3)\ A_{n+3} + 8\ (n+2)$$

$A_{n+2} - 24\ A_{n+1} \big) / ((n+2)\ (n+3)\ (n+4)), n >= 0, [A_0 = 1, A_1 = 2,$

$A_2 = 2, A_3 = \dfrac{8}{3}]]$

Yeah, this approch does not always
yield nice outputs. Although we
get a different representation for
the two latter expressions (this
does not happen with my Maple
package), the algorithm is able to
prove that they are identical.

(%i79) `FPS(f-g,z,n);`

(%o79) *0*

4 Conclusion

As I said, I just presented a few
examples. I hope you enjoyed
seeing those. My Maxima package
will soon be available on the
Github repository of
maxima-packages. Feel free to use
it and contact me if you have any
question.