Modeling Complex Systems Assignment 1

Frederick Carlson and Shannon O'Connor

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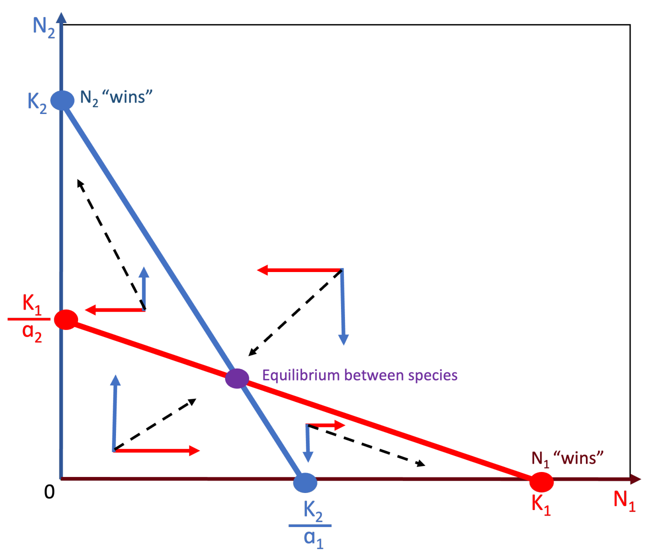
Assignment #1 of 3

Question 1

Answer:

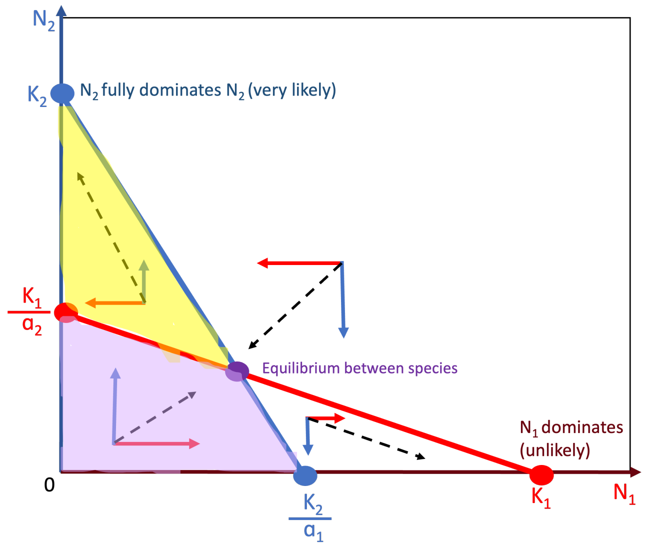


This case shows an unstable equilibrium point for the Lotka-Volterra competition model. This means that the system can arrive at 3 possible scenarios: either species may win and "take all", or there may be equilibrium between species. Where the system ends up depends on where it starts.

Question 2

Answer:

Periodically reducing N_1 and N_2 to ρN_1 and ρN_2 could preserve healthy levels of both N_1 and N_2 if done when N_1 and N_2 are within certain ranges. Assuming this is a scenario where there is an unstable equilibrium point (as shown in Question 1), reducing N_1 and N_2 to ρN_2 and ρN_1 would be beneficial if done when N_2 is greater than $\frac{K_1 - N_1}{\alpha_2}$ and less than $K_2 - \alpha_1 N_1$, and N_2 is greater than 0 and less than $\frac{K_2\alpha_2 - K_1}{\alpha_1\alpha_2 - 1}$. In this region of the graph (shaded yellow in the plot below, between the two isoclines to the left of their intersection), the system is moving towards N_2 dominating.

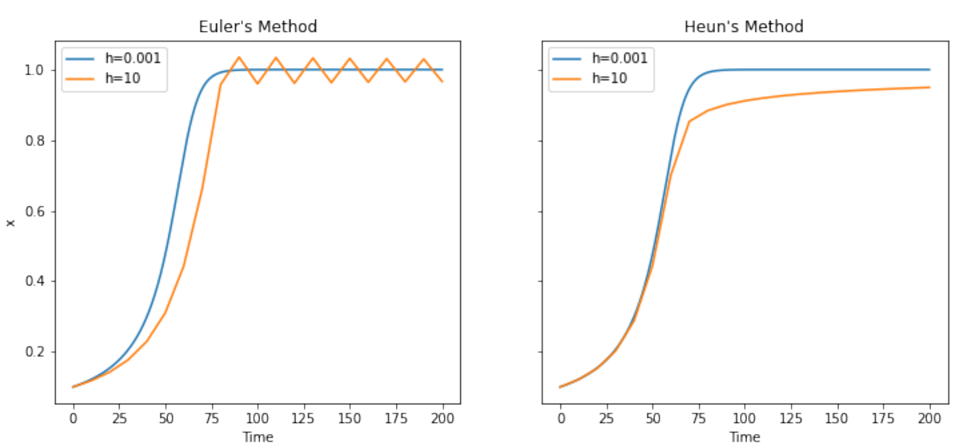


Reducing both N_1 and N_2 in this scenario would bring the populations into the purple region of the graph where N_1 is still greater than 0 but N_2 is now less than $\frac{K_1-N_1}{\alpha_2}$. For this to be effective, ρ must be set such that N_2 is reduced to below $\frac{K_1-N_1}{\alpha_2}$ and N_1 is greater than 0. In this region of the plot, both N_1 and N_2 are increasing, and the system is moving towards the equilibrium point.

Question 3

Answer:

Euler's forward method is very similar to a discrete time model. In Euler's method, small increases and decreases in x (a state variable) are accumulated over changes in time (typically written as Δt or step-size h). This is a discretized formula whose solution approaches the true solution of the original (continuous) differential equation as the change in time approaches 0 ($\lim \Delta T \to 0$ or $h \to 0$).
As the discrete time steps get smaller and smaller, the model becomes a closer and closer approximation of a continuous-time model. This is similar to how the summed areas of smaller and smaller triangles within a circle can become a more and more accurate approximation of the area of the circle. In a discrete model, the change in time (Δt) is always +1.0, but in the Euler method, Δt becomes smaller and smaller.

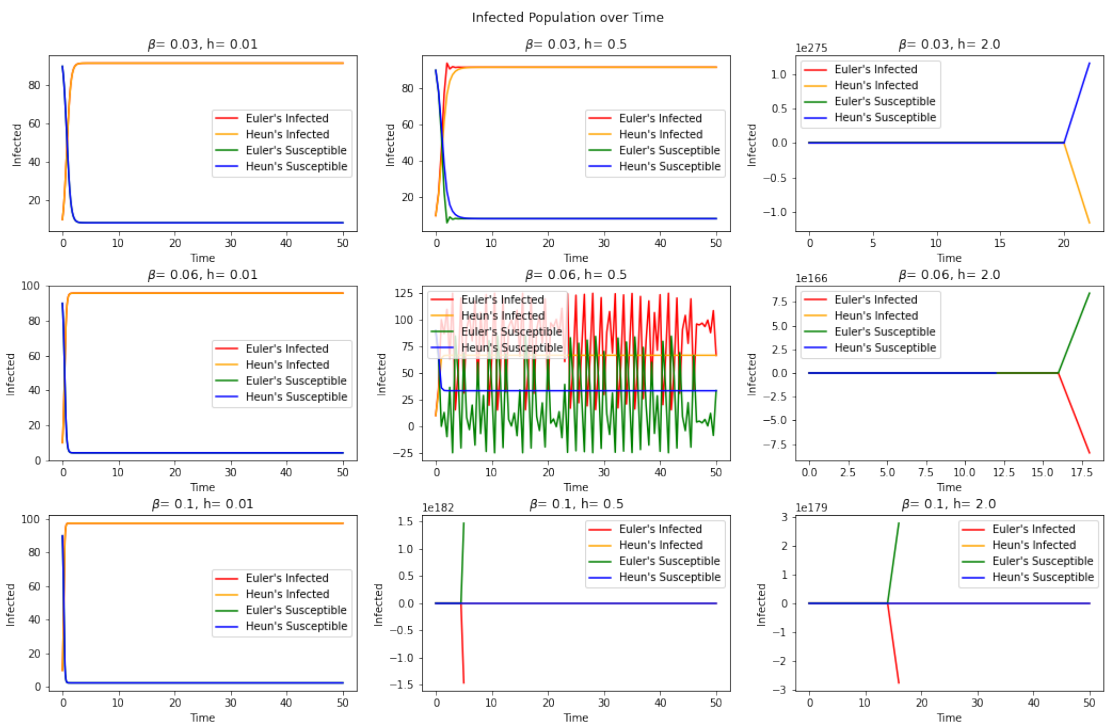


Heun's method adds a step to Euler's method to better approximate the continuous model by considering where Euler's method would go and therefore considering more points in its approximation. In the above plots, Euler's method initially overshoots more when h is large, and then bounces back and forth around the asymptote of the continuous graph. Heun's method shows less difference in the overall shape of the graph with a low (h=0.001) vs high (h=10) h value, indicating a better approximation.

Question 4

Answer:

Both Heun's and Euler's methods perform similarly when infection rate β and step size h are both low and even when β is high but h is really low. When $h = 0.01$, both methods perform similarly for all β. However, when h is larger, Euler's method performs much less well. Heun's method also performs worse when h is larger but is less affected than Euler's method. This is shown in the plot below, where both results give results that make sense when h=0.01, but both methods give odd results at higher h and β values. When $h = 0.5$ and $\beta = 0.06$, Heun's method seems to perform better than Euler's. However neither method performs well when $h = 2$ regardless of Beta, and neither method performs well when $h = 0.5$ and $\beta = 0.1$. Both step size and the parameter β affect how the methods perform, and this is iMportant to consider when applying either method to other models.



Question 5

Answer:

The Taylor Series Expansion of $f(x(t))$:

$$x(t+h) = x(t) + h \cdot f'(x(t)) + \frac{h^2}{2} \cdot f''(x(t)) + Oh^3$$

Take derivative of x(t):

$$x'(t+h) = f(x'(t)) + h \cdot f''(x(t)) + \frac{h^2}{2} \cdot f'''(x(t)) + Oh^2$$

Heun's method:

$$x(t+h) \approx x(t) + \left(\frac{f(x) + [x(t) + f(x) \cdot h]}{2} \right) \cdot h$$

This can be rewritten:

$$x(t+h) \approx x(t) + \frac{h}{2} \cdot [F'(x(t)) + F'(x(t+h))]$$

Now, can substitute $x'(t+h)$ from the Taylor Series:

$$x(t+h) \approx x(t) + \frac{h}{2} \cdot [F'(x(t)) + F'(x(t)) + h \cdot F''(x(t)) + \frac{h^2}{2} \cdot F'''(x(t)) + Oh^2]$$

The $\frac{h^2}{2} \cdot F'''(x(t))$ term can be dropped.
The terms up to the second order are the same for both the Taylor Series and Heun's approximation. This is why the global precision of Heun's method is h^2. The higher terms are basically irrelevant.

Question 6

Answer:

When prey is abundant, then predator population increases. This leads to a decrease in prey followed by a decline in predators, and this cycle repeats. These characteristics of the model mean we have a direct cycle relationship that can be tested for stability relatively easily. The Lotka-Volterra model is subject to chaos because it is sensitive to its initial conditions. As we saw in class, the initial populations of prey and predator affect the trajectory of the model.
The SIS model is noncyclic because of a reinfection $\rightarrow$ recovery $\rightarrow$ susceptibility sub-loop. This model also does not show sensitivity to initial conditions. Chaos is not possible and therefore neither are cycles.
The SIRS and SIR models are typically noncyclic for similar reasons. The SIR model also is uni-directional. Once an individual within the model enters R, that individual cannot return to S or I. Because the classic SIR model is within a closed population, this eliminates the possibility for there to be cycles. The SIRS model, with very specific parameters, can show a similar cyclical relationship as the predator-prey model, but this is unlikely. More complex SIR or SIRS models or those with wild parameters may show chaotic behavior. For example, if the rate of loss of immunity in a SIRS model is very high, the SIRS model may show oscillations that move into the chaotic range.

Question 7

Our ODE models a very simplified version of cartel economics in which P(t) is the price of the product at time t, Q(t) is the quantity of the product produced by the cartel at time t, and C(t) is the cost of producing the product at time t.

$$\frac{dP}{dt} = \alpha(P - C)$$
$$\frac{dQ}{dt} = \beta(P - C)$$
$$\frac{dC}{dt} = \gamma C$$

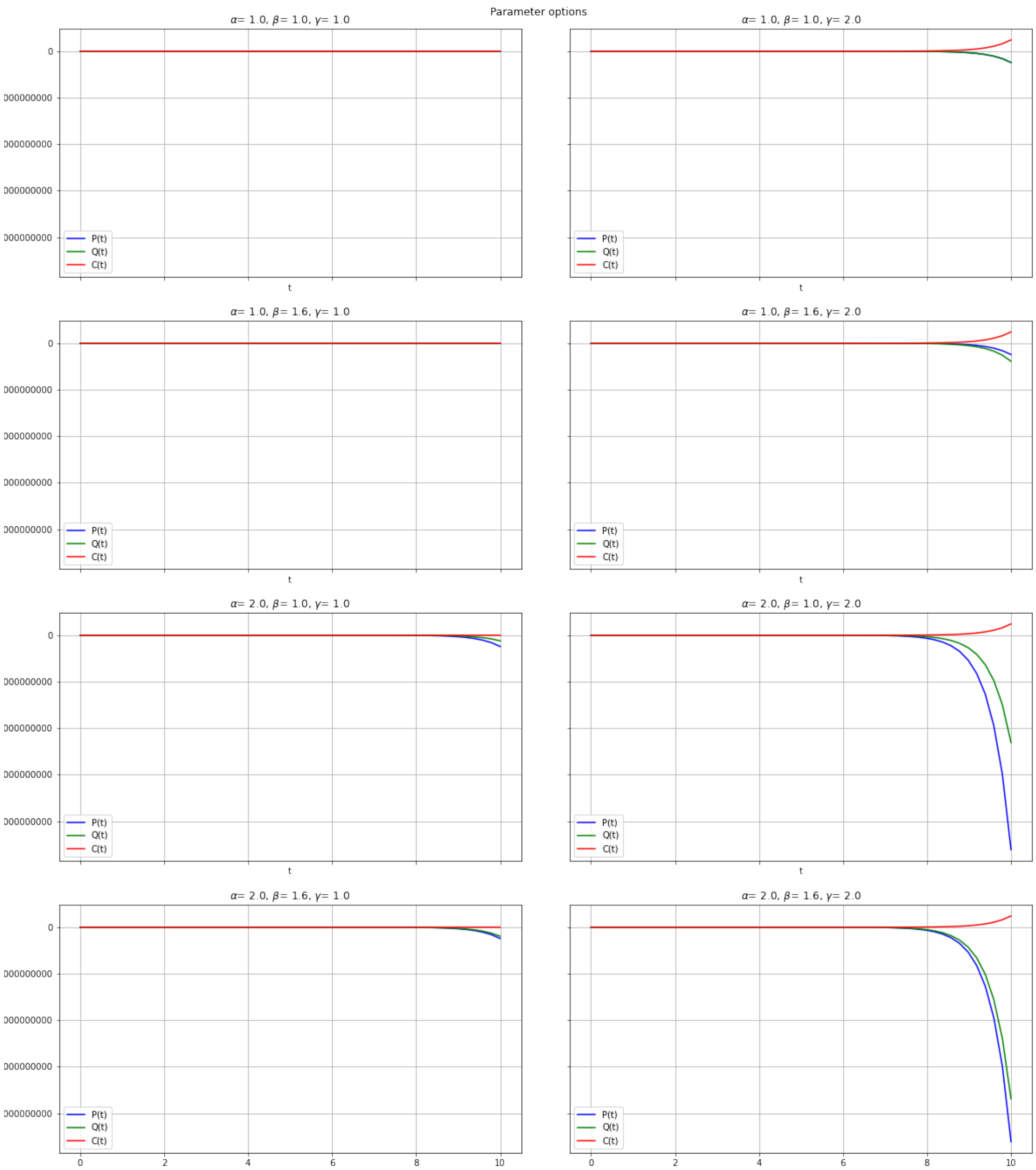
Equation one models the rate of change of price over time. The term (P - C) represents the profit margin, i.e., the difference between the price and the cost. The constant α represents the responsiveness of the price to the profit margin.

Equation two models the rate of change of quantity over time. The term (P - C) again represents the profit margin. The constant β represents the responsiveness of the price to the profit margin.

Equation three models the rate of change of cost over time. The term γC implies that the cost is growing at a rate proportional to its current value.

In this system, α, β, and γ are parameters that represent the speed at which the price, quantity, and cost change in response to the difference between the price and the cost. In other words, these parameters reflect the sensitivity of price, quantity and cost to the ratio of price:cost.

The system reflects that while the price to cost ratio has an effect on price and quantity produced, the cost of production is independent, changing at a rate of γC and not $\gamma(P - C)$. Increases in price will respond very quickly to increases in cost, but increases in cost will not respond to increases in price. This scenario reflects real world conditions: a business/cartel can easily increase price to respond to increases in cost, but a business typically has less control over cost. Changes in quantity are also manipulable by the cartel: based on the difference between price and cost, the cartel can choose to increase or decrease the quantity produced in the interest of profit.
This model simplifies real-world conditions by assuming that cost of producing product at time t has the same effects on price and quantity of product produced regardless of it the cost is driven by cost of materials or cost of labor.



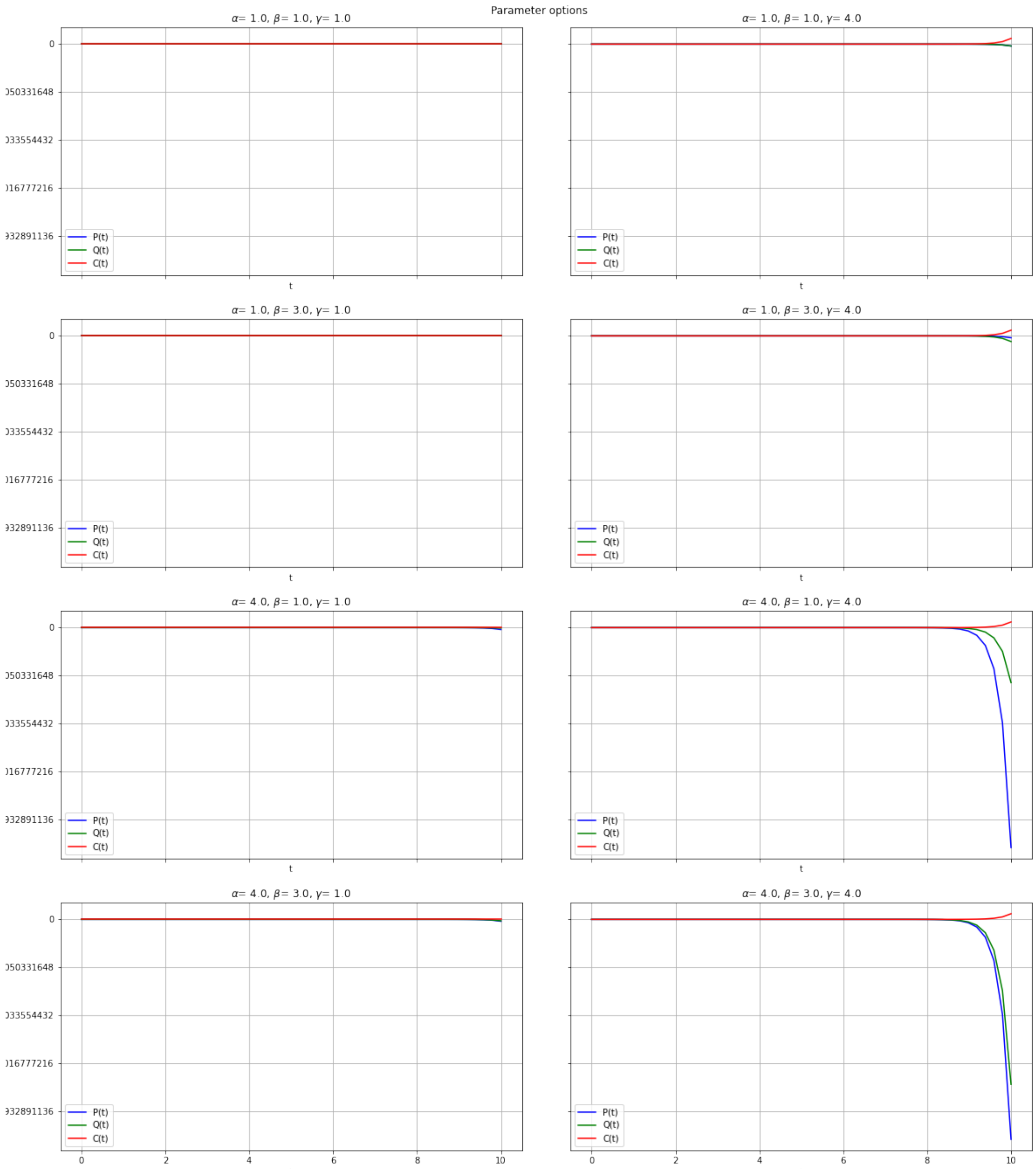
The above plot shows how this model performed with various parameters and initial conditions where all state variables are set to 1. As shown, changing the parameters has a significant effect on the behavior of the model. Changing the initial conditions did not seem to have a major effect. When α and γ were both set to 1, quantity, cost and price all remain in line with each other, whether

β is set to 1 or 1.6. This makes sense, as cost is remaining the same in this scenario, which means price and quantity also will remain the same.
When $\alpha = 2.0$, $\beta = 1.0$, and $\gamma = 1.0$, the state variables stay close to each other but price and quantity eventually fall very slightly below cost, with price decreasing more than quantity. With the same α and γ values but $\beta = 1.6$, quantity and price decrease by the same amount. In this scenario, price increases at a rate of 2x the difference between price and cost, while cost is still constant and quantity directly reflects the difference between price and cost.

When $\alpha = 1.0$, $\beta = 1.0$, and $\gamma = 2.0$, cost eventually increases at the same time quantity and price decrease in line with each other. With the same α and γ values but $\beta = 1.6$, the pattern is similar but with quantity decreasing slightly more so than price.

When $\alpha = 2.0$, $\beta = 1.0$, and $\gamma = 2.0$, the eventual decrease in price and quantity are much more dramatic than with other parameter settings, while the increase in cost is similar. In this case, the decrease in price is more dramatic than the decrease in quantity. When $\alpha = 2.0$, $\beta = 1.6$, and $\gamma = 2.0$, the pattern is similar but with much less of a difference between the decrease in price and quantity. The below plot shows the same system ran with more extreme parameters. This changed the magnitudes of the state variables, but did not change the shape of the plots.

We do not believe that this base economic model has cycles, mostly because of the cost term (dC/dt). In this term, the cost either grows or decays.



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