Deterministic Min-Cut in Near-Linear Time

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Background

- Edge connectivity (global min-cut) is a problem making vigorous approaches.
- Since Karger (2000) provided a breakthrough using Graph sparsification via *random sampling*, a near-linear Monte-Carlo algorithm was built up. His framework is inherited by most of the meaningful papers.
- However, approaches for de-randomizing Karger's graph sparsification methods were rare.

Background

- Kawarabayashi and Thorup (2018) seemed to notice that **Low-Conductance Cut** would be effective to find a global mincut, and they managed to establish near-linear time algorithm for **unweighted case**.
- Saranurak, Li, Peng, and many other authors made steady publications about **Expander Decomposition** and their applications on graph algorithms. (CGL+19, GRST20, etc)

Notations

Most of trivial notations are omitted.

- $G = (V, E, w)$: graph to be investigated for min-cut. **weighted** and may be a **multigraph**.
- λ: min-cut of G.
- $\partial S := E(S, V - S)$. Note that $\lambda = \min_{\emptyset \subsetneq S \subsetneq V} w(\partial S)$.
- $\deg_G(v) := w(\partial\{v\})$.
- $\mathrm{vol}_G(S) := \sum_{v \in S} \deg(v)$.

Notations (Cont'd)

- L_G: Graph laplacian of G.
- 1_S: for $S \subseteq V$, indicator vector v such that $v_s = 1 \iff s \in S$. Note that $w(\partial S) = 1_S^T L_G 1_S$.

Recall: Karger's approach

In global, this paper suggests de-randomization of Karger's sampling routine. To specify where would we do a surgery, let's recall Karger's main approach.

Recall: Karger's approach

Thm. (Kar00) This was the only randomized part from Karger.

- For a weighted graph G, Let H be an unweighted graph with m' and min-cut λ', based on $V(G)$.
- Suppose any min-cut of G corresponds to $7/6$-min cut of H, considered as vertex separation.

Then, there is a deterministic $O(m'\lambda' \log n)$ algorithm giving a tree packing with size $O(\lambda')$, and at least one of them 2-respects the min-cut of G.

Recall: Karger's approach

Thm. (Kar99) We can achieve the condition with $m' = O(n \log n)$ and $\lambda' = O(\log n)$ with high probability, by random sampling.

Note that Kar99 could achieve stronger condition; it fully guarantees that **any cuts** in G does not vary by multiplier $1 \pm \varepsilon$.

Thm. (GMW19) Given a tree T, we may find the minimum cut 2-respecting T in $O(m \log n)$ time. Hence, edge connecticity is tractible in $O(m \log^2 n + n \log^3 n)$ time (MC).

Li's approach

Sparsifier Thm. (Li21) For any $\varepsilon \in (0,1]$, we may *determinsitically* compute (H, W) where H is an *unweighted* graph based on V and $W = \varepsilon^4 \lambda / f(n)$ such that

- For any min-cut ∂S^* of G, we have $W \cdot |\partial S^*| \leq (1+\varepsilon)\lambda$. **($\varepsilon$-cap)**
- For any cut ∂S of H, we have $W \cdot |\partial S| \geq (1-\varepsilon)\lambda$. **($\varepsilon$-support)**
- Within $m \cdot \varepsilon^{-4} f(n)$ time.
  With $f(n) = 2^{O(\log n)^{5/6} (\log \log n)^{O(1)}} = n^{o(1)}$.

Easy to see that putting $\varepsilon = 0.01$ satisfies the condition stated on Kar00, requiring $m' = m \cdot n^{o(1)} = m^{1+o(1)}$, and $\lambda' \leq (1+\varepsilon)\varepsilon^{-4} f(n) = n^{o(1)}$.

So, rest of the paper is dedicated to prove **Sparsifier Thm**, via strategies from Expander Decomposition.

Details are hard, so let me take some time for informal sketch here.

Sketch of the proof

Our main objective for de-randomization is to maintain a "pessimistic estimator" Ψ, which is a function of "pre-determined edges" to a probability upper bound.

- Ψ measures the "upper bound" to fail the cut approximation,
- $\Psi(\cdot)$ always lies in $[0, 1)$.
- Being valid, there's always a choice for augmentation that not increasing Ψ.
- $\Psi(\text{all edges})$ must be smaller than 1, to indicate success.

Sketch of the proof

However, there are exponentially many cuts to manage. Any of cuts should not fail to be approximated. So we separate cuts by **small (unbalanced)** cuts and **large (balanced)** cuts.

- For "small cuts", we make a structural representation for them, which enables efficient initialization/update for Ψ. Hence we obtain a sparsifier $\widehat{H}$ at least preserves small cuts.
- For "large cuts", we give up about $(1 \pm \varepsilon)$ ratio approximation, but employ another approximation parameter $\gamma = n^{o(1)}$, as long as "large cut" values do not fall below the min-cut. The large-cut sparsifier $\widetilde{H}$ even do not need to be a subgraph of G.

Sketch of the proof

Now we *overlay* $\widehat{H}$ and $\widetilde{H}$ properly, with some mixing multipliers. This procedure would increase the cut value for $\widehat{H}$, but it is possible to keep deviation from $\widehat{H}$ is small enough.

Let's realize this by investigating the case when G is an *unweighted* ϕ-expander.

Expander Case

Expander Case

It is beneficial to study the expander case, though it does not cover universal case.

The central **Sparsifier Thm** goes like:

Sparsifier Thm. If G is an unweighted ϕ-expander, we may compute an unweighted graph H and $W = \varepsilon^3\lambda/n^{o(1)}$ such that the graph WH ε-caps any min-cut, and ε-supports any cuts. Computation takes almost-linear time.

Expander Case

Note that $w(\partial_G S) = \sum_{u,v \in S} 1_u^T L_G 1_v$. Hence if WH approximates $(L_G)_{uv}$ by additive error $\varepsilon'\lambda$, then $w(\partial_G S)$ is approximable by multiplier $(1 + |S|^2 \varepsilon')$.

Hence this approximation is fine if S or $V - S$ is small. Inspired by this, we define *balancedness* by following:

Definition. S (or ∂S) is unbalanced if $\min(\mathrm{vol}(S), \mathrm{vol}(V - S)) \leq \alpha\lambda/\phi$ for some $\alpha = n^{o(1)}$.

Suppose $\mathrm{vol}(S) \leq \alpha\lambda/\phi$. As $\deg_G(v) \geq \lambda$, easily driven that $|S| \leq \alpha/\phi$.

Unbalanced cuts includes min-cut

Since G is ϕ-expander, any min cut S^* with $\mathbf{vol}(S) \leq \mathbf{vol}(V - S)$,

$$\frac{w(\partial S)}{\mathbf{vol}(S)} = \frac{\lambda}{\mathbf{vol}(S)} \geq \phi \implies \mathbf{vol}(S) \leq \frac{\lambda}{\phi}$$

For $\alpha \geq 1$. Thus, it suffices to approximate $(L_G)_{uv}$ by additive error $(\phi/\alpha)^2\varepsilon\lambda$. For simplicity, just assume $u \neq v$. Now we'd set up the pessimistic estimators.

Pessimistic Estimator for Unbalanced cuts

Suppose the random sampling with probability $p = \Theta(\alpha \log n / \varepsilon^2 \phi \lambda)$ and weight selected edges by $\widehat{W} := 1/p$. For u, v for some unbalanced cut S,

$$\Pr[\left| w_G(u,v) - \frac{w_H(u,v)}{p} \right| > \frac{\phi^2 \varepsilon \lambda}{\alpha^2}]$$

is to be cared. Suppose $w_G(u,v) = \alpha\lambda/\phi$ (which is upper bound, and the worst case) to apply Chernoff's bound. Define $\delta := \varepsilon\phi/\alpha$

$$\Pr[|w_H - p w_G| > \delta \cdot p \cdot w_G] < 2\exp(-\delta^2 p \cdot \alpha\lambda/3\phi) = 2\exp(-\Theta(\log n)).$$

Which could be set smaller than $1/n^2$, even when $u = v$.

Pessimistic Estimator for Unbalanced Cuts

So we put $\Psi_{u,v} := 2\exp(-\delta^2 \cdot p \cdot w_G(u,v)/3)$ as upper bound and $\Psi := \sum_{u,v} \Psi_{u,v}$.

Note that Ψ is initially smaller than 1, and for each step adding an edge, we may maintain Ψ non-increasing and efficiently update Ψ.

At the end, the resulted graph $(1+\varepsilon)$-caps any unbalanced cuts, including min-cuts.

For balanced cut

Now we define a "lossy" approximator $\widetilde{H}$, which is just a $\Theta(1)$-expander on V with each vertex has degree $\Theta(\deg_G(v)/\lambda)$, and edges weighted by $\widetilde{W} = \Theta(\varepsilon\phi\lambda)$.

If we overlay $\widehat{H}$ with $\widetilde{H}$ to result H, then we may show both of them:

- Though affected by $\tilde{H}$, any balanced cut is $O(\varepsilon)$-capped.
- Due to edges in $\hat{H}$, any balanced cut is $O(\varepsilon)$-supported.

Note that unbalanced cuts are already ε-supported.

General Case

would be handwritten.