Shorter Tours by Nicer Ears:

From connected T-join to graphic TSP variants

Jongseo Lee
October 16 2023

Introduction

Review: Christofides Algorithm for TSP

- TSP = connected + every vertex has even degree

- Connectivity: Find MST ($\leq OPT$)
- Parity: Find the minimum-cost T-join ($\leq 0.5OPT$)
  - Definition: An edge set J is called T-join if the set of vertices that have an odd degree is exactly the set T.
  - Finding T-join: solve min-cost perfect matching on T.

Connected T-join Problem

- Input
  - Undirected (connected) graph $G = (V, E)$
  - $T \subseteq V$ with even size
- Goal
  - Find a minimum cardinality set $F \subseteq 2G$ such that $(V(G), F)$ is connected and F is a T-join

- If $T = \emptyset$, this is equivalent to the graphic TSP.
- If $T = \{s, t\}$, this is equivalent to $s - t$ path TSP.

Bi-Connected Components

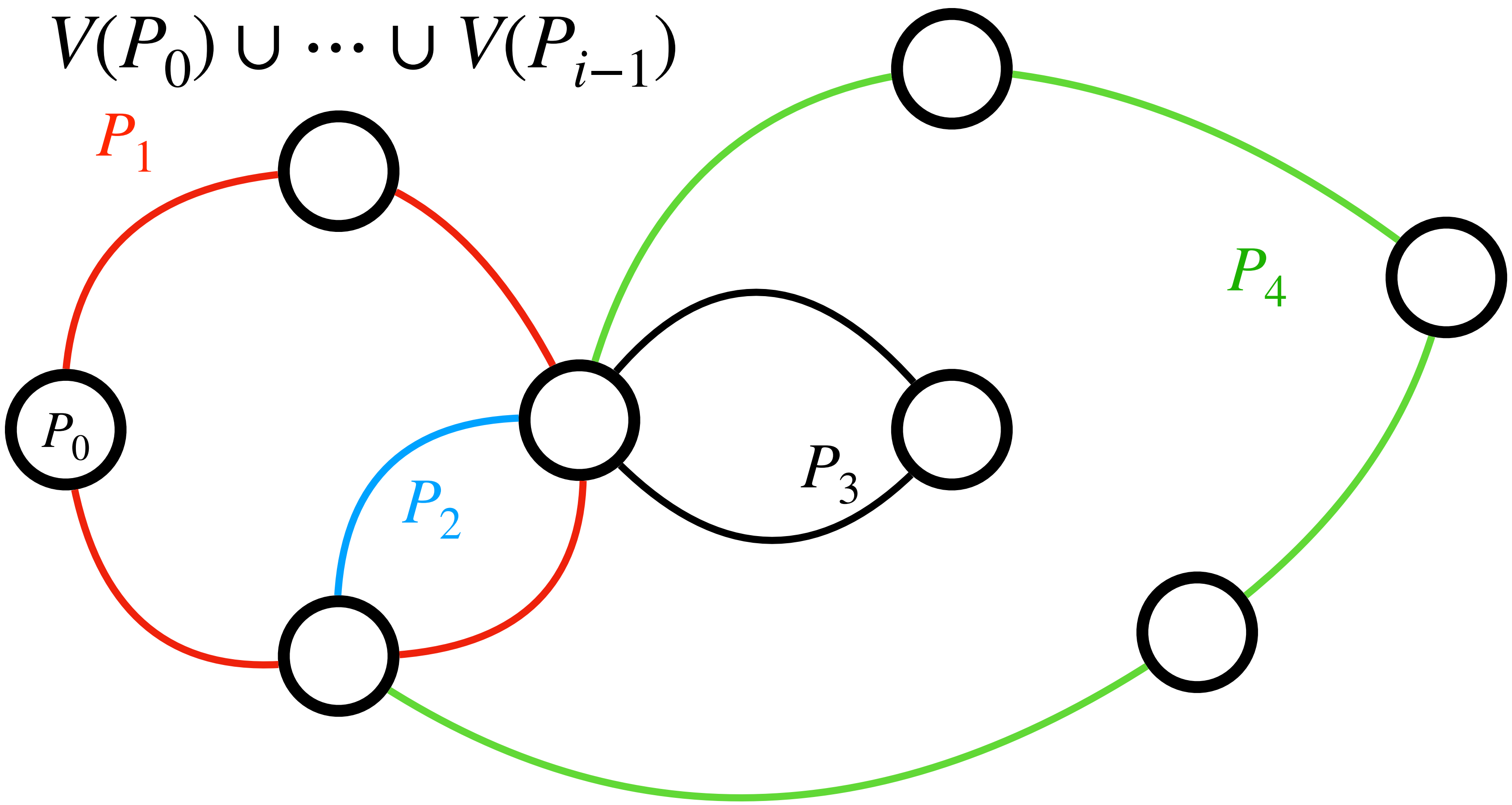
Lemma. Let G_1, G_2 be 2-connected graphs with $V(G_1) \cap V(G_2) = \{v\}$. Let $G := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and $T \subseteq V(G)$ with $|T|$ even. For $i = 1,2$, define T_i be the even set among $(T \cap V(G_i)) - \{v\}$ and $(T \cap V(G_i)) \cup \{v\}$. Solving connected T-join in (G, T) is equivalent to solving connected T-join in (G_1, T_1) and (G_2, T_2).

- In other words, we can consider each BCC separately.

Ear-Decomposition

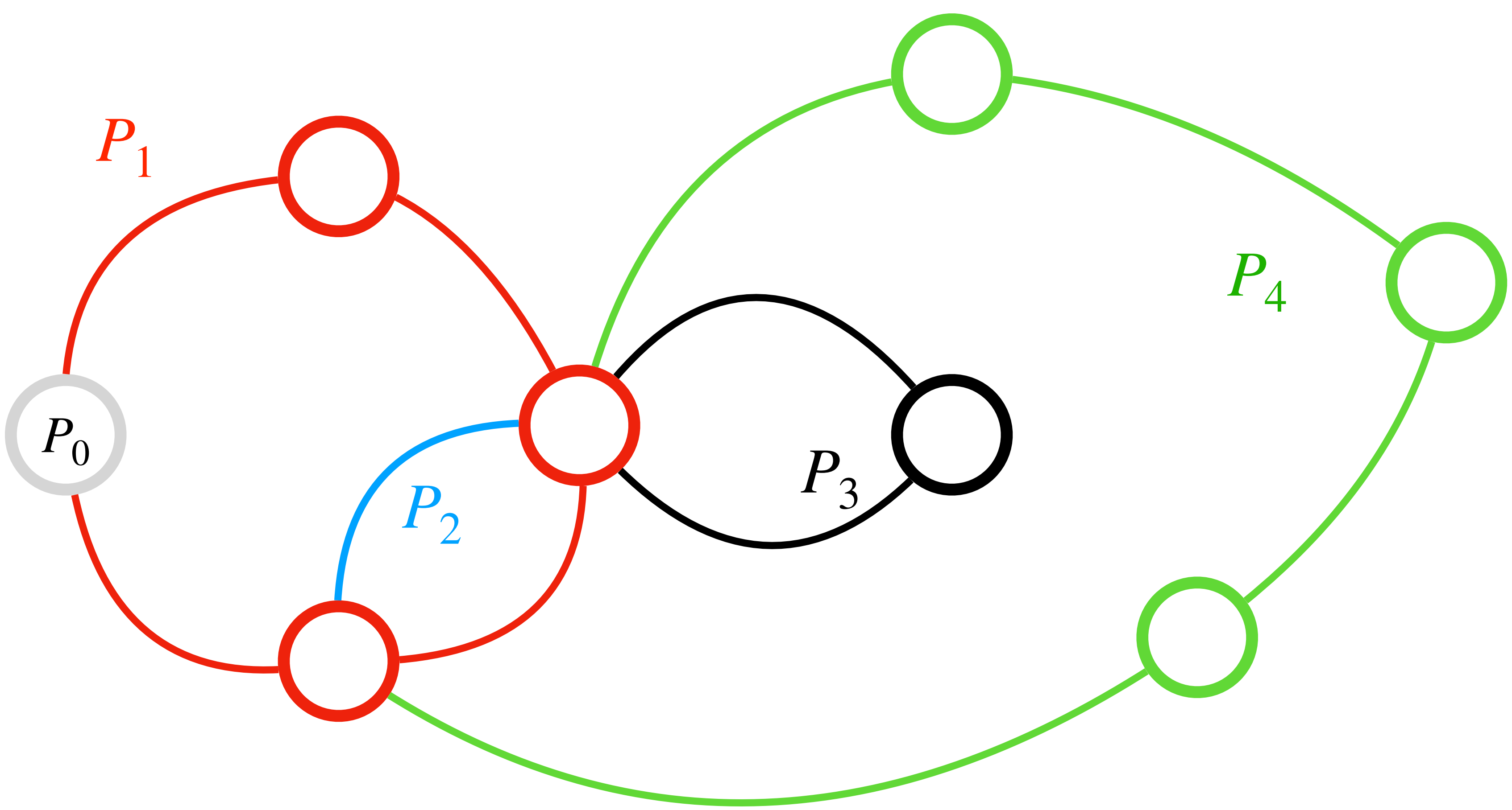
Ear-Decomposition

- An **ear-decomposition** is a sequence $P_0, P_1, \cdots, P_k$ where P_0 is a graph consisting of only one vertex (and no edge), and for each $i \in [k]$ we have:
  - (closed ear) P_i is a circuit sharing exactly one vertex with $V(P_0) \cup \cdots \cup V(P_{i-1})$, or
  - (open ear) P_i is a path sharing exactly its two different endpoints with $V(P_0) \cup \cdots \cup V(P_{i-1})$



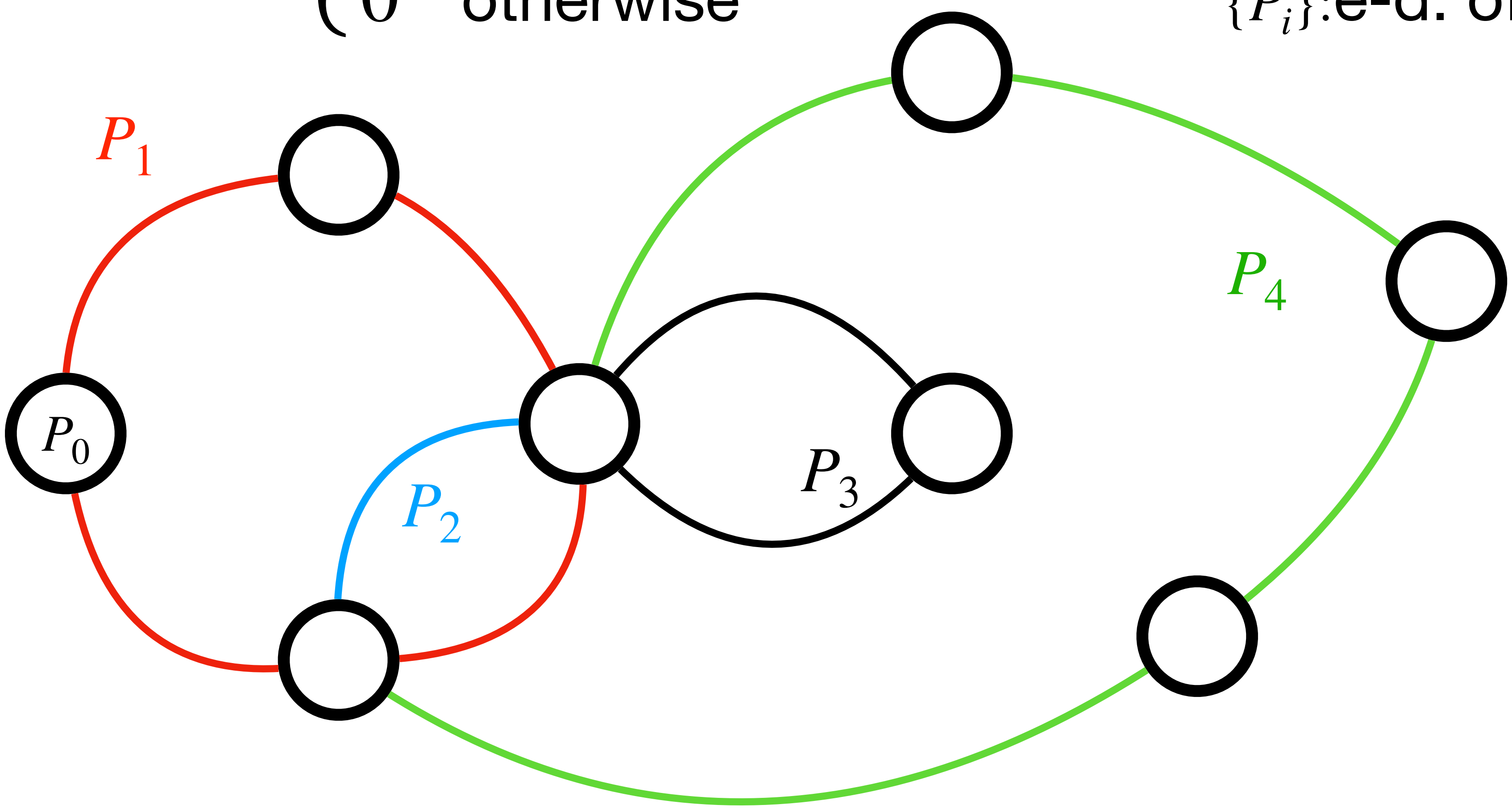
Ear-Decomposition

- ear = endpoint + internal vertices
- $in(Q)$:= set of internal vertices of an ear Q (colored vertices)
  - $|in(Q)| = |E(Q)| - 1$
- If $q \in in(Q)$ is an endpoint of P, say P is attached to Q (at q).



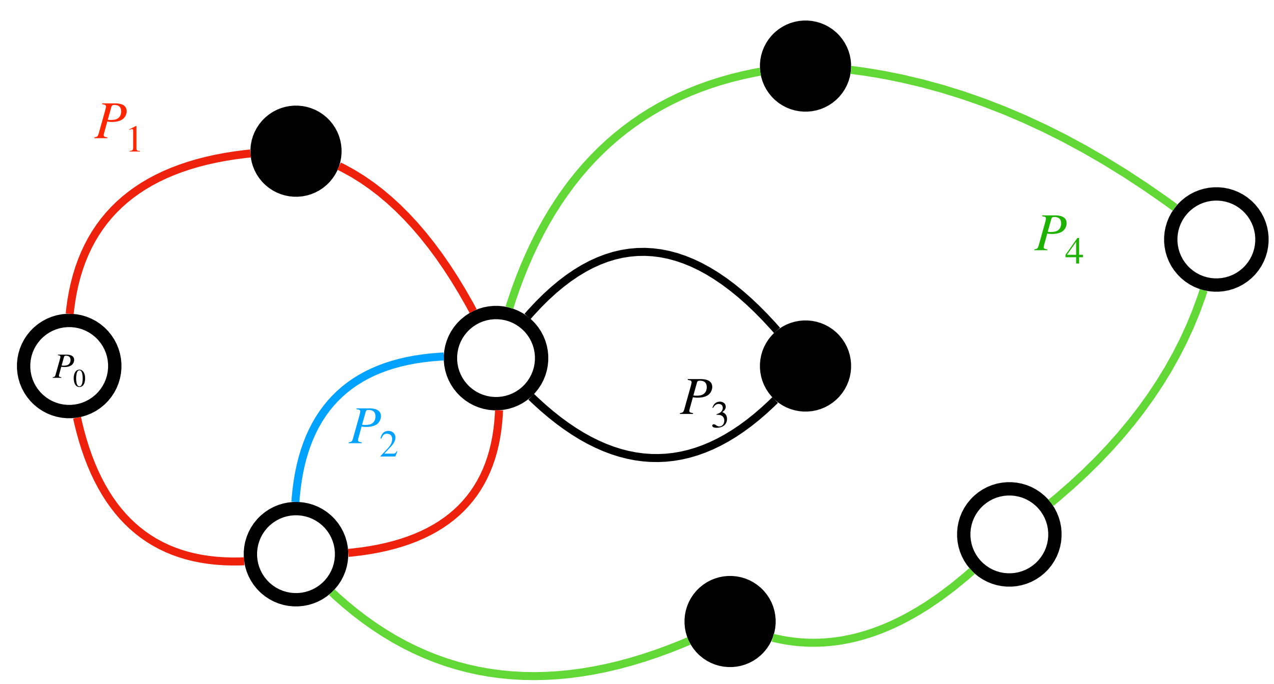
Ear-Decomposition

- l-ear: an ear of length l (i.e. number of edges = l)
- Short ear: 2-ear or 3-ear
- Nontrivial ear: ear of length greater than 1
- Pendant ear: nontrivial and no nontrivial ear attached to it
- Even ear: an ear with even length
- $\phi(P) = \begin{cases} 1 & |E(P)| \text{ is even} \\ 0 & \text{otherwise} \end{cases}$; $\phi(G) := \min_{\{P_i\}:\text{e-d. of } G} \sum_{P \in \{P_i\}} \phi(P)$



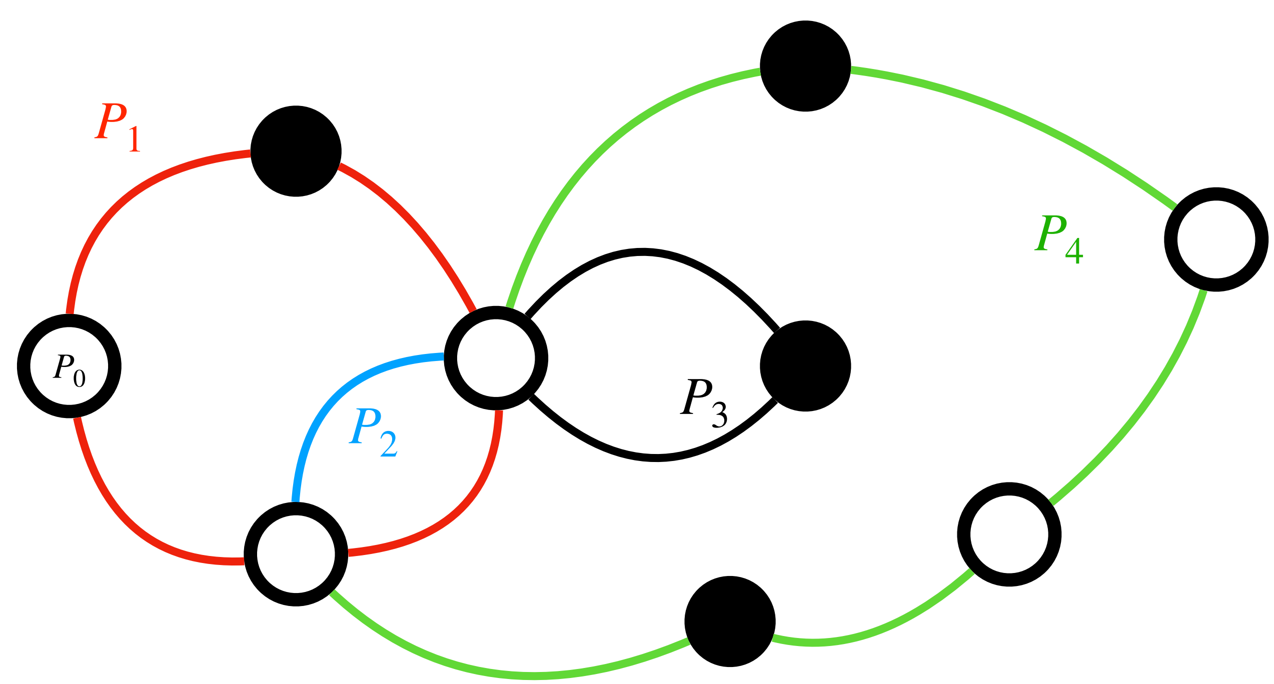
Connected T-join from Ear-Decomposition

- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even



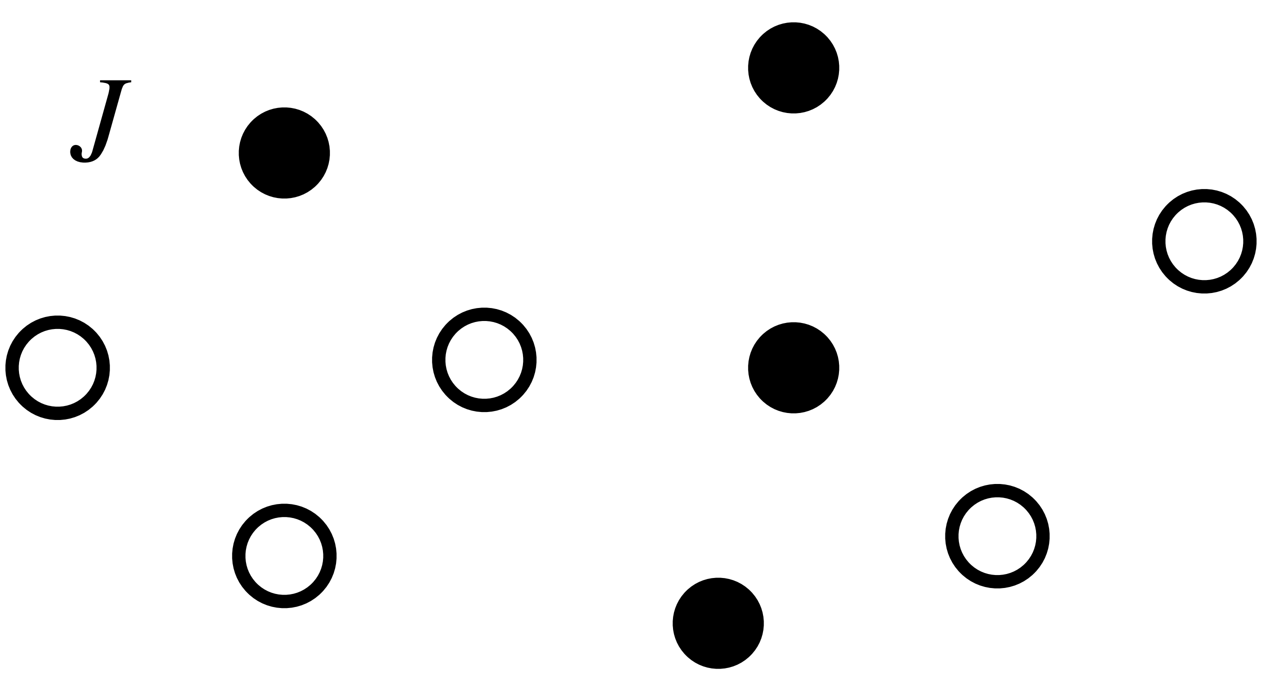
Connected T-join from Ear-Decomposition

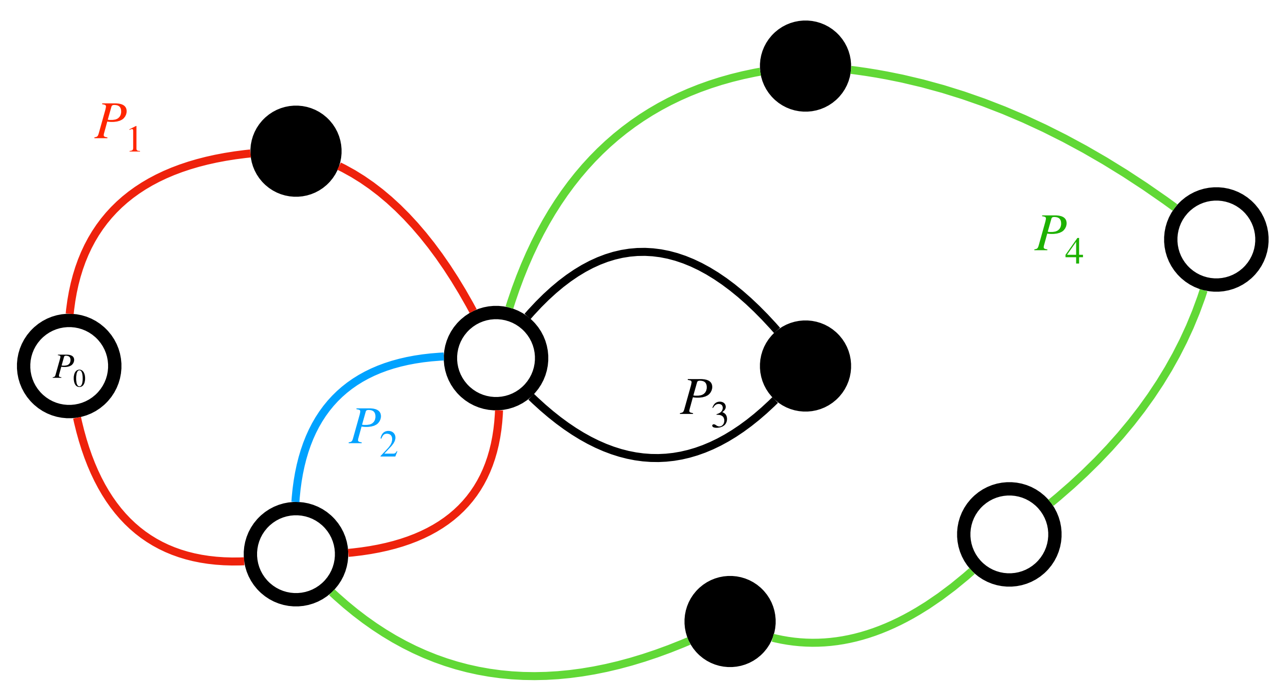
- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$



Connected T-join from Ear-Decomposition

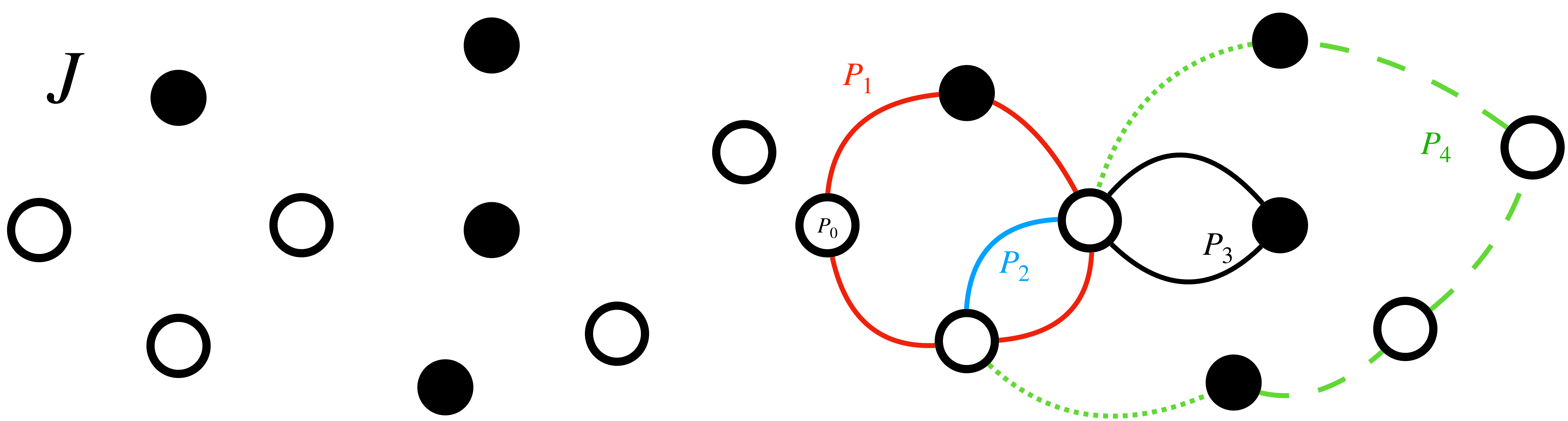
- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \varnothing$ initially)





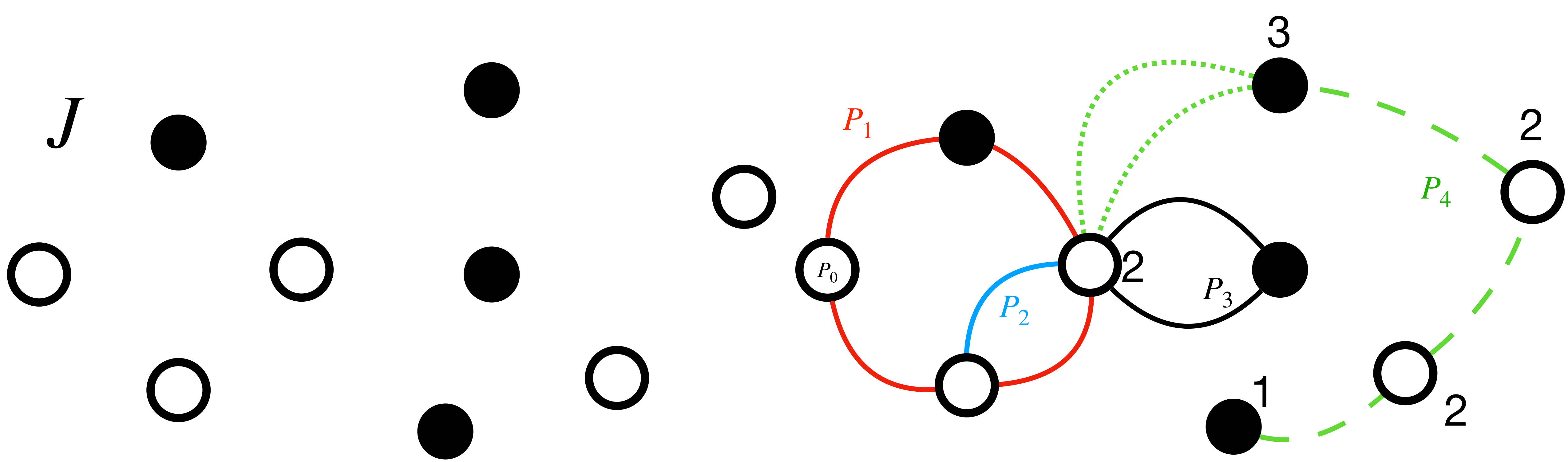
Connected T-join from Ear-Decomposition

- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \emptyset$ initially)
  - Find a connected $(T \cap in(P))$-join J_P in P



Connected T-join from Ear-Decomposition

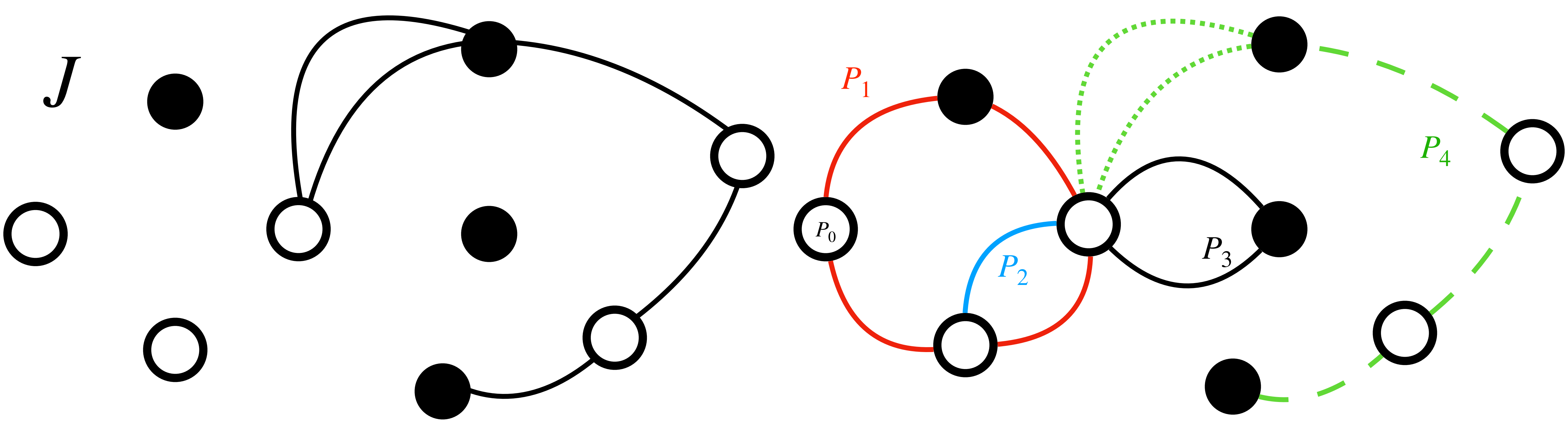
- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \emptyset$ initially)
  - Find a connected $(T \cap in(P))$-join J_P in P



Connected T-join from Ear-Decomposition

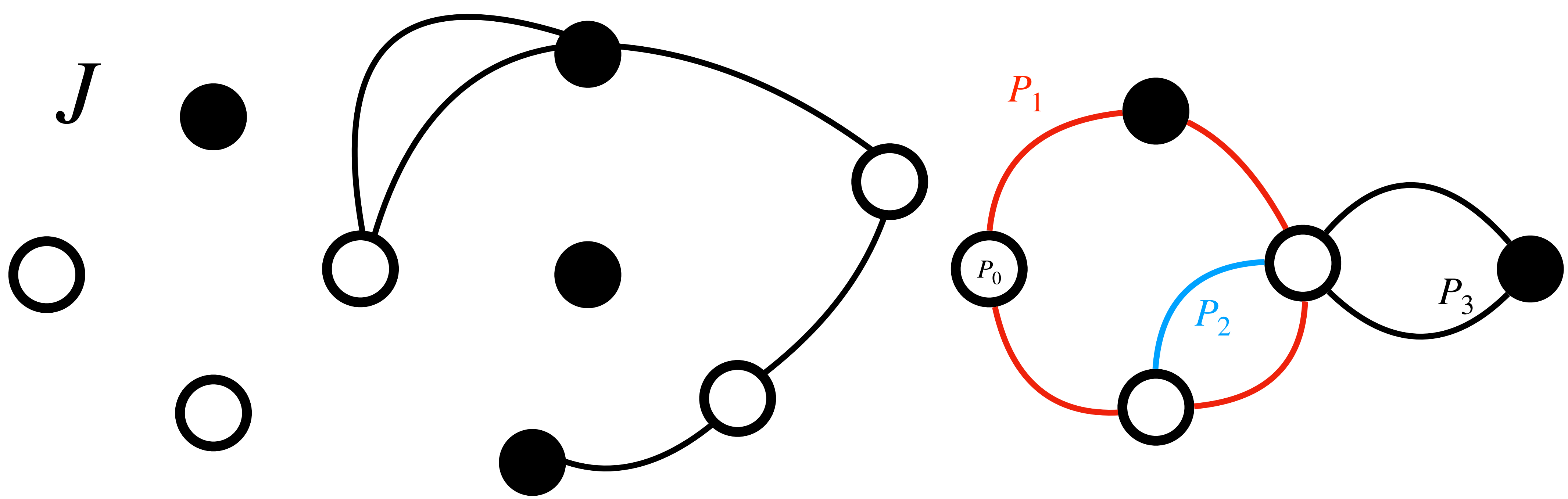
- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \varnothing$ initially)
  - Find a connected $(T \cap in(P))$-join J_P in P
  - $J := J \cup J_P$

J



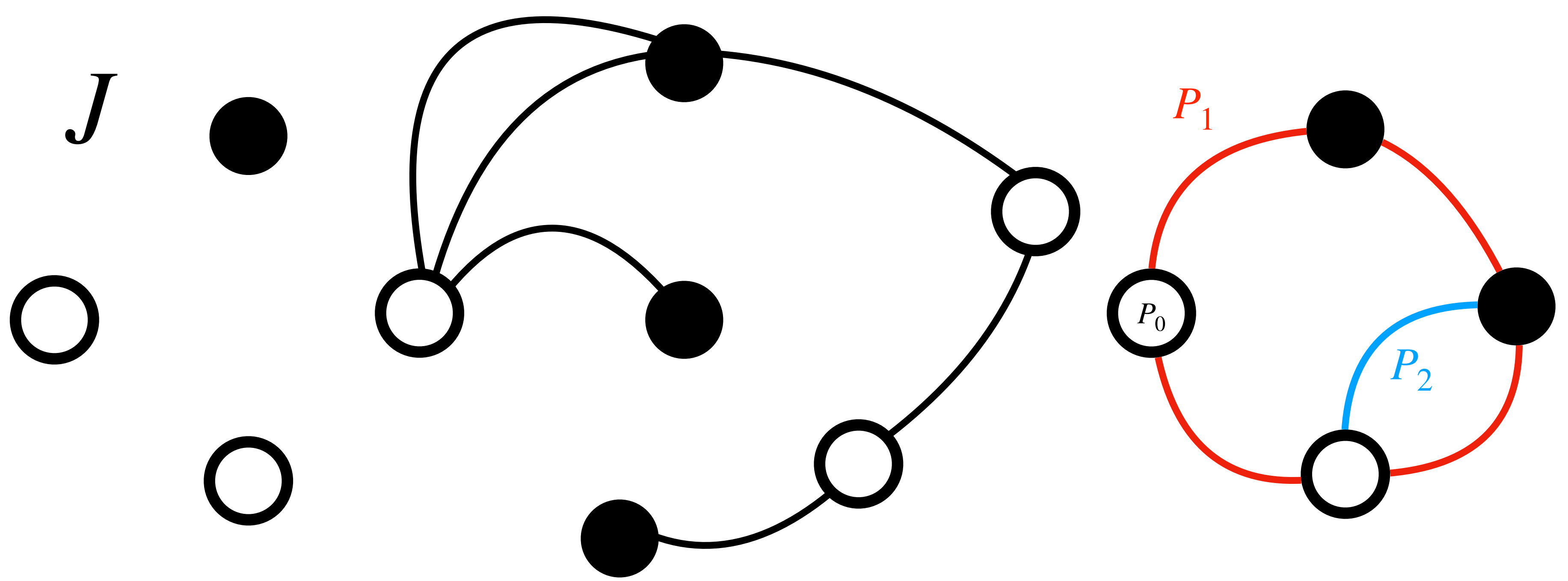
Connected T-join from Ear-Decomposition

- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \varnothing$ initially)
    - Find a connected $(T \cap in(P))$-join J_P in P
    - $J := J \cup J_P$
    - Delete P from G, modify T appropriately



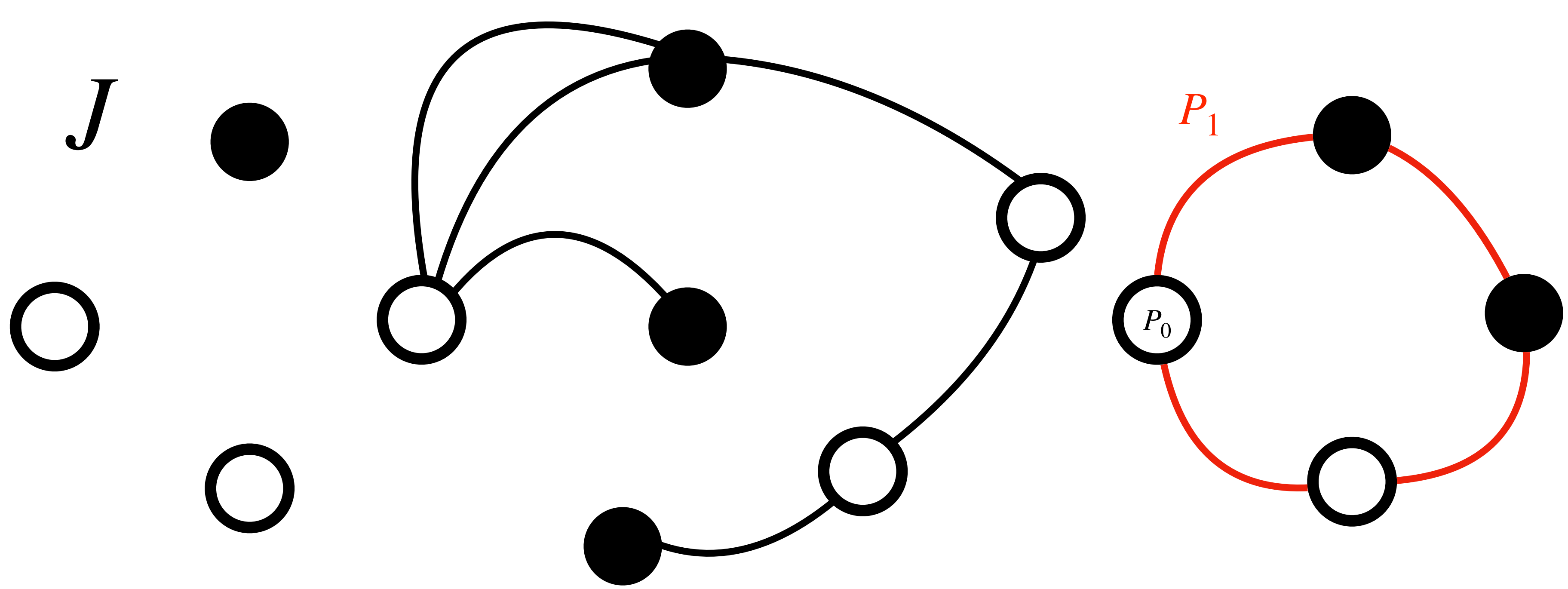
Connected T-join from Ear-Decomposition

- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \emptyset$ initially)
  - Find a connected $(T \cap in(P))$-join J_P in P
  - $J := J \cup J_P$
  - Delete P from G, modify T appropriately



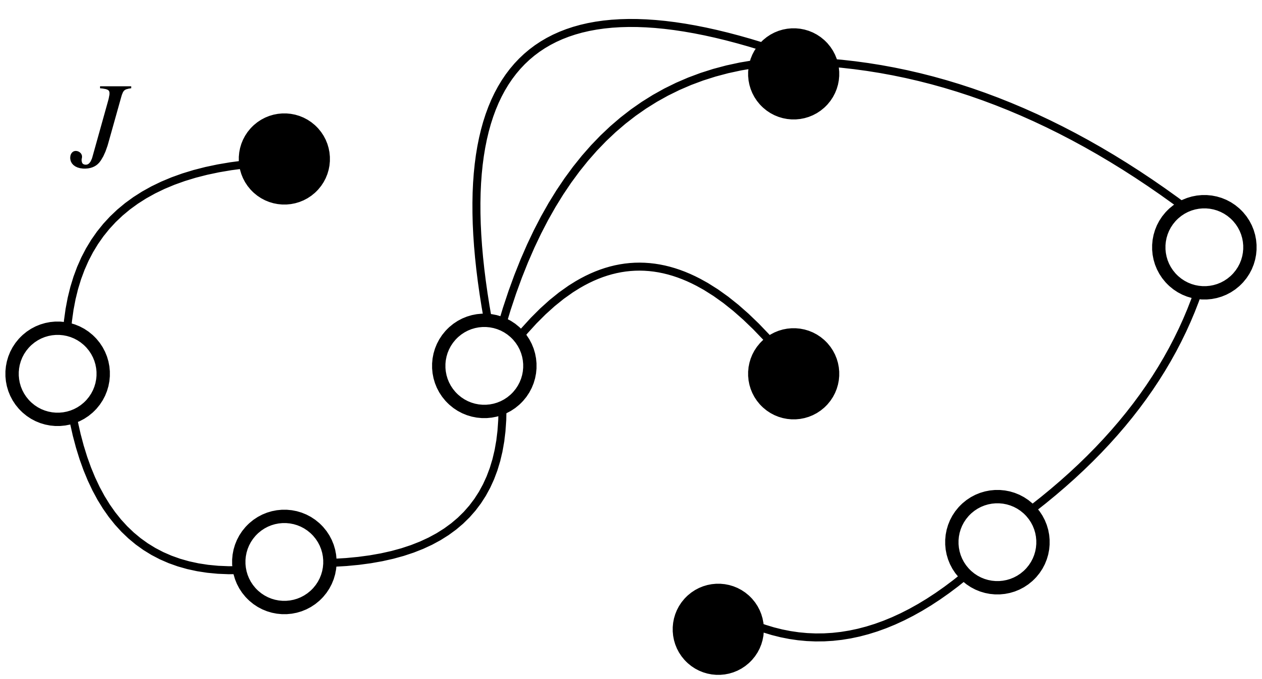
Connected T-join from Ear-Decomposition

- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \emptyset$ initially)
  - Find a connected $(T \cap in(P))$-join J_P in P
  - $J := J \cup J_P$
  - Delete P from G, modify T appropriately



Connected T-join from Ear-Decomposition

- G: 2-edge-connected graph with an ear decomposition $\{P_i\}$
- T: $T \subseteq V(G)$, $|T|$ even
- Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
- For each ear P: ($J := \emptyset$ initially)
  - Find a connected $(T \cap in(P))$-join J_P in P
  - $J := J \cup J_P$
  - Delete P from G, modify T appropriately



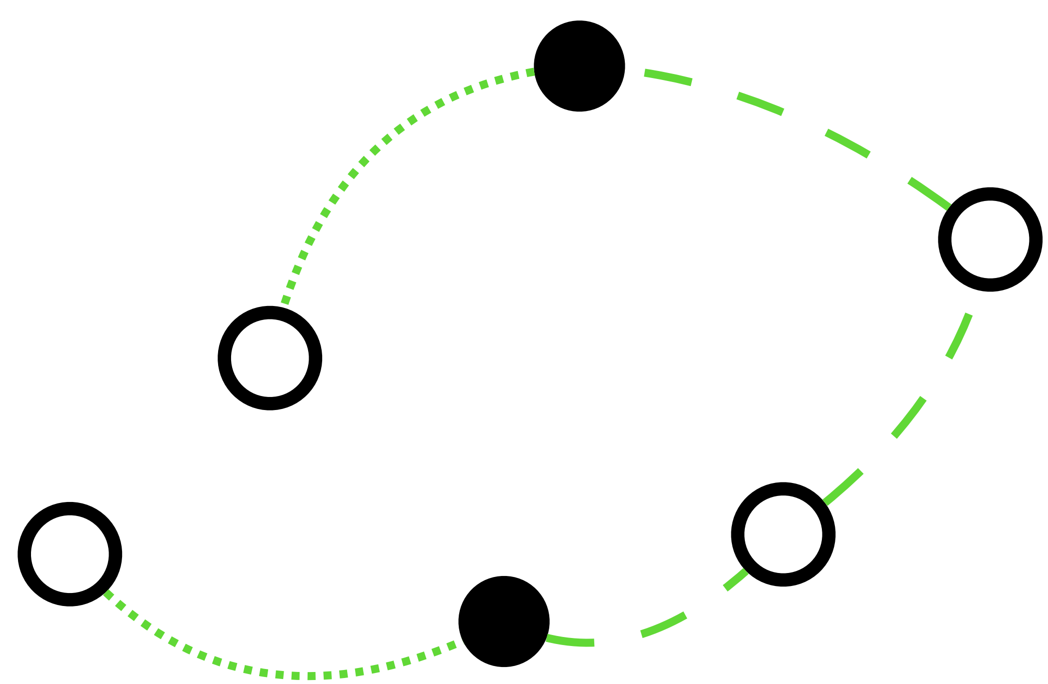
Connected T-join from Ear-Decomposition

- **Lemma.** For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.
- $\gamma(P) = \begin{cases} 1 & P \text{ is short and } in(P) \cap T = \emptyset \\ 0 & \text{ohterwise} \end{cases}$
- Terms ϕ and γ are the knobs that we will "try" to control

Connected T-join from Ear-Decomposition

- **Lemma.** For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

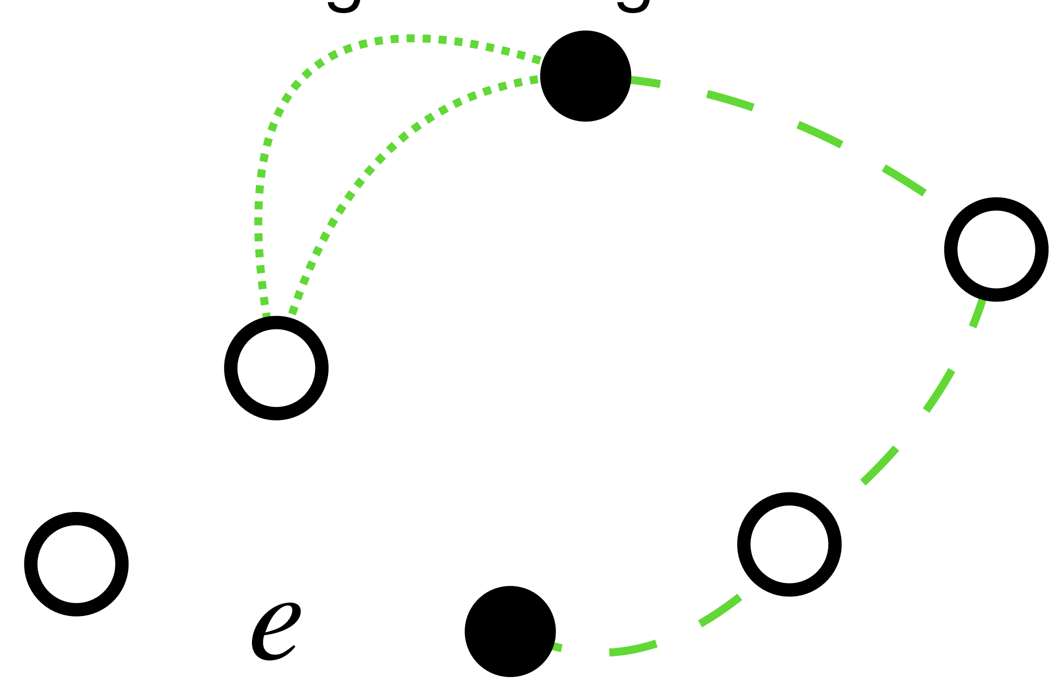
- Proof Sketch
  - Subdivide P into two types of subpaths R, B by vertices of $in(P) \cap T$



Connected T-join from Ear-Decomposition

- **Lemma.** For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

- Proof Sketch
  - Subdivide P into two types of subpaths R, B by vertices of $in(P) \cap T$
  - Suppose $|E_R| \leq |E_B|$.
  - $F := E(P) \uplus E_R - \{e\}$
  - $S := T \Delta T_R$ where T_R is the set of vertices having odd degree in $(V(P), E_R)$



Connected T-join from Ear-Decomposition

- **Lemma.** For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

- **Theorem.** There's a polynomial-time algorithm finds connected T-join with at most $\frac{3}{2}(|V(G)| - 1) + \pi_2 - \frac{1}{2}\phi(G)$ edges where π_2 is the number of 2-ears

- Proof.
  - Find the ear-decomposition of G with $\phi(G)$ even ears [Frank 1993]
  - Apply the Lemma repeatedly → obtain a connected T-join with size at most $\frac{3}{2}(|V(G)| - 1) + \frac{1}{2}\phi(G) - l$
    - l: # of nontrivial and not short ears
  - $l \geq \phi(G) - \pi_2$

Nice Ear-Decomposition

Nice Ear-Decomposition

- Let G be a graph. An ear-decomposition of G is called **nice** if
  1. the number of even ears is $\phi(G)$
  2. all short ears are pendant
  3. internal vertices of different short ears are non-adjacent in G

- An **eardrum** in G is the set M of components of an induced subgraph in which every vertex has degree at most 1.
  - i.e. eardrum = isolated vertices + induced matching

- Given a nice ear-decomposition and $T \subseteq V(G)$ with $|T|$ even, an ear P is **clean** if it is
  - short (thus pendant)
  - $in(P) \cap T = \emptyset$

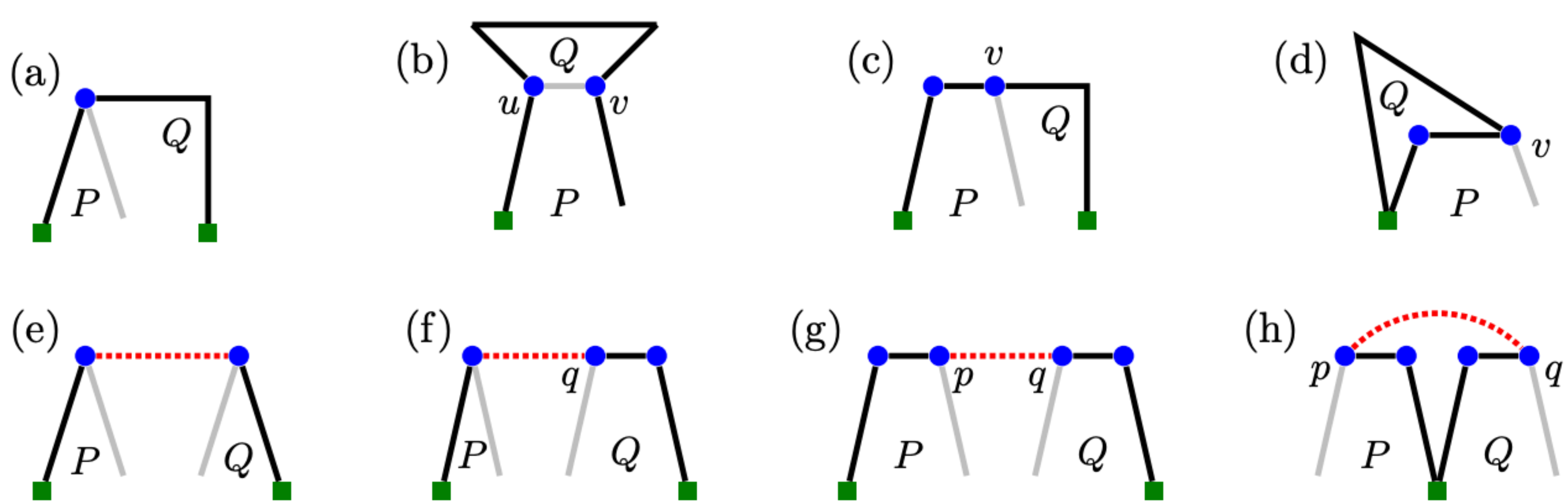
- Eardrum $M := G[\{\text{clean ears}\}]$ is called "associated" with the ear-decomposition and T

Computing Nice Ear-Decomposition

- **Lemma.** For any 2-vertex-connected graph G, there exists a nice ear-decomposition, and such an ear-decomposition can be computed in $O(|V(G)||E(G)|)$ time.

Computing Nice Ear-Decomposition

- **Lemma.** For any 2-vertex-connected graph G, there exists a nice ear-decomposition, and such an ear-decomposition can be computed in $O(|V(G)||E(G)|)$ time.
- Proof Sketch
  - Take any open ear-decomposition with $\phi(G)$ even ears
  - Modify ear-decomposition to satisfy 2 and 3: decrease the number of non-trivial ears and not increase the number of even ears
    - (a): Make all 2-ears pendant
    - (b), (c), (d): Make all 3-ears pendant
    - (e): there's no edges connecting internal vertices of 2-ears
    - (f), (g), (h): deal with problematic 3-ears to satisfy 3



Finding Another Nice Ear-Decomposition

- **Lemma.** Let G: 2-edge-connected graph, $T \subseteq V(G)$ with $|T|$ even. Let a nice ear-decomposition and an associated eardrum M be given. For $f \in M$,
  - P_f : the ear with f as the set of internal vertices
  - Q_f : any path in G having f as the set of internal vertices
- Then, replacing the ears $\{P_f\}$ by the ears $\{Q_f\}$ and changing the set of 1-ears accordingly, we get a nice ear-decomposition again with the same associated eardrum.

Finding Another Nice Ear-Decomposition

- Which $\{Q_f\}$ would be useful to replace $\{P_f\}$ with?
  - $(V(G), \cup_{f \in M} E(Q_f))$ has as few components as possible
    - Intuitively, as pay small price as possible to make whole graph connected

Finding Another Nice Ear-Decomposition

- Which $\{Q_f\}$ would be useful to replace $\{P_f\}$ with?
  - $(V(G), \cup_{f \in M} E(Q_f))$ has as few components as possible
    - Intuitively, as pay small price as possible to make whole graph connected
    - Ideally, if this graph is forest...

- Let G be a graph, M be a eardrum in G. Let $\mathscr{P}_f (f \in M)$ denote the set of $(|f| + 1)$-paths in G where $in(P) = f$. An **earmuff** (for M in G) is a set of paths $\{P_f : f \in F\}$ where $F \subseteq M$, $P_f \in \mathscr{P}_f$, and $(V(G), \cup_{f \in F} E(P_f))$ is a forest.

Finding Another Nice Ear-Decomposition

- Let G be a graph, M be a eardrum in G. Let $\mathscr{P}_f(f \in M)$ denote the set of $(|f|+1)$-paths in G where $in(P) = f$. An **earmuff** (for M in G) is a set of paths $\{P_f : f \in F\}$ where $F \subseteq M$, $P_f \in \mathscr{P}_f$, and $(V(G), \cup_{f \in F} E(P_f))$ is a forest.

- Among earmuffs, the one with maximum $|F|$ is called **maximum earmuff** and its size is denoted by $\mu(G, M)$.

Finding Another Nice Ear-Decomposition

- **Lemma.** Maximum earmuff can be found in polynomial-time.
- Proof.
  - Represent each path $P \in \mathscr{P}_f(f \in M)$ by the set $e_P \in \binom{V(G) - V_M}{2}$
  - Let M_1 be the cycle matroid of the complete graph on $V(G) - V_M$
  - Let M_2 be the partition matroid on $V(G) - V_M$ with constraints $|I \cap \mathscr{P}_f| \leq 1$ for each $f \in M$
  - Finding such an earmuff is equivalent to finding the largest common independent set

Algorithms

Notations and Bounds

- $L_\mu(G, M) := |V(G)| - 1 + |M| - \mu(G, M)$
- **Fact.** $L_\mu(G, M) \leq OPT$

Recall: Considering Pendant Ears

- **Lemma 1.** For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

Recall: Considering Pendant Ears

- **Lemma 1.** For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

- In the same way, one can prove:
- **Lemma 2.** For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{1}{2}|in(P)| + \frac{1}{2}\phi(P)$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

Recall: An Algorithm Using Ear-Decomposition

- **Algorithm 1.**
  - Input
    - G, T, an ear decomposition of $\phi(G)$ edges
  - Consider ears in reverse order: $P_k, P_{k-1}, \cdots, P_1$
  - For each ear P: ($J := \emptyset$ initially)
    - Find a connected $(T \cap in(P))$-join J_P in P
    - $J := J \cup J_P$
    - Delete P from G, modify T appropriately

- **Theorem.** Algorithm 1 finds a connected T-join with at most $\frac{3}{2}(|V(G)| - 1) + \pi_2 - \frac{1}{2}\phi(G)$ edges where π_2 is the number of 2-ears

Using Maximum Earmuff

- **Algorithm 2.**
  - Input
    - G, T, M, a nice ear-decomposition of G with maximum earmuff
  - $V_M := \cup M$; the set of internal vertices of clean ears
  - V_1: set of internal vertices of pendant but not clean ears
  - $V_0 := V(G) - (V_1 \cup V_M)$ (Note: V_0 is 2-edge-connected)

    1. E_1: union of the edge sets of clean ears
    - $|E_1| = \frac{3}{2}|V_M| + \frac{1}{2}\phi_M$ and $(V_M \cup V_0, E_1)$ has $|V_0| - \mu(G, M)$ components

    2. Add a set E_2 of $|V_0| - \mu(G, M) - 1$ edges of $G[V_0]$ to make $(V_M \cup V_0, E_1 \cup E_2)$ connected

    3. Apply Lemma 1 to all the remaining pendant ears and obtain E_3
    - Now, $(V(G), E_1 \cup E_2, \cup E_3)$ is connected

    4. Correctly the parities of the vertices in V_0 by adding minimum T_0-join E_4
    - T_0: set of vertices in V_0 having wrong degree
  - Output $(V(G), E_1 \cup E_2 \cup E_3 \cup E_4)$

Using Maximum Earmuff

Lemma 1. For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

- Recall Notation
  - $L_\mu(G, M) := |V(G)| - 1 + |M| - \mu(G, M) \leq OPT$
- **Theorem.** Algorithm 2 finds a connected T-join with at most $L_\mu(G, M) + \frac{1}{2}(|V(G)| + \phi(G) - 1) - \pi$ edges where π is the number of pendant edges
- Proof Sketch
  - $|E_1| = \frac{3}{2}|V_M| + \frac{1}{2}\phi_M$
  - $|E_2| = |V_0| - \mu(G, M) - 1$

Using Maximum Earmuff

Lemma 1. For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) + \gamma(P) - 1$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

- Recall Notation
  - $L_\mu(G, M) := |V(G)| - 1 + |M| - \mu(G, M) \leq OPT$
- **Theorem.** Algorithm 2 finds a connected T-join with at most $L_\mu(G, M) + \frac{1}{2}(|V(G)| + \phi(G) - 1) - \pi$ edges where π is the number of pendant edges
- Proof Sketch
  - $|E_1| = \frac{3}{2}|V_M| + \frac{1}{2}\phi_M$
  - $|E_2| = |V_0| - \mu(G, M) - 1$
  - For each ear P, we added at most $\frac{3}{2}|in(P)| + \frac{1}{2}\phi(P) - 1$ edges
  - $|E_3| \leq \frac{3}{2}|V_1| + \frac{1}{2}\phi_1 - (\pi - |M|)$
    - ϕ_1: number of even pendant ears that are not clean

Using Maximum Earmuff

Lemma 2. For a pendant ear P, there exists $F \subseteq E(P)$ and $S \subseteq V(G) - in(P)$ such that $|F| \leq \frac{1}{2}|in(P)| + \frac{1}{2}\phi(P)$ and $F \cup J$ is a connected T-join for every connected S-join J of $G - in(P)$.

- Recall Notation
  - $L_\mu(G, M) := |V(G)| - 1 + |M| - \mu(G, M) \leq OPT$
- **Theorem.** Algorithm 2 finds a connected T-join with at most $L_\mu(G, M) + \frac{1}{2}(|V(G)| + \phi(G) - 1) - \pi$ edges where π is the number of pendant edges
- Proof Sketch
  - $|E_1| = \frac{3}{2}|V_M| + \frac{1}{2}\phi_M$
  - $|E_2| = |V_0| - \mu(G, M) - 1$
  - $|E_3| \leq \frac{3}{2}|V_1| + \frac{1}{2}\phi_1 - (\pi - |M|)$
    - ϕ_1: number of even pendant ears that are not clean
  - Using Lemma 2, we corrected the parity of vertices in V_0.
  - $|E_4| \leq \frac{1}{2}(|V_0| - 1 + \phi_0)$
    - $\phi_0 := \phi(G[V_0])$

Using Maximum Earmuff

- Recall Notation
    - $L_\mu(G, M) := |V(G)| - 1 + |M| - \mu(G, M) \leq OPT]$
- **Theorem.** Algorithm 2 finds a connected T-join with at most $L_\mu(G, M) + \frac{1}{2}(|V(G)| + \phi(G) - 1) - \pi$ edges where π is the number of pendant edges
- Proof Sketch
    - $|E_1| = \frac{3}{2}|V_M| + \frac{1}{2}\phi_M$
    - $|E_2| = |V_0| - \mu(G, M) - 1$
    - $|E_3| \leq \frac{3}{2}|V_1| + \frac{1}{2}\phi_1 - (\pi - |M|)$
        - ϕ_1: number of even pendant ears that are not clean
    - $|E_4| \leq \frac{1}{2}(|V_0| - 1 + \phi_0)$
        - $\phi_0 := \phi(G[V_0])$
    - Add all together and compute

Putting All Together

- Algorithm 1 → $cost_1 \leq \frac{3}{2}(|V(G)| - 1) + \pi - \frac{1}{2}\phi(G)$ (Note: $\pi_2 \leq \pi$)
- Algorithm 2 → $cost_2 \leq OPT + \frac{1}{2}(|V(G)| + \phi(G) - 1 - 2\pi)$

Putting All Together

- Algorithm 1 → $cost_1 \leq \frac{3}{2}(|V(G)| - 1) + \pi - \frac{1}{2}\phi(G)$ (Note: $\pi_2 \leq \pi$)
  - This is good when π is small
- Algorithm 2 → $cost_2 \leq OPT + \frac{1}{2}(|V(G)| + \phi(G) - 1 - 2\pi)$
  - This is good when π is large

Putting All Together

- Algorithm 1 → $cost_1 \leq \frac{3}{2}(|V(G)| - 1) + \pi - \frac{1}{2}\phi(G)$ (Note: $\pi_2 \leq \pi$)
    - This is good when π is small
- Algorithm 2 → $cost_2 \leq OPT + \frac{1}{2}(|V(G)| + \phi(G) - 1 - 2\pi)$
    - This is good when π is large
- **Theorem.** There's a $\frac{3}{2}$-approximation algorithm for the connected T-join problem.

Putting All Together

- Algorithm 1 → $cost_1 \leq \frac{3}{2}(|V(G)| - 1) + \pi - \frac{1}{2}\phi(G)$ (Note: $\pi_2 \leq \pi$)
    - This is good when π is small
- Algorithm 2 → $cost_2 \leq OPT + \frac{1}{2}(|V(G)| + \phi(G) - 1 - 2\pi)$
    - This is good when π is large
- **Theorem.** There's a $\frac{3}{2}$-approximation algorithm for the connected T-join problem.
- Proof Sketch
    - Find a nice ear-decomposition with a maximum earmuff
    - If $\pi \leq \frac{1}{2}\phi(G)$, use Algorithm 1
        - $cost \leq \frac{3}{2}(|V(G)| - 1) \leq \frac{3}{2}OPT$ since $\pi - \frac{1}{2}\phi(G) \leq 0$
    - If $\pi > \frac{1}{2}\phi(G)$, use Algorithm 2
        - $cost \leq \frac{3}{2}OPT$ since $\phi(G) - 2\pi \leq 0$ and $OPT \geq |V(G)| - 1$

Applications

- Graphic Path TSP: $\frac{3}{2}$-approximation algorithm
  - Simply connected $\{s, t\}$-join problem
  - NOTE: There's a very simple algorithm for this problem too (which found later)

- Graphic TSP: $\frac{7}{5}$-approximation algorithm
  - Combine the new algorithm with the previous work

- 2-ECSS Problem: $\frac{4}{3}$-approximation algorithm
  - Input: A connected graph G
  - Output: 2-edge-connected spanning multi-subgraph with minimum number of edges

Reference

- A. Sebö and J. Vygen (2012), Shorter Tours by Nicer Ears: 7/5-approximation for graphic TSP, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs, Combinatorica 34
- A. Frank (1993), Conservative weightings and ear-decompositions of graphs, Combinatorica 13