Modeling Complex Systems Assignment 3

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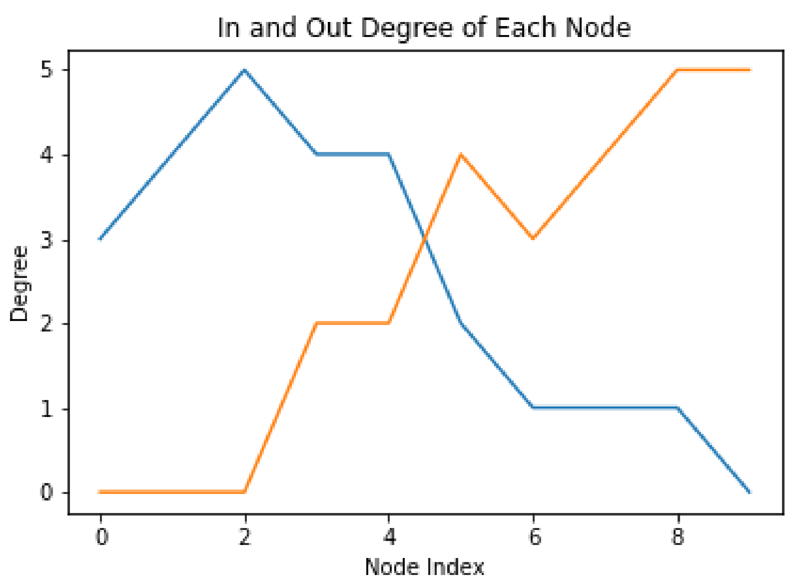
Question 1

A)

This network is acyclic because all the edges in it are unidirectional and flow in the same general direction. The overall direction of the model is going from higher index nodes to lower index nodes. In our network, we did notice that making any one edge bidirectional would make the network cyclic. Reversing the direction of some edges would make the network cyclic, but the edge reversed must have the connections needed to form a cycle once reversed.

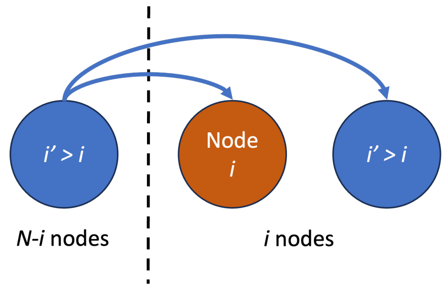
B)

The average in degree for our network was 2.5, and the average out degree was also 2.5. This was determined by finding the in- and out- degree of each node and summing them, then dividing by the total number of nodes. Our network had 10 nodes. The image below shows a plot of the node index vs the in and out degree of each node.



C)

The expected number of edges that run from nodes $i' > i$ to nodes $i' \leq i$ is based on how many nodes can be the source and how many nodes can be the target of an edge that meets this condition, and the overall probability of an edge existing, which is given by p.



The graphic above shows the structure of these edges, where the source node must be $i' > i$ and the target node must be $i' \leq i$. There are $N - i$ nodes that can be the source node for these edges, and i nodes that could be the target node. Therefore, we would expect there to be $(N - i)(i)$ edges that meet this condition, assuming there is a 100% probability of any possible edge forming. However, in this model, there is a probability p of any edge forming. Therefore, the expected number of edges that run from nodes $i' > i$ to nodes $i' \leq i$ is $(N - i)(i)(p)$, or:

$$(Ni - i^2)p$$

D)

Assuming N is even, the number of expected edges that run from nodes $i' > i$ to nodes $i' \leq i$ is the smallest at $i = N$ (which will always be 0) and the largest at $i = \frac{N}{2}$. For example, in our model, we used $N = 10$, and the node at index 10 has 0 expected edges running from the nodes $i' > i$ to the nodes $i' \leq i$, and the node at index 5 has 12.5. This is somewhat intuitive, as there are no nodes with an index greater than N, and $i = \frac{N}{2} + 1$ has the maximal possible nodes that can form edges that meet the condition described here.

Question 2

A)

In a heterogeneous mean field system for a SIS epidemic model on a configuration model network, the change in infected individuals I of degree k at each time step t, $\dot{I_k}$, is given by:

$$\dot{I_k} = S_k \cdot k \cdot \Theta \cdot \beta - \alpha \cdot I_k$$

where:

- S_k represents the number of susceptible individuals of degree k, who could become infected individuals of degree k, I_k
- Θ is the mean-field quantity
- β is the transmission rate
- α is the recovery rate

S_k nodes can become I_k nodes based on how many connections they have (the value of k), the strength of the connection field and the transmission rate. I_k nodes become the S_k nodes based entirely on the given recovery rate.
The value of Θ represents the probability that a random edge around a susceptible node connects to an infectious node, and is calculated as:

$$\Theta = \frac{\sum_k kI_k}{\sum_k kN_k}$$

B)

In a scenario where there is a vaccine for the infection, around the vaccinated nodes, the transmission rate (both to and from) β is reduced by a multiplicative factor $(1 - \rho)$. This would change the equation for the change in the infected nodes. First, we now need two Θ values:

$$\Theta = \frac{\sum_k kI_k}{\sum_k kN_k}$$

$$\Theta_v = \frac{\sum_k kI_k^v}{\sum_k kN_k}$$

This allows us to differentiate the proportion of vaccinated infected nodes around a given susceptible node from the proportion of unvaccinated infected nodes around a given susceptible node. This is important because the transmission rate is different from that of vaccinated infected nodes. The change in unvaccinated infected nodes at time t is given by:

$$\dot{I_k} = (S_k \cdot k \cdot \Theta \cdot \beta) + (S_k \cdot k \cdot \Theta_v \cdot (1 - \rho) \cdot \beta) - \alpha \cdot I_k$$

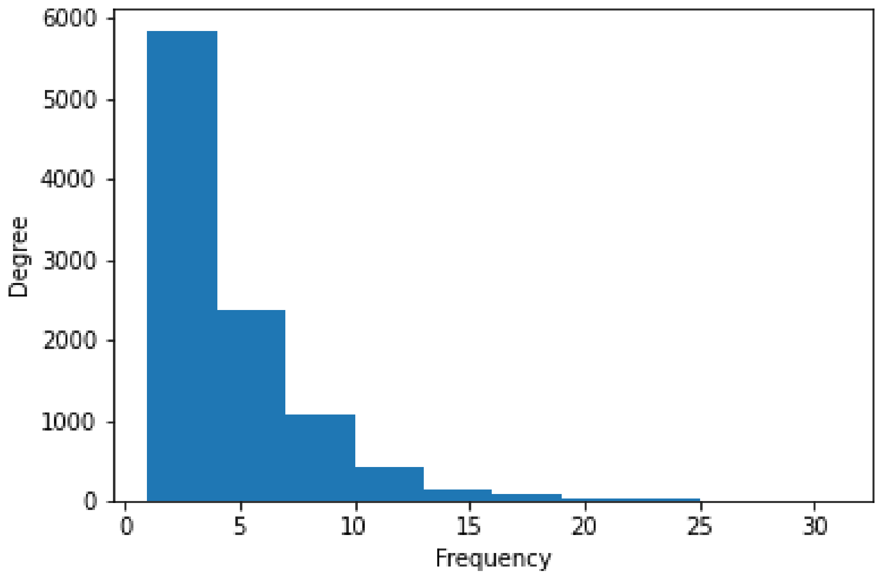
For unvaccinated individuals, susceptible nodes become infected at a rate of β from contact with fellow unvaccinated nodes, and at a rate of $\beta \cdot (1 - \rho)$ from contact with vaccinated nodes. The change in vaccinated infected individuals I of degree k at each time step t, $\dot{I_k}$, would be given by:

$$\dot{I_k^v} = (S_k^v \cdot k \cdot \Theta_v \cdot (1 - \rho) \cdot \beta) + (S_k^v \cdot k \cdot \Theta \cdot (1 - \rho) \cdot \beta) - \alpha \cdot I_k^v$$

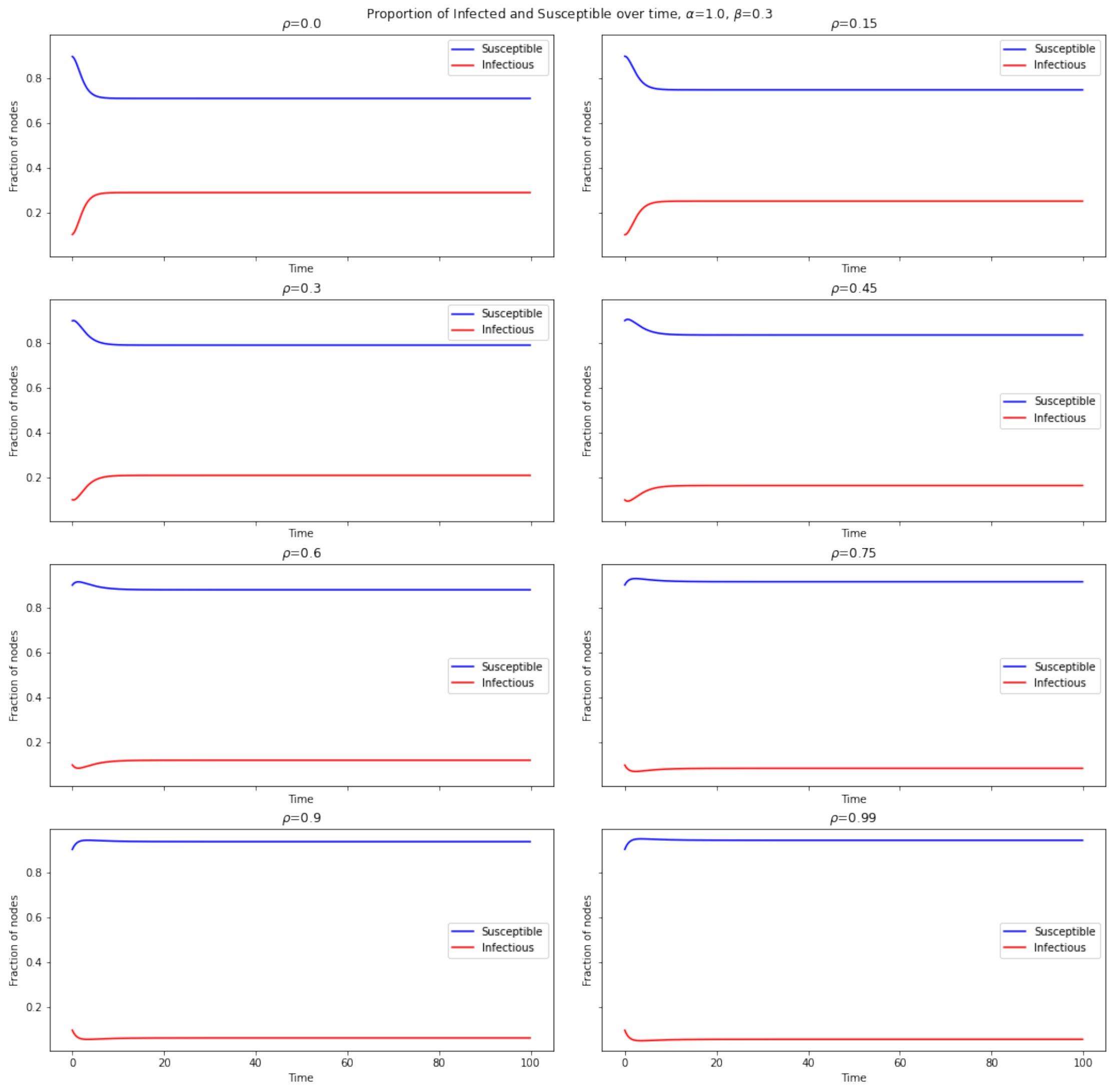
Vaccinated susceptible individuals become infectious at a rate of $\beta \cdot (1 - \rho)$ from both vaccinated and unvaccinated infected individuals.

C)

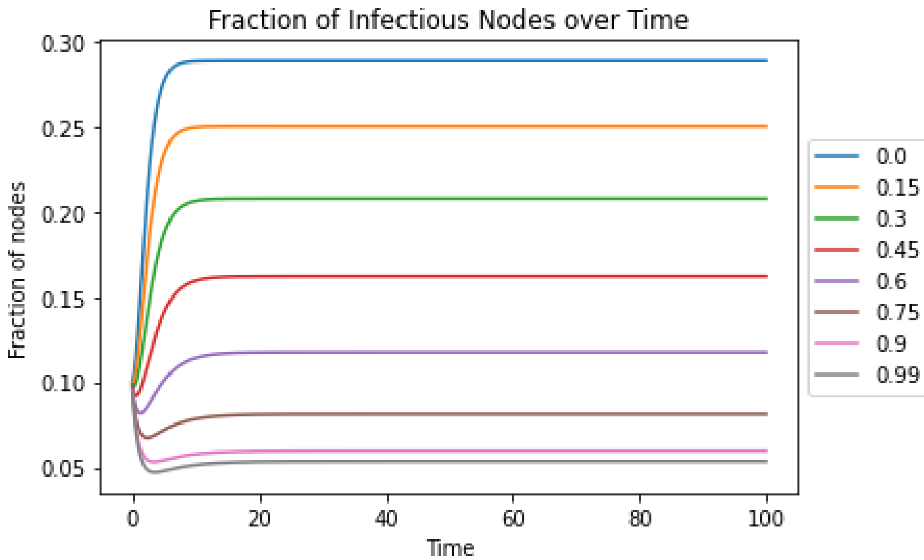
To see what value ρ was required to eradicate the epidemic when a random 40% of the population is vaccinated, we first created our configuration network from a geometric distribution. A degree distribution histogram for our network is shown below.



We found that with a random 40% of the population getting vaccinated, it took a pretty high ρ value for the infection to not take off. Even at 0.99 the infection reaches a steady state at $\approx 5.1\%$ of the population, which is significantly lower than when ρ is 0 (reaches a steady state at $\approx 28.8\%$ of the population) but still not fully eradicated. The figure below shows the plots of our model with ρ values ranging from 0 to 0.99.

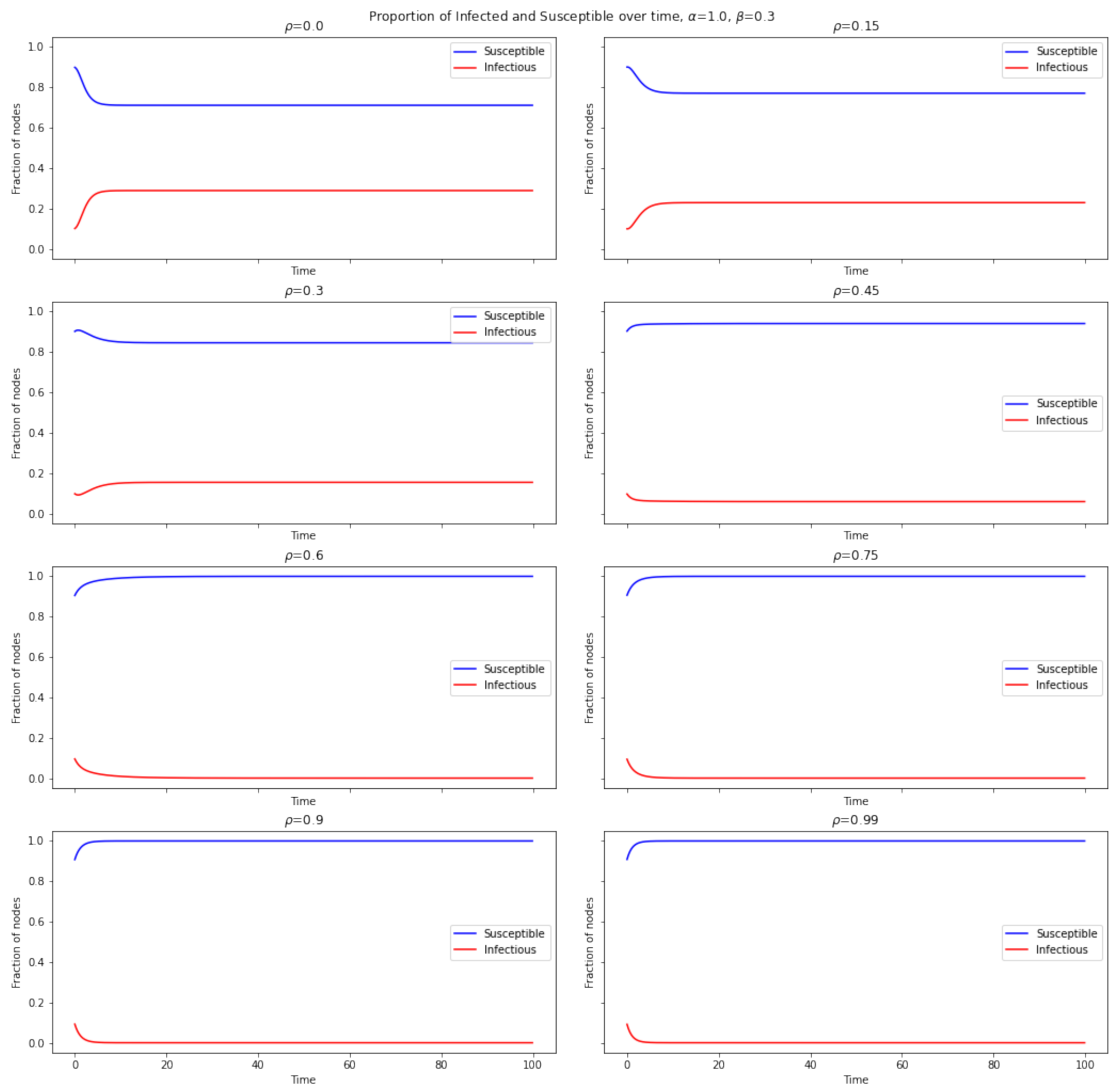

Proportion of Infected and Susceptible over time, α=1.0, β=0.3
ρ=0.0
ρ=0.15
ρ=0.3
ρ=0.45
ρ=0.6
ρ=0.75
ρ=0.9
ρ=0.99
Susceptible
Infectious
Fraction of nodes
Time

The steady state reached with each other ρ value are: $\rho = 0.15$: 25.0%, $\rho = 0.3$: 20.6%, $\rho = 0.45$: 16.0%, $\rho = 0.6$: 11.4%, $\rho = 0.75$: 7.8%, $\rho = 0.9$: 5.8%. The figure below shows the proportion of infected individuals over time with each value ρ in one plot, to better show the effect of increasing vaccine efficacy.

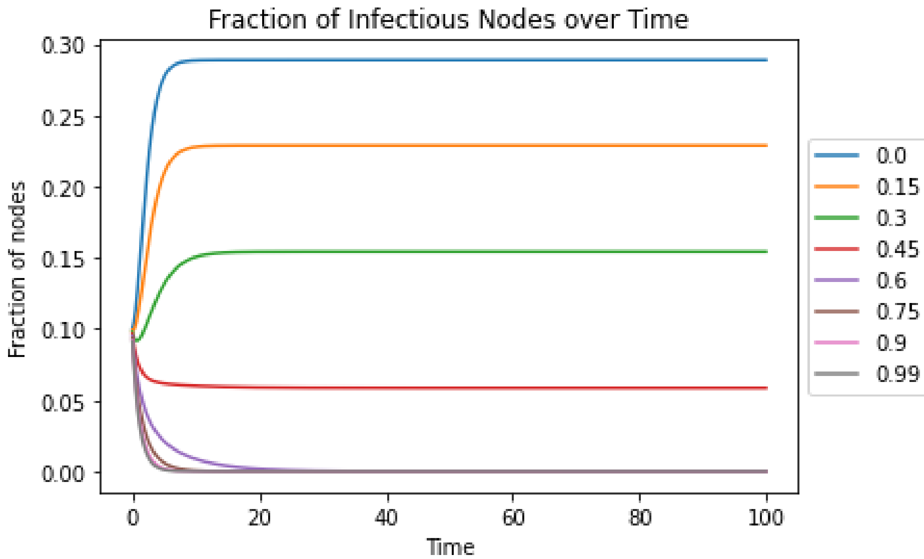


D)

When the top 40% of nodes are vaccinated, it takes a lower ρ value to prevent the epidemic from taking off. The figure below shows the plots of our model with the top 40% of nodes vaccinated with ρ values ranging from 0 to 0.99.

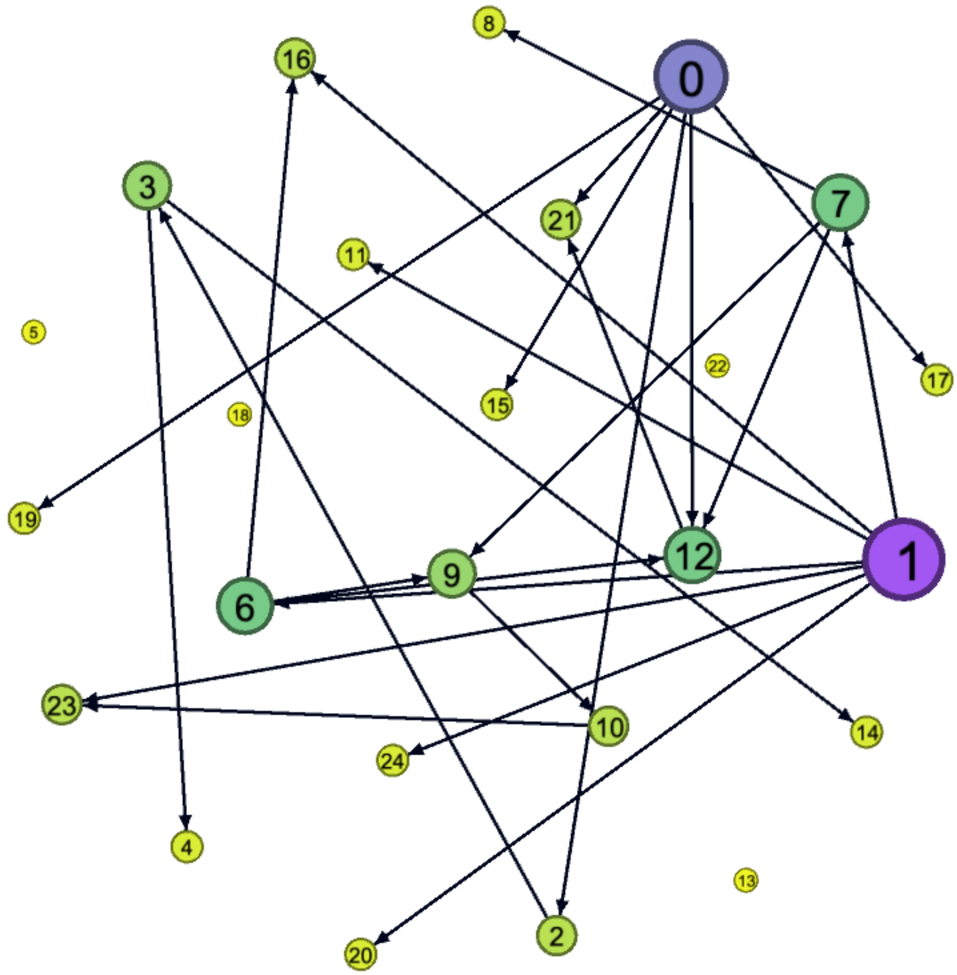


With $\rho = 0.45$, the infection reaches a steady state at only 6%, and with $\rho \geq 0.6$, the infection reaches a steady state at less than 1% of the population. The figure below shows the proportion of infected individuals over time with each value ρ in one plot, to better show the effect of increasing vaccine efficacy.

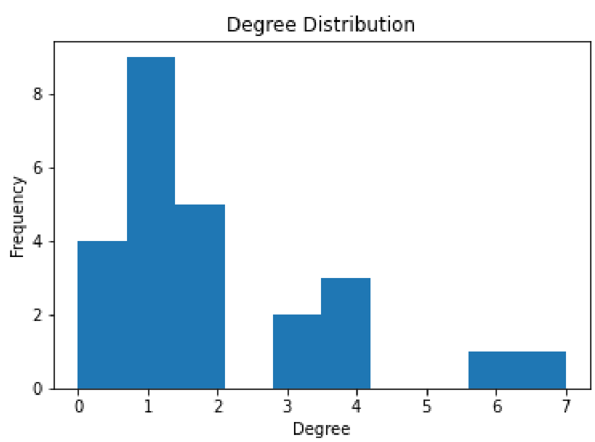


Question 3

The CIELNET network was created from Project Ciel, an operation in which the Canadian police targeted a drug-smuggling network importing liquid hashish from Jamaica to Montreal from May 1996 to June 1997. The network has a hierarchical structure which is typical of Canadian drug-smuggling operations. Overall, 75 people were surveilled during the operation and 25 were identified as active in the smuggling ring. These 25 individuals are the 25 nodes in the CIELNET network, and the edges represent communication exchanges between these individuals. The edges are weighted and directed, with higher weights representing more active communication channels. The most connected nodes (Nodes 1 and 0) are the top of the hierarchy and the main targets of the investigation. The image below shows a representation of the network created in Gephi. The more connected nodes are more purple and larger.



The average degree of the network is 2.0, and a histogram of the dgeree distribution is shown below.



Question 4

A)

In our implementation of the voter model, at every timestep t, a random node is selected from the CIELNET network. Then, a list of that node's neighbors is generated, and if that node has at least 1 neighbor, one neighbor is chosen at random and the original node takes on the color of that neighbor. Each node is initially assigned either blue or red with given proportions. We ran our model 100 times for a max of 11,000 time steps with 5 different initial conditions. Our model always reached a consensus within the 11,000 time steps. The results from each initial condition averaged over 100 runs are shown in the table below.

Initial percentage red	**Average time to reach consensus**
50%	10,263.93
60%	9,526.74
70%	8,965.71
80%	6,819.05
90%	4,470.65

When the network is initially 50% red and 50% blue, it takes on average 10,263.93 steps to reach consensus, and when the network is initially 90% red and 10% blue, it takes on average 4470.65 steps to reach consensus. The network takes the longest to reach consensus when there is a 50/50 red-blue split, and takes less time the less even the initital conditions are. This makes sense, as 90% red and 10% blue is already very close to a consensus, while a 50/50 split is as far from a consensus as one can get.

B)

We used `networkx.configuration_model` to create a configuration model with the same degree sequence as the CIELNET network. We then ran our voter model on this network with the same initial conditions as with the original network. In running the model for 100 runs with the different initial conditions, we noticed that there seemed to be more variability in the average over 100 runs when the same initial conditions were ran multiple times. Because of this, we decided to average this model over 300 runs for each initial condition, to get a more reliable result. The results from each initial condition averaged over 300 runs are shown in the table below.

Initial percentage red	**Average time to reach consensus**
50%	10,892.337
60%	10,643.483
70%	10,101.450
80%	8,793.173
90%	5691.010

The configuration model shows a similar pattern to the original network, but the initial conditions seem to affect the time it takes to reach consensus more subtly. There is less of a difference between the 50/50, 60/40 and 70/30 splits, though the 50/50 split still takes the longest to arrive at consensus.

References

Morselli, C. (2021, August 29). CIELNET/Project Ciel. Analytic Technologies. https://sites.google.com/site/ucinetsoftware/datasets/covert-networks/cielnet This is the link to the network used in Questions 3 and 4.