Ramanujan Coverings of Graphs

Aeren

Eigenvalue, Eigenvector, Spectrum

DEFINITION

Let V be a vector space over a field $K(= \mathbb{R} \text{ or } \mathbb{C})$ and $T: V \to V$ a linear operator.

1. $\lambda \in \mathbb{C}$ is an **eigenvalue** of T if there is a non-zero vector $v \in V$ such that $T(v) = \lambda v$.
2. A non-zero vector $v \in V$ is an **eigenvector** of f if there's a $\lambda \in \mathbb{C}$ such that $T(v) = \lambda v$.
3. The **spectrum** $\mathrm{Spec}(T)$ of T is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not bijective where I is the identity operator.

- If λ is an eigenvalue of T, then $\lambda \in \mathrm{Spec}(T)$ since $T - \lambda I$ is not injective.

- If V is infinite dimensional, there might be some $\lambda \in \mathrm{Spec}(T)$ which is not an eigenvalue.

- Let V be the vector space over $\mathbb{C}$ consisting of complex sequences. Let $R: V \to V$ be the right-shifting operator $(a_1, a_2, a_3 \ldots) \mapsto (0, a_1, a_2, \ldots)$. R has no eigenvalue since solving $Rv = \lambda v$ yields $v = (0,0, \ldots)$. On the other hand, $0 \in \mathrm{Spec}(R)$ since $R - 0I = R$ is not surjective.

Normed Vector Space

DEFINITION
A **normed vector space** V is a vector space V over a field $K(=\mathbb{R} \text{ or } \mathbb{C})$ endowed with a function $\|\cdot\|: V \to \mathbb{R}$, called the **norm**, such that
1. $\|v\| \geq 0$ for all $v \in V$
2. $\|v\| = 0$ if and only if $v = 0$
3. For all $v \in V$ and $a \in K$, $|a| \cdot \|v\| = \|av\|$
4. For all $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$

- The n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the standard Euclidean norm is a normed vector space.

- The vector space of square-summable complex sequences with the norm defined by $\|(a_1, a_2, \ldots)\| = \sqrt{\sum_{i=1}^{\infty} |a_i|^2}$ is a normed vector space. (This space is called $l^2(\mathbb{C})$)

Bounded Linear Operator

DEFINITION
Let V, W be normed vector spaces. A linear operator $T: V \to W$ is said to be **bounded** if there exists a non-negative real r such that $\|T(v)\|_W \leq r\|v\|_V$ for all $v \in V$. Smallest such r is called the **operator norm** of T, denoted as $\|T\|$.

- If V is finite dimensional, a linear operator $T: V \to W$ is always bounded.
- Let $C = \{c \in \mathbb{R}^{\mathbb{N}}: c_i = 0 \text{ for all but finitely many } i\}$.
- First, endow C with the norm $\|c\|_1 = \sum_{i=1}^{\infty} |c_i|$.
- Consider the linear operator $T: C \to \mathbb{R},\ c \mapsto \sum_{i=1}^{\infty} c_i$,
- $|T(c)| = |\sum_{i=1}^{\infty} c_i| \leq \sum_{i=1}^{\infty} |c_i| = \|c\|_1$, so $\|T\| = 1$.
- Second, endow C with the norm $\|c\|_2 = \max_{i=1}^{\infty} |c_i|$.
- Consider the same linear operator.
- For $c = (1, -1, 0, 0, \dots)$, $|T(c)| = 1 > r\|c\|_2 = 0$ for all $r \in \mathbb{R}$. Therefore, the linear operator is not bounded.

Bounded Linear Operator

THEOREM
For a bounded linear operator T, $\mathrm{Spec}(T)$ is non-empty compact subset of $\{c \in \mathbb{C} : |c| \leq \|T\|\}$.

- Let $X = l^2(\mathbb{C})$ and take the right shift operator $R: X \to X$.
- The operator preserves the norm, so it is bounded with $\|R\| = 1$.
- We'll show that $\mathrm{Spec}(R) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
- Let λ be a complex number and $T = R - \lambda I$. Since we've already shown that $0 \in \mathrm{Spec}(R)$, we'll assume that $\lambda \neq 0$.
- Note that solving the equation $(-\lambda c_1, c_1 - \lambda c_2, c_2 - \lambda c_3, \dots) = (-\lambda d_1, d_1 - \lambda d_2, d_2 - \lambda d_3, \dots)$ yields $c = d$ so $R - \lambda I$ is always injective.
- On the other hand, solving the equation $(-\lambda c_1, c_1 - \lambda c_2, c_2 - \lambda c_3, \dots) = (d_1, d_2, d_3, \dots)$ for constant d yields $c_n = -\frac{d_1 + \lambda d_2 + \cdots + \lambda^{n-1} d_n}{\lambda^n}$.
- If $|\lambda| > 1$, such c is always in X, so $R - \lambda I$ is surjective. If not, then taking $d = (1,0,0, \dots)$ yields $c = \left(-\frac{1}{\lambda}, -\frac{1}{\lambda^2}, -\frac{1}{\lambda^3}, \dots\right)$, which is not in X, which breaks the surjectivity.

Spectral Radius Of A Bounded Linear Operator

DEFINITION

The **spectral radius** $\sigma(T)$ of a bounded linear operator T is defined as

$$\sigma(T) = \max_{\lambda \in \mathrm{Spec}(T)} |\lambda|$$

THEOREM

Let $A \in \mathbb{C}^{n \times n}$. Then $\sigma(A) < 1$ if and only if

$$\lim_{k \to \infty} A^k = 0$$

On the other hand, if $\sigma(A) > 1$, $\lim_{k \to \infty} \left\| A^k \right\| = \infty$.

THEOREM (Gelfand)

For a normed vector space V and a bounded linear operator $T: V \to V$,

$$\sigma(T) = \lim_{k \to \infty} \left\| T^k \right\|^{1/k}$$

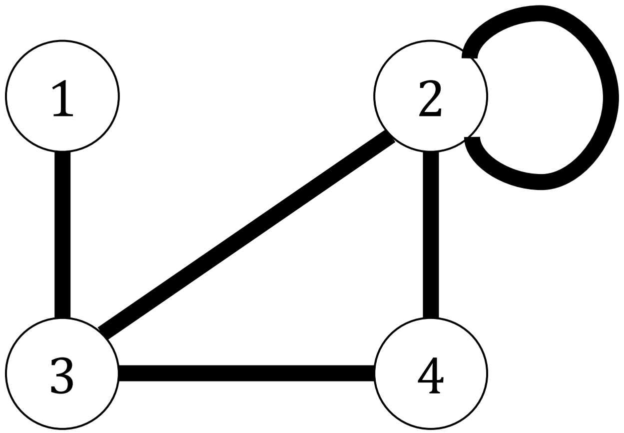
Perron-Frobenius Eigenvalue, Spectral Gap

- We assume that G is a finite, connected, undirected graph on n vertices with adjacency matrix A_G.

THEOREM (Spectral Theorem)
Let H be a Hermitian matrix (A square complex matrix whose conjugate transpose equals itself).
1. All the eigenvalues of H are real.
2. H is diagonalizable.

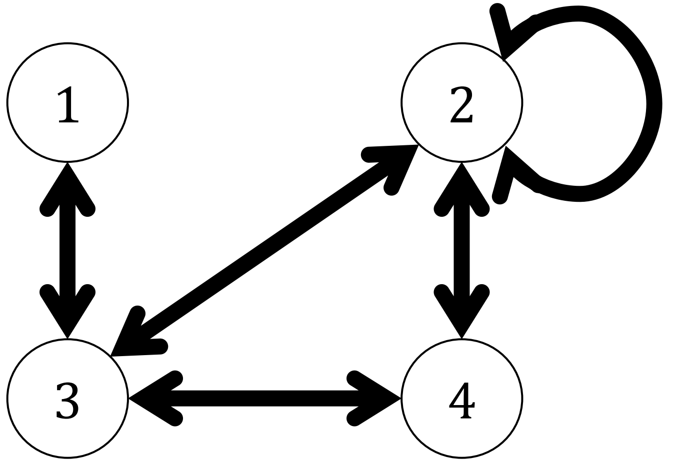
THEOREM (Perron and Frobenius)
Let M be a real square matrix with non-negative entries and D_M be the directed graph with adjacency matrix M.
If D_M is strongly connected(such M is called **irreducible**), there exists a real eigenvalue λ of M such that for all eigenvalues λ'(possibly complex) of M, $\lambda \geq |\lambda'|$.

Perron-Frobenius Eigenvalue, Spectral Gap



G

$$A_G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$



D_{A_G}

Eigenvalues of A_G: $\{3.05896, -1.43091, 0.698857, -0.326909\}$

- Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A_G. $\mathfrak{pf}(G) \stackrel{\text{def}}{=} \lambda_1$ is the **Perron-Frobenius eigenvalue** of A_G.
- The eigenvalues are symmetric along 0 if and only if G is bipartite, and $\lambda_1 > |\lambda_n|$ otherwise.
- We also define $\lambda(G) = \max(|\lambda_2|, |\lambda_n|)$.
- The difference $\mathfrak{pf}(G) - \lambda(G)$ is called the **spectral gap.**

Expansion

DEFINITION

Let G be a finite undirected graph with n vertices.

1. The **edge expansion** (or **isoperimetric number** or **Cheeger constant**) is

$$h(G) = \min_{\substack{0<|S|\leq\frac{n}{2} \\ S\subset V(G)}} \frac{|\partial S|}{|S|}$$

where $\partial S = \{\{u,v\} \in E(G) : u \in S, v \in V(G)\setminus S\}$.

2. The **vertex expansions** (or **vertex isoperimetric numbers** or **magnifications**) are

$$h_{out}(G) = \min_{\substack{0<|S|\leq\frac{n}{2} \\ S\subset V(G)}} \frac{|\partial_{out} S|}{|S|}$$

$$h_{in}(G) = \min_{\substack{0<|S|\leq\frac{n}{2} \\ S\subset V(G)}} \frac{|\partial_{in} S|}{|S|}$$

where $\partial_{out} S = \{u \in V(G)\setminus S : \exists v \in S \text{ such that } \{u,v\} \in E(G)\}$ and $\partial_{in} S = \{u \in S : \exists v \in V(G)\setminus S \text{ such that } \{u,v\} \in E(G)\}$.

Expansion

- Expansion is known to be related to how fast a random walk on a finite connected regular graph converges via the expander mixing lemma.

THEOREM (Cheeger Inequality)

Let G be a d-regular graph and g be its spectral gap. Then following inequalities hold.

$$\frac{1}{2}g \leq h(G) \leq \sqrt{2dg}$$

$$h_{out}(G) \leq \left(\sqrt{4g} - 1\right)^2 - 1$$

$$h_{in}(G) \leq \sqrt{8g}$$

Adjacency Operator

- We note that an adjacency matrix of a finite graph is just a linear map between n dimensional vector space given an ordered basis.

- In order to extend the definition to the infinite case, we think of a vector $a = (a_1, \dots, a_n)$ as a function $f: V(G) \to \mathbb{R}$ such that for the vertex v_i $(i = 1, \dots, n)$, $f(v_i) = a_i$.

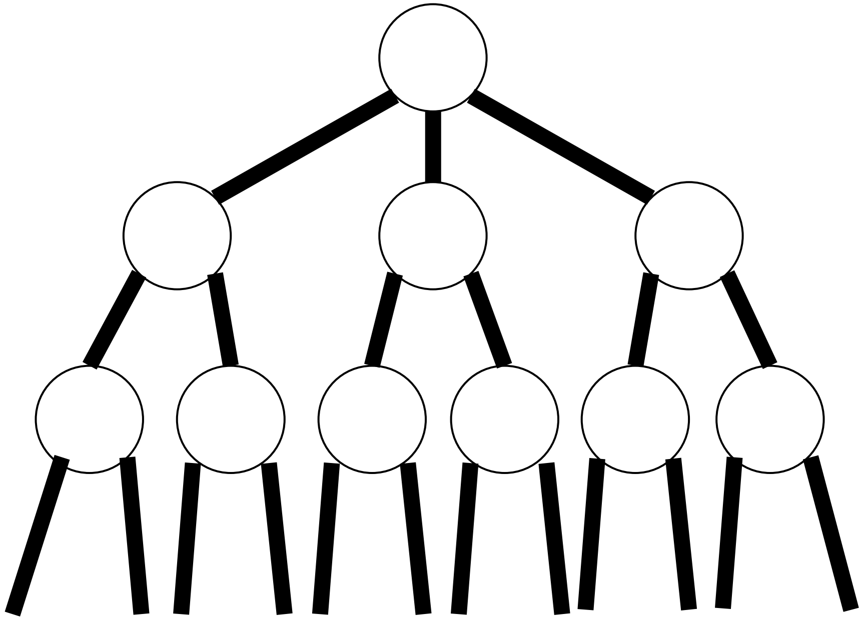
DEFINITION
Let G be an undirected graph (possibly infinite) with bounded degree.
$l^2\big(V(G)\big)$ is the normed real vector space of functions $f: V(G) \to \mathbb{R}$ such that $\sum_{v \in V(G)} f(v)^2$ converges, endowed with the l^2-norm $\left(\|f\| = \sqrt{\sum_{v \in V(G)} f(v)^2} \right)$.
The **adjacency operator** of G is the bounded linear operator $A: l^2\big(V(G)\big) \to l^2\big(V(G)\big)$ such that

$$A(f)(v) = \sum_{(v,w) \in E(G)} f(w)$$

Spectral Radius

DEFINITION
The **spectral radius** $\sigma(G)$ of a graph G (possibly infinite) with bounded degree is the spectral radius of its adjacency operator.

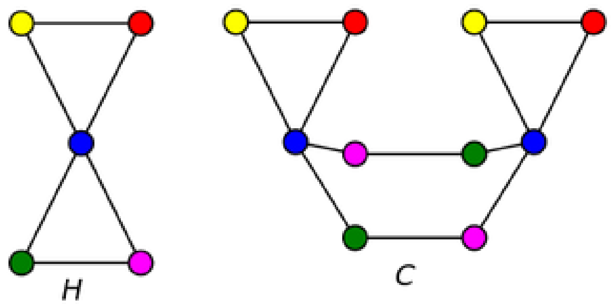
- If T_d is the unique (up to isomorphism) d-regular tree for $d \geq 2$, then $\sigma(T_d) = 2\sqrt{d-1}$.

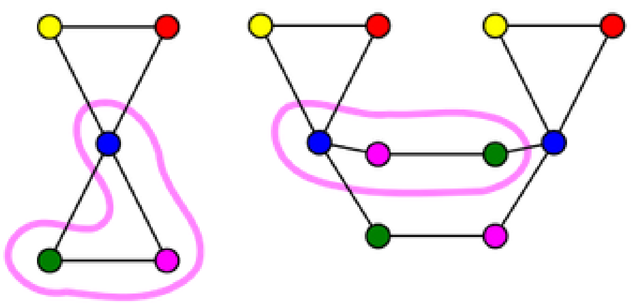


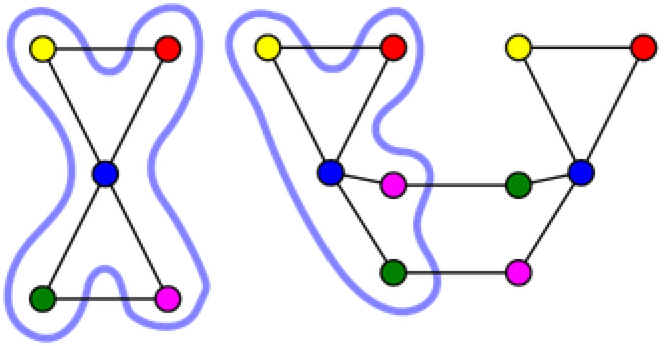
Covering Graph

DEFINITION
Let H and C be graphs (possibly infinite).

1. Let $f: V(C) \to V(H)$ be a surjection. f is said to be a **covering map** from C to H if for all vertex $v \in V(C)$, the induced map $f: N(v) \to N(f(v))$ is bijective.
2. If there exists a covering map from C to H, C is said to be a **cover** of H.
3. A covering map C is said to be an ***r*-cover** if $f^{-1}(\{w\})$ is of size r for all $w \in V(H)$.







- In particular, a cover of a bipartite graph is again bipartite.

Universal Covering Graph

DEFINITION
Let H be a connected graph. A **universal cover** of H is a tree that is also a cover of H.

- Universal covering graph of a fixed graph is unique. (up to isomorphism)

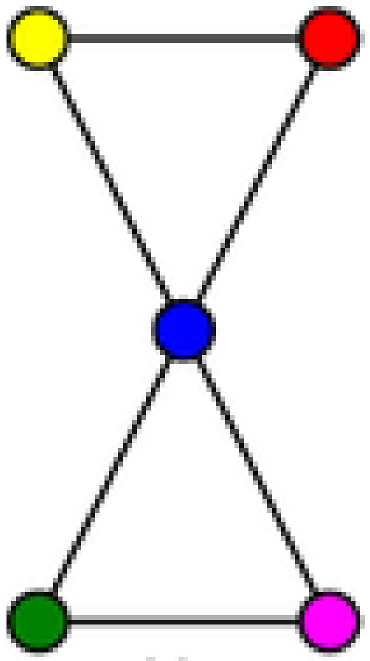
- The universal covering graph of a d-regular graph is the unique d-regular tree.

- The universal covering graph of a tree is isomorphic to itself.

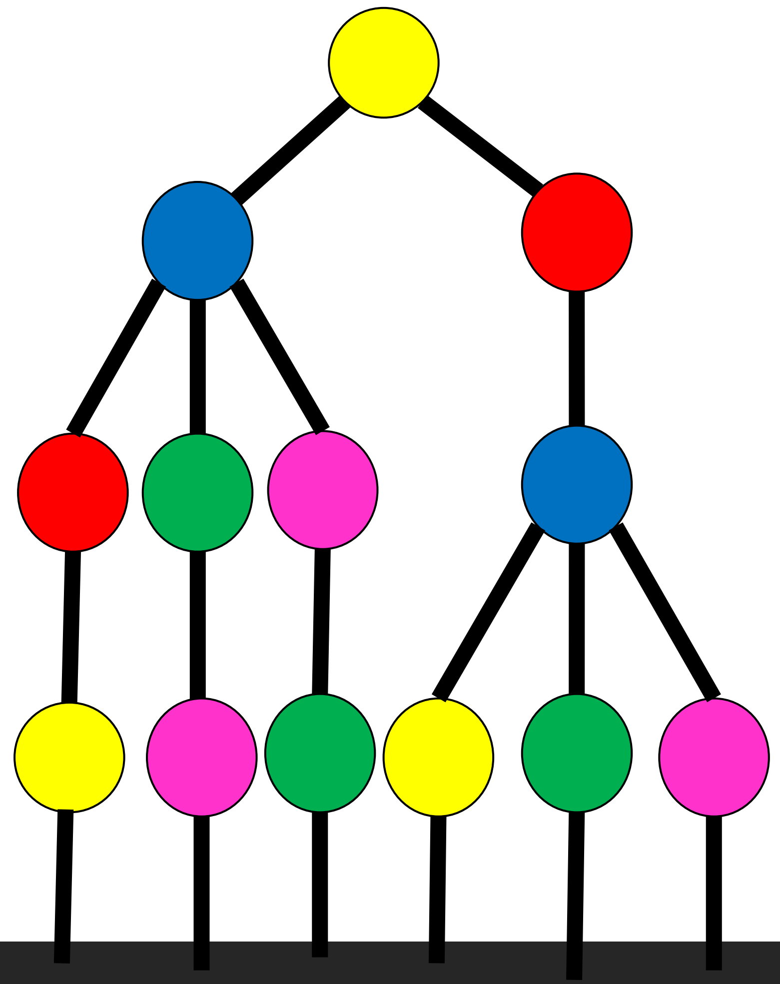
- The universal covering graph of a graph with cycle is always infinite.

- The universal covering graph of a finite graph is always of bounded degree.

- A graph has isomorphic universal cover to its covers.



H



Ramanujan Graph

- Let $\rho(G)$ be the spectral radius of the universal covering graph of G.

- $\rho(G) = 2\sqrt{d-1}$ for a d-regular graph in particular.

THEOREM (Noga)
Let G be a d-regular graph. The following inequality holds:

$$\lambda(G) \geq \rho(G) - \frac{\rho(G) - 1}{\lfloor D(G)/2 \rfloor}$$

where $D(G)$ is the diameter of G.

- We can see that $\lambda(G)$ is not much smaller than $\rho(G)$ for a regular graph G.

- We call the range $[-\rho(G), \rho(G)]$ the **Ramanujan interval**.

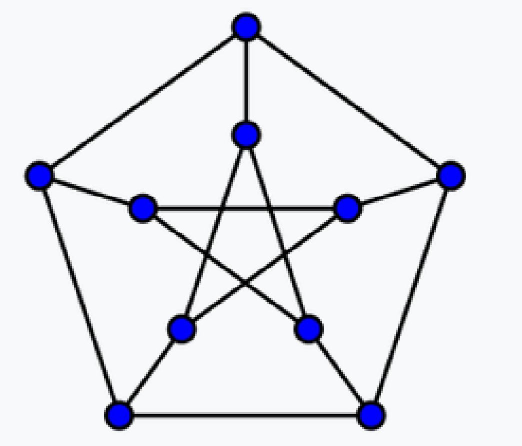
Ramanujan Graph

DEFINITION
A graph G is a **Ramanujan graph** if $\lambda(G)$ lies within the Ramanujan interval.

DEFINITION
A bipartite graph G is a **bipartite Ramanujan graph** if all of its eigenvalues except for the largest and the smallest one lies within the Ramanujan interval.

- The complete graph K_{d+1} (which is d-regular) has spectrum $d, -1, \dots, -1$, so $\lambda(G) = 1 \leq \rho(G) = 2\sqrt{d-1}$ and thus it is a Ramanujan graph for all $d > 1$.
- The complete bipartite graph $K_{d,d}$ (which is again d-regular) has spectrum $d, 0, \dots, 0, -d$, so it is a bipartite Ramanujan graph for all $d \geq 1$.

- The Peterson graph has spectrum $3,1,1,1,1,1,-2,-2,-2,-2$
so it is a Ramanujan graph.



Ramanujan Cover

- Consider the equivalence relation of all finite connected graphs by isomorphic universal covering.
- For example, all the finite connected d-regular graphs are equivalent.
- Such equivalence classes contains at least one Ramanujan graph: K_{d+1}.
- Some classes (such as the class containing (k, l)-biregular tree) contains none.
- Still, it makes sense to look for the “optimal expanders”.
- For a fixed class, our strategy goes like this

1) we pick a graph with as small number of “bad eigenvalues” as possible.
2) extend the graph by only adding mostly “good eigenvalues”.

Ramanujan Cover

- Let H be a finite cover of G with m vertices with covering map $T: V(H) \to V(G)$.
- If $f: V(G) \to \mathbb{R}$ is an eigenfunction of G, then $f \circ T$ is an eigenfunction of H with the same eigenvalue.
- Therefore, out of m eigenvalues of H, n of them are from G, referred to as **old eigenvalues** and the remaining $m - n$ of them will be referred as **new eigenvalues.**

DEFINITION
Let H be a finite cover of a finite graph G.
1. We say that H is a **Ramanujan cover** of G if all new eigenvalues of H lies in the Ramanujan interval.
2. We say that H is a **one-sided Ramanujan cover** of G if all new eigenvalues of H are bounded from above by $\rho(G)$.

Ramanujan Cover

THEOREM (Marcus, Spielman, Srivastava)
Every connected loopless graph has an one-sided Ramanujan 2-cover.

COROLLARY
There exists infinitely many r-regular bipartite Ramanujan graphs for every r.

THEOREM (Main Objective)
Every connected loopless graph has an one-sided Ramanujan r-cover for every r.

COROLLARY
Every connected bipartite graph has a Ramanujan r-cover for every r.

Group Representation

DEFINITION
A **representation** of a group Γ on a vector space V over a field K is a group homomorphism

$$\Gamma \to \mathrm{GL}(V)$$

such V is called the **representation space** and the dimension of V is called the **dimension** of the representation.

- Let $C_3 = \{1, c, c^2\}$ be the cyclic group of order 3 and $z = e^{2\pi i/3}$ a complex number. Then the homomorphism

$$\rho: C_3 \to GL(\mathbb{C}^2)$$

given by

$$\rho(c^n) = \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix}$$

is a group representation.

Generalization

- We first choose an orientation of each edges in $E(G)$ and write it $E^+(G)$, and write $E^-(G)$ for the reversed oriented edges.

- We now identify $E(G)$ as $E^+(G) \cup E^-(G)$.

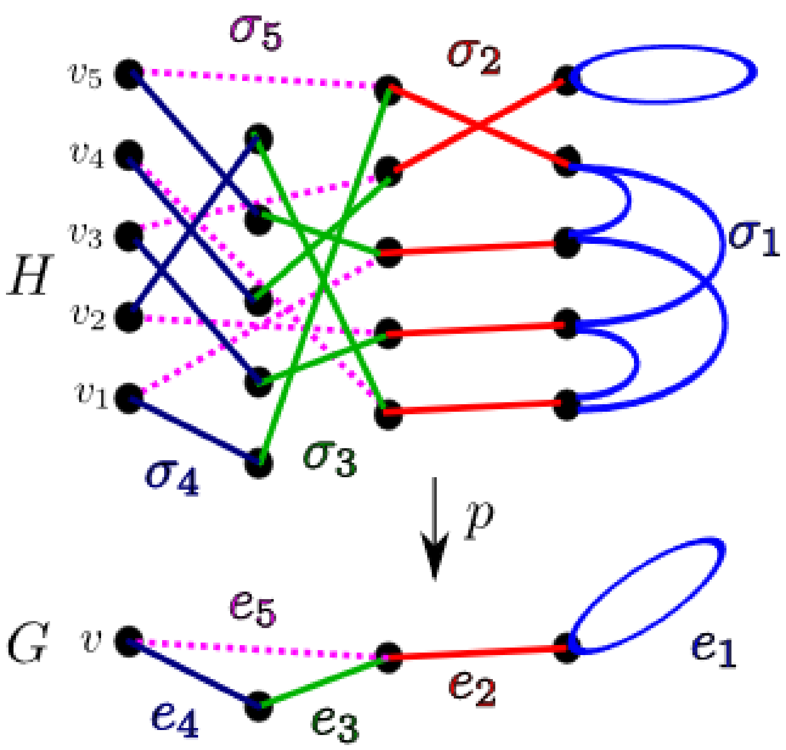
- Also, whenever we have an edge e in $E^\pm(G)$, we write the corresponding reversed edge in $E^\mp(G)$ as $-e$.

- Let H be an r-cover of G.

- For each $v \in G$, we identify each vertices of H which maps to v by the covering map as $v_1, \ldots, v_r$.

- The edges of H are now given by the function $\sigma: E(G) \to S_r$ satisfying $\sigma(-e) = \sigma^{-1}(e)$. We denote $\sigma(e)$ as σ_e.

- In order words, for each edge $e \in E^+(G)$, we introduce in H the r edges from $t(e)_i$ to $h(e)_{\sigma_e(i)}$.



DEFINITION

Let Γ be a finite group. A **Γ-labelling** of the graph G is a function $\gamma: E(G) \to \Gamma$ satisfying $\gamma(-e) = \gamma^{-1}(e)$.

Generalization

- Let $\pi\colon \Gamma \to \mathrm{GL}(\mathbb{C}^d)$ be a representation of Γ.

- For any Γ-labelling γ of G, $A_{\gamma,\pi} \in M_{nd}(\mathbb{C})$ is the matrix obtained from A_G as follows.

- For all $u, v \in V(G)$, replace the (i, j) entry of A_G by the $d \times d$ block $\sum_{e:u\to v} \pi(\gamma(e))$.

DEFINITION

The $A_{\gamma,\pi}$ obtained above is called a **$(\boldsymbol{\Gamma}, \boldsymbol{\pi})$-cover** of G.

- $A_{\gamma,\pi}$ obtained this way is always Hermitian. Thus, $\mathrm{Spec}(A_{\gamma,\pi})$ is a subset of $\mathbb{R}$.

DEFINITION

The (Γ, π)-cover $A_{\gamma,\pi}$ of G is said to be **Ramanujan** if $\mathrm{Spec}(A_{\gamma,\pi}) \subseteq [-\rho(G), \rho(G)]$ and **one-sided Ramanujan** if $\mathrm{Spec}(A_{\gamma,\pi}) \subseteq [-\infty, \rho(G)]$.

THEOREM

If $\gamma\colon E(G) \to S_r$ is an S_r-labelling of G, then the new spectrum of the r-cover of G associated with γ is equal to the spectrum of $A_{\gamma,\mathrm{std}}$, where std is the standard $r - 1$ dimensional representation of S_r.

Generalization

- For which pair (Γ, π) is it guaranteed that every connected loopless graph has a one-sided/full Ramanujan cover?

DEFINITION
Let Γ be a finite group and $\pi: \Gamma \to \mathrm{GL}(\mathbb{C}^d)$ its representation. We say that (Γ, π) satisfies

1. $(\mathcal{P}1)$ if all exterior power $\Lambda^m \pi$ are irreducible and non-isomorphic $(0 \leq m \leq d)$.
2. $(\mathcal{P}2)$ if $\pi(\Gamma)$ is a complex-reflection group.

THEOREM (Main Objective)
Let Γ be a finite group and $\pi: \Gamma \to \mathrm{GL}(\mathbb{C}^d)$ its representation such that (Γ, π) satisfies $(\mathcal{P}1)$ and $(\mathcal{P}2)$. Then every connected, loopless graph G has a one-sided Ramanujan (Γ, π)-cover.

Overview of the Proof

THEOREM
Let $f, g \in \mathbb{R}[x]$ be polynomials of degree n such that for all $\lambda \in [0,1]$, $(1-\lambda)f + \lambda g$ is real-rooted. Then for all $1 \leq i \leq n$, the i-th root of $(1-\lambda)f + \lambda g$ moves monotonically when λ moves from 0 to 1.

- For an r-cover H of G, consider the polynomial

$$\phi_H = \frac{\det(xI - A_H)}{\det(xI - A_G)} = \det\left(xI - A_{\sigma,\mathrm{std}}\right)$$

- Let $\Delta_r(G)$ be the simplex of all ϕ_H.
- Every point $p \in \Delta_r(G)$ is associated with a polynomial ϕ_p, and a probability distribution of all r-covers of G.
- First, we find a point p_{ram} where the roots of ϕ_p lies within the Ramanujan interval.
(It turns out that the center of the simplex, which corresponds to the uniform distribution, satisfies this)
- Second, we find a convex region in $\Delta_r(G)$ which contains p_{ram} and one of the vertex of the simplex. where each polynomial corresponding to each points is real-rooted.
- Using the theorem above, we gradually reach a vertex of the simplex, which corresponds to an r-cover of G, whose roots lies within the Ramanujan interval.

Open Problems

Does every connected loopless graph has a Ramanujan r-cover for every r?

Are there infinitely many r-regular Ramanujan graphs for every r?

Is there any more pairs (Γ, π) of a finite group and its irreducible representation which guarantees the existence of one-sided Ramanujan (Γ, π)-cover for every finite graphs?
So far, not a single counterexample is known.

Previous question but with full Ramanujan (Γ, π)-cover.

Does every result also hold for graphs with loops?