Our goal is to explore various monte carlo method for computing

$$\alpha = \int_0^1 h(x)dx$$

where

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

Of course, the exact value shall be

$$\alpha = 1.98.$$

Pretended not to know the exact value, we have used OMC with exact sampling of uniform random variable, denoted by omc_integral(n).

1. Find F_1, the cdf of p_1 given by

$$p_1(x) = \frac{1}{C}(2 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x)),$$

   for some constanst C.

2. Find F_1^{-1}.

3. Implement Pseudocode
   inverse_transform_sampling(F^{-1}, n):

   - Generate iid $U(0,1)$ random variables
   $$\{Y_i : i = 1, \ldots, n\}.$$
   - Compute
   $$\{X_i = F^{-1}(Y_i) : i = 1, \ldots, n\}.$$

4. Implement Pseudocode
   importance_sampling_integral(n):

   - Generate iid p_1 samples, denoted by
   $$\{X_i : i = 1, 2, \ldots, n\}.$$
   - Compute the average of the integrand h adjusted by likelyhood ratio (also referred to radon-nikodym derivative) p/p_1, i.e.
   $$\bar{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i) \cdot \frac{p(X_i)}{p_1(X_i)}.$$

5. Demonstrate the convergence rate of the above importance sampling.

6. Could you find a pdf p_2 better than p_1?