Pseudorandom Generators For Group Products

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Derandomization Problem Of Permutation Branching Program

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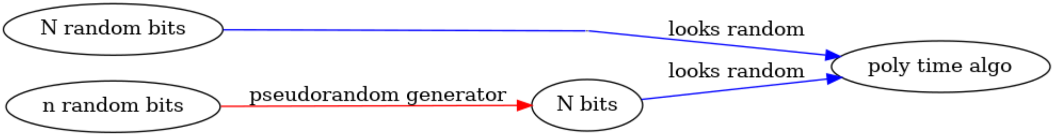
- Randomness is an important computational resource.
- Many computational problems are known to have much more efficient randomized algorithms than the deterministic ones.
- Thus it is natural to attempt to reduce the amount of "randomness", i.e. the number of completely random bits used.

Derandomization Problem Of Permutation Branching Program

- Let $n \ll N$. A **pseudorandom generator** accepts a bitstring of length n generated by the n truly random bits, called the **seed**, and yields, in polynomial time, a bitstring of length N, which must be indistinguishable from N truly random bits for any polynomial time algorithm.



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- As an example, it is known that we only need $O(\log^2 N)$ truly random bits to construct a pseudorandom generator producing $O(poly(N))$ bits in $SPACE(\log(N))$
- Constructing a pseudorandom generator for a class of algorithms is called the **derandomization problem**.

Derandomization Problem Of Permutation Branching Program

DEFINITION

A **branching program** with n variables $x_1, \cdots, x_n$ is a directed acyclic multigraph where

1. one of the node is marked as an **input node**,
2. outdegree of each nodes is either 2, called an **internal node**, or 0, called a **terminal**,
3. each outward edges of an internal node are labelled 0 and 1 respectively,
4. each internal nodes are labelled with one of the x_i, and
5. each terminals are marked as **accepting** or **rejecting**.

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A branching program **accepts** the bitstring $S \in \{0,1\}^n$ if the terminal reachable by following the edge labelled S_i for each internal nodes with label x_i is accepting. Otherwise, it **rejects** the bitstring.

Derandomization Problem Of Permutation Branching Program

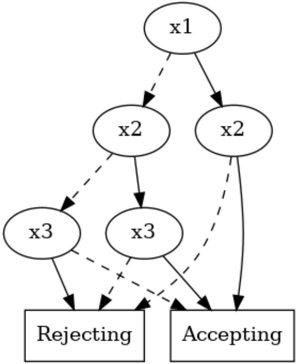

Figure: Branching program for $f(x_1, x_2, x_3) = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (x_1 \wedge x_2) \vee (x_2 \wedge x_3)$

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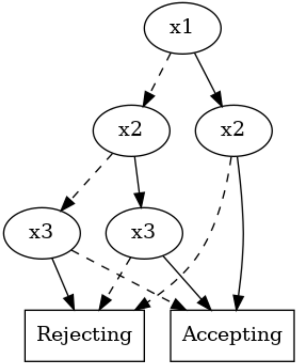

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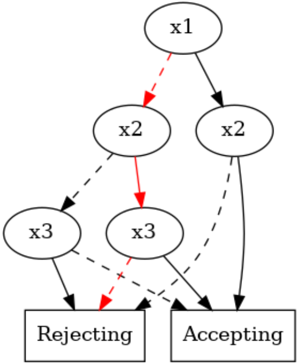

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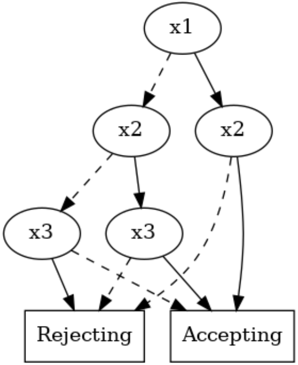

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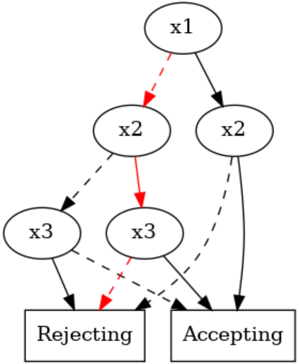

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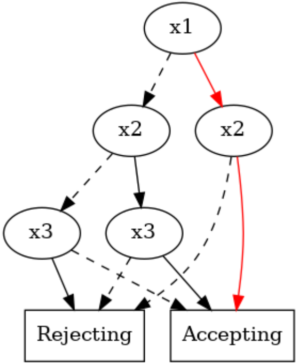

Figure: Accepts 110

Derandomization Problem Of Permutation Branching Program

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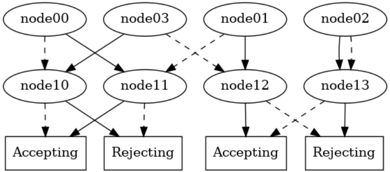
- It is known that it's suffice to find the construction of a pseudorandom generator for polynomial size read-once branching programs in order to solve the derandomization problem for space bounded computations.
- This paper proves that the **Impagliazzo-Nisan-Wigderson generator**(INW generator) solves this problem for a subclass of branching programs, called **permutation branching programs**.

Derandomization Problem Of Permutation Branching Program

DEFINITION

A **permutation branching program** with n variables $x_1, \cdots, x_n$ of width k is a branching program such that

1. nodes are divided into levels $0, 1, \cdots, n$, each containing k vertices numbered $1, 2, \cdots, k$, such that level $0 \leq i < n$ nodes are labelled x_{i+1},
2. one of the level 0 node is marked as an input vertex and level n nodes are terminals, and
3. edges are divided into levels $0, 1, \cdots, n-1$ such that level i edges of label $b \in \{0, 1\}$ forms a permutation from level i nodes to level $i+1$ nodes.



Pseudorandom Generators For Group Products

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- Consider the problem of finding a pseudorandom generator such that for every fixed permutation $\pi \in S_k$ and every $1 \leq i \leq k$, it approximates the probability that a random input with input node i reaches the terminal $\pi(i)$ for a fixed read-once permutation branching program.

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- Now we may attempt to restate the problem using finite groups.

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- Given a group word $w = (g_1, \cdots, g_n)$, we define the probability distribution Rnd^w on G as

$$\mathrm{Rnd}^w(g) = \frac{1}{2^n} |\{(x_1, \cdots, x_n) \in \{0,1\}^n : g = g_1^{x_1} \cdots g_n^{x_n}\}|$$

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.
- It's not hard to see that the derandomization problem for Rnd^w is equivalent to that of the permutation branching program.

INW Generator

DEFINITION

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Recall that an expander graph has property that each "small" subset of vertices has "large" boundary.

INW Generator

- Given a 2^d-regular multigraph, we may label (u, e), for each vertex u and an edge e incident to u so that the labels of (u, e) forms a permutation of $\{0, 1\}^d$ for u.

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- For $y \in \{0, 1\}^r$ and $y' \in \{0, 1\}^d$, let $\nu(y, y')$ be a neighbor of y reachable by the edge e where (y, e) is labelled y'.

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DEFINITION

Let $\Gamma_1, \Gamma 2 : \{0, 1\}^r \to \{0, 1\}^n$ be functions and F be a 2^d-regular multigraph with vertex set $\{0, 1\}^r$. The **expander product** of Γ_1 and Γ_2 by F is the function $\Gamma_1 \otimes_F \Gamma_2 : \{0, 1\}^{r+d} \to \{0, 1\}^{2n}$ defined by

$$(\Gamma_1 \otimes_F \Gamma_2)(y, y') = (\Gamma_1(y), \Gamma_2(\nu(y, y')))$$

.

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- In order to construct such family, we'll first look at some basic operations on expander graphs.

INW Generator

DEFINITION

For a D-regular undirected graph G, the **rotation map** $\mathrm{Rot}_G : [N] \times [D] \to [N] \times [D]$ is defined as follows: $\mathrm{Rot}_G(v, i) = (w, j)$ if the i-th edge incident to v leads to w and this edge is the j-th edge incident to w.

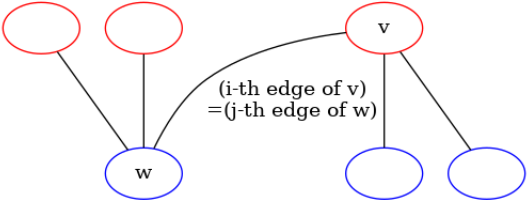

Figure: $\mathrm{Rot}_G(v, i) = (w, j), \mathrm{Rot}_G(w, j) = (v, i)$

INW Generator

First Operation(Power)

Let G be a D-regular multigraph on $[N]$. The t-th power of G is the D^t-regular multigraph G^t whose rotation map is given by $\mathrm{Rot}_{G^t}(v_0, (k_1, \cdots, k_t)) = (v_t, (l_t, \cdots, l_1))$ where these values are computed via the rule $\mathrm{Rot}_G(v_{i-1}, k_i) = (v_i, l_i)$.

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The t-th power is just the graph whose normalized adjacency matrix is the t-th power of the normalized adjacency matrix of the operand.

INW Generator

Since the t-th power of a matrix has eigenvalues powered by t, the following is immediate:

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THEOREM

If G is an (N, D, λ)-expander, then G^t is an (N, D^t, λ^t)-expander. Moreover, Rot_{G^t} is computable in time $\mathrm{poly}(\log N, \log D, t)$ with t oracle queries to Rot_G.

INW Generator

Second Operation(Tensor Product)

Let G_1 be a D_1-regular multigraph on $[N_1]$ and G_2 a D_2-regular multigraph on $[N_2]$. The **tensor product** $G_1 \otimes G_2$ is the $D_1 \cdot D_2$-regular multigraph on $[N_1] \times [N_2]$ given by $\mathrm{Rot}_{G_1 \otimes G_2}((v,w),(i,j)) = ((v',w'),(i',j'))$ where $\mathrm{Rot}_{G_1}(v,i) = (v',i')$ and $\mathrm{Rot}_{G_2}(w,j) = (w',j')$.

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The tensor product is just the graph whose normalized adjacency matrix is the product of the respective normalized adjacency matrices of operands.

INW Generator

Tensor product of matrices has a nice property that its eigenvalues are the multiset of pairwise products of respective eigenvalues of operands. Therefore, the largest eigenvalue $1 \cdot 1 = 1$ and the second largest is $\max(\lambda_1 \cdot 1, 1 \cdot \lambda_2)$. Therefore, the following holds:

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THEOREM

If G_1 is an (N_1, D_1, λ_1)-expander and G_2 is an (N_2, D_2, λ_2)-expander, then $G_1 \otimes G_2$ is an $(N_1 \cdot N_2, D_1 \cdot D_2, \max(\lambda_1, \lambda_2))$-expander. Moreover, $\mathrm{Rot}_{G_1 \otimes G_2}$ is computable in time $\mathrm{poly}(\log N_1 N_2, \log D_1 D_2)$ with one oracle queries to each Rot_{G_1} and Rot_{G_2}.

INW Generator

Third Operation(Zig-zag Product)

If G_1 is a D_1-regular graph on $[N]$ and G_2 is a D_2-regular graph on $[D_1]$, then their **zig-zag product** $G_1 \text{ⓩ} G2$ is a D_2^2-regular graph on $[N] \times [D_1]$ whose rotation map is as follows:
$\mathrm{Rot}_{G_1 \text{ⓩ} G_2}((v,k),(i,j))$:

1. Let $(k', i') = \mathrm{Rot}_{G_2}(k, i)$.
2. Let $(w, l') = \mathrm{Rot}_{\mathrm{G}_1}(v, k')$.
3. Let $(l, j') = \mathrm{Rot}_{\mathrm{G}_2}(l', j)$.
4. Output $((w, l), (j', i'))$.

INW Generator

THEOREM

If G_1 is an (N, D_1, λ_1)-expander and G_2 is a (D_1, D_2, λ_2)-expander, then $G_1 ⓩ G_2$ is a $(N_1 D_1, D_2^2, \lambda_1 + \lambda_2 + \lambda_2^2)$-expander. Moreover, $\mathrm{Rot}_{G_1 ⓩ G_2}$ can be computed in time $\mathrm{poly}(\log N, \log D_1, \log D_2)$ with one oracle query to Rot_{G_1} and two oracle queries to Rot_{G_2}.

INW Generator

Let H be a (D^8, D, λ)-expander for some D and λ. For $t \geq 1$, we define a (D^{8t}, D^2, λ_t)-expander G_t as follows:

1. $G_1 = H^2$.
2. $G_2 = H \otimes H$.
3. For $t \geq 3$,
$$G_t = \left(G_{\lceil \frac{t-1}{2} \rceil} \otimes G_{\lceil \frac{t-1}{2} \rceil}\right)^2 \text{Ⓩ} H$$
.

INW Generator

THEOREM

For every $t \geq 1$, G_t is an (D^{8t}, D^2, λ_t)-expander with $\lambda_t \in \lambda + O(\lambda^2)$. Moreover, Rot_{G_t} can be computed in time $\mathrm{poly}(t, \log D)$ with $\mathrm{poly}(t)$ oracle queries to Rot_H.

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- For the time complexity, note that the depth of the recursion is $\log t$ and evaluation of Rot_{G_t} requires 4 evaluations of rotation maps for smaller graphs.

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- Therefore, total number of recursive calls is at most $4^{\log t} = t^2$.

INW Generator

Corollary

There is a universal constant $c_0 > 0$ such that for every constant $0 < \lambda < 1$ and $d = c_0 \lceil \log 1/\lambda \rceil$, there exists a sequence F_m of $(2^{d \cdot m}, 2^d, \lambda)$-expanders, where neighbors in F_m are computable in $O(d \cdot m)$ space and $\mathrm{poly}(d \cdot m)$ time.

INW Generator

We're now ready to present the construction of INW generator.

- For $0 < \lambda < 1$ and an integer $n \geq 1$, (λ, n)-INW generator is obtained recursively as follows.
- Let $\Gamma_0 : \{0,1\}^d \to \{0,1\}^d$ be the identity mapping.
- Then $\Gamma_{i+1} = \Gamma_i \otimes_{F_i} \Gamma_i$ where F_i is the $(2^{d(i+1)}, 2^d, \lambda)$-expander from the previous corollary.
- This gives (λ, n)-INW generator for every $n = d2^k$ where $k > 0$.
- We obtain the generator for arbitrary n by taking first n bit from (λ, n')-INW generator where $n' = d2^k \geq n$ is taken to be the smallest.
- Hence, (λ, n)-INW generator giving n bits of output has seed length $O(\log n \cdot \log 1/\lambda)$.

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With the INW generator Γ, we have the distribution

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THEOREM

Let G be any finite group of size at least 4 and $0 < \delta < 1$. Let $\lambda = \delta/(2^{c_1|G|^{12}} \cdot \sqrt{|G|})$ where c_1 is the universal constant from the corollary. Then (λ, n)-INW generator Γ uses seeds of length $O(\log n \cdot (|G|^{12} + \log 1/\delta))$ to product n bits such that for every group word w of length n,

$$\|\mathrm{Rnd}^w - D_\Gamma^w\| \leq \delta$$

. Moreover, the output of the generator is computable in space linear in the seed length.

Conclusion

- The author mentions that he didn't try to optimize the constant factor involved in the construction.
- The general derandomization problem for branching programs remains open.

The End