Analytic Geometry

DEF (*Dot Product*)The dot product between two vectors u and v is defined as

$$u^\top v = u_1v_1 + u_2v_2 + \ldots + u_nv_n = \sum_{i=1}^{n} u_iv_i$$

DEF (*Bilinearity*) The dot product is bilinear, i.e. for any vectors u, v, w and scalar a,

$$\begin{aligned} au^\top v &= au^\top v = u^\top av \\ u + v^\top w &= u^\top w + v^\top w \\ w^\top u + v &= w^\top u + w^\top v \end{aligned}$$

DEF (*Commutativity*)The dot product is commutative, i.e. $u^\top v = v^\top u$

DEF (*Inner Product*) The inner product between two vectors u and v is defined as

$$\langle u, v\rangle$$

Dot product is a special case of inner product.

DEF (*ℓ_2 Norm*) The ℓ_2 norm of a vector v is defined as

$$\|v\|_2 = \sqrt{v^\top v} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

Also called the Euclidean norm.

DEF (*ℓ_2 properties*) For all vectors u, v and scalar a,

- The ℓ_2 norm is non-negative, i.e. $\|v\|_2 \geq 0$.
- $\|au\|_2 = |a|\,\|u\|_2$ for any scalar a.
- $\|u\|_2$ is zero if and only if u is the zero vector.
- The triangle inequality holds, i.e. $\|u + v\|_2 \leq \|u\|_2 + \|v\|_2$.
- $\|x - y\|_2 = \|y - x\|_2$, also called symmetry.
- $\|u + v\|_2^2 = \|u\|_2^2 + 2u^\top v + \|v\|_2^2$
- $\cos\theta = \frac{u^\top v}{\|u\|_2\|v\|_2}$ (can be proved using the law of cosines)

THM (*Cauchy-Schwarz Inequality*) For any vectors u, v, the following inequality holds:

$$|\langle u, v\rangle| \leq \|u\|_2\,\|v\|_2$$

DEF (*Line*) A line is a set of points

$$\{x : x = u + tv \text{ for some } t \in \mathbb{R}\}$$

where u is a point on the line and $v \neq 0$ is the direction vector.

DEF (*Plane*) A plane is a set of points

$$\{x : v^\top x - u = 0\}$$

where v is the normal vector to the plane and u is the shift from the origin.

DEF (*Projection*) The vector $\|u\|_2 \cos\theta \frac{v}{\|v\|_2}$ is a projection of u onto v.

DEF (*Distance between a point and a plane*) The distance between a point z and a plane $v^\top x - u = 0$ is given by

$$\frac{|v^\top z - u|}{\|v\|_2}$$

DEF (*Distance between a point and a line*) The distance between a point z and a line $x = u + tv$ is given by

$$\left\| z - u - \frac{z - u^\top v}{\|v\|_2^2} v \right\|_2$$

DEF (*Singular Value Decomposition (SVD)*) The SVD of a matrix X is $U\Sigma V^T$, where $U^\top U = I, V^\top V = I$ and Σ is a diagonal matrix:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n \end{bmatrix}$$

and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.

THM (*Eckart-Young Theorem*) Let $\Sigma_k = \operatorname{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ where $k \leq d$. The matrix $U\Sigma_k V^T$ is the optimal solution to the following problem:

$$\min_{\hat{X}} \left\| X - \hat{X} \right\|_F \text{ s.t. } \operatorname{rank}(\hat{X}) \leq k$$

The matrices $Z = U\Sigma_k$ and $W = V^T$ are the optimal solution to

$$\min_{Z,W} \|X - ZW\|_F \text{ s.t. } Z \in \mathbb{R}^{n\times k}, W \in \mathbb{R}^{k\times d}$$

Calculus

DEF (*Derivative*) The derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at x_0 is defined as:

$$(D_x f)(x_0) = \left(\frac{d}{dx} f\right)(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

DEF (*Directional Derivative*) The directional derivative of a function $f : \mathbb{R}^d \to \mathbb{R}$ along the direction v at $x_0 \in \mathbb{R}^d$ is defined as:

$$(D_v f)(x_0) = \lim_{h\to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$$

DEF (*Partial Derivative*) The partial derivative is a directional derivative along the direction of coordinate axes. For a function $f : \mathbb{R}^d \to \mathbb{R}$, the partial derivative with respect to the i-th coordinate is denoted as:

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_d) - f(x_1, \ldots, x_i, \ldots, x_d)}{h}$$

DEF (*Gradient*) The gradient of a function is the vector consisting of all partial derivatives denoted by ∇f or $\frac{\partial f}{\partial x}$. For a function $f : \mathbb{R}^d \to \mathbb{R}$,

EX (*Gradient Identities*)

- Given a function $f(a) = b^\top a$, $(\nabla f)(a) = b$
- Given a function $f(a) = \|a\|_2^2$, $(\nabla f)(a) = 2a$

EX (*Hessian*) The Hessian of a function $f: \mathbb{R}^d \to \mathbb{R}$ is the matrix of second partial derivatives:

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_d} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_d x_1} & \frac{\partial^2 f}{\partial x_d x_2} & \cdots & \frac{\partial^2 f}{\partial x_d^2} \end{bmatrix}$$

EX (*Hessian Example*) Given a function $f(x, y) = x^2 - y^2$, the Hessian is:x:

$$\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

The 2 indicates that it looks like a cup along the x-axis and the -2 indicates that it looks like an upside-down cup along the y-axis.

Optimisation

DEF (*Minimum*) The minimum of a function $f: \mathbb{R}^d \to \mathbb{R}$ is written as $\min_x f(x)$, and has the property that $\min_x f(x) \leq f(y)$ for all $y \in \mathbb{R}^d$. The value x^* such that $f(x^*) = \min_x f(x)$ is called the minimizer.

DEF (*Convexity*) A function $f: \mathbb{R}^d \to \mathbb{R}$ is convex if for any $0 \leq \alpha \leq 1$, we have

$$f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y)$$

In general,

$$f(\sum_{i=1}^{k} \alpha_i x_i) \leq \sum_{i=1}^{k} \alpha_i f(x_i)$$

DEF (*Concavity*) A function f is concave if $-f$ is convex.

THM (*First Order Condition (Convexity)*) If f is convex then,

- $f(x) \geq f(y) + \nabla f(y)^\top (x - y)$

DEF (*Positive Semidefinite*) A matrix A is positive semidefinite if for all vectors $v \neq 0$ we have $v^\top A v \geq 0$. Also written as $A \succeq 0$.

THM (*Convex Function Implies Positive Semidefinite Hessian*) If f is convex, then $\nabla^2 f(x) \succeq 0$.

DEF (*Positive Definite*) A matrix A is positive definite if for all vectors $v \neq 0$ we have $v^\top A v > 0$. Also written as $A \succ 0$.

DEF (*Affine*) A function f is affine if $f(x) = Ax + b$ for some matrix A and vector b.

THM (*Affine Transform Preservation*) If f is convex, then $g(x) = f(Ax + b)$ is also convex.

THM (*Non-negative Weighted Sum*) If $f_1, f_2, \ldots, f_k$ are convex functions, then $f(x) = \sum_{i=1}^{k} \beta_i f_i(x)$ is convex for all $\beta_i \geq 0$.

EX (*Gradient of Quadratic Form*) $\nabla_x (x^\top A x) = (A + A^\top) x$

DEF (*Strictly Convex*) A function $f: \mathbb{R}^d \to \mathbb{R}$ is called strictly if for $0 \leq \alpha \leq y$ we have

$$f(\alpha x + (1 - \alpha) y) < \alpha f(x) + (1 - \alpha) f(y)$$

for any $x \neq y$.

DEF (*First Order Condition (Strict Convexity)*) If f is strictly convex then,

- $f(x) > f(y) + \nabla f(y)^\top (x - y)$

THM (*Unique Minimizer*) If f is strictly convex, then f has a unique minimizer.

THM (*Jensen's Inequality*) If a function f is convex,

$$f(\mathbb{E}_{x \sim p}[x]) \leq \mathbb{E}_{x \sim p}[f(x)]$$

DEF (*Gradient Descent*) The gradient descent algorithm is given by

$$x_{t+1} = x_t - \eta \nabla f(x_t)$$

where η is the learning rate/step size.

DEF (*Convergence*) Given $\epsilon > 0$, we say that an algorithm converges to a point x^* if

$$f(x_t) - f(x^*) \leq \epsilon$$

DEF (*Convergence Rate*) The convergence rate of an algorithm is the rate at which the algorithm converges to the optimal point. There are three types of convergence rates:

- Sublinear: $f(x_t) - f(x^*) \leq \frac{c}{t^2}$ ($\epsilon = O(\frac{1}{t^2})$,$t = O(\frac{1}{\sqrt{\epsilon}})$)
- Linear: $f(x_t) - f(x^*) \leq c r^t$ ($\epsilon = O(r^t)$,$t = O(\log \frac{1}{\epsilon})$)
- Quadratic: $f(x_t) - f(x^*) \leq c r^{2^t}$ ($\epsilon = O(r^{2^t})$,$t = O(\log \log \frac{1}{\epsilon})$)

Where $0 < r < 1$.

DEF (*Subgradient*) A subgradient at x is a vector g that satisfies

$$f(y) \geq f(x) + g^\top (y - x)$$

for any y, and the set of subgradients at x is denoted as $\partial f(x)$. $\nabla f(x) \in \partial f(x)$ if f is differentiable at x. In other words subgradients are tangents that are below the function.

EX (*Constrained Optimisation Problem*) An example of a constrained optimisation problem is

$$\min_x x^2 \text{ s.t. } -2.5 \leq x \leq -0.5$$

DEF (*Feasible Solution*) A feasible solution is a point that satisfies all the constraints.

DEF (*Lagrangian*) If you have an optimisation problem of the form

$$\min_x f(x) \text{ s.t. } h(x) \leq 0$$

the **Lagrangian** is defined as

$$f(x) + \lambda h(x)$$

for some $\lambda \geq 0$ (Lagrange multiplier).

ALG (*Solving the Lagrangian*)

- Solve $g(\lambda) = \min_x [f(x) + \lambda h(x)]$

- Find $\hat{\lambda}$ such that $\min_x[f(x) + \hat{\lambda}h(x)]$ gives a feasible solution
- Suppose $\hat{x}$ is the solution to the above, and $x^* = \arg\min_{x:h(x)\leq 0} f(x)$ (the optimal solution), then

$$f(\hat{x}) = f(\hat{x}) + \hat{\lambda}h(\hat{x}) \leq f(x^*) + \hat{\lambda}h(x^*) \leq f(x^*)$$

Probability

DEF (*Gaussian Distribution*) We write $x \sim \mathcal{N}(\mu, \Sigma)$ to denote that x is a random variable with mean μ and covariance Σ. It means that the probability density function of x is given by

$$p(x) = \frac{1}{2\pi^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

DEF (*Statistical Independence*) Two variables x and y are independent if

$$p(x, y) = p(x)p(y)$$

Equivalently,

$$p(x|y) = p(x)$$

. The independence of x and y is denoted by $x \perp y$.
DEF (*Statistical Independence [general]*) If $\{x_1, \ldots, x_n\} \perp \{y_1, \ldots, y_m\}$ then

$$p(x_1, \cdots, x_n, y_1, \cdots, y_m) = p(x_1, \cdots, x_n)p(y_1, \cdots, y_m)$$

EX (*Factorisation of a joint distribution*) Suppose $x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{Z}$. If $\{x, y\} \perp z$ then

$$p(x, y, z) = p(x, y)p(z)$$

The original joint distribution (of size $|\mathcal{X}| \times |\mathcal{Y}| \times |\mathcal{Z}|$) can be factorised into two distributions of size $|\mathcal{X}| \times |\mathcal{Y}|$ and $|\mathcal{Z}|$.

DEF (*Mutual Independence*) A set of variables $\{x_1, \ldots, x_n\}$ are mutually independent if

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p(x_i)$$

DEF (*Pairwise Independence*) A set of variables $\{x_1, \ldots, x_n\}$ are pairwise independent if

$$p(x_i, x_j) = p(x_i)p(x_j)$$

for all $i \neq j$.
THM (*Mutual Independence implies Pairwise Independence*) If a set of variables $\{x_1, \ldots, x_n\}$ are mutually independent, then they are pairwise independent. Converse is not true!

DEF (*Conditional Independence*) The variables x and y are conditionally independent given z if

$$p(x, y|z) = p(x|z)p(y|z)$$

This is denoted by $x \perp y|z$.
DEF (*Marginilisation*) The marginal distribution of x is obtained by summing out all other variables.

$$p(x) = \sum_y p(x, y)$$
$$p(x|z) = \sum_y p(x, y|z)$$
$$p(x, y|z) = p(x|z)p(y|z)$$

DEF (*Bayes Rule*) Bayes rule is a way to invert conditional probabilities.

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

DEF (*Chain Rule*) Any joint distribution can be factorised into a product of conditional distributions.

$$p(x_1, \ldots, x_n) = p(x_1)p(x_2|x_1)p(x_3|x_1, x_2)\cdots p(x_n|x_1, \ldots, x_{n-1})$$

DEF (*(directed) Graph Representation*) A directed graph is a set of nodes connected by edges. Each vertex is a random variable and each edge represents a direct dependency. It is directed and acyclic (DAG). A distribution factorises according to the graph if

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p(x_i|\text{parents}(x_i))$$

DEF (*Graph structures*)
- Chain $x \perp z|y$

$$x \longrightarrow y \longrightarrow z$$

- Common cause $x \perp z|y$

$$x \longleftarrow y \longrightarrow z$$

- v-structure $x \perp z$ but $x \not\perp z|y$

$$x \longrightarrow y \longleftarrow z$$