Note on Jacobian

In the case of a nonlinear partial differential equation like the generalized heat conduction equation with $q(\mathbf{x}, T)$ and $k(\mathbf{x}, T, \mu)$, the Newton-Raphson method is often employed for numerical solutions. To use the Newton-Raphson method, one needs to linearize the nonlinear equation around an initial guess, typically resulting in a system of linear equations of the form $J\Delta T = -R$, where J is the Jacobian matrix, ΔT is the correction to the temperature field, and R is the residual vector.

For the given equation in its weak form:

$$\int_\Omega \nabla v \cdot k(\mathbf{x}, T, \mu)\ \nabla T \, d\Omega - \int_\Omega vq(\mathbf{x}, T) \, d\Omega = 0$$

The residual R associated with a test function v can be written as:

$$R(v) = \int_\Omega \nabla v \cdot k(\mathbf{x}, T, \mu) \nabla T \, d\Omega - \int_\Omega vq(\mathbf{x}, T) \, d\Omega$$

To linearize this, you will need to evaluate the Jacobian J, which is the derivative of the residual R with respect to T. In general, the component J_{ij} of the Jacobian matrix is given by:

$$J_{ij} = \frac{\partial R_i}{\partial T_j}$$

For the generalized heat conduction equation, this typically becomes:

$$J_{ij} = \int_\Omega \nabla N_i \cdot \frac{\partial k}{\partial T} \frac{\partial T}{\partial T_j} \nabla T \, d\Omega + \int_\Omega \nabla N_i \cdot k(\mathbf{x}, T, \mu) \nabla N_j \, d\Omega - \int_\Omega N_i \frac{\partial q}{\partial T} \frac{\partial T}{\partial T_j} \, d\Omega$$

$$J_{ij} = \int_\Omega \nabla N_i \cdot \frac{\partial k}{\partial T} N_j \ \nabla T \, d\Omega + \int_\Omega \nabla N_i \cdot k(\mathbf{x}, T, \mu) \nabla N_j \, d\Omega - \int_\Omega N_i \frac{\partial q}{\partial T} N_j \, d\Omega$$

Here N_i and N_j are the shape functions associated with nodes i and j. Note that the terms $\frac{\partial k}{\partial T}$ and $\frac{\partial q}{\partial T}$ come into play because the thermal conductivity k and the heat source q are functions of T.

The Newton-Raphson method iteratively updates the temperature field T using $J\Delta T = -R$ until the residual R is below a certain tolerance.