$$P = \Phi(-d_2) K e^{-r\tau} - \Phi(-d_1) S_0 \; ; \quad K > S_0$$

as $\sigma \to 0$

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right]$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right)$$

As $\sigma \to 0$, b/c $K > S_0$

$$d_1 \to -\infty \; , \quad d_2 \to -\infty$$

$$\Phi(-d_2) \to 1 \qquad \Phi(-d_1) \to 1$$

$$P \to K e^{-r\tau} - S_0$$

As $\sigma \to \infty$ $\quad d_1 \to \infty \; , \quad d_2 \to -\infty$

$$P \to K e^{-r\tau}$$

$C = \Phi(d_1) S_0 - \Phi(d_2) e^{-rt} K$

as $T \to \infty$ $\quad d_1 \to \infty, \quad d_2 \to -\infty$

$C \to S_0$

✓

<u>Goal</u>

We observed in BSM

$$C(T) \uparrow \quad \text{as} \quad T \uparrow$$

Now we want to generalize this fact to any stock model.

<u>prop</u> Suppose

① f is convex, $\uparrow$

② X is a <u>sub-martingale</u>,

then

$$g(t) \triangleq \mathbb{E}[f(X_t)] \quad \text{is} \quad \uparrow$$

<u>Recall</u>

① X is said to be mtgl if

$$\mathbb{E}[X_{t+h} \mid \mathcal{F}_t] = X_t$$

② X is said to be submtgl, if

$$\mathbb{E}[X_{t+h} \mid \mathcal{F}_t] \geq X_t$$

Pf $g(t+h) = \mathbb{E}[f(X_{t+h})]$

$= \mathbb{E}[\mathbb{E}[f(X_{t+h}) | \mathcal{F}_t]]$ Tower Property

$\geq \mathbb{E}[f(\mathbb{E}[X_{t+h} | \mathcal{F}_t])]$ b/c convexity

$\geq \mathbb{E}[f(X_t)]$ b/c ① sub-mtgl ② f is $\uparrow$

$= g(t)$

ex Let ① $C_t = \mathbb{E}[e^{-rt}(S_t - k)^+]$

② $(e^{-rt} S_t)_{t \geq 0}$ is mtgl.

Then $C_t \uparrow$

Pf ① $X_t = e^{-rt} S_t - k e^{-rt}$.

② $C_t = \mathbb{E}[(X_t)^+] = \mathbb{E}[f(X_t)^+]$

where $f(x) = (x)^+ =$  = convex.

increasing.

Thus $t \longmapsto C_t$ is $\uparrow$

IV

Let ① $f: \sigma \mapsto BSM(\sigma; \underbrace{r, S_0, T, K, \text{otype}}_{\text{other paras}})$

② Market price

$$M(r, S_0, T, K, \text{otype})$$

Then $\hat{\sigma}$ is called IV. if

$$f(\hat{\sigma}) = M(r, S_0, T, K, \text{otype})$$

i.e. $$\hat{\sigma} = f^{-1} \circ M(r, S_0, T, K, \text{otype})$$

Volatility smile

Let T, otype fixed, choose different K's one can compute different IV $\hat{\sigma}_K$

$$K \longmapsto \hat{\sigma}_K = f^{-1} \circ M(K; r, S_0, T, \text{otype})$$

as volatility smile.

---

volatility surface

choosing different (T, K), otype is fixed then we call

$$(T, K) \longmapsto \hat{\sigma}_{T,K} = f^{-1} \circ M(T, K; r, S_0, \text{otype})$$

as volatility surface

Q Fix underlying asset, there should be one volatility σ. But option data gives many $(\hat{\sigma}_{T,K})$, which one is the "correct" one?

This leads to calibration

<> <>

Model calibration

- Model parameter
$$\theta = (\theta_1, \theta_2 \cdots \theta_m)$$
- price engine for n instruments
$$\theta \longmapsto f_i(\theta), \quad i = 1, 2 \cdots n$$
→ i^{th} instrument valuation

Denote it by
$$\theta \longmapsto f(\theta) = \begin{pmatrix} f_1(\theta) \\ \vdots \\ f_n(\theta) \end{pmatrix}$$
- Market price for n instruments
$$M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}$$

Then calibrated Model($\hat{\theta}$) shall be
$$\hat{\theta} = \arg\min_{\theta} H(f(\theta), M)$$
where H is some error function:

① $H(x, y) = \left(\sum_{i=1}^{n} w_i |x_i - y_i|^2 \right)^{\frac{1}{2}}$

② $H(x, y) = \left(\sum_{i=1}^{n} w_i \left| \frac{x_i - y_i}{y_i} \right|^2 \right)^{1/2}$ → SSRE

③ $H(x, y) = \left(\sum_{i=1}^{n} w_i |\ln x_i - y_i|^2 \right)^{\frac{1}{2}}$