# THE DIRECTED GRID THEOREM

#### KEN-ICHI KAWARABAYASHI AND STEPHAN KREUTZER

ABSTRACT. The grid theorem, originally proved by Robertson and Seymour in Graph Minors V in 1986, is one of the most central results in the study of graph minors. It has found numerous applications in algorithmic graph structure theory, for instance in bidimensionality theory, and it is the basis for several other structure theorems developed in the graph minors project.

In the mid-90s, Reed and Johnson, Robertson, Seymour and Thomas (see [33, 21]), independently, conjectured an analogous theorem for directed graphs, i.e. the existence of a function  $f:\mathbb{N}\to\mathbb{N}$  such that every digraph of directed tree-width at least f(k) contains a directed grid of order k. In an unpublished manuscript from 2001, Johnson, Robertson, Seymour and Thomas gave a proof of this conjecture for planar digraphs. But for over a decade, this was the most general case proved for the Reed, Johnson, Robertson, Seymour and Thomas conjecture.

In this paper, nearly two decades after the conjecture was made, we are finally able to confirm the Reed, Johnson, Robertson, Seymour and Thomas conjecture in full generality and to prove the directed grid theorem.

As consequence of our results we are able to improve results in Reed et al. in 1996 [35] (see also [32]) on disjoint cycles of length at least l and in [24] on quarter-integral disjoint paths. We expect many more algorithmic results to follow from the grid theorem.

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#### 1. Introduction

Structural graph theory has proved to be a powerful tool for coping with computational intractability. It provides a wealth of concepts and results that can be used to design efficient algorithms for hard computational problems on specific classes of graphs occurring naturally in applications. Of particular importance is the concept of tree width, introduced by Robertson and Seymour as part of their seminal graph minor series [37]<sup>1</sup>. Graphs of small tree width can recursively be decomposed into subgraphs of constant size which can be combined in a tree like way to yield the original graph. This property allows to use algorithmic techniques such as dynamic programming, divide and conquer etc, to solve many hard computational problems efficiently on graphs of small tree width. In this way, a huge number of problems has been shown to become tractable, e.g. solvable in linear or polynomial time, on graph classes of bounded tree width. See e.g. [3, 4, 5, 13] and references therein. But methods from structural graph theory, especially graph minor theory, also provide a powerful and vast toolkit of concepts and ideas to handle graphs of large tree width and to understand their structure.

One of the most fundamental theorems in this context is the grid theorem, proved by Robertson and Seymour in [38]. It states that there is a function  $f:\mathbb{N}\to\mathbb{N}$  such that every graph of tree with at least f(k) contains a  $k\times k$ -grid as a minor. This function, initially being enormous, has subsequently been improved and is now polynomial [6]. The grid theorem is important both for structural graph theory as well as for algorithmic applications. For instance, algorithmically it is the basis of an algorithm design principle called bidimensionality theory, which has been used to obtain many approximation algorithms, PTASs, subexponential algorithms and fixed-parameter algorithms on graph classes excluding a fixed minor. These include feedback vertex set, vertex cover, minimum maximal matching, face cover, a series of vertex-removal parameters, dominating set, edge dominating set, R-dominating set, connected dominating set, connected edge dominating set, connected R-dominating set and unweighted TSP tour. See [8, 9, 10, 11, 16, 15] and references therein.

Furthermore, the grid theorem also plays a key role in Robertson and Seymour's graph minor algorithm and their solution to the disjoint paths problem [39] (also see [23]) in a technique known as the *irrelevant vertex technique*. Here, a problem is solved by showing that it can be solved efficiently on graphs of small tree width and otherwise, i.e. if the tree width is large and therefore the graph contains a large grid, that a vertex deep in the middle of the grid is irrelevant for the problem solution and can therefore be deleted. This yields a natural recursion that eventually leads to the case of small tree width. Such applications also appear in some other problems, see [18, 27, 28].

Furthermore, with respect to graph structural aspects, the excluded grid theorem is the basis of the seminal structure and decomposition theorems in graph minor theory such as in [40].

The structural parameters and techniques discussed above all relate to undirected graphs. However, in various applications in computer science, the most natural model are directed graphs. Given the enormous success width parameters had for problems defined on undirected graphs, it is natural to ask whether they can

<sup>&</sup>lt;sup>1</sup>Strictly speaking, Halin [19] came up with the same notion in 1976, but it went unnoticed until it was rediscovered by Robertson and Seymour [38] in 1984.

also be used to analyse the structure of digraphs and the complexity of NP-hard problems on digraphs. In principle it is possible to apply the structure theory for undirected graphs to directed graphs by ignoring the direction of edges. However, this implies an information loss and may fail to properly distinguish between simple and hard input instances (for example, the disjoint paths problem is NP-complete for directed graphs even with only two source/terminal pairs [17], yet it is solvable in polynomial time for any fixed number of terminals for undirected graphs [23, 39]). Hence, for computational problems whose instances are digraphs, methods based on undirected graph structure theory may be less useful.

As a first step towards a structure theory specifically for directed graphs, Reed [34] and Johnson, Robertson, Seymour and Thomas [21] proposed a concept of directed tree width and showed that the k-disjoint paths problem is solvable in polynomial time for any fixed k on any class of graphs of bounded directed tree width. Reed [33] and Johnson et al. [21] also conjectured a directed analogue of the grid theorem.

Conjecture 1.1. (Reed and Johnson, Robertson, Seymour, Thomas) There is a function  $f: \mathbb{N} \to \mathbb{N}$  such that every digraph of directed tree width at least f(k) contains a cylindrical grid of order k as a butterfly minor

Actually, according to [21], this conjecture was formulated by Robertson, Seymour and Thomas, together with Alon and Reed at a conference in Annecy, France in 1995. Here, a directed grid consists of k concentric directed cycles and 2k paths connecting the cycles in alternating directions. See Figure 3 for an illustration and Definition 3.3 for details. A butterfly minor of a digraph G is a digraph obtained from a subgraph of G by contracting edges which are either the only outgoing edge of their tail or the only incoming edge of their head. See Definition 2.1 for details.

In an unpublished manuscript, Johnson et al. [22] proved the conjecture for planar digraphs. Very recently, we started working on this conjecture, and made some progress. For example, in [26], this result was generalised to all classes of directed graphs excluding a fixed undirected graph as an undirected minor. This includes classes of digraphs of bounded genus. Another related result was established in [24], where a half-integral directed grid theorem was proved. More precisely, it was shown that there is a function  $f: \mathbb{N} \to \mathbb{N}$  such that every digraph G of directed tree width at least f(k) contains a half-integral grid of order k. Here, essentially, a half-integral grid in a digraph G is a cylindrical grid in the digraph obtained from G by duplicating every vertex, i.e. adding for each vertex an isomorphic copy with the same in- and out-neighbours. However, despite the conjecture being open for nearly 20 years now, no progress beyond the results mentioned before has been obtained. The main result of this paper, building on [26, 24] is to finally solve this long-standing open problem.

**Theorem 1.2.** There is a function  $f : \mathbb{N} \to \mathbb{N}$  such that every digraph of directed tree width at least f(k) contains a cylindrical grid of order k as a butterfly minor

We believe that this grid theorem for digraphs is a first but important step towards a more general structure theory for directed graphs based on directed tree width, similar to the grid theorem for undirected graphs being the basis of more general structure theorems. Furthermore, it is likely that the duality of directed tree width and directed grids will make it possible to develop algorithm design techniques such as bidimensionality theory or the irrelevant vertex technique for

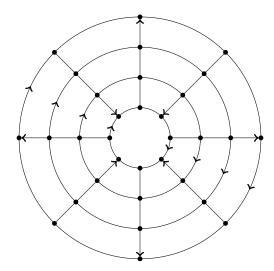


FIGURE 1. Cylindrical grid  $G_4$ .

directed graphs. We are particularly optimistic that this approach will prove very useful for algorithmic versions of Erdős-Pósa type results and in the study of the directed disjoint paths problem. As mentioned above, the half-integral directed grid theorem in [24] has been used to show that a variant of the quarter-integral directed disjoint paths problem can be solved in polynomial time. It is conceivable that our grid theorem here will allow us to show that the half-integral directed disjoint paths problem can be solved in polynomial time. Here, the half-integral directed disjoint paths problem is the problem to decide for a given digraph G and k pairs  $(s_1, t_1), \ldots, (s_k, t_k)$  of vertices whether there are directed paths  $P_1, \ldots, P_k$ such that  $P_i$  links  $s_i$  to  $t_i$  and such that no vertex of G is contained in more than two paths from  $\{P_1,\ldots,P_k\}$ . While we are optimistic that the directed grid theorem will provide the key for proving that the problem is solvable in polynomial time, this requires much more work and significant new ideas and we leave this for future work. Note that in a sense, half-integral disjoint paths are the best we can hope for, as the directed disjoint paths problem is NP-complete even for only k=2 pairs of source/target pairs [17].

However, the directed grid theorem may also prove relevant for the integral directed disjoint paths problem. In a recent breakthrough, Cygan et al. [7] showed that the planar directed disjoint paths problem is fixed-parameter tractable using an irrelevant vertex technique (but based on a different type of directed grid). They show that if a planar digraph contains a grid-like subgraph of sufficient size, then one can delete a vertex in this grid without changing the solution. The bulk of the paper then analyses what happens if such a grid is not present. If one could prove a similar irrelevant vertex rule for the directed grids used in our paper, then the grid theorem would immediately yield the dual notion in terms of directed tree width for free. The directed disjoint paths problem beyond planar graphs therefore is another prime algorithmic application we envisage for directed grids.

Another obvious application of our result is to Erdős-Pósa type results such as Younger's conjecture proved by Reed et al. in 1996 [35]. In fact, in their proof of

Younger's conjecture, Reed et al. construct a kind of a directed grid. This technique was indeed a primary motivation for considering directed tree width and a directed grid minor as a proof of the directed grid conjecture would yield a simple proof for Younger's conjecture. In fact our result immediately gives the following corollaries.

(1) Our grid theorem implies the following stronger result than Reed et al. in 1996 [35] (see also [32]): for every  $\ell$  and every integer  $n \geq 0$ , there exists an integer  $t_n = t_n(\ell)$  such that for every digraph G, either G has n pairwise vertex disjoint directed cycles of length at least  $\ell$  or there exists a set T of at most  $t_n$  vertices such that G - T has no directed cycle of length at least  $\ell$ .

The undirected version was proved by Birmelé, Bondy and Reed [2], and very recently, Havet and Maia [20] proved the case  $\ell = 3$  for directed graphs.

(2) The half-integral directed grid theorem in [24] has been used to show that a variant of the 1/4-integral directed disjoint paths problem can be solved in polynomial time. By our new result, we can improve this to 1/3-integral.

Organisation and high level overview of the proof structure. In Section 3, we state our main result and present relevant definitions. In Sections 5 and 6, then, we present the proof of our main result.

At a very high level, the proof works as follows. It was already shown in [34] that if a digraph G has high directed tree width, it contains a *directed bramble* of very high order (see Section 3). From this bramble one either gets a subdivision of a suitable form of a directed clique, which contains the cylindrical grid as butterfly minor, or one can construct a structure that we call a web (see Definition 5.11).

Our main technical contributions of this paper are in Sections 5 and 6. In Section 5 we show that this web can be ordered and rerouted to obtain a nicer version of a web called a *fence*. Actually, we need a much stronger property for this fence. Let us observe that a fence is essentially a cylindrical grid with one edge of each cycle deleted. In Section 5, we also prove that there is a linkage from the bottom of the fence back to its top (in addition, we require some other properties that are too technical to state here).

Hence, in order to obtain a cylindrical grid, all that is needed is to find such a linkage that is disjoint from (a subfence of) the fence. The biggest problem here is that the linkage from the bottom of the fence back to its top can go anywhere in the fence. Therefore, we cannot get a subfence that is disjoint from this linkage. This means that we have to create a cylindrical grid from this linkage, together with some portion of the fence. This, however, is by far the most difficult and also the most novel part of the proof, which we present in Section 6, and takes more than 20 pages.

Let us mention that our proof is constructive in the sense that we can obtain the following theorem, which may be of independent interest.

**Theorem 1.3.** There is a function  $f : \mathbb{N} \to \mathbb{N}$  such that given any directed graph and any fixed constant k, in polynomial time, we can obtain either

- (1) a cylindrical grid of order k as a butterfly minor, or
- (2) a directed tree decomposition of width at most f(k).



FIGURE 2. Butterfly contracting the dotted edge in the digraph on the left.

Note that the second conclusion follows from the result in [22], which says that for fixed l, there is a polynomial time algorithm to construct a directed tree decomposition of a given directed graph G of width 3l, if G has directed tree width at most l. So for Theorem 1.3, if the directed tree width of a given directed graph is at least 3f(k), we obtain the first conclusion from the constructive proof of Theorem 1.2. Otherwise, we obtain the second conclusion by the result in [22].

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## 2. Preliminaries

In this section we fix our notations and briefly review relevant concepts from graph theory. We refer to, e.g., [12] for details. For any  $n \in \mathbb{N}$  we define  $[n] := \{1, \ldots, n\}$ . For any set U and  $k \in \mathbb{N}$  we define  $[U]^{\leq k} := \{X \subseteq U : |X| \leq k\}$ . We define  $[U]^{=k}$  etc. analogously. We write  $2^U$  for the power set of U.

2.1. Background from graph theory. Let G be a digraph. We refer to its vertex set by V(G) and its edge set by E(G). If  $(u,v) \in E(G)$  is an edge then u is its tail and v its head. Unless stated explicitly otherwise, all paths in this paper are directed. We therefore simply write path for directed path.

The following non-standard notation will be used frequently throughout the paper. If  $Q_1$  and  $Q_2$  are paths and e is an edge whose tail is the last vertex of  $Q_1$  and whose head is the first vertex of  $Q_2$  then  $Q_1eQ_2$  is the path  $Q=Q_1+e+Q_2$  obtained from concatenating e and  $Q_2$  to  $Q_1$ . We will usually use this notation in reverse direction and, given a path Q and an edge  $e \in E(Q)$  write "Let  $Q_1$  and  $Q_2$  be subpaths of Q such that  $Q=Q_1eQ_2$ ." Hereby we define the subpath  $Q_1$  to be the initial subpath of Q up to the tail of e and  $Q_2$  to be the suffix of Q starting at the head of e.

In this paper we will work with a version of directed minors known as butterfly minors (see [21]).

**Definition 2.1** (butterfly minor). Let G be a digraph. An edge  $e = (u, v) \in E(G)$  is butterfly-contractible if e is the only outgoing edge of u or the only incoming edge of v. In this case the graph G' obtained from G by butterfly-contracting e is the graph with vertex set  $(V(G) - \{u, v\}) \cup \{x_{u,v}\}$ , where  $x_{u,v}$  is a fresh vertex. The edges of G' are the same as the edges of G except for the edges incident with u or v. Instead, the new vertex  $x_{u,v}$  has the same neighbours as u and v, eliminating parallel edges. A digraph H is a butterfly-minor of G if it can be obtained from a subgraph of G by butterfly contraction.

See Figure 2 for an illustration of butterfly contractions. We illustrate butterfly-contractions by the following example, which will be used frequently in the paper.

**Example 2.2.** Let G be a digraph. Let  $P = P_1 e P_2$  be a directed path in G consisting of two subpaths  $P_1, P_2$  joined by an edge e with tail in  $P_1$  and head in  $P_2$ . If every edge in  $E(G) \setminus E(P)$  incident to a vertex  $v \in V(P_1)$  has v as its head and every edge in  $E(G) \setminus E(P)$  incident to a vertex  $u \in (P_2)$  has u as its tail then P can be butterfly-contracted into a single vertex, as then every edge in  $E(P_1)$  is the only outgoing edge of its tail and every edge in  $E(P_2) \cup \{e\}$  is the only incoming edge of its head.

**Definition 2.3** (intersection graph). Let  $\mathcal{P}$  and  $\mathcal{Q}$  be sets of pairwise disjoint paths in a digraph G. The intersection graph  $\mathcal{I}(\mathcal{P},\mathcal{Q})$  of  $\mathcal{P}$  and  $\mathcal{Q}$  is the bipartite (undirected) graph with vertex set  $\mathcal{P} \cup \mathcal{Q}$  and an edge between  $P \in \mathcal{P}$  and  $Q \in \mathcal{Q}$  if  $P \cap Q \neq \emptyset$ .

We will also frequently use Ramsey's theorem (see e.g.[12]).

**Theorem 2.4** (Ramsey's Theorem). For all integers  $q, l, r \geq 1$ , there exists a (minimum) integer  $R_l(r,q) \geq 0$  so that if Z is a set with  $|Z| \geq R_l(r,q)$ , Q is a set of |Q| = q colours and  $h: Q \rightarrow [Z]^l$  is a function assigning a colour from Q to every l-element subset of Z then there exist  $T \subseteq Z$  with |T| = r and  $x \in Q$  so that h(X) = x for all  $X \subseteq T$  with |X| = l.

We will also need the following lemma adapted from [31]. In the following lemma, we denote by  $K_s$ , for some  $s \geq 1$ , the biorientation of the complete graph on s vertices.

**Lemma 2.5.** There is a computable function  $f_{clique}: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$  such that for all  $n, k \geq 0$ , if  $G := K_{f_{clique}(n,k)}$  and  $\gamma: E(G) \to [V(G)]^{\leq k}$  such that  $\gamma(e) \cap e = \emptyset$  for all  $e \in E(G)$  then there is  $H \cong K_n \subseteq G$  such that  $\gamma(e) \cap V(H) = \emptyset$  for all  $e \in E(H)$ .

*Proof.* Let  $R(n) := R_2(n, 2)$  denote the *n*-th Ramsey number as defined above. We first define the function  $f_{clique}$  inductively as follows. For all  $n \ge 0$  let  $f_{clique}(n, 0) := n$  and for k > 1 let

$$f_{clique}(n,k) := R(\max\{f_{clique}(n,k-1), f_{clique}(n-1,k)\}) + 1.$$

We prove the lemma by induction on k. For k=0 there is nothing to show. So let k>0. Choose a vertex  $v\in V(G)$  and colour all edges e in  $G_v:=G-v$  by v if  $v\in \gamma(e)$  and by  $\bar{v}$  otherwise. Let  $l:=\max\{f_{clique}(n,k-1),f_{clique}(n-1,k)\}$ . By Ramsey's theorem, as  $|G_v|\geq R(l)$  there is a set  $X\subseteq V(G_v)$  of size l such that all edges between elements of X are coloured v or there is such a set where all edges are coloured  $\bar{v}$ .

In the first case, let G' be the subgraph of G-v induced by X. Then,  $|\gamma(e)| \le k-1$  for all  $e \in E(G')$  and as  $|X| \ge f_{clique}(n,k-1)$ , we can apply the induction hypothesis to find the desired clique  $H \cong K_n$  in G'.

So suppose X induces a subgraph where all edges are labelled  $\bar{v}$ . Let G' := G[X] and  $\gamma'(e) := \gamma(e)$  for all  $e \in E[G']$ . As  $|G'| \ge f_{clique}(n-1,k)$ , by the induction hypothesis, G' contains a subgraph  $H' \cong K_{n-1}$  such that  $\gamma(e) \cap V(H') = \emptyset$  for all  $e \in E(H')$ . Hence,  $H := G[V(H') \cup \{v\}]$  is the required subgraph of G isomorphic to  $K_n$  with  $\gamma(e) \cap V(H) = \emptyset$  for all  $e \in E(H)$ .

**Lemma 2.6.** For all integers  $n, k \geq 0$ , if  $G := K_{n \cdot (k+1)}$  and  $\gamma : V(G) \rightarrow [V(G)]^{\leq k}$  such that  $v \notin \gamma(v)$  for all  $v \in V(G)$  then there is  $H \cong K_n \subseteq G$  such that  $\gamma(v) \cap V(H) = \emptyset$  for all  $v \in V(H)$ .

*Proof.* We construct H greedily. In each step we choose a vertex from G, add it to H and delete  $v \cup \gamma(v)$  from G.

We also need the next result by Erdős and Szekeres [14].

**Theorem 2.7** (Erdős and Szekeres' Theorem). Let s,t be integers and let n=(s-1)(t-1)+1. Let  $a_1,\ldots,a_n$  be distinct integers. Then there exist  $1 \leq i_1 < \cdots < i_s \leq n$  such that  $a_{i_1} < \cdots < a_{i_s}$  or there exist  $1 \leq i_1 < \cdots < i_t \leq n$  such that  $a_{i_1} > \cdots > a_{i_t}$ .

2.2. Linkages, Separations, Half-Integral and Minimal Linkages. A linkage  $\mathcal{P}$  is a set of mutually vertex-disjoint directed paths in a digraph. For two vertex sets  $Z_1$  and  $Z_2$ ,  $\mathcal{P}$  is a  $Z_1$ - $Z_2$  linkage if each member of  $\mathcal{P}$  is a directed path from a vertex in  $Z_1$  to some other vertex in  $Z_2$ . The order of the linkage, denoted by  $|\mathcal{P}|$ , is the number of paths. In slightly sloppy notations, we will sometimes identify a linkage  $\mathcal{P}$  with the subgraph consisting of the paths in  $\mathcal{P}$ . Furthermore, we define  $V(\mathcal{P}) := \bigcup \{V(P) : P \in \mathcal{P}\}$  and  $E(\mathcal{P}) := \bigcup \{E(P) : P \in \mathcal{P}\}$ .

**Definition 2.8** (well-linked sets). Let G be a digraph and  $A \subseteq V(G)$ . A is well linked, if for all  $X, Y \subseteq A$  with |X| = |Y| = r there is an X - Y-linkage of order r.

A separation (A,B) in an undirected graph is a pair  $A,B\subseteq G$  such that  $G=A\cup B$ . The order is  $|V(A\cap B)|$ . A separation in a directed graph G is an ordered pair (X,Y) of subsets of V(G) with  $X\cup Y=V(G)$  so that no edge has the tail in  $X\setminus Y$  and the head in  $Y\setminus X$ . Its order is  $|X\cap Y|$ . We shall frequently need the following version of Menger's theorem.

**Theorem 2.9** (Menger's Theorem). Let G = (V, E) be a digraph with  $A, B \subseteq V$  and let  $k \ge 0$  be an integer. Then exactly one of the following holds:

- $\bullet$  there is a linkage from A to B of order k or
- there is a separation (X,Y) of G of order less than k with  $A\subseteq X$  and  $B\subseteq Y$ .

Let  $A, B \subseteq V(G)$ . A half-integral A-B linkage of order k in a digraph G is a set  $\mathcal{P}$  of k A-B-paths such that no vertex of G is contained in more than two paths in  $\mathcal{P}$ . The next lemma collects simple facts about half-integral linkages which are needed below.

**Lemma 2.10.** Let G be a graph and  $A, B, C \subseteq V(G)$ .

- (1) If G contains a half-integral A-B linkage of order k then G contains an A-B-linkage of order  $\frac{k}{2}$ .
- (2) If |B| = k and G contains an A-B-linkage  $\mathcal{L}$  of order k and a B-C-linkage  $\mathcal{L}'$  of order k then G contains an A-C-linkage of order  $\frac{k}{2}$ .

*Proof.* Part (2) follows immediately from Part (1) as  $\mathcal{L}$  and  $\mathcal{L}'$  can be combined to a half-integral A-C-linkage (this follows as |B| = k and therefore the endpoints of  $\mathcal{L}$  and  $\mathcal{L}'$  in B coincide).

For Part (1), suppose towards a contradiction that G does not contain an A-B-linkage of order  $\frac{k}{2}$ . Hence, by Menger's theorem, there is a separation (X,Y) of G

of order  $<\frac{k}{2}$  such that  $A \subseteq X$  and  $B \subseteq Y$ . But then there cannot be a half-integral A-B-linkage of order k as every vertex in  $X \cap Y$  can only be used twice.  $\square$ 

We now define *minimal linkages*, which play an important role in our proof.

**Definition 2.11** (minimal linkages). Let G be a digraph, let  $C, D \subseteq V(G)$  and let  $H \subseteq G$  be a subgraph. For  $k \geq 1$ , a C-D-linkage  $\mathcal{L}$  of order k is minimal with respect to H, or H-minimal, if for all edges  $e \in \bigcup_{P \in \mathcal{L}} E(P) \setminus E(H)$  there is no C-D-linkage of order k in the graph  $(\mathcal{L} \cup H) - e$ .

The following lemma will be used later on.

**Lemma 2.12.** Let G be a digraph. Let  $\mathcal{P}$  be a linkage and let  $\mathcal{L}$  be a linkage such that  $\mathcal{L}$  is  $\mathcal{P}$ -minimal. Then  $\mathcal{L}$  is  $\mathcal{P}'$ -minimal for every  $\mathcal{P}' \subseteq \mathcal{P}$ .

*Proof.* It suffices to show the lemma for the case where  $\mathcal{P}' = \mathcal{P} \setminus \{P\}$  for some path P. The general case then follows by induction.

Suppose  $\mathcal{L}$  is not  $\mathcal{P}'$ -minimal. Hence, there is an edge  $e \in E(\mathcal{L}) \setminus E(\mathcal{P}')$  such that there is an A-B-linkage  $\mathcal{L}'$  of order k in  $(\mathcal{P}' \cup \mathcal{L}) - e$ . Clearly, this edge has to be in  $E(P) \cap E(\mathcal{L})$  as it would otherwise violate the minimality of  $\mathcal{L}$  with respect to  $\mathcal{P}$ .

Furthermore,  $\mathcal{L}'$  must use every edge in  $E(\mathcal{L}) \setminus E(\mathcal{P})$  as again it would otherwise violate the  $\mathcal{P}$ -minimality of  $\mathcal{L}$ . Let  $Q \subseteq P$  be the maximum directed subpath of  $P \cap \mathcal{L}$  containing e and let  $s, t \in V(G)$  be its first and last vertex, respectively.

If s and t are both end vertices of paths in  $\mathcal{L}$  then this implies that  $Q \in \mathcal{L}$  and no vertex of Q is adjacent in  $\mathcal{L} \cup \mathcal{P}'$  to any vertex of  $\mathcal{P}'$ . Hence in  $\mathcal{L} \cup \mathcal{P} - e$  there is no path from s to t, contradicting the choice of  $\mathcal{L}'$ .

It follows that at least one of s, t is not an endpoint of a path in  $\mathcal{L}$ . We assume that s is this vertex. The case for t is analogous. So suppose s is not an end vertex of any path in  $\mathcal{L}$ . Let  $e_s$  be the edge in  $E(\mathcal{L})$  with head s. As any two paths in  $\mathcal{P}$  are pairwise disjoint, the edge  $e_s$  can not be in  $E(\mathcal{P})$ .

By construction of Q, no vertex in  $V(Q) \setminus \{s,t\}$  is incident to any edge in  $E(\mathcal{L}) \cup E(\mathcal{P})$  other than the edges in Q. Furthermore, as explained above,  $e_s$  must be in  $E(\mathcal{L}')$  as it is not in  $E(\mathcal{P})$  (and hence if  $e_s$  were not in  $E(\mathcal{L}')$ , then  $\mathcal{L}'$  would be  $\mathcal{P}$ -minimal). As s is not an end vertex of a path in  $\mathcal{L}$ , and hence not an end vertex of a path in  $\mathcal{L}'$ , this implies that there must be an outgoing edge of s in  $\mathcal{L}'$ . But this must be on the path Q. Hence,  $\mathcal{L}'$  must include both Q and the edge  $e_s$ , a contradiction.

Note that the converse is not true. I.e. if  $\mathcal{L}$  is  $\mathcal{P}$ -minimal and  $\mathcal{L}' \subset \mathcal{L}$  then  $\mathcal{L}'$  may no longer be  $\mathcal{P}$ -minimal. In the rest of the paper we will mainly use the following property of minimal linkages.

**Lemma 2.13.** Let G be a digraph and  $\mathcal{P} \subseteq G$  be a subgraph. Let  $\mathcal{R}$  be a  $\mathcal{P}$ -minimal linkage between two sets A and B. Let  $R \in \mathcal{R}$  be a path and let  $e \in E(R) \setminus E(\mathcal{P})$ . Let  $R_1, R_2$  be the two components of R - e such that the tail of e lies in  $R_1$ . Then there are at most  $r := |\mathcal{R}|$  paths from  $R_1$  to  $R_2$  in  $\mathcal{P} \cup \mathcal{R}$ .

*Proof.* As  $\mathcal{R}$  is  $\mathcal{P}$ -minimal, there are no r-pairwise vertex disjoint A-B paths in  $(\mathcal{P} \cup \mathcal{R}) - e$ . Let S be a minimal A-B separator in  $(\mathcal{P} \cup \mathcal{R}) - e$ . Hence, |S| = r - 1 and S contains exactly one vertex from every  $R' \in \mathcal{R} \setminus \{R\}$ .

Towards a contradiction, suppose there were r pairwise vertex-disjoint paths from  $R_1$  to  $R_2$  in  $(\mathcal{P} \cup \mathcal{R}) - e$ . At most r-1 of these contain a vertex from S

and hence there is an  $R_1$ - $R_2$  path R' in  $(\mathcal{P} \cup \mathcal{R}) - e - S$ . But then  $R_1 \cup R' \cup R_2$  contains an A-B path in  $(\mathcal{P} \cup \mathcal{R}) - e - S$ , contradicting the fact that S is an A-B separator in  $(\mathcal{P} \cup \mathcal{R}) - e$ . Hence, there are at most r - 1 disjoint paths from  $R_1$  to  $R_2$  in  $(\mathcal{P} \cup \mathcal{R}) - e$  and therefore at most r pairwise vertex-disjoint  $R_1$ - $R_2$  paths in in  $(\mathcal{P} \cup \mathcal{R})$ .

## 3. Directed Tree-Width

The main result of this paper is the grid theorem for directed tree width. We briefly recall the definition of directed tree width from [21].

By an arborescence we mean a directed graph R such that R has a vertex  $r_0$ , called the root of R, with the property that for every vertex  $r \in V(R)$  there is a unique directed path from  $r_0$  to r. Thus every arborescence arises from a tree by selecting a root and directing all edges away from the root. If  $r, r' \in V(R)$  we write r' > r if  $r' \neq r$  and there exists a directed path in R with initial vertex r and terminal vertex r'. If  $e \in E(R)$  we write r' > e if either r' = r or r' > r, where r is the head of e.

Let G be a digraph and let  $Z \subseteq V(G)$ . We say that a set  $S \subseteq (V(G) - Z)$  is Z-normal if there is no directed walk in G - Z with the first and the last vertex in S that uses a vertex of  $G - (Z \cup S)$ . It follows that every Z-normal set is the union of the vertex sets of strongly connected components of G - Z. It is straightforward to check that a set S is Z-normal if, and only if, the vertex sets of the strongly connected components of G - Z can be numbered  $S_1, S_2, \ldots, S_d$  in such a way that

- (1) if  $1 \le i < j \le d$ , then no edge of G has head in  $S_i$  and tail in  $S_j$ , and
- (2) either  $S = \emptyset$ , or  $S = S_i \cup S_{i+1} \cup \cdots \cup S_j$  for some integers i, j with  $1 \le i \le j \le d$ .

**Definition 3.1.** A directed tree decomposition of a digraph G is a triple  $(T, \beta, \gamma)$ , where T is an arborescence,  $\beta: V(T) \to 2^{V(G)}$  and  $\gamma: E(T) \to 2^{V(G)}$  are functions such that

- (1)  $\{\beta(t): t \in V(T)\}\$  is a partition of V(G) into nonempty sets and
- (2) if  $e \in E(T)$ , then  $\bigcup \{\beta(t) : t \in V(T), t > e\}$  is  $\gamma(e)$ -normal.

For any  $t \in V(T)$  we define  $\Gamma(t) := \beta(t) \cup \bigcup \{\gamma(e) : e \sim t\}$ , where  $e \sim t$  if e is incident with t.

The width of  $(T, \beta, \gamma)$  is the least integer w such that  $|\Gamma(t)| \leq w + 1$  for all  $t \in V(T)$ . The directed tree width of G is the least integer w such that G has a directed tree decomposition of width w.

The sets  $\beta(t)$  are called the *bags* and the sets  $\gamma(e)$  are called the *guards* of the directed tree decomposition. It is easy to see that the directed tree width of a subdigraph of G is at most the directed tree width of G.

**Example 3.2.** As an example of directed tree decompositions, consider the graph in Figure 4 a) and its directed tree decomposition in b). Here, the bags  $\beta(t)$  are illustrated by circles and the guards  $\gamma(e)$  by square boxes. The width of the decomposition is 2.

We now recall the concept of cylindrical grids as defined in [33, 21].

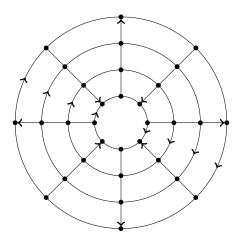


FIGURE 3. Cyclindrical grid  $G_4$ .

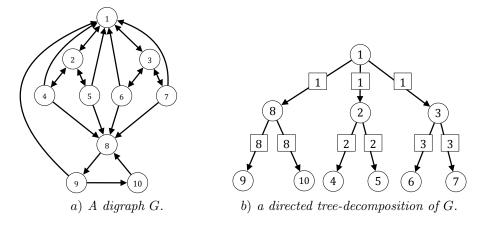


FIGURE 4. a) A sample digraph and b) a corresponding directed tree-decomposition of width 2.

**Definition 3.3** (cylindrical grid). A cylindrical grid of order k, for some  $k \geq 1$ , is a digraph  $G_k$  consisting of k directed cycles  $C_1, \ldots, C_k$ , pairwise vertex disjoint, together with a set of 2k pairwise vertex disjoint paths  $P_1, \ldots, P_{2k}$  such that

- each path  $P_i$  has exactly one vertex in common with each cycle  $C_i$ ,
- the paths  $P_1, \ldots, P_{2k}$  appear on each  $C_i$  in this order and
- for odd i the cycles  $C_1, \ldots, C_k$  occur on all  $P_i$  in this order and for even i they occur in reverse order  $C_k, \ldots, C_1$ .

See Figure 3 for an illustration of  $G_4$ .

**Remark 3.4.** Let us define an elementary cylindrical wall  $W_k$  of order k to be the digraph obtained from the cylindrical grid  $G_k$  by replacing every vertex v of degree 4 in  $G_k$  by two new vertices  $v_s$ ,  $v_t$  connected by an edge  $(v_s, v_t)$  such that  $v_s$  has the same in-neighbours as v and  $v_t$  has the same out-neighbours as v.

A cylindrical wall of order k is a subdivision of  $W_k$ , i.e. a digraph obtained from  $W_k$  by replacing edges by pairwise internally vertex disjoint directed paths in the obvious way. Clearly, every cylindrical wall of order k contains a cylindrical grid of order k as a butterfly minor. Conversely, a cylindrical grid of order k as a butterfly minor contains a cylindrical wall of order  $\frac{1}{2}k$  as subgraph.

What we actually show in this paper is that every digraph of large directed tree width contains a cylindrical wall of high order as subgraph.

Directed tree width has a natural duality, or obstruction, in terms of directed brambles (see [33, 34]) and we will actually prove our main result in terms of brambles rather than directed tree width.

**Definition 3.5.** Let G be a digraph. A bramble in G is a set  $\mathcal{B}$  of strongly connected subgraphs  $B \subseteq G$  such that if  $B, B' \in \mathcal{B}$  then  $B \cap B' \neq \emptyset$  or there are edges e, e' such that e links B to B' and e' links B' to B.

A cover of  $\mathcal{B}$  is a set  $X \subseteq V(G)$  of vertices such that  $V(B) \cap X \neq \emptyset$  for all  $B \in \mathcal{B}$ . Finally, the order of a bramble is the minimum size of a cover of  $\mathcal{B}$ . The bramble number  $\operatorname{bn}(G)$  of G is the maximum order of a bramble in G.

The next lemma is mentioned in [34] and can be proved by converting brambles into havens and back using [21, (3.2)].

**Lemma 3.6.** There are constants c, c' such that for all digraphs G,  $\operatorname{bn}(G) \leq c \cdot dtw(G) \leq c' \cdot \operatorname{bn}(G)$ .

Using this lemma we can state our main theorem equivalently as follows, which is the result we prove in this paper.

**Theorem 3.7.** There is a computable function  $f : \mathbb{N} \to \mathbb{N}$  such that for all digraphs G and all  $k \in \mathbb{N}$ , if G contains a bramble of order at least f(k) then G contains a cylindrical grid of order k as a butterfly minor.

## 4. Getting a web

The main objective of this section is to show that every digraph containing a bramble of high order either contains a cylindrical wall of order k or contains a structure that we call a web.

**Definition 4.1** ((p,q)-web). Let  $p,q,d \geq 0$  be integers. A (p,q)-web  $(\mathcal{P},\mathcal{Q})$  with avoidance d in a digraph G consists of two linkages  $\mathcal{P} = \{P_1, \ldots, P_p\}$  and  $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$  such that

- (1)  $\mathcal{P}$  is an A-B linkage for two distinct vertex sets  $A, B \subseteq V(G)$  and  $\mathcal{Q}$  is a C-D linkage for two distinct vertex sets  $C, D \subseteq V(G)$ ,
- (2) for  $1 \leq i \leq q$ ,  $Q_i$  intersects all but at most  $\frac{1}{d} \cdot p$  paths in  $\mathcal{P}$  and
- (3)  $\mathcal{P}$  is  $\mathcal{Q}$ -minimal.

We say that  $(\mathcal{P}, \mathcal{Q})$  has avoidance d = 0 if  $Q_i$  intersects all paths in  $\mathcal{P}$ , for all  $1 \leq i \leq q$ .

The set  $C \cap V(Q)$  is called the top of the web, denoted  $top((\mathcal{P}, \mathcal{Q}))$ , and  $D \cap V(\mathcal{Q})$  is the bottom  $bot((\mathcal{P}, \mathcal{Q}))$ . The web  $(\mathcal{P}, \mathcal{Q})$  is well linked if  $C \cup D$  is well linked.

The notion of top and bottom refers to the intuition, used in the rest of the paper, that the paths in  $\mathcal{Q}$  are thought of as *vertical* paths and the paths in  $\mathcal{P}$  as *horizontal*. In this section we will prove the following theorem.

**Theorem 4.2.** For every  $k, p, l, c \ge 1$  there is an integer l' such that the following holds. Let G be a digraph of bramble number at least l'. Then G contains a cylindrical grid of order k as a butterfly minor or a  $(p', l \cdot p')$ -web with avoidance c, for some  $p' \ge p$ , such that the top and the bottom of the web are elements of a well linked set  $A \subseteq V(G)$ .

The starting point for proving the theorem are brambles of high order in directed graphs. In the first step we adapt an approach developed in [26], based on [36], to our setting.

**Lemma 4.3.** Let G be a digraph and  $\mathcal{B}$  be a bramble in G. Then there is a path  $P := P(\mathcal{B})$  intersecting every  $B \in \mathcal{B}$ .

*Proof.* We inductively construct the path P as follows. Choose a vertex  $v_1 \in V(G)$  such that  $v_1 \in B_1$  for some  $B_1 \in \mathcal{B}$  and set  $P := (v_1)$ . During the construction we will maintain the property that there is a bramble element  $B \in \mathcal{B}$  such that the last vertex v of P is the only element of P contained in B. Clearly this property is true for the path  $P = (v_1)$  constructed so far.

As long as there still is an element  $B \in \mathcal{B}$  such that  $V(P) \cap V(B) = \emptyset$ , let v be the last vertex of P and  $B' \in \mathcal{B}$  be such that  $P \cap B' = \{v\}$ . By definition of a directed bramble, there is a path in  $G[V(B \cup B')]$  from v to a vertex in B. Choose P' to be such a path so that only its endpoint is contained in B and all other vertices of P' are contained in B'. Hence, P' only shares v with P and we can therefore combine P and P' to a path ending in B to obtain a desired path.  $\square$ 

**Lemma 4.4.** Let G be a digraph,  $\mathcal{B}$  be a bramble of order  $k \cdot (k+2)$  and  $P = P(\mathcal{B})$  be a path intersecting every  $B \in \mathcal{B}$ . Then there is a set  $A \subseteq V(P)$  of order k which is well linked.

*Proof.* We first construct a sequence of subpaths  $P_1, \ldots, P_{2k}$  of P and brambles  $\mathcal{B}_1, \ldots, \mathcal{B}_{2k} \subseteq \mathcal{B}$  as follows. Let  $P_1$  be the minimal initial subpath of P such that  $\mathcal{B}_1 := \{B \in \mathcal{B} : B \cap P_1 \neq \emptyset\}$  is a bramble of order  $\lfloor \frac{k+1}{2} \rfloor$ . Now suppose  $P_1, \ldots, P_i$  and  $\mathcal{B}_1, \ldots, \mathcal{B}_i$  have already been constructed. Let v be the last vertex of  $P_i$  and let s be the successor of v on P. Let  $P_{i+1}$  be the minimal subpath of P starting at s such that

$$\mathcal{B}_{i+1} := \{ B \in \mathcal{B} : B \cap \bigcup_{l \le i} V(P_l) = \emptyset \text{ and } B \cap P_{i+1} \ne \emptyset \}$$

has order  $\lfloor \frac{k+1}{2} \rfloor$ . As long as i < 2k this is always possible as  $\mathcal{B}$  has order k(k+2). Now let in this way  $P_1, \ldots, P_{2k}$  and  $\mathcal{B}_1, \ldots, \mathcal{B}_{2k}$  be constructed. For  $1 \le i \le k$  let  $a_i$  be the first vertex of  $P_{2i}$ . We define  $A := \{a_1, \ldots, a_k\}$ .

We show next that A is well linked. Let  $X, Y \subseteq A$  such that |X| = |Y| = r. W.l.o.g. we assume that  $X \cap Y = \emptyset$ . Obviously  $r \leq \lfloor \frac{k}{2} \rfloor$ . Suppose there is no linkage of order r from X to Y. By Menger's theorem (Theorem 2.9) there is a set  $S \subseteq V(G)$  of order |S| < r such that there is no path from X to Y in  $G \setminus S$ .

Let  $X := \{a_{j_1}, \ldots, a_{j_r}\}$  and  $Y := \{a_{i_1}, \ldots, a_{i_r}\}$ . As |S| < r there is an index l and  $B \in \mathcal{B}_{2j_l}$  such that  $S \cap \left(V(P_{2j_l}) \cup V(B)\right) = \emptyset$  and similarly there is an index s and  $B' \in \mathcal{B}_{2i_s-1}$  such that  $S \cap \left(V(P_{2i_s-1}) \cup \{a_{i_s}\} \cup V(B')\right) = \emptyset$ . But as any two bramble elements of  $\mathcal{B}$  touch this implies that there is a path through  $B \cup B'$  from a vertex of  $v \in V(P_{2j_l})$  to a vertex  $w \in V(P_{2i_s-1})$  and hence, as  $a_{j_l}$  is the starting

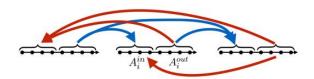


FIGURE 5. A 4-linked path system of order 3.

vertex of  $P_{2j_l}$  and  $a_{i_s}$  is the starting vertex of  $P_{2i_s}$  a path linking  $a_{j_l}$  to  $a_{i_s}$  avoiding S which is a contradiction.

For the rest of the section fix a digraph G, a bramble  $\mathcal B$  of order  $\operatorname{bn}(G) > (2 \cdot k \cdot h) \cdot ((2 \cdot k \cdot h) + 2)$ , for some  $k,h \geq 1$ , a path P and a well linked set  $A \subseteq V(P)$  with  $|A| = 2 \cdot k \cdot h$  as in the previous lemma. We split P into subpaths  $P_1, \ldots, P_h$  as follows.  $P_1$  is the minimum initial subpath containing 2k elements of A. If  $P_1, \ldots, P_i$  are already defined we let  $P_{i+1}$  be the minimum initial subpath of  $P \setminus \bigcup_{j \leq i} P_j$  that contains 2k elements of A. For  $1 \leq i \leq h$  we let  $A_i := V(P_i) \cap A$ . Hence, all  $A_i$  are of size 2k and between any two  $A_i$  and  $A_j$  there are 2k pairwise vertex-disjoint paths linking  $A_i$  to  $A_j$ . For all  $1 \leq i \leq h$  we split  $A_i$  into two sets  $A_i^{in}$  and  $A_i^{out}$  of order k such that  $A_i^{in}$  contains the first k vertices of  $A_i$  appearing on  $P_i$  when traversing  $P_i$  from beginning to end. The next lemma immediately follows from the well linkness of A.

**Lemma 4.5.** For all  $1 \le i, j \le h$  there is a linkage  $L_{i,j}$  of order k linking  $A_i^{out}$  and  $A_i^{in}$ .

We will now define the first of various sub-structures we are guaranteed to find in digraphs of large directed tree-width.

**Definition 4.6** (path system). Let G be a digraph and let  $l, p \geq 1$ . An l-linked path system of order p is a sequence  $S := (\mathcal{P}, \mathcal{L}, \mathcal{A})$ , where

- $\mathcal{A} := \left(A_i^{in}, A_i^{out}\right)_{1 \leq i \leq p}$  such that  $A := \bigcup_{1 \leq i \leq p} A_i^{in} \cup A_i^{out} \subseteq V(G)$  is a well linked set and  $|A_i^{in}| = |A_i^{out}| = l$ , for all  $1 \leq i \leq p$ ,
- $\mathcal{P} := (P_1, \dots, P_p)$  is a sequence of pairwise vertex disjoint paths and for all  $1 \leq i \leq p$ ,  $A_i^{in}, A_i^{out} \subseteq V(P_i)$  are so that all  $v \in A_i^{in}$  occur on  $P_i$  before any  $v' \in A_i^{out}$  and the first vertex of  $P_i$  is in  $A_i^{in}$  and the last is in  $A_i^{out}$  and
- $\mathcal{L} := (L_{i,j})_{1 \leq i \neq j \leq p}$  is a sequence of linkages such that for all  $1 \leq i \neq j \leq p$ ,  $L_{i,j}$  is a linkage of order l from  $A_i^{out}$  to  $A_j^{in}$ .

The system S is clean if for all  $1 \le i \ne j \le p$  and all  $Q \in L_{i,j}$ ,  $Q \cap P_s = \emptyset$  for all  $1 \le s \le p$  with  $s \notin \{i, j\}$ .

See Figure 5 for an illustration of path systems. The next lemma follows from Lemma 4.4 and 4.5.

**Lemma 4.7.** Let G be a digraph and  $l, p \geq 1$ . There is a computable function  $f_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$  such that if G contains a bramble of order  $f_1(l, p)$  then G contains an l-linked path system S of order p.

We now show how to construct a clean path system from an arbitrary path system.

**Lemma 4.8.** Let G be a digraph. There is a computable function  $f_3: \mathbb{N}^4 \to \mathbb{N}$  such that for all integers  $l, p, k, c \geq 1$ , if G contains a bramble of order  $f_3(l, p, k, c)$  then G contains a clean l-linked path system S of order p or a well linked  $(p', k \cdot p')$ -web of avoidance c, for some  $p' \geq p$ .

*Proof.* Let  $l, p, k, c \ge 1$ . For all  $0 \le i \le p$  let  $l_i := c^i \cdot l$ . Let n := p. Furthermore, let  $p_0 := p$  and for  $0 < i \le p$  let  $p_i := f_{clique}((p_{i-1}+1)\cdot(1+2k\cdot l_i)^{n-i},2k\cdot l_i)$ , where  $f_{clique}$  is the function defined in Lemma 2.5. Finally, define  $f_3(l,p,k,c) := f_1(l_n,p_n)$ , where  $f_1$  is the function defined in Lemma 4.7.

Let G be a digraph containing a bramble of order  $f_3(l, p, k, c)$ . By Lemma 4.7, G contains an  $l_n$ -linked path system  $S := (\mathcal{P}, \mathcal{L}, \mathcal{A})$  of order  $p_n$ . By backwards induction on i, we define for all  $0 \le i \le n$ 

- sets  $\mathcal{Y}_i, \mathcal{P}_i \subseteq \mathcal{P}$  with  $|\mathcal{Y}_i| = n i$  and  $|\mathcal{P}_i| = p_i$  and
- for all  $P_s, P_t \in \mathcal{Y}_i \cup \mathcal{P}_i$  with  $s \neq t$  a  $\mathcal{P}_i$ -minimal  $A_s^{out}$ - $A_t^{in}$ -linkage  $L_{s,t}^i$  of order  $l_i$  such that no linkage  $L_{s,t}^i$  hits any path in  $\mathcal{Y}_i$  (except possibly at its endpoints).

Clearly,  $\mathcal{Y}_0$  contains n = p paths and induces a clean  $l = l_0$ -linked path system of order p.

Initially, we set  $\mathcal{P}_n := \mathcal{P}$  and  $\mathcal{Y}_n := \emptyset$ . Furthermore, for all  $1 \leq i \neq j \leq p_n$ , we choose a  $\mathcal{P}$ -minimal  $A_i^{out}$ - $A_j^{in}$ -linkage  $L_{i,j}^n$ . Clearly this satisfies the conditions above.

Now suppose  $\mathcal{Y}_i, \mathcal{P}_i$  and the  $L_{s,t}^i$  satisfying the conditions above have already been defined.

Step 1. We label each pair  $s \neq t$  with  $P_s, P_t \in \mathcal{P}_i$  by

$$\gamma(s,t) := \big\{ P_n \in \mathcal{P}_i : n \neq s, t \text{ and } \begin{array}{l} P_n \text{ hits at least } (1 - \frac{1}{\varsigma}) l_i \text{ paths in } L^i_{s,t} \text{ or } \\ P_n \text{ hits at least } (1 - \frac{1}{\varsigma}) l_i \text{ paths in } L^i_{t,s} \end{array} \big\}.$$

If there is a pair s,t such that  $|\gamma(s,t)| \geq 2 \cdot k \cdot l_i$  then for at least one of  $L^i_{s,t}$  or  $L^i_{t,s}$ , say  $L^i_{s,t}$ , there is a set  $\gamma \subseteq \gamma(s,t)$  of order  $k \cdot l_i$  such that all  $P_n \in \gamma$  hit at least  $(1-\frac{1}{c})l_i$  paths in  $L^i_{s,t}$ . As by Lemma 2.12, the  $\mathcal{P}_i$ -minimal linkage  $L^i_{s,t}$  is also  $\gamma$ -minimal,  $(L^i_{s,t},\gamma)$  form a  $(l_i,k \cdot l_i)$ -web with avoidance c. This yields the second outcome of the lemma. Note that the web is well linked as the vertical paths  $\gamma$  are formed by paths from  $\mathcal{P}$  and, by definition of path systems, these start and end in elements of the well linked set A.

Otherwise, all sets  $\gamma(s,t)$  contain at most  $2k \cdot l_i$  paths. Note that, by construction,  $\gamma$  is symmetric, i.e.  $\gamma(s,t) = \gamma(t,s)$  for all s,t, and therefore  $\gamma$  defines a labelling of the clique  $K_{|\mathcal{P}_i|}$  so that we can apply Lemma 2.5. As  $|\mathcal{P}_i| \geq f_{clique}((p_{i-1}+1)(1+2k \cdot l_i)^{n-i}, 2k \cdot l_i)$ , by Lemma 2.5, there is a set  $\mathcal{P}'_i \subseteq \mathcal{P}_i$  of order  $(p_{i-1}+1)(1+2k \cdot l_i)^{n-i}$  such that for no pair  $s \neq t$  with  $P_s, P_t \in \mathcal{P}'_i$  there is a path  $P \in \mathcal{P}'_i \setminus \{P_s, P_t\}$  hitting  $(1-\frac{1}{c})l_i$  paths in  $L^i_{s,t}$  or  $L^i_{t,s}$ .

So far we have found a set  $\mathcal{P}'_i$  such that for no linkage  $L^i_{s,t}$  for two paths  $P_s, P_t \in \mathcal{P}'_i$  there is another path  $P \neq P_s, P_t$  hitting at least  $(1 - \frac{1}{c})l_i$  paths in  $L^i_{s,t}$ . However, such a path P could still exist for a linkage  $L_{s,t}$  or  $L_{t,s}$  between a  $P_s \in \mathcal{Y}_i$  and a  $P_t \in \mathcal{P}'_i$ . We address this problem in the second step.

Step 2. Let  $(Y_1, \ldots, Y_{n-i}) = \mathcal{Y}_i$  be an enumeration of all paths in  $\mathcal{Y}_i$ . Inductively, we will construct sets  $\mathcal{Q}_j^i \subseteq \mathcal{P}_i'$ , for  $0 \leq j \leq n-i$ , such that  $|\mathcal{Q}_j^i| = (p_{i-1} + p_{i-1})$ 

 $1)(1+2kl_i)^{n-i-j}$  and for no  $1 \leq s \leq j$  (with  $j \geq 1$ ) and  $P_t \in \mathcal{Q}_j^i$  there is a path  $P \in \mathcal{Q}_j^i \setminus \{P_t\}$  such that P hits at least  $(1-\frac{1}{c})l_i$  paths in  $L_{s,t}^i$  or  $L_{t,s}^i$ .

Let  $\mathcal{Q}_0^i := \mathcal{P}_i'$  which clearly satisfies the conditions. So suppose  $\mathcal{Q}_j^i$ , for some j < n - i, has already been defined. Set s := j + 1. For every  $P_t \in \mathcal{Q}_j^i$  define

$$\gamma(t) := \big\{ P \in \mathcal{Q}^i_j \setminus \{P_t\} : \begin{array}{l} P \text{ hits at least } (1 - \frac{1}{c}) l_i \text{ paths in } L^i_{s,t} \text{ or } \\ P \text{ hits at least } (1 - \frac{1}{c}) l_i \text{ paths in } L^i_{t,s} \end{array} \big\}.$$

Again, if there is a  $P_t \in \mathcal{Q}^i_j$  such that  $|\gamma(t)| \geq 2 \cdot k \cdot l_i$  then choose  $\gamma \subseteq \gamma(t)$  of size  $|\gamma| = k \cdot l_i$  such that for one of  $L^i_{s,t}$  or  $L^i_{t,s}$ , say  $L^i_{s,t}$ , every  $P \in \gamma$  hits at least  $(1 - \frac{1}{c})l_i$  paths in  $L^i_{s,t}$ . Then  $(\gamma, L^i_{s,t})$  is a well linked web as requested.

Otherwise, as  $|\mathcal{Q}_{j}^{i}| = (p_{i-1}+1)(1+2k \cdot l_{i})^{n-i-j}$ , by Lemma 2.6, there is a subset  $\mathcal{Q}_{j+1}^{i}$  of order  $(p_{i-1}+1)(1+2k \cdot l_{i})^{n-i-(j+1)}$  such that for no  $P_{t} \in \mathcal{Q}_{j+1}^{i}$ , there is a path  $P \in \mathcal{Q}_{j+1}^{i} \cup \mathcal{Y}_{i}$  hitting at least  $(1-\frac{1}{c})l_{i}$  paths in  $L_{s,t}^{i}$  or  $L_{t,s}^{i}$ .

Now suppose  $\mathcal{Q}_{n-i}^i$  has been defined. We choose a path  $P_n \in \mathcal{Q}_{n-i}^i$  and set  $\mathcal{Y}_{i-1} := \mathcal{Y}_i \cup \{P_n\}$  and define  $\mathcal{P}_{i-1} := \mathcal{Q}_{n-i}^i \setminus \{P_n\}$ . By construction,  $|\mathcal{Y}_{i-1}| = n - (i-1)$  and  $|\mathcal{P}_{i-1}| = p_{i-1}$ . Furthermore, for every pair  $P_s, P_t \in \mathcal{Y}_{i-1} \cup \mathcal{P}_{i-1}$  there is a linkage L from  $A_s^{out}$  to  $A_t^{in}$  of order  $\frac{1}{c} \cdot l_i = l_{i-1}$  such that  $P_n$  does not hit any path in L and, by induction hypothesis, neither does any  $P' \in \mathcal{Y}_i$ . Hence, for any such pair  $P_s, P_t$  we can choose a  $\mathcal{P}_{i-1}$ -minimal  $A_s^{out} - A_t^{in}$ -linkage avoiding every path in  $\mathcal{Y}_{i-1}$ .

Hence,  $\mathcal{Y}_{i-1}$  and  $\mathcal{P}_{i-1}$  satisfy the conditions above and we can continue the induction.

If we do not get a web as the second outcome of the lemma then after p iterations we have constructed  $\mathcal{Y}_0$  and  $\mathcal{P}_0$  and the linkages  $L^0_{s,t}$  for every  $P_s, P_t \in \mathcal{Y}_0$  with  $s \neq t$ . Clearly,  $\mathcal{Y}_0$  induces a clean  $l = l_0$ -linked path system of order  $p_0 = p$  which is the first outcome of the lemma.

The following lemma completes the proof of Theorem 4.2.

**Lemma 4.9.** For every  $k, p, l, c \ge 1$  there is an integer l' such that the following holds. Let S be a clean l'-linked path system of order k. Then either G contains a cylindrical grid of order k as a butterfly minor or a well linked  $(p', l \cdot p')$ -web with avoidance c, for some  $p' \ge p$ .

*Proof.* Let  $K := k \cdot (k-1)$ . We define a function  $f : [K] \to \mathbb{N}$  with  $f(t) := (c \cdot K \cdot l)^{(K-t+1)} p$  and set l' := f(1). For all t we define  $g(t) := \frac{f(t)}{K \cdot l}$ .

Let  $S := (\mathcal{P}, \mathcal{L}, \mathcal{A})$  be a clean l'-linked path system of order k, where  $\mathcal{P} := (P_1, \ldots, P_k), \mathcal{L} := (L_{i,j})_{1 \leq i \neq j \leq k}$  and  $\mathcal{A} := (A_i^{in}, A_i^{out})_{1 \leq i \leq k}$ .

We fix an ordering of the pairs  $\{(i,j): 1 \leq i \neq j \leq k\}$ . Let  $\sigma: [K] \to \{(i,j): 1 \leq i \neq j \leq k\}$  be the bijection between [K] and  $\{(i,j): 1 \leq i \neq j \leq k\}$  induced by this ordering. We will inductively construct linkages  $L^r_{i,j}$ , where  $r \leq K$ , such that

- (1) for all s < r,  $L^r_{\sigma(s)}$  contains a single path P from  $A^{out}_i$  to  $A^{in}_j$ , where  $(i,j) := \sigma(s)$ , and P does not share an internal vertex with any path in some  $L^r_{s,t}$  with  $\{s,t\} \neq \{i,j\}$ ,
- (2)  $|L_{\sigma(r)}^r| = f(r)$
- (3) for all q > r we have  $|L^r_{\sigma(q)}| = g(r) = \frac{f(r)}{K \cdot l}$ , and  $L^r_{\sigma(q)}$  is  $L^r_{\sigma(r)}$ -minimal.

For r=1 we choose a linkage  $L^1_{\sigma(1)}$  satisfying Condition 2 and for q>1 we choose the other linkages as in Condition 3.

Now suppose the linkages have already been defined for r. Let  $(i,j) := \sigma(r)$ . If there is a path  $P \in L^r_{i,j}$  which, for all q > r, is disjoint to at least  $\frac{g(r)}{c}$  paths in  $L^r_{\sigma(q)}$ , define  $L^{r+1}_{i,j} = \{P\}$ . Let  $(s,t) := \sigma(r+1)$  and let  $L^{r+1}_{s,t}$  be an  $A^{out}_s - A^{in}_t$ -linkage of order  $\frac{g(r)}{c} = f(r+1)$  such that no path in  $L^{r+1}_{s,t}$  hits P other than at its endpoints. Such a linkage exists by the choice of P. For each q > r+1 and  $(s',t') = \sigma^{-1}(q)$  choose an  $A^{out}_{s'} - A^{in}_{t'}$ -linkage  $L^{r+1}_q$  of order  $g(r+1) = \frac{g(r)}{(c \cdot K \cdot l)}$  such that every path in it has no inner vertex in P and which is  $L^{r+1}_{s,t}$ -minimal. So in this case, we can construct linkages  $L^{r+1}_{i,j}$  as desired.

Otherwise, for all paths  $P \in L^r_{i,j}$  there are i',j' with  $\sigma^{-1}(i',j') > r$  such that P hits more than  $(1-\frac{1}{c})g(r)$  paths in  $L^r_{i',j'}$ . As  $|L^r_{i,j}| = f(r) = g(r) \cdot K \cdot l$ , by the pigeon hole principle, there is a q > r such that at least  $\frac{f(r)}{K} = g(r) \cdot l$  paths in  $L^r_{i,j}$  hit all but at most  $\frac{g(r)}{c}$  paths in  $L^r_{\sigma(q)}$ . Let  $Q \subseteq L^r_{i,j}$  be the set of such paths. As a result,  $(L^r_{\sigma(q)}, Q)$  forms a  $(g(r), \frac{f(r)}{K})$ -web with avoidance c. As  $\frac{f(r)}{K} = g(r) \cdot l$  and the endpoints of the paths in Q are in the well linked set  $A^{out}_i \cup A^{in}_j \subseteq A$ , this case gives the second possible output of the lemma.

Hence, we assume that the previous case never happens and eventually r=K. We now have paths  $P_1,\ldots,P_k$  and between any two  $P_i,P_j$  with i< j a path  $L'_{i,j}$  from  $A^{out}_i$  to  $A^{in}_j$  and a path  $L'_{j,i}$  from  $A^{out}_j$  to  $A^{in}_i$ . Furthermore, for all  $(i,j)\neq (i',j')$  these paths are pairwise vertex disjoint except possibly at their endpoints in case they begin or end in the same path in  $\{P_1,\ldots,P_k\}$ . By definition of path systems,  $A^{in}_i$  occurs on  $P_i$  before  $A^{out}_i$ . Hence, by Example 2.2,  $\bigcup_{1\leq i\leq k}P_i\cup\bigcup_{1\leq i\neq j\leq k}L'_{i,j}$  contains a cylindrical grid of order k as a butterfly minor.  $\square$ 

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let  $k, p, l, c \geq$  be given as in the statement of the theorem. Let  $l_1$  be the number l' defined for k, p, l, c in Lemma 4.9. Let f be the function as defined in Lemma 4.8. Let  $m := \max\{k, p\}$ . We define  $l' := f(l_1, m, l, c)$ . By Lemma 4.8, if G contains a bramble of order l', then G contains a clean  $l_1$ -linked path system  $(\mathcal{P}, \mathcal{L}, \mathcal{A})$  of order m or a well linked  $(p', l \cdot p')$ -web with avoidance c, for some  $p' \geq m$ . As  $m \geq p$ , the latter yields the second outcome of the Theorem.

If instead we obtain the path system, then Lemma 4.9 implies that G contains a cylindrical grid of order  $m \geq k$ , which is the first outcome of the Theorem, or a well linked  $(p', l \cdot p')$ -web with avoidance c, for some  $p' \geq m \geq p$ . This yields the second outcome of the Theorem.

We close the section with a simple lemma allowing us to reduce every web to a web with avoidance 0.

**Lemma 4.10.** Let p', q', d be integers and let  $p \geq \frac{d}{d-1}p'$  and  $q \geq q' \cdot \binom{p}{\frac{1}{d}p}$ . If a digraph G contains a (p,q)-web  $(\mathcal{P},\mathcal{Q})$  with avoidance d then it contains a (p',q')-web with avoidance 0.

*Proof.* For all  $Q \in \mathcal{Q}$  let  $\mathcal{A}(Q) \subseteq \mathcal{P}$  be the paths  $P \in \mathcal{P}$  with  $P \cap Q = \emptyset$ . By definition of avoidance in webs,  $|\mathcal{A}(Q)| \leq \frac{1}{d}p$ . Hence, by the pigeon hole principle, there is a set  $\mathcal{A} \subseteq \mathcal{P}$  and a set  $\mathcal{Q}' \subseteq \mathcal{Q}$  of at least q' paths such that  $\mathcal{A}(Q) = \mathcal{A}$ 

for all  $Q \in \mathcal{Q}'$ . Let  $\mathcal{P}' := \mathcal{P} \setminus \mathcal{A}$ . Hence,  $P \cap Q \neq \emptyset$  for all  $P \in \mathcal{P}'$  and  $Q \in \mathcal{Q}'$ . As  $p \geq \frac{d}{d-1} \cdot p'$  we have  $p - \frac{1}{d}p \geq p'$  and hence  $|\mathcal{P}'| \geq p'$ . Furthermore,  $\mathcal{P}'$  is  $\mathcal{Q}'$ -minimal. For, Lemma 2.12 implies that  $\mathcal{P}$  is  $\mathcal{Q}'$ -minimal. But  $\mathcal{P}'$  is obtained from  $\mathcal{P}$  by deleting only those paths P which have an empty intersection with every  $Q \in \mathcal{Q}'$ . Clearly, deleting these does not destroy minimality. Therefore,  $(\mathcal{P}', \mathcal{Q}')$  contains a (p', q')-web with avoidance 0.

## 5. From Webs to Fences

The objective of this section is to show that if a digraph contains a large well linked web, then it also contains a big fence whose bottom and top come from a well linked set. We give a precise definition of a fence and then state the main theorem of this section. The results obtained in this section are inspired by results in [35], which we generalise and extend.

**Definition 5.1** (fence). Let p,q be integers. A (p,q)-fence in a digraph G is a sequence  $\mathcal{F} := (P_1, \ldots, P_{2p}, Q_1, \ldots, Q_q)$  with the following properties:

- (1)  $P_1, \ldots, P_{2p}$  are pairwise vertex disjoint paths of G and  $\{Q_1, \ldots, Q_q\}$  is an A-B-linkage for two distinct sets  $A, B \subseteq V(G)$ , called the top and bottom, respectively. We denote the top A by  $top(\mathcal{F})$  and the bottom B by  $bot(\mathcal{F})$ .
- (2) For  $1 \le i \le 2p$  and  $1 \le j \le q$ ,  $P_i \cap Q_j$  is a path (and therefore non-empty).
- (3) For  $1 \leq j \leq q$ , the paths  $P_1 \cap Q_j, \ldots, P_{2p} \cap Q_j$  appear in this order on  $Q_j$ , and the first vertex of  $Q_j$  is in  $V(P_1)$  and the last vertex is in  $V(P_{2p})$ .
- (4) For  $1 \leq i \leq 2p$ , if i is odd then  $P_i \cap Q_1, \ldots, P_i \cap Q_q$  are in order in  $P_i$ , and if i is even then  $P_i \cap Q_q, \ldots, P_i \cap Q_1$  are in order in  $P_i$ .

The fence  $\mathcal{F}$  is well linked if  $A \cup B$  is well linked.

The main theorem of this section is to show that any digraph with a large web where bottom and top come from a well linked set contains a large well linked fence.

**Theorem 5.2.** For every  $p, q \ge 1$  there is are p', q' such that any digraph G containing a well linked (p', q')-web contains a well linked (p, q)-fence.

To prove the previous theorem we first establish a weaker version where instead of a fence we obtain an acyclic grid. We give the definition first.

**Definition 5.3** (acyclic grid). An acyclic (p,q)-grid is a (p,q)-web  $\mathcal{P} = \{P_1, \ldots, P_p\}$ ,  $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$  with avoidance d = 0 such that

- (1) for  $1 \le i \le p$  and  $1 \le j \le q$ ,  $P_i \cap Q_j$  is a path  $R_{ij}$ ,
- (2) for  $1 \le i \le p$ , the paths  $R_{i1}, \ldots, R_{iq}$  are in order in  $P_i$ , and
- (3) for  $1 \leq j \leq q$ , the paths  $R_{1j}, \ldots, R_{pj}$  are in order in  $Q_j$ .

The definition of top and bottom as well as well linkedness is taken over from the underlying web.

**Theorem 5.4.** For all integers  $t, d \geq 1$ , there are integers p, q such that every digraph G containing a well linked (p,q)-web  $(\mathcal{P},\mathcal{Q})$  with avoidance d contains a well linked acyclic (t,t)-grid.

Theorem 5.2 is now easily obtained from Theorem 5.4 using the following lemma, which is (4.7) in [35]. It is easily seen that in the construction in [35] the top and bottom of the fence are subsets of the top and bottom of the acyclic grid it is constructed from.

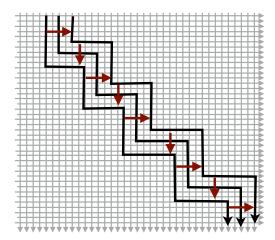


FIGURE 6. Constructing a Fence in a Grid.

**Lemma 5.5.** For every integer  $p \ge 1$ , there is an integer  $p'' \ge 1$  such that every digraph with a (p'', p'')-grid has a (p, p)-fence such that the top and bottom of the fence are subsets of the top and bottom of the grid.

As we will be using this result frequently, we demonstrate the construction in Figure 6. Essentially, we construct the fence inside the grid by starting in the top left corner and then taking alternatingly vertical and horizontal parts of the acyclic grid. This yields the vertical paths of the fence (marked as thick black lines from the top left to the bottom right in the figure). To get the alternating horizontal paths we use the short paths marked in red in the figure. Note that by this construction, the horizontal and vertical paths of the new fence each contain subpaths of the horizontal as well as vertical paths of the original grid.

We now turn towards the proof of Theorem 5.4. We first need some definitions.

**Definition 5.6.** Let  $Q^*$  be a linkage and let  $Q \subseteq Q^*$  be a sub linkage of order q. Let P be a path intersecting every path in Q.

- (1) Let  $r \geq 0$ . An edge  $e \in E(P) E(\mathcal{Q}^*)$  is r-splittable with respect to  $\mathcal{Q}$  (and  $\mathcal{Q}^*$ ) if there is a set  $\mathcal{Q}' \subseteq \mathcal{Q}$  of order r such that  $Q \cap P_1 \neq \emptyset$  and  $Q \cap P_2 \neq \emptyset$  for all  $Q \in \mathcal{Q}'$ , where  $P_1, P_2$  are the two subpaths of P e such that  $P = P_1 e P_2$ .
- (2) A subset  $Q' \subseteq Q$  of order q' is a segmentation of P (with respect to  $Q^*$ ) if there are edges  $e_1, \ldots, e_{q'-1} \in E(P) E(Q^*)$  with  $P = P_1 e_1 \ldots P_{q'-1} e_{q'-1} P_{q'}$ , for suitable subpaths  $P_1, \ldots, P_{q'}$ , such that Q' can be ordered as  $(Q_1, \ldots, Q_{q'})$  and  $V(Q_i) \cap V(P) \subseteq V(P_i)$ .

We next lift the previous definition to pairs  $(\mathcal{P}, \mathcal{Q})$  of linkages.

**Definition 5.7.** Let  $\mathcal{P}$  and  $\mathcal{Q}^*$  be linkages and let  $\mathcal{Q} \subseteq \mathcal{Q}^*$  be a sub linkage of order q. Let r > 0.

(1) An r-split of  $(\mathcal{P}, \mathcal{Q})$  of order q' (with respect to  $\mathcal{Q}^*$ ) is a pair  $(\mathcal{P}', \mathcal{Q}')$  of linkages of order  $r = |\mathcal{P}'|$  and  $q' = |\mathcal{Q}'|$  with  $\mathcal{Q}' \subseteq \mathcal{Q}$  such that  $\mathcal{P}'$  can be ordered  $\mathcal{P}' := (P_1, \ldots, P_r)$  in such a way that there is a path  $P \in \mathcal{P}$  and edges  $e_1, \ldots, e_{r-1} \in E(P) \setminus E(\mathcal{Q}^*)$  with  $P = P_1 e_1 P_2 \ldots e_{r-1} P_r$  and

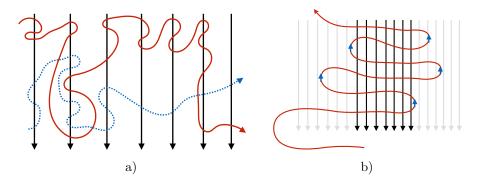


FIGURE 7. a) A 2-segmentation of order 7 and b) a 6-split of order 7.

every  $Q \in \mathcal{Q}'$  can be segmented into subpaths  $Q_1, \ldots, Q_r$  such that  $Q = Q_1e'_1 \ldots e'_{r-1}Q_r$  and  $V(Q) \cap V(P_i) \subseteq V(Q_{r+1-i})$ .

(2) An r-segmentation of  $\mathcal{P}$  of order q' (with respect to  $\mathcal{Q}$  and  $\mathcal{Q}^*$ ) is a pair  $(\mathcal{P}', \mathcal{Q}')$ , where  $\mathcal{P}'$  is a linkage of order r and  $\mathcal{Q}' \subseteq \mathcal{Q}$  is a linkage of order q' such that  $\mathcal{Q}'$  is a segmentation of every path  $P_i$  into segments  $P_1^i e_1 P_2^i \dots e_{q'-1} P_{q'}^i$  and for every  $Q \in \mathcal{Q}'$  and all  $i \neq j$ , if Q intersects  $P_i$  in segment  $P_i^j$ , for some l, then Q intersects  $P_j$  in segment  $P_l^j$ . We say that Q is an r-segmentation of  $\mathcal{P}$  (with respect to  $\mathcal{Q}^*$ ).

An r-split (P, Q) and an r-segmentation (P, Q) are well linked if the set of start and end vertices of paths in Q is a well linked set.

Note that in an r-split  $(\mathcal{P}', \mathcal{Q}')$  of  $(\mathcal{P}, \mathcal{Q})$ , all paths in  $\mathcal{P}'$  are obtained from a single path in  $\mathcal{P}$ . The previous concepts are illustrated in Figure 7. Part a) of the figure illustrates a segmentation, where the vertical paths are the paths in  $\mathcal{Q}$  which segment the two paths in  $\mathcal{P}$ . In Part b), a single path P is split at 5 edges, marked by the arrows on P. The paths in  $\mathcal{Q}$  involved in the split are marked by solid black vertical paths whereas the paths in  $\mathcal{Q}$  which do not split P are displayed in light grey.

**Remark 5.8.** To simplify the presentation we agree on the following notation when working with r-splits as in the previous definition. If  $\mathcal{P}$  only contains a single path P we usually simply write an r-split of  $(P, \mathcal{Q})$  instead of  $(\{P\}, \mathcal{Q})$ . Furthermore, as the order in an r-split is important, we will often write r-splits as  $((P_1, \ldots, P_r), \mathcal{Q}')$ .

We first show the following lemma, which is essentially shown in [35].

**Lemma 5.9.** Let  $r, s \geq 0$ . Let  $Q^*$  be a linkage and let  $Q \subseteq Q^*$  be a sub linkage of order q. Let P be a path intersecting every path in Q. If  $q \geq r \cdot s$  then P contains an r-splittable edge with respect to Q and  $Q^*$  or there is an s-segmentation  $Q' \subseteq Q$  with respect to  $Q^*$ .

*Proof.* Let  $(Q_1, \ldots, Q_{r \cdot s}) \subseteq \mathcal{Q}$  be a set of pairwise disjoint paths. For  $1 \leq j \leq r \cdot s$ , let  $F_j$  be the minimal subpath of P that includes  $V(P \cap Q_j)$ . Note that as  $\mathcal{Q}^*$  is a set of pairwise vertex disjoint paths, if  $Q \neq Q' \in \mathcal{Q}^*$  and  $v \in V(P) \cap V(Q)$  and  $v' \in V(P) \cap V(Q')$  then the minimal subpath of P containing v and v' must contain an edge  $e \notin E(\mathcal{Q}^*)$ .

Suppose first that for some edge  $e \in E(P) \setminus E(\mathcal{Q}^*)$  there are at least r distinct values of j, say  $j_1, \ldots, j_r$ , such that e belongs to  $F_{j_1}, \ldots, F_{j_r}$ . Then e is r-spittable as witnessed by  $\mathcal{Q}' := \{Q_{j_1}, \ldots, Q_{j_r}\}$ .

Thus we may assume that every edge of  $E(P) \setminus E(Q^*)$  occurs in  $F_j$  for fewer than r values of j. Consequently there are s values of j, say  $j_1, \ldots, j_s$ , so that  $F_{j_1}, \ldots, F_{j_s}$  are pairwise vertex-disjoint. Thus  $Q' := (Q_{j_1}, \ldots, Q_{j_s})$  is an s-segmentation of P.

The lemma has the following consequence which we will use frequently below.

**Corollary 5.10.** Let H be a digraph and let  $\mathcal{Q}^*$  be a linkage in H and let  $\mathcal{Q} \subseteq \mathcal{Q}^*$  be a linkage of order q. Let  $P \subseteq H$  be a path intersecting every path in  $\mathcal{Q}$ . Let  $c \geq 0$  be such that for every edge  $e \in E(P) - E(\mathcal{Q}^*)$  there are no c pairwise vertex disjoint paths in H - e from  $P_1$  to  $P_2$ , where  $P = P_1 e P_2$ . For all  $s, r \geq 0$ , if  $q \geq (r + c) \cdot s$  then

- a) there is an s-segmentation  $Q' \subseteq Q$  of P with respect to  $Q^*$  or
- b) a 2-split  $((P_1, P_2), \mathcal{Q}'')$  of  $(P, \mathcal{Q})$  of order r with respect to  $\mathcal{Q}^*$ .

Proof. By Lemma 5.9, there is a s-segmentation of P or an (r+c)-splittable edge  $e \in E(P) - E(\mathcal{Q}^*)$ . In the second case, let  $P = P_1 e P_2$  and let  $\mathcal{Q}' \subseteq \mathcal{Q}$  of order (r+c) witnessing that e is (r+c)-splittable. Thus, every path in  $\mathcal{Q}'$  intersects  $P_1$  and  $P_2$ . As there are at most c disjoint paths from  $P_1$  to  $P_2$  in H-e, at most c of the paths in  $\mathcal{Q}'$  hit  $P_1$  before they hit  $P_2$ . Hence, there is a subset  $\mathcal{Q}'' \subseteq \mathcal{Q}'$  of order r such that for all  $Q \in \mathcal{Q}''$  the last vertex of  $V(P_2 \cap Q)$  occurs before the first vertex of  $V(P_1 \cap Q)$ . Hence,  $((P_1, P_2), \mathcal{Q}'')$  is a 2-split of  $(P, \mathcal{Q})$  of order r.

We will mostly apply the previous lemma in a case where  $H \subseteq G$  is a subgraph induced by two linkages  $\mathcal{P}$  and  $\mathcal{Q}$ , and  $P \in \mathcal{P}$ .

We now present one of our main constructions showing that for every x, y, q every web of high enough order either contains an x-segmentation of order q or a y-split of order q. This construction will again be used in Section 6 below. We first refine the definition of webs from Section 4. The difference between the webs (with avoidance 0) used in Section 4 and the webs with linkedness c defined here is that we no longer require that in a web  $(\mathcal{P}, \mathcal{Q}), \mathcal{P}$  is  $\mathcal{Q}$ -minimal. Instead we require that in every path P, if we split P at an edge e, i.e.  $P = P_1 e P_2$ , then there are at most c paths from  $P_1$  to  $P_2$  in  $\mathcal{P} \cup \mathcal{Q}$ . This is necessary as in the various constructions below, minimality will not be preserved but this forward path property is preserved. We give a formal definition now.

**Definition 5.11** ((p,q)-web). Let  $p,q,c \geq 0$  be integers and let  $\mathcal{Q}^*$  be a linkage. A (p,q)-web with linkedness c with respect to  $\mathcal{Q}^*$  in a digraph G consists of two linkages  $\mathcal{P} = \{P_1, \ldots, P_p\}$  and  $\mathcal{Q} = \{Q_1, \ldots, Q_q\} \subseteq \mathcal{Q}^*$  such that

- (1) Q is a C-D linkage for two distinct vertex sets  $C, D \subseteq V(G)$  and  $\mathcal{P}$  is an A-B linkage for two distinct vertex sets  $A, B \subseteq V(G)$ ,
- (2) for 1 < i < q,  $Q_i$  intersects every path  $\mathcal{P} \in \mathcal{P}$ .
- (3) for every  $P \in \mathcal{P}$  and every edge  $e \in E(P) \setminus E(\mathcal{Q}^*)$  there are at most c disjoint paths from  $P_1$  to  $P_2$  in  $\mathcal{P} \cup \mathcal{Q}$  where  $P_1, P_2$  are the subpaths of P such that  $P = P_1 e P_2$ .

The set  $C \cap V(\mathcal{Q})$  is called the top of the web, denoted  $top((\mathcal{P}, \mathcal{Q}))$ , and  $D \cap V(\mathcal{Q})$  is the bottom  $bot((\mathcal{P}, \mathcal{Q}))$ . The web  $(\mathcal{P}, \mathcal{Q})$  is well linked if  $C \cup D$  is well linked.

**Remark 5.12.** Note that every (p,q)-web with avoidance 0 is a (p,q)-web with linkedness p.

**Lemma 5.13.** For all  $p, q, r, s, c \ge 0$  and all  $x, y \ge 0$  with  $p \ge x$  there is a number  $q' := (p \cdot q \cdot (q+c))^{2^{(x-1)y+1}}$  such that if G contains a (p, q')-web  $W := (\mathcal{P}, \mathcal{Q})$  with linkedness c, then G contains a y-split  $(\mathcal{P}', \mathcal{Q}')$  of  $(\mathcal{P}, \mathcal{Q})$  of order q or an x-segmentation  $(\mathcal{P}', \mathcal{Q}')$  of  $(\mathcal{P}, \mathcal{Q})$  of order q. Furthermore, if W is well linked then so is  $(\mathcal{P}', \mathcal{Q}')$ .

*Proof.* We set  $q' := (p \cdot q \cdot (q+c))^{2^{(x-1)y+1}}$ . We will provide an algorithmic proof of the lemma. Fix  $\mathcal{Q}^* := \mathcal{Q}$  for the rest of the proof. All applications of Corollary 5.10 will be with respect to this original linkage  $\mathcal{Q}^*$ .

For all  $0 \le i \le (x-1)y+1$  we define  $q_i := (p \cdot q \cdot (q+c))^{2^{(x-1)y+1-i}}$ . Let  $(\mathcal{P},\mathcal{Q})$  be a  $(p,q_0)$ -web. For  $0 \le i \le (x-1)y+1$  we will construct a sequence  $\mathcal{M}_i := (\mathcal{P}^i,\mathcal{Q}^i,\mathcal{S}^i_{seg},\mathcal{S}^i_{split})$ , where  $\mathcal{P}^i,\mathcal{Q}^i,\mathcal{S}^i_{seg},\mathcal{S}^i_{split}$  are linkages of order  $p,q_i,x_i$  and  $y_i$  respectively and such that  $\mathcal{Q}^i \subseteq \mathcal{Q}^*$  is an  $x_i$ -segmentation of  $\mathcal{S}^i_{seg}$  and  $(\mathcal{S}_{split},\mathcal{Q}^i)$  is a  $y_i$ -split of  $(\mathcal{P},\mathcal{Q})$ . Recall that in particular this means that the paths in  $\mathcal{S}_{split}$  are the subpaths of a single path in  $\mathcal{P}$  split at edges  $e \in E(P) \setminus E(\mathcal{Q}^*)$ .

We first set  $\mathcal{M}_0 := (\mathcal{P}^0 := \mathcal{P}, \mathcal{Q}^0 := \mathcal{Q}, \mathcal{S}^0_{seg} := \emptyset, \mathcal{S}^0_{split} := \emptyset)$  and define  $x_0 = y_0 := 0$ . Clearly, this satisfies the conditions on  $\mathcal{M}_0$  defined above.

Now suppose that  $\mathcal{M}_i$  has already been defined for some  $i, x_i \geq 0$  and  $y_i > 0$ . If  $\mathcal{S}^i_{split} = \emptyset$ , we first choose a set  $\mathcal{Q}^+ \subset \mathcal{Q}_i$  of order  $q_i/p \geq q_{i+1}(q_{i+1}+c)$  and a path  $P \in \mathcal{P}_i$  such that every path in  $\mathcal{Q}^+$  intersects P before any other path in  $\mathcal{P}_i$ . Note that this is possible by the pigeon hole principle. We set  $\mathcal{S}_{split} = \{P\}$ . Otherwise, if  $\mathcal{S}^i_{split} \neq \emptyset$ , we set  $\mathcal{S}_{split} := \mathcal{S}^i_{split}$  and  $\mathcal{Q}^+ := \mathcal{Q}_i$ .

Now, let  $P \in \mathcal{S}_{split}$ . We apply Corollary 5.10 to  $P, \mathcal{Q}^+$  with respect to  $\mathcal{Q}^*$  setting  $x = q_{i+1}$  and  $y = (q_{i+1} + c)$ . If we get a  $q_{i+1}$ -segmentation  $\mathcal{Q}_1 \subseteq \mathcal{Q}^+$  of P with respect to  $\mathcal{Q}^*$  we set

$$\mathcal{P}^{i+1} := \mathcal{P}^i \setminus \{P\}, \qquad \mathcal{Q}^{i+1} := \mathcal{Q}_1, \qquad \mathcal{S}^{i+1}_{seg} := \mathcal{S}_{seg} \cup \{P\} \quad \text{ and } \quad \mathcal{S}^{i+1}_{split} := \emptyset.$$

Otherwise, we get a 2-split  $((P_1, P_2), \mathcal{Q}_2)$  of order  $q_{i+1}$  where  $\mathcal{Q}_2 \subseteq \mathcal{Q}^+$ . Then we set

$$\begin{split} \mathcal{P}^{i+1} &:= & \mathcal{P}^i \setminus \{P\} \cup \{P_1, P_2\}, \\ \mathcal{Q}^{i+1} &:= & \mathcal{Q}_2, \\ \mathcal{S}^{i+1}_{seg} &:= & \mathcal{S}^i_{seg} \quad \text{and} \\ \mathcal{S}^{i+1}_{split} &:= & \mathcal{S}^i_{split} \setminus \{P\} \cup \{P_1, P_2\}. \end{split}$$

It is easily verified that the conditions for  $\mathcal{M}^{i+1} := (\mathcal{P}^{i+1}, \mathcal{Q}^{i+1}, \mathcal{S}^{i+1}_{seg}, \mathcal{S}^{i+1}_{split})$  are maintained. In particular, the linkedness c of  $(\mathcal{P}^{i+1}, \mathcal{Q}^{i+1})$  is preserved as deleting or splitting paths cannot increase forward connectivity (in contrast to the minimality property).

Note that in the construction above, in each step we either increase  $x_i$  (and decrease  $y_i$ ) or we increase  $y_i$ . Hence, after at most  $i \leq (x-1) \cdot y + 1$  steps, either we have constructed a set  $\mathcal{S}_{seg}^i$  of order x or a set  $\mathcal{S}_{split}^i$  of order y. This concludes the proof.

Consider the case that the outcome of the previous lemma is a y-split. This case is illustrated in Figure 7 b). We call the structure that we obtain in this case a pseudo-fence.

**Definition 5.14** (pseudo-fence). A(p,q)-pseudo-fence is a pair  $(\mathcal{P} := (P_1, \dots, P_{2p}), \mathcal{Q})$  of pairwise disjoint paths, where  $|\mathcal{Q}| = q$ , such that each  $Q \in \mathcal{Q}$  can be divided into segments  $Q_1, \dots, Q_{2p}$  occurring in this order on Q such that for all i, each  $P_i$  hits all  $Q \in \mathcal{Q}$  in their segment  $Q_i$  and P does not hit any Q in another segment. Furthermore, for all  $1 \leq i \leq p$ , there is an edge  $e_i$  connecting the endpoint of  $P_{2i}$  to the start point of  $P_{2i-1}$ .

The top of (P, Q) is the set of start vertices and the bottom the set of end vertices of Q.

The next lemma follows immediately from the definitions.

**Proposition 5.15.** Let  $(\mathcal{P}', \mathcal{Q}')$  be a y-split of order q of some pair  $(\mathcal{P}, \mathcal{Q})$  of linkages. Then  $(\mathcal{P}', \mathcal{Q}')$  form a (y, q)-pseudo-fence.

In the following three lemmas (which generalize the results in [35]), we show how in each of the two cases of the previous we get an acyclic grid. We first need some preparation.

**Lemma 5.16.** There is a function  $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$  with the following properties. Let  $\mathcal{P}$  be a linkage and let R be a path intersecting every path in  $\mathcal{P}$ . Furthermore, let  $\sigma: \mathcal{P} \to 2^{\mathcal{P}}$  be a function such that

- (a)  $|\sigma(P)| \leq K$ , for some fixed integer K and for  $P \in \mathcal{P}$ ,
- (b) for every  $P \in \mathcal{P}$ ,  $P \in \sigma(P)$  and for all  $P' \in \sigma(P)$  we have  $\sigma(P') = \sigma(P)$ , and
- (c) for all  $P \in \mathcal{P}$ ,  $\sigma(P)$  can be ordered as  $(P_1, \ldots, P_s)$ , for some  $s \leq K$ , such that for all  $1 \leq i \leq s$ , after the first vertex on R it has in common with  $P_i$ , R does not intersect any  $P_j$  with j < i.

For all  $k \geq 0$ , if  $|\mathcal{P}| \geq f(k, K)$ , then there is a sequence  $(P_1, \ldots, P_k)$  of distinct paths  $P_1, \ldots, P_k \in \mathcal{P}$  such that for all  $1 \leq i < k$ , R contains a subpath  $R_i$  from a vertex in  $P_i$  to a vertex in  $P_{i+1}$  which is internally vertex disjoint from  $\bigcup_{1 \leq i \leq k} (P_i \cup \sigma(P_i))$ . Furthermore, the sequence can be chosen either in a way that

- (1) the first vertex of  $\bigcup_{1 \le i \le k} P_i$  on R is contained in  $V(P_1)$  or
- (2) the last vertex of  $\bigcup_{1 \le i \le k} P_i$  on R is contained in  $V(P_k)$ .

*Proof.* We define f(1,K) = 1. For i > 1 let  $g_i := K + (f(i-1,K)+i) \cdot (K^2 + 1)^{f(i-1,K)+i}$  and let  $f(i,K) := 1 + K^2 \cdot g_i$ .

Suppose  $|\mathcal{P}| \geq f(k, K)$ . We first construct a sequence  $(P_1, \ldots, P_k)$  satisfying Condition 1. The construction for Condition 2 is analogous.

Let  $P \in \mathcal{P}$  be the path such that the first vertex, when traversing from beginning to end, R has in common with any path in  $\mathcal{P}$  is on P. Let  $\sigma(P) = (P^1, \ldots, P^{p_K})$  be ordered by the order given in Condition (c). Let  $x_1, \ldots, x_s$  be the vertices of  $V(\sigma(P)) \cap V(R)$  ordered in the order in which they occur on R. We define  $x_{s+1}$  to be the last vertex of R. Note that, by Condition (c), there are indices  $i_1, \ldots, i_{p_K}$  such that for all  $1 \leq j \leq p_K$ , the vertices R has in common with  $P_j$  are  $\{x_l : i_{j-1} < l \leq i_j\}$  (where we define  $i_0 := 0$ ). For any  $1 \leq i \leq s$  let  $R_i$  be the subpath of R from  $x_i$  to  $x_{i+1}$  (including both) and let  $\mathcal{B}'_i := \{P' \in \mathcal{P} : V(P') \cap V(R_i) \neq \emptyset\}$ .

For all  $1 \leq i \leq p_K$  let  $\mathcal{P}_j^* := \{P \in \mathcal{P} : P \in \mathcal{B}_l' \text{ for some } i_{j-1} < l \leq i_j \text{ and } \sigma(P) \cap \mathcal{B}_l' = \emptyset \text{ for all } l \leq i_{j-1}\} \text{ and let } n_i := |\mathcal{P}_i^*|. \text{ As } |\mathcal{P}| \geq f(k, K) \text{ and } |\sigma(P)| \leq K \text{ for all } P, \text{ there is a minimal index } t \leq p_K \text{ such that } n_i \geq f(k, K) - K^2 \cdot g_k \geq g_k.$  Let  $\mathcal{P}^* := \mathcal{P}_t \text{ and } \mathcal{B}_j := \mathcal{B}_j' \cap \mathcal{P}^* \text{ for all } i_{t-1} \leq j \leq i_t. \text{ i}$ 

We now consider two cases depending on  $\mathcal{B}'_i$ .

Suppose first that  $|\mathcal{B}'_i| \geq f(k-1,K)$  for some  $i_{t-1} < i \leq i_t$ . Then we apply the induction hypothesis to  $\mathcal{B}_i$  and  $R_i$  to get a sequence  $(H_2, \ldots, H_k)$  of distinct paths  $H_j \in \mathcal{B}_i$ , for  $2 \leq j \leq k$ , such that for all  $2 \leq j < k$ ,  $R_i$  contains a subpath  $R_i^*$  from a vertex in  $H_j$  to a vertex in  $H_{j+1}$  which is internally vertex disjoint from  $\bigcup_{2 \le i \le k} (H_i \cup \sigma(H_i))$  and furthermore among  $H_2, \ldots, H_k$ , the first path  $R_i$  intersects is  $H_2$ . Clearly, none of the  $R_i^*$  intersects any path in  $\sigma(P)$ . Let  $R_1, \ldots, R_{t-1}$  be subpaths of R such that  $R_j$  links  $P_j$  to  $P_{j+1}$ . Such paths exists and by construction of  $\mathcal{P}^*$  they are internally vertex disjoint from  $H_2, \ldots, H_k$ . Hence,  $(P_1, \ldots, P_t, H_2, \ldots, H_k)$  is a sequence as required by the lemma, possibly longer than needed. Let  $H'_1, \ldots, H'_k$  be the first k paths in this sequence. Note that the subpaths  $R_i$  of R constructed above linking  $H'_i$  to  $H'_{i+1}$  are internally vertex disjoint from  $\sigma(H'_1), \ldots, \sigma(H'_k)$ . Hence we can simply attach the paths in  $\bigcup (\sigma(H'_1), \ldots, \sigma(H'_k)) \setminus \{H'_1, \ldots, H'_k\}$  to the paths  $H'_1, \ldots, H'_k$  so that they form paths  $S_1, \ldots, S_k$  satisfying the requirements of the lemma and Condition 1. More precisely, we construct  $S_1, \ldots, S_k$  as follows. For  $1 \leq l \leq k$  let  $S_l^0 := S_l$  and set  $\mathcal{A}_0 := \bigcup \left(\sigma(H_1'), \dots, \sigma(H_k')\right) \setminus \{H_1', \dots, H_k'\}.$  Then, as long as  $\mathcal{A}_l \neq \emptyset$ , we choose a path  $H \in \mathcal{A}_l$  such that there is a path  $S_i^l$ , for some  $1 \leq j \leq k$ , such that H and  $S_i^l$ can be connected by an edge  $e \in E(P_l)$  for some  $P_l \in \mathcal{P}'$  of which both, H and  $S_i^l$ are a subpath. Note that this is always possible by construction of the sets. Then we set  $S_j^{l+1} := S_j^l + e + H$ ,  $S_{j'}^{l+1} := S_{j'}^l$ , for all  $j' \neq j$  and  $A_{l+1} := A_l \setminus \{H\}$ .

Once  $A_l = \emptyset$  for some l we set  $S_i := S_i^l$ , for all i, to obtain a tuple satisfying Condition 1 and the requirement of the lemma.

Case 2. So suppose that  $|\mathcal{B}_i| < f(k-1,K)$  for all  $i_{t-1} < i \le i_t$ . Let  $h_{K+k} := 1$  and  $h_i := (K^2 + 1) \cdot h_{i+1}$ , for all  $1 \le i < K + k$ . We will define a sequence  $(\mathcal{C}_i, \mathcal{X}_i, \mathcal{P}_i)$ , where  $\mathcal{C}_i \subseteq \{\mathcal{B}_i : i_{t-1} < i \le i_t\}$ ,  $\mathcal{X}_i \subseteq \mathcal{P}^*$  and  $\mathcal{P}_i := (P_1, \dots, P_{l_i})$  with  $P_j \in \mathcal{P}^*$  for all  $1 \le j \le l_i$  such that

- $|\bigcup C_i| \geq h_i$ ,
- for every path  $P' \in \mathcal{X}_i$  at least one path of  $\sigma(P')$  occurs in every  $\mathcal{B} \in \mathcal{C}_i$  and
- for every  $1 \leq j < l_i$ , for no path  $P_j \in \mathcal{P}_i$  any path in  $\sigma(P_j)$  occurs in any  $\mathcal{B} \in \mathcal{C}_i$  and there is a subpath  $R'_i$  of R with first vertex in  $P_j$ , last vertex in  $P_{j+1}$  and internally vertex disjoint from  $\bigcup_{\mathcal{B} \in \mathcal{C}_i} \{P' : P' \in \mathcal{B}\} \cup \{P_1, \dots, P_{l_i}\}$ . Furthermore, if  $\mathcal{P}_i$  is not empty then the first vertex R has in common with  $\bigcup \mathcal{P}_i$  is on  $P_1$ .

Initially we set  $C_0 := \{ \mathcal{B}_i : i_{t-1} \leq i \leq i_t \}, \ \mathcal{X}_0 = \mathcal{P}_0 = \emptyset$ , which clearly satisfies the criteria.

Now suppose  $C_i$ ,  $\mathcal{R}_i$ ,  $\mathcal{X}_i$  have already been defined. Let  $\mathcal{B} = \mathcal{B}_r \in C_i$  be such that r is minimal, i.e.  $\mathcal{B}$  is the first set in  $C_i$  such that R intersects a path in  $\mathcal{B}_i$ . Let P' be the first path in  $\mathcal{B}$  that R intersects.

Let  $P'_1, \ldots P'_m$  be an ordering of  $\sigma(P')$  in the order in which R intersects the paths in  $\sigma(P')$ . Let  $\mathcal{Q} \subseteq \bigcup \mathcal{C}_i$  be the set of paths Q such that  $\sigma(P')$  and Q do not occur together in any  $\mathcal{B} \in \mathcal{C}_i$ .

- Suppose first that  $|\mathcal{Q}| \geq K^2 \cdot h_{i+1}$ . For all  $1 \leq a \leq m$  let  $\mathcal{Q}_a := \{Q \in \mathcal{Q} : R \text{ intersects } Q \text{ before the last vertex it has in common with } P_a \text{ and no path in } \sigma(Q) \text{ intersects } R \text{ before the last vertex } R \text{ has in common with } P_{a-1} \}$ . For a = 1 we include into  $\mathcal{Q}_1$  all paths  $Q \in \mathcal{Q}$  R intersects before the last vertex it has in common with  $P_1$  and we add all paths  $Q \in \mathcal{Q}$  to  $\mathcal{Q}_m$  such that R intersects any  $Q' \in \sigma(Q)$  only after the last vertex it has in common with  $P_m$ . As  $|\mathcal{Q}| \geq K^2 \cdot h_{i+1}$  and  $|\sigma(Q)| \leq K$  for all Q there must be a minimal index  $c \leq m$  such that  $|\mathcal{Q}_c| \geq h_{i+1}$ . We set  $\mathcal{C}_{i+1} := \{\mathcal{B}_1 \cap \mathcal{Q}_c, \ldots, \mathcal{B}_m \cap \mathcal{Q}_c\} \setminus \{\emptyset\}, \mathcal{X}_{i+1} := \mathcal{X}_i \text{ and } \mathcal{P}_{i+1} := (P_1, \ldots, P_{l_i}, P'_1, \ldots, P'_c)$ . Clearly, this satisfies the conditions above.
- Otherwise, we set  $C_{i+1} := \{ \mathcal{B}' \in C_i : \sigma(P') \cap \mathcal{B}' \neq \emptyset \}$ ,  $\mathcal{X}_{i+1} := \mathcal{X}_i \cup \{P\}$  and  $\mathcal{P}_{i+1} := \mathcal{P}_i$ . Again, this satisfies the conditions above.

This completes the construction. We stop the process as soon as  $|\mathcal{P}_i| = k$ . We claim that after at most  $i \leq f(k-1,K) + k$  steps the process stops with a set  $\mathcal{P}_i$  of size k.

Towards this aim, note first that in every step, we either increase  $|\mathcal{X}_i|$  or  $|\mathcal{P}_i|$ . Hence, if we do not find a  $\mathcal{P}_i$  of size k, then after at most f(k-1,K)+k-1 steps we have constructed a set  $\mathcal{X}_i$  of size f(k-1,K). In this case,  $\mathcal{X}_i = \mathcal{B}$  for all  $\mathcal{B} \in \mathcal{C}_i$  by construction. But then, in every construction step j < i at most  $K^2 \cdot g_j$  paths Q where removed from  $\bigcup \mathcal{C}_i$ . Hence,  $\mathcal{P} = \bigcup \mathcal{C}_0$  contains at most  $|\mathcal{X}_i| + \sum_{j \leq i} K^2 \cdot h_i \leq K + (f(k-1,K)+k) \cdot K^2 \cdot h_{f(k-1,K)+k} = K + (f(k-1,K)+k) \cdot (K^2+1)^{f(k-1,K)+k}$  paths. But this contradicts the choice of  $|\mathcal{P}| \geq f(k,K) > K + (f(k-1,K)+k) \cdot (K^2+1)^{f(k-1,K)+k}$ .

In this generality the previous lemma will be needed in Section 6.4. In this section, we only need the following simpler version which follows from the previous lemma by setting  $\gamma(P) := \{P\}$  for all P.

**Corollary 5.17.** There is a function  $f: \mathbb{N} \to \mathbb{N}$  with the following properties. Let  $\mathcal{P}$  be a linkage and let R be a path intersecting every path in  $\mathcal{P}$ . For all  $k \geq 0$ , if  $|\mathcal{P}| \geq f(k)$  then there is a sequence  $(P_1, \ldots, P_k)$  of distinct paths  $P_1, \ldots, P_k \in \mathcal{P}$  such that for all  $1 \leq i < k$ , R contains a subpath  $R_i$  from a vertex in  $P_i$  to a vertex in  $P_{i+1}$  which is internally vertex disjoint from  $\bigcup_{1 \leq i \leq k} P_i$ .

Furthermore, the sequence can be chosen either  $\bar{in}$  a way that

- (1) the first vertex of  $\bigcup_{1 \le i \le k} P_i$  on R is contained in  $V(P_1)$  or
- (2) the last vertex of  $\bigcup_{1 \le i \le k} P_i$  on R is contained in  $V(P_k)$ .

We first consider the case of splits.

**Lemma 5.18.** Let f be the function defined in Corollary 5.17. For all  $k \geq 0, q \geq f(k)$  and  $p \geq {q \choose k}k!k^2$ , if G contains a p-split ( $S_{split}, Q$ ) of order q, then G contains a (k, k)-grid (P', Q'). Furthermore,  $Q' \subseteq Q$  and for every path  $P' \in P'$ , every subpath S of P' with both endpoints on a path in Q' but internally vertex disjoint from Q' is also a subpath of a path in P. Finally, if ( $S_{split}, Q$ ) is well linked then so is (P', Q').

*Proof.* Let  $S_{split} = (P_1, \ldots, P_p)$  be ordered in the order in which the paths in  $\mathcal{Q}$  traverse the paths in  $S_{split}$ . By definition, for all  $1 \leq i \leq p$ , the path  $P_i$  intersects every path in  $\mathcal{Q}$ . Hence, as  $|\mathcal{Q}| \geq f(k)$ , by Corollary 5.17, for all  $1 \leq i \leq p$ , there is a sequence  $\mathcal{Q}^i := (Q_1^i, \ldots, Q_k^i)$  of paths  $Q_j^i \in \mathcal{Q}$  such that for all  $1 \leq j < k$  there is

a subpath  $P_j^i$  of  $P_i$  internally vertex disjoint from  $\bigcup \mathcal{Q}^i$  which links  $Q_j^i$  to  $Q_{j+1}^i$ . As  $|\mathcal{P}| \geq {q \choose k} k! k^2$ , for at least  $k^2$  values  $i_1, \ldots, i_{k^2}$  the sequence  $\mathcal{Q}^{i_j}$  is the same. Let  $\mathcal{Q}' := \mathcal{Q}^{i_j}$  for some (and hence all)  $1 \leq j \leq k^2$ . By renumbering  $\mathcal{Q}$  we can assume that  $\mathcal{Q}' := (Q_1, \ldots, Q_{k^2})$ .

Hence, for all  $1 \leq l < k$  and  $1 \leq j \leq k^2$ , the path  $P_{i_j}$  contains a subpath  $F_{l,i_j}$  linking  $Q_l$  and  $Q_{l+1}$  which is internally vertex disjoint from  $\bigcup \mathcal{Q}'$ . For all  $1 \leq l < k$  and  $0 \leq j < k$  let  $R_j$  be the union of the paths  $F_{l,j\cdot k+l}$  and the subpaths of  $Q_{l'}$ , for 1 < l' < k linking the endpoint of  $F_{l'-1,j\cdot k+l'}$  and the start vertex of  $F_{l',j\cdot k+l'+1}$ . Hence,  $R_i$  is such that  $R_i \cap Q_l$  is a path for all  $1 \leq l \leq k$  and  $R_i$  traverses  $\mathcal{Q}'$  in the order  $(Q_1, \ldots, Q_k)$ .

Therefore,  $(\{R_1, \ldots, R_k\}, \{Q_{j_1}, \ldots, Q_{j_k}\})$  form a (k, k)-grid. As we do not split any path in  $\mathcal{Q}$ , well linkedness is preserved.

Clearly, the extra condition of the lemma is also satisfied as each subpath  $F_{l,i}$  is a subpath of a path in  $\mathcal{P}$ .

We now consider the case where the result of Lemma 5.13 is a segmentation.

**Lemma 5.19.** Let f be the function defined in Corollary 5.17. Let t be an integer and let  $q \geq \binom{f(3t)}{3t} \cdot 12t^2$  and  $r \geq f(3t) \cdot q!$ . If G contains an r-segmentation  $(S_{seg}, \mathcal{Q})$  of order q, then G contains a (t, t)-grid  $W' = (\mathcal{P}', \mathcal{Q}')$  such that  $\mathcal{P}' \subseteq S_{seg}$ . Furthermore, if the set of start and end vertices of  $\mathcal{Q}$  is well linked, then so is W'. More precisely, the set of start and end vertices of  $\mathcal{Q}'$  are subsets of the start and end vertices of  $\mathcal{Q}$ .

Finally, the grid  $(\mathcal{P}', \mathcal{Q}')$  can be chosen so that one (but not both) of the following properties is satisfied. Let  $\mathcal{P}' := (P_1, \ldots, P_t)$  be an ordering of  $\mathcal{P}'$  in order in which they occur on the paths  $\mathcal{Q}'$  of the grid.

- (1) For every  $Q \in \mathcal{Q}'$ , the first path  $P \in \mathcal{P}$  hit by Q is  $P_1$ .
- (2) For every  $Q \in \mathcal{Q}'$ , the last path  $P \in \mathcal{P}$  hit by Q is  $P_t$ .

*Proof.* Let  $S_{seq} := (P_1, \ldots, P_r)$  and  $Q := (Q_1, \ldots, Q_q)$ .

Note that in some sense this case is symmetric to the case of Lemma 5.18 in that here each  $P_i$  can be split into segments  $P_{i,1}, \ldots, P_{i,q}$  so that  $Q_j$  intersects  $P_i$  only in  $P_{i,j}$ . So in principle the same argument with the role of  $\mathcal{P}$  and  $\mathcal{Q}$  exchanged applies to get a grid. However, in this case the paths in  $\mathcal{Q}$  are split so that the well linkedness is not be preserved. We therefore need some extra arguments to restore well linkedness.

By definition of a segmentation, for all  $1 \leq i \leq r$ , the path  $P_i$  can be divided into disjoint subpaths  $P_{i,1}, \ldots, P_{i,q}$  such that  $Q_j$  intersects  $P_i$  only in the subpath  $P_{i,j}$ . As  $r \geq f(3t)q!$  there is a subset  $\mathcal{P}' \subseteq \{P_1, \ldots, P_r\}$  of order f(3t) such that the subpaths occur in the same order on each path in  $\mathcal{P}'$ . By renumbering the paths we may assume that  $\mathcal{P}' = \{P_1, \ldots, P_{f(3t)}\}$ .

As every  $Q_i \in \mathcal{Q}$  intersects every  $P \in \mathcal{P}'$  and  $|\mathcal{P}'| = f(3t)$ , Corollary 5.17 implies that for all  $1 \leq i \leq q$  there is a sequence  $\mathcal{P}^i := (P_1^i, \dots, P_{3t}^i)$  of paths from  $\mathcal{P}'$  such that for all  $1 \leq j < 3t$ ,  $Q_i$  contains a subpath  $F_{i,j}$  linking  $P_j^i$  to  $P_{j+1}^i$  which is internally vertex disjoint from  $\bigcup \mathcal{P}^i$ . Furthermore, the sequence  $\mathcal{P}^i$  can be chosen so that either the first vertex on  $Q_i$  the path  $Q_i$  has in common with  $\bigcup \mathcal{P}^i$  is on  $P_3^i$  or the last vertex on  $Q_i$  the path  $Q_i$  has in common with  $\bigcup \mathcal{P}^i$  is on  $P_{3t}^i$ . Depending on which of the two options we choose we can satisfy Condition 1 or 2 of the lemma. In the sequel, we present the proof in case we choose the first option. The case for

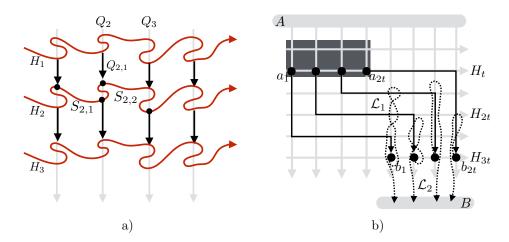


FIGURE 8. Illustration of the construction in the proof of Lemma 5.19.

the second option is completely analogous. So suppose that the first vertex on  $Q_i$  the path  $Q_i$  has in common with  $\bigcup \mathcal{P}^i$  is on  $P_1^i$ .

As  $q \geq \binom{f(3t)}{3t} \cdot 12t^2$ , there are at least  $12t^2$  values  $i_1, \ldots, i_{12t^2}$  such that  $\mathcal{P}^{i_j} = \mathcal{P}^{i_{j'}}$  for all  $1 \leq j \leq j' \leq 12t^2$ . For ease of presentation we assume that  $i_j = j$ , i.e.  $\mathcal{P}^i = \mathcal{P}^{i'}$  for all  $1 \leq i \leq i' \leq 12t^2$ . For all  $1 \leq j \leq 3t$  let  $H_j := P_j^1$  and let  $\mathcal{H} := (H_1, \ldots, H_{3t})$ .  $\mathcal{H}$  will be the set of horizontal paths in the grid we construct, i.e.  $\mathcal{H}$  will play the role of  $\mathcal{P}'$  in the statement of the lemma. Hence this implies the condition of the lemma that  $\mathcal{P}' \subseteq \mathcal{S}_{seq}$ .

For all  $1 \leq i \leq 3t$  and  $1 \leq j \leq 12t^2$ , we define  $H_{i,j}$  as the segment of  $H_i$  that contains  $H_i \cap Q_j$ .

By construction, every  $Q_i$  contains a subpath  $Q_{i,j}$  linking  $H_{j,i}$  and  $H_{j+1,i}$  with no inner vertex in  $\bigcup \mathcal{H}$ , for all  $1 \leq j < 3t$ . Let  $S_{i,j}$  be the subpath of  $H_i$  from the endpoint of  $Q_{j,i-1}$  to the start point of  $Q_{j+1,i}$ . See Figure 8 a) for an illustration.

Now we construct a grid as follows. The horizontal paths are just the paths in  $\mathcal{H}$ . For all  $1 \leq l \leq 4t$  let  $V_l := \bigcup \{Q_{(3t)(l-1)+i,i}: 1 \leq i < 3t\} \cup \bigcup \{S_{i,(3t)(l-1)+i}: 1 \leq i < 3t\}$ . By construction, each  $V_l$  intersects all  $H_j$  in a path and the paths  $H_1, \ldots, H_{3t}$  occur on  $V_l$  in this order, for all  $1 \leq l \leq 4t$ . Furthermore, for each  $1 \leq i \leq 3t$ , the paths  $V_1, \ldots, V_{4t}$  occur on all  $H_i$  in this order. Hence  $(\mathcal{H}, \mathcal{V})$  is an acyclic (3t, 4t)-grid, but it is not yet well linked.

Recall from above the definition of A and B. By construction of the sequence  $\mathcal{P}^i$ , for all  $V_i$  the first vertex  $v_i$  of  $V_i$  in  $\bigcup \mathcal{H}$  is on  $H_1$  and hence there is an initial subpath  $R_i$  of some path  $Q \in \mathcal{Q}$  (which was used to construct  $V_i$ ) from A to  $v_i$  with no inner vertex in  $\bigcup (\mathcal{H} \cup \mathcal{V})$ . By construction,  $V'_i := R_i \cup V_i$  is a path starting in A and  $(\mathcal{H}, \{V'_1, \ldots, V'_{4t}\})$  is still an acyclic grid of the same order but starting in a well linked set.

For  $1 \leq i \leq 2t$  let  $a_i$  be a vertex in  $V(V'_i \cap H_{t+1})$  and let  $b_i$  be the end vertex of  $V'_{2t+i}$ . Then  $\mathcal{H} \cup \mathcal{V}$  contains a linkage  $\mathcal{L}_1$  from  $\{a_1, \ldots, a_{2t}\}$  to  $\{b_1, \ldots, b_{2t}\}$  of order 2t. See Figure 8 b) for an illustration (where t = 2).

Now, each  $b_i$  is on a path  $Q_i$  and hence  $Q_i$  contains a subpath  $T_i$  from  $b_i$  to its endpoint in B. By construction, this path  $T_i$  does not intersect any  $V_1, \ldots, V_{2t}$  and does not contain any vertex of the initial subpaths of the  $H_i$  from the beginning to

their intersection with  $V_{2t}$ . Hence,  $T_1, \ldots, T_{2t}$  forms a linkage  $\mathcal{L}_2$  from  $b_1, \ldots, b_{2t}$  to B. The linkage  $\mathcal{L}_2$  is illustrated by the dotted lines in Figure 8 b).

By construction,  $\mathcal{L}_1$  and  $\mathcal{L}_2$  can be combined to form a half-integral linkage from  $\{a_1,\ldots,a_{2t}\}$  to B which does not intersect any  $V_i$ , with  $1 \leq i \leq 2t$ , at any vertex on  $V_i$  before  $a_i$  and does not intersect any  $H_i$  at any vertex before  $H_i$  intersects  $V_{2t}$ , i.e. it does not intersect the area marked in dark grey in Figure 8 b). By Lemma 2.10, there is also an (integral) linkage  $\mathcal{L}'$  from  $\{a_1,\ldots,a_{2t}\}$  to B in the graph  $\bigcup \mathcal{L}$  of order t. Hence,  $\mathcal{L}'$  contains t pairwise vertex disjoint paths  $L_1,\ldots,L_t$  from a subset  $C:=\{a_{i_1},\ldots,a_{i_t}\}$  to B, such that  $L_l$  has  $a_{i_l}$  as the start vertex. By deleting all vertical paths  $V_j$  with  $j \neq i_l$ , for all  $1 \leq i \leq t$ , and joining  $V_{i_j}$  and  $L_j$  to form a new path  $V''_{i_j}$ , we obtain a well linked acyclic (t,t)-grid  $(\{H_1,\ldots,H_t\},\{V''_{i_1},\ldots,V''_{i_t}\})$ .

We are now ready to prove Theorem 5.4.

Proof of Theorem 5.4. Let t, d be integers and let a (p, q)-web of avoidance d in a digraph G be given. We will determine the minimal value for p and q in the course of the proof. By Lemma 4.10, G contains a  $(p_1, q_1)$ -web with avoidance 0 as long as

(1) 
$$p \ge \frac{d}{d-1}p_1 \quad \text{and} \quad q \ge q_1 \binom{p}{\frac{1}{d}p}.$$

As noted above, any such  $(p_1, q_1)$ -web is a  $(p_1, q_1)$ -web with linkedness  $c = p_1$ . By Lemma 5.13, if

(2)

$$p_1 = p_2,$$
  $q_1 \ge (p_2 \cdot q_2 \cdot (q_2 + c))^{2^{(x-1)y+1}},$   $x = p_2,$  and  $y = f(3t)q_2!$ 

then G contains a  $p_2$ -split of order  $q_2$  or a  $f(3t)q_2$ !-segmentation of order  $q_2$ . In the first case, if

(3) 
$$q_2 \ge f(t)$$
 and  $p_2 \ge \begin{pmatrix} q_2 \\ t \end{pmatrix} \cdot t! \cdot t^2$ ,

where f is the function of Corollary 5.17, then Lemma 5.18 implies that G contains an acyclic well linked (t,t)-grid. In the other case, if

$$(4) q_2 \ge \binom{f(3t)}{3t} 12t^2$$

then Lemma 5.19 implies that G contains an acyclic well linked (t,t)-grid as required. Clearly, for any  $t \geq 0$  we can always choose the numbers  $p, p_1, p_2, q, q_1, q_2, x, y$  so that all inequalities above are satisfied, which concludes the proof.

As noted in the beginning of this section, Theorem 5.2 follows from Theorem 5.4. So far we have shown that every digraph which contains a large well linked web also contains a large well linked fence  $(\mathcal{P}, \mathcal{Q})$ . The well linkedness of  $(\mathcal{P}, \mathcal{Q})$  implies the existence of a minimal bottom-up linkage as defined in the following definition.

**Definition 5.20.** Let  $(\mathcal{P}, \mathcal{Q})$  be a fence. A  $(\mathcal{P}, \mathcal{Q})$ -bottom-up linkage is a linkage  $\mathcal{R}$  from  $bot(\mathcal{P}, \mathcal{Q})$  to  $top(\mathcal{P}, \mathcal{Q})$ . It is called minimal  $(\mathcal{P}, \mathcal{Q})$ -bottum-up linkage, if  $\mathcal{R}$  is  $(\mathcal{P}, \mathcal{Q})$ -minimal.

We close this section by establishing a simple routing principle in fences which will be needed below. This is (3.2) in [35].

**Lemma 5.21.** Let  $(P_1, \ldots, P_{2p}, Q_1, \ldots, Q_q)$  be a (p,q)-fence in a digraph G, with the top A and the bottom B. Let  $A' \subseteq A$  and  $B' \subseteq B$  with |A'| = |B'| = r for some  $r \leq p$ . Then there are vertex disjoint paths  $Q'_1, \ldots, Q'_r$  in  $\bigcup_{1 \leq i \leq 2p} P_i \cup \bigcup_{1 \leq j \leq q} Q_j$  such that  $(P_1, \ldots, P_{2p}, Q'_1, \ldots, Q'_r)$  is a (p,r)-fence with top A' and bottom B'.

## 6. From Fences to Cylindrical Grids

So far we have seen that every digraph of sufficiently high directed tree-width either contains a cylindrical grid or a well linked fence. In this section we complete the proof of our main result by showing that if G contains a well linked fence of sufficient order, then it contains a cylindrical grid of large order as a butterfly minor. The main result of this section is the following theorem, which completes the proof of Theorem 1.2.

**Theorem 6.1.** Let G be a digraph. For every  $k \geq 1$  there are integers  $p, r \geq 1$  such that if G contains a (p, p)-fence  $\mathcal{F}$  and a minimal  $\mathcal{F}$ -bottom-up linkage  $\mathcal{R}$  of order r then G contains a cylindrical grid of order k as a butterfly minor.

Let  $\mathcal{F}$  and  $\mathcal{R}$  be as in the statement of Theorem 6.1. We prove the theorem by analysing how  $\mathcal{R}$  intersects  $\mathcal{F}$ . Essentially, we follow the paths in  $\mathcal{R}$  from the bottom of  $\mathcal{F}$  (i.e., start vertices) to its top (i.e., end vertices) and somewhere along the way we will find a cylindrical grid of large order as a butterfly minor, either (i) because  $\mathcal{R}$  avoids a sufficiently large subfence, or (ii) because it contains subpaths that "jump" over large fractions of the fence or (iii) because  $\mathcal{R}$  and  $\mathcal{Q}$  intersect in a way that they generate a cylindrical grid (as a butterfly minor) locally. We will show each case in the following subsections, respectively.

6.1. Bottom up linkages which avoid a subfence. We first prove the easiest part, namely when  $\mathcal{R}$  "avoids" a sufficiently large subfence of  $\mathcal{F}$  (i.e., case (i)), as this is needed in the arguments below.

**Definition 6.2** (subfence). Let  $\mathcal{F} := (\mathcal{P}, \mathcal{Q})$  be a fence. A sub-fence of  $\mathcal{F}$  is a fence  $(\mathcal{P}', \mathcal{Q}')$  such that every  $P \in \mathcal{P}'$  and every  $Q \in \mathcal{Q}''$  is contained in  $\bigcup \mathcal{P} \cup \bigcup \mathcal{Q}$ . A sub-grid of a grid is defined analogously.

To deal with case (i) we first prove a technical lemma which essentially is [35, (3.3)].

**Lemma 6.3.** For every t, if  $\mathcal{F} := (\mathcal{P}, \mathcal{Q})$  is a (2q, q)-fence, where q := (t-1)(2t-1)+1, and  $\mathcal{R}$  is an  $\mathcal{F}$ -bottom-up linkage of order q such that no path in  $\mathcal{R}$  has any internal vertex in  $\mathcal{P} \cup \mathcal{Q}$ , then  $\mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}$  contains a cylindrical grid of order t as a butterfly minor.

Proof. Let  $(\mathcal{P}, \mathcal{Q})$  be a (2q, q)-fence and let  $\mathcal{R}$  be a linkage of order q from the bottom  $B := bot(\mathcal{F})$  of the fence to the top  $A := top(\mathcal{F})$  such that no internal vertex of  $\mathcal{R}$  is in  $\mathcal{P} \cup \mathcal{Q}$ . Let  $a_1, \ldots, a_q$  be the elements of A and  $b_1, \ldots, b_q$  be the elements of B such that  $\mathcal{Q}$  links  $a_i$  and  $b_i$ , for all  $1 \le i \le q$ . For all  $1 \le j \le q$  let  $i_j$  be such that  $\mathcal{R}$  contains a path linking  $b_j$  and  $a_{i_j}$ . By Theorem 2.7, there is a sequence  $j_1 < j_2 < \cdots < j_t$  such that  $i_{j_s} < i_{j_{s'}}$  whenever s < s' or there is a sequence  $j_1 < j_2 < \cdots < j_{2t}$  such that  $i_{j_s} > i_{j_{s'}}$  whenever s < s'. In either case, let  $\mathcal{R}'$  be the paths in  $\mathcal{R}$  linking  $b_{j_s}$  to  $a_{i_{j_s}}$  for all  $1 \le s \le t$  (or  $1 \le s \le 2t$  respectively).

In the first case, by Lemma 5.21, there are vertex disjoint paths  $\mathcal{P}'$  and  $\mathcal{Q}'$  in  $\mathcal{P} \cup \mathcal{Q}$  such that  $(\mathcal{P}', \mathcal{Q}')$  is a (2t, t)-fence with the top  $\{a_{i_{j_s}} : 1 \leq s \leq t\}$  and the bottom  $\{b_{i_s} : 1 \leq s \leq t\}$  and it is easily seen that  $\mathcal{Q}'$  can be chosen so that it links  $a_{i_{j_s}}$  to  $b_{i_s}$ . Hence,  $(\mathcal{P}', \mathcal{Q}')$  together with  $\mathcal{R}$  yields a cylindrical grid of order t, obtained by contracting the paths in  $\mathcal{R}$  into a single edge.

In the second case, again by Lemma 5.21, there are vertex disjoint paths  $\mathcal{P}'$  and  $\mathcal{Q}'$  in  $\mathcal{P} \cup \mathcal{Q}$  such that  $(\mathcal{P}', \mathcal{Q}')$  is a (2t, 2t)-fence with the top  $\{a_{i_{j_s}}: 1 \leq s \leq 2t\}$  and the bottom  $\{b_{i_s}: 1 \leq s \leq 2t\}$  and it is easily seen that  $\mathcal{Q}'$  can be chosen so that it links  $a_{i_{j_{2t+1-s}}}$  to  $b_{i_s}$ . Let  $\mathcal{Q}' = (Q_1, \ldots, Q_{2t})$  be ordered from left to right. To obtain a cylindrical grid of order t, we take for each  $P \in \mathcal{P}'$  the minimal subpath  $P^*$  of P containing all vertices of  $V(P) \cap \bigcup_{1 \leq i \leq t} V(Q_i)$ . Hence, from each such  $P \in \mathcal{P}'$  we only take the "left half". Let  $\mathcal{P}^* := \{P^*: P \in \mathcal{P}'\}$ . Then  $(\mathcal{P}^*, \{Q_1, \ldots, Q_t\})$  form a fence of order t. Furthermore, for all  $1 \leq i \leq t$ ,  $Q_i \cup R_{j_i} \cup Q_{2t+1-i} \cup R_{j_{2t+1-s}}$  constitutes a cycle  $C_i$ . Here,  $R_{i_s}$  is the path in  $\mathcal{R}'$  linking  $b_{i_s}$  to  $a_{i_{j_{2t+1-s}}}$ . Furthermore,  $C_i$  and  $C_j$  are pairwise vertex disjoint whenever  $i \neq j$ . Hence,  $C_1, \ldots, C_t$  and  $\mathcal{P}^*$  together contain a cylindrical grid of order t as butterfly minor.

The previous lemma shows that whenever we have a fence and a bottom-up linkage  $\mathcal{R}$  disjoint from the fence, this implies a cylindrical grid of large order as a butterfly minor. We show in the next lemma that it suffices if the linkage  $\mathcal{R}$  is only disjoint from a sufficiently large subfence rather than from the entire fence. This lemma finishes Case (i) and will be applied frequently in the sequel to ensure that the bottom-up linkage hits every part of a very large fence.

**Lemma 6.4.** For every  $p \ge 1$  let t' := 2((p-1)(2p-1)+1) and t := 3t'. Let G be a digraph containing a (t,t)-fence  $\mathcal{F}$  consisting of linkages  $(\mathcal{P},\mathcal{Q})$  and a linkage  $\mathcal{R}$  of order t from the bottom of  $\mathcal{F}$  to the top. Furthermore, let  $\mathcal{P}',\mathcal{Q}' \subseteq \mathcal{F}$  be linkages such that

- (1)  $(\mathcal{P}', \mathcal{Q}')$  form a (t', t')-fence  $\mathcal{F}'$ ,
- (2)  $V(P') \cap V(R') = \emptyset$  for any path  $P' \in \mathcal{P}'$  and for any path  $R' \in \mathcal{R}$ ,
- (3)  $V(Q') \cap V(R') = \emptyset$  for any path  $Q' \in \mathcal{Q}'$  and for any path  $R' \in \mathcal{R}$  and
- (4)  $\mathcal{F}'$  is "in the middle" of the fence  $\mathcal{F}$ , i.e. if  $\mathcal{P} = (P_1, \dots, P_{2t})$  is ordered from top to bottom and  $\mathcal{Q} = (Q_1, \dots, Q_t)$  is ordered from left to right, then  $\mathcal{F}' \cap (P_1 \cup \dots \cup P_{2t'} \cup P_{2t-t'} \cup \dots \cup P_{2t'} \cup Q_1 \cup \dots \cup Q_{t'} \cup Q_{t-t'} \cup \dots \cup Q_t) = \emptyset$ .

Then G contains a cylindrical grid of order p as a butterfly minor.

*Proof.* Choose a set  $A := \{a_1, \ldots, a_{t'}\}$  and  $B := \{b_1, \ldots, b_{t'}\}$  of vertices from the top and the bottom of  $\mathcal{F}$  such that  $\mathcal{R}$  links A to B. Further, let  $A' := \{a'_1, \ldots, a'_{t'}\}$  and  $B' := \{b'_1, \ldots, b'_{t'}\}$  be the top and the bottom of  $\mathcal{F}'$  such that  $\mathcal{Q}'$  contains a path linking  $a'_i$  to  $b'_i$ , for all i. We fix a plane embedding of  $\mathcal{F}$  and assume that the vertices  $a_1, \ldots, a_{t'}$  are ordered so that they appear from left to right on the top of  $\mathcal{F}$  and likewise for  $b_1, \ldots, b_{t'}, a'_1, \ldots, a'_{t'}$  and  $b'_1, \ldots, b'_{t'}$ .

As  $\mathcal{F}'$  is in the middle of  $\mathcal{F}$ , by Lemma 5.21, there is a linkage  $\mathcal{L}$  in  $G[\mathcal{P} \cup \mathcal{Q}]$  linking B' to B. Furthermore, there is an A-A'-linkage  $\mathcal{L}'$  of order t'. Note that  $\mathcal{L}$  and  $\mathcal{L}'$  are pairwise vertex disjoint and moreover each path in  $\mathcal{L} \cup \mathcal{L}'$  does not hit any vertex in  $\mathcal{F}'$  except for the vertices in A'/cupB'. Hence, the linkages  $\mathcal{L}, \mathcal{R}$  and  $\mathcal{L}'$  can be combined to form a half-integral linkage from B' to A'. By Lemma 2.10, this yields an integral linkage  $\mathcal{L}''$  from a subset  $B'' \subseteq B'$  to a subset  $A'' \subseteq A'$  of

order  $t'' := \frac{t'}{2}$ . By Lemma 5.21 there are paths  $\mathcal{P}'', \mathcal{Q}''$  in  $G[\mathcal{P}' \cup \mathcal{Q}']$  such that  $(\mathcal{P}'', \mathcal{Q}'')$  yields a (t'', t'')-fence with the top A'' and the bottom B''. Now we can apply Lemma 6.3 to obtain the desired cylindrical grid of order p as a butterfly minor.

In the sequel, we will also need the analogous statement of Lemma 6.4 for acyclic grids instead of fences. The only difference between the two cases is that in a fence, when routing from the top to the bottom, we can route paths to the left as well as to the right (as the paths in  $\mathcal{P}$  hit the vertical paths in  $\mathcal{Q}$  in alternating directions) whereas in a grid we can only route from left to right. Otherwise, the same proof as before establishes the next lemma.

**Lemma 6.5.** For  $p \ge 1$  let t' := 2((p-1)(2p-1)+1) and t := 3t'. Let G be a digraph containing a (t,t)-grid W and a linkage  $\mathcal{R}$  of order t' from the bottom of W to its top. Furthermore, let  $\mathcal{P}', \mathcal{Q}' \subseteq W$  be linkages such that

- (1)  $(\mathcal{P}', \mathcal{Q}')$  constitutes a(t', t')-grid W,
- (2)  $V(P') \cap V(R') = \emptyset$  for any path  $P' \in \mathcal{P}'$  and for any path  $R' \in \mathcal{R}$ ,
- (3)  $V(Q') \cap V(R') = \emptyset$  for any path  $Q' \in Q'$  and for any path  $R' \in \mathcal{R}$ ,
- (4) W' is "in the middle" of the grid W, i.e. if  $\mathcal{P} = (P_1, \ldots, P_t)$  is ordered from top to bottom and  $\mathcal{Q} = (Q_1, \ldots, Q_t)$  ordered from left to right, then  $W' \cap (P_1 \cup \cdots \cup P_{t'} \cup P_{t-2t'} \cup \cdots \cup P_t \cup Q_1 \cup \cdots \cup Q_{t'} \cup Q_{t-2t'} \cup \cdots \cup Q_t) = \emptyset$ , and
- (5) the linkage  $\mathcal{R}$  joins the last third of the bottom vertices  $(b_{2t/3+1}, \ldots, b_t)$  to the first third of the top vertices  $(a_1, \ldots, a_{t/3})$ .

Then G contains a cylindrical grid of order p as a butterfly minor.

6.2. Taming jumps. The previous results solve the easy cases in our argument, i.e. where the bottom-up linkage  $\mathcal{R}$  avoids a large part of the fence (i.e., case (i)). It has the following consequence that we will use in all our arguments below. Suppose  $\mathcal{F}$  is a huge fence and  $\mathcal{R}$  is an  $\mathcal{F}$ -bottom-up linkage. If  $\mathcal{R}$  avoids any tiny subfence, where "tiny" essentially means  $4k^2$ , then this implies that  $\mathcal{F} \cup \mathcal{R}$  contains a cylindrical grid of order k as a butterfly minor. Hence, for that not to happen, almost all paths of  $\mathcal{R}$  must hit every tiny subfence of  $\mathcal{F}$ . In particular, this observation will be used in the next lemma to show that  $\mathcal{R}$  not only must hit every small subfence, but it must in fact go through  $\mathcal{F}$  in a very nice way, namely going up "row by row". This indeed finishes Case (ii). We give a formal definition and prove this lemma next.

**Definition 6.6.** Let  $\mathcal{F} := (\mathcal{P}, \mathcal{Q})$  with  $\mathcal{P} := (P_1, \dots, P_{2p})$  and  $\mathcal{Q} := (Q_1, \dots, Q_q)$  be a (p,q)-fence. For  $1 \leq i \leq 2p$ , the i-th row  $\operatorname{row}_i(\mathcal{F})$  of  $\mathcal{F}$  is  $P_i \cup \bigcup_{1 \leq j \leq q} Q_j^i$ , where  $Q_j^i$  is the subpath of  $Q_j$  starting at the first vertex of  $Q_j$  after the last vertex of  $V(Q_j \cup P_{i-1})$  and ending in the last vertex of  $V(Q_j \cap P_i)$  on  $Q_j$ . Here, for i = 1, we take the initial subpath of  $Q_j$  up to the last vertex of  $Q_j \cap P_1$ .

Hence, the *i*-th row of a fence is the union of the vertical paths between  $P_{i-1}$  and  $P_i$ , including  $P_i$  but none of  $P_{i-1}$ , so that rows are disjoint.

**Definition 6.7.** Let  $\mathcal{F}$  be a fence and let  $\mathcal{R}$  be an  $\mathcal{F}$ -bottom-up linkage. Let  $R \in \mathcal{R}$  be a path. For some i > j with  $i - j \geq 2$ , a jump from i to j in R is a subpath  $J \subseteq E(R) \setminus E(\mathcal{F})$  of R such that the start vertex u of J is in row i of  $\mathcal{F}$  and its end vertex v in row j. The length of the jump J is i - j.

Note that if J is a jump linking u and v, then there is no path from u to v in  $\mathcal{F}$ . The following lemma finishes case (ii).

**Lemma 6.8.** For every  $t, t' \geq 1$  there are integers  $p, q, r \geq 1$  such that if  $\mathcal{F} := (\mathcal{P}, \mathcal{Q})$  is a (p,q)-fence and  $\mathcal{R}$  is a minimal  $\mathcal{F}$ -bottom-up-linkage of order r then  $\mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}$ 

- (1) contains a cylindrical grid of order t as butterfly minor or
- (2) a sub-fence  $(\mathcal{P}', \mathcal{Q}')$  of  $(\mathcal{P}, \mathcal{Q})$  of order t' and a  $(\mathcal{P}', \mathcal{Q}')$ -minimal  $(\mathcal{P}', \mathcal{Q}')$ -bottom-up-linkage  $\mathcal{R}'$  of order t' such that  $\mathcal{R}'$  goes up row by row, i.e. for every  $R \in \mathcal{R}'$ , the last vertex of R in the i-th row of  $(\mathcal{P}', \mathcal{Q}')$  lies in R before the first vertex of the j-th row for all j < i 1.

*Proof.* Given t we let t'' := 6((t-1)(2t-1)+1). Given t', let  $d_{2t''} := 2t''$  and, for  $0 \le l < 2t''$ , let  $d_i := 6t'^2d_{i+1}$ . Let  $p := 2d_0$  and  $q := d_0$ .

We prove the lemma by eliminating jumps in  $\mathcal{R}$ . In the first step, we eliminate jumps of very large length, i.e. that jump over a large part of the fence.

Step 1. Taming long jumps. For  $0 \le l \le 2t''$ , we inductively construct a sequence  $J_l \subseteq \bigcup \mathcal{R} - E(\mathcal{F})$  of jumps of order l, subfences  $\mathcal{F}_l$  of  $\mathcal{F}$  of order at least  $d_l$  and  $\mathcal{F}_l$ -minimal  $\mathcal{F}_l$ -bottom-up-linkages  $\mathcal{R}_l$  of order  $d_l$  as follows. We set  $J_0 := \emptyset$ ,  $\mathcal{F}_0 := \mathcal{F}$  and  $\mathcal{R}_0 := \mathcal{R}$ .

Now suppose  $\mathcal{F}_l, \mathcal{R}_l$  and  $J_l$  have already been defined. If there is a path  $R \in \mathcal{R}_l$  which contains a jump J in  $\mathcal{F}_l$  of length at least  $d_l$ , then let  $i_l$  be the row containing its start vertex  $u_l$  and  $j_l$  be the row containing its end  $v_l$ . By construction,  $i_l - j_l \geq d_{l+1} + 6$ . We take a sub-fence  $\mathcal{F}_{l+1}$  of  $\mathcal{F}_l$  of order  $d_{l+1}$  contained in the rows  $i_l - 3$  to  $j_l + 3$ , not including  $P_{i_l - 3}$  and  $P_{j_l + 3}$ . We set  $J_{l+1} := J_l \cup \{J\}$ . Furthermore, let  $\mathcal{R}_{l+1}$  be an  $\mathcal{F}_{l+1}$ -minimal  $\mathcal{F}_{l+1}$ -bottom-up-linkage in  $G - \bigcup J_{l+1}$  of order  $d_{l+1}$ . Otherwise, i.e. if there is no such  $R \in \mathcal{R}_l$ , the construction stops here.

Now suppose we have constructed a sequence  $J_l$ ,  $\mathcal{F}_l$ ,  $\mathcal{R}_l$  for all  $l \leq 2t''$ . Then we have found a sub-fence  $\mathcal{F}_{2t''}$  of order 2t'' contained in some rows i to j of  $\mathcal{F}$ , where i > j, and a set  $\mathcal{J}$  of 2t'' jumps  $J_1, \ldots, J_{2t''}$  such that the jump  $J_l$  that starts at  $u_l$  and ends in  $v_l$  satisfies the followings:

- $i_l > i + 2$  and  $j_l < j 2$ , for all  $1 \le l \le 2t''$  and
- $i_l > i_{l+1} + 3$  and  $j_l < j_{l+1} 3$ , for all  $1 \le l < 2t''$ .

Let  $s_1, \ldots, s_{2t''}$  be 2t'' vertices in  $top(\mathcal{F}_{t''})$  ordered from left to right in  $\mathcal{F}_{2t''}$  and let  $s'_1, \ldots, s'_{2t''}$  be 2t'' vertices in  $totof(\mathcal{F}_{t''})$  such that  $s_r$  and  $s'_r$  are in the same path  $Q_r \in \mathcal{Q}$ , for all r. It is easily seen that there is a linkage  $\mathcal{L}_1 \subseteq E(\mathcal{F})$  of order 2t'' linking  $\{v_1, \ldots, v_{2t''}\}$  to  $\{s_1, \ldots, s_{2t''}\}$  and a linkage  $\mathcal{L}_2 \subseteq E(\mathcal{F})$  linking  $\{s'_1, \ldots, s'_{2t''}\}$  to  $\{u_1, \ldots, u_{2t''}\}$ . Obviously,  $V(\mathcal{L}_1) \cap V(\mathcal{L}_2) = \emptyset$  and they are internally disjoint from  $\mathcal{F}_{t''}$ . Hence,  $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{J}$  is a half-integral linkage of order 2t'' from  $\{s'_1, \ldots, s'_{2t''}\}$  to  $\{s_1, \ldots, s_{2t''}\}$ . By Lemma 2.10,  $\mathcal{L}$  contains an integral linkage  $\mathcal{L}'$  of order  $\frac{2t''}{2} = t''$  from  $\{s'_1, \ldots, s'_{2t''}\}$  to  $\{s_1, \ldots, s_{2t''}\}$ . By Lemma 6.4,  $\mathcal{F}_{2t''} \cup \mathcal{L}'$  contains a cylindrical grid of order t as a butterfly minor, which is the first outcome of the lemma.

Step 2. Taming short jumps. So now suppose that the construction stops after l < 2t'' steps. Hence, we now have a sub-fence  $\mathcal{F}' := \mathcal{F}_l = (\mathcal{P}', \mathcal{Q}')$  of order  $d := d_l$  and we can take an  $\mathcal{F}'$ -minimial  $\mathcal{F}'$ -bottom-up linkage  $\mathcal{R}' \subseteq \mathcal{R}_l$  of order t'. By construction,  $\mathcal{R}'$  does not contain any long jump, i.e. any jump of length  $d = d_{l+1}$ . Let  $\mathcal{P}' = (P_1, \ldots, P_{2d})$  be the canonical ordering of  $\mathcal{P}'$  ordered from top to

bottom. Let  $\mathcal{P}'' := \{P_{6t'd_{l+1} \cdot i + (i \mod 2)} \in \mathcal{P}' : i \leq 2t'\}$ . As  $d = 6t'^2d_{i+1}, |\mathcal{P}''| \geq 2t'$ .

Note that  $\mathcal{P}''$  contains paths in alternating directions as we have added  $i \mod 2$  in each step.

Let Q'' be a linkage in  $\bigcup \mathcal{F}'$  of order t' linking the set of end vertices of  $\mathcal{R}'$  to the set of start vertices of  $\mathcal{R}'$ . Such a linkage exists by Lemma 5.21. Then  $\mathcal{F}'' := (\mathcal{P}'', Q'')$  is a subfence of  $\mathcal{F}'$  of order t'.

We claim that  $\mathcal{R}'$  traverses  $\mathcal{F}''$  row by row. Towards a contradiction, suppose there are i, j such that j < i - 1 and  $\mathcal{R}'$  contains a path R which hits a vertex in row j of  $\mathcal{F}''$  before it hits a vertex in row i. Note that the distance between i and j in  $\mathcal{F}'$  is at least  $(i-j) \cdot 6t'd_{l+1}$ . By construction,  $\mathcal{R}'$  does not contain any jump of length  $d_{l+1}$ . Hence, as R goes from the bottom of  $\mathcal{F}'$  to its top, before R can hit a vertex in row j it must hit at least 4t' rows between row i and row j. Furthermore, R continues from the vertex v it hits in row i to the top of  $\mathcal{F}'$ . Hence, after it hits row i it must hit at least 2t' rows between i and i-2t' in  $\mathcal{F}'$ . But then, if e is the first edge of R after v that is not in  $E(\mathcal{F}')$  (this edge must exist as R goes up) and if  $R_1, R_2$  are the two components of R - e with  $R_1$  being the initial subpath of R, then there are t' vertex disjoint paths in  $\mathcal{R}' \cup \mathcal{F}'$  between  $R_1$  and  $R_2$  which, by Lemma 2.13, contradicts the assumption that  $\mathcal{R}'$  is  $\mathcal{F}'$ -minimal.

6.3. Avoiding a pseudo-fence. In this subsection we prove two lemmas needed later on in the proof. Essentially, the two lemmas deal with the case that we have a fence  $\mathcal{F} = (\mathcal{P}, \mathcal{Q})$  and a bottom-up linkage which avoids the paths in  $\mathcal{Q}$ . This is proved in Lemma 6.11 below. We also need a variant of it where instead of a fence we only have a *pseudo-fence*. Recall the definition of a pseudo-fence from Definition 5.14.

**Lemma 6.9.** For every  $p \geq 1$  there is an integer t' such that if G is a digraph containing a (t'', t')-pseudo-fence  $W = (\mathcal{P}, \mathcal{Q})$ , where  $t'' := 3 \cdot t' \cdot \binom{t'}{t'_2}$ , and a linkage  $\mathcal{R}$  of order t' from the bottom of W to the top of W such that no internal vertex of any path in  $\mathcal{R}$  is contained in  $V(\mathcal{Q})$ , then G contains a cylindrical grid of order p as a butterfly minor.

*Proof.* Let  $\mathcal{P} := (P_1, \dots, P_{2t''})$  be ordered from top to bottom, i.e. in the order in which the paths in  $\mathcal{P}$  appear on the paths in  $\mathcal{Q} := (Q_1, \dots, Q_{t'})$ .

Let U be the subgraph of W containing  $P_1, \ldots, P_{2t'}$  and for each  $Q \in \{Q_1, \ldots, Q_{t'}\}$  the minimal subpath of Q containing all vertices of

$$V(\mathcal{Q}) \cap \bigcup_{1 \leq i \leq 2t'} V(P_i).$$

Analogously, let D be the lower part of W, i.e. the part formed by  $P_{2t''-2t'}, \ldots, P_{2t''}$  and the minimal subpaths of the  $Q_i$  containing all vertices  $Q_i$  has in common with  $P_{2t''-2t'}, \ldots, P_{2t''}$ . Finally, let M be the middle part, i.e. the subgraph of W induced by the paths  $P_{2t'+1}, \ldots, P_{2t''-2t'-1}$  and the subpaths of the  $Q_i$  connecting U to D.

We will write  $\mathcal{P}_U, \mathcal{P}_M', \mathcal{P}_D$  for the paths in  $\mathcal{P}$  contained in U, M, D, respectively. Similarly, we write  $\mathcal{Q}_U, \mathcal{Q}_M, \mathcal{Q}_D$  for the corresponding subpaths of  $\mathcal{Q}$ . See Figure 9 a) for an illustration of the construction in this proof. We first establish a simple routing property in pseudo-fences.

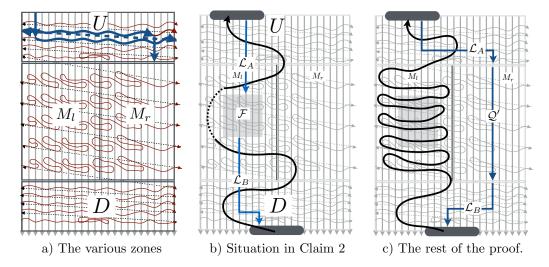


FIGURE 9. Illustration for the Proof of Lemma 6.9.

Claim 1. Let A be a set of start vertices of paths in  $\mathcal{Q}$  and let A' be a set of start vertices of paths in  $\mathcal{Q}_M$  such that  $k := |A| = |A'| \le t'$ . Then there is an A-A'-linkage  $\mathcal{L}$  of order  $\frac{k}{2}$ .

Proof. Let  $A := (a_1, \ldots, a_k)$  and  $A' := (a'_1, \ldots, a'_k)$ . Let  $Q_1, \ldots, Q_k$  be the paths in  $Q_U$  with start vertices  $a_1, \ldots, a_k$  and let  $Q'_1, \ldots, Q'_k$  be the paths in  $Q_U$  with end vertices  $a'_1, \ldots, a'_k$ . For each  $1 \le i \le k$  let  $P_i^1 := P_{2i}$  and  $P_i^2 := P_{2i-1}$ . This is possible as  $k \le t'$ . By definition of a pseudo-fence, there is an edge from the end vertex of  $P_i^1$  to the first vertex of  $P_i^2$ .

Then, for all  $1 \leq i \leq k$ ,  $Q_i \cup P_i^1 \cup P_i^2 \cup Q_i'$  contains a path  $L_i$  from  $a_i$  to  $a_i'$ . Furthermore, no vertex of G is contained in more than two paths of  $\mathcal{L}' := \{L_1, \ldots, L_k\}$ . Hence,  $\mathcal{L}'$  is a half-integral linkage from A to A'. By Lemma 2.10, there also is an integral A-A'-linkage of order  $\frac{k}{2}$ .

Clearly, the analogous statement holds for sets A of end vertices of  $\mathcal{Q}_M$  and A' of end vertices of  $\mathcal{Q}$ .

We now divide M into two parts,  $M_l$  and  $M_r$ . For every  $P \in \mathcal{P}'_M$  let  $\mathcal{Q}(P) \subseteq \mathcal{Q}$  be set of the first  $\frac{t'}{2}$  paths in  $\mathcal{Q}$  that P intersects. By the pigeon hole principle, as  $|\mathcal{P}'_M| \geq 2t'(\frac{t'}{2})$  there is a subset  $\mathcal{P}_M \subseteq \mathcal{P}'_M$  of order  $\geq 2t'$  such that  $\mathcal{Q}(P) = \mathcal{Q}(P')$  for all  $P, P' \in \mathcal{P}_M$ . We define  $M_l := \mathcal{Q}(P)$  for some (and hence all)  $P \in \mathcal{P}_M$ , and the minimal initial subpaths of the  $P \in \mathcal{P}_M$  containing all vertices of  $\mathcal{Q}(P)$ .  $M_r$  contains the other paths of  $\mathcal{Q}_M$  and the parts of the paths of  $\mathcal{P}_M$  not contained in  $M_l$ . We write  $\mathcal{P}_{M_l}$ ,  $\mathcal{P}_{M_r}$  and  $\mathcal{Q}_{M_r}$ ,  $\mathcal{Q}_{M_l}$  for the corresponding paths.

To simplify the presentation we renumber the paths in  $\mathcal{P}_U, \mathcal{P}_M, \mathcal{P}_D$  such that  $\mathcal{P}_U := \{P_1, \dots, P_{2t'}\}, \mathcal{P}_M := \{P_{2t'+1}, \dots, P_{4t'}\}$  and  $\mathcal{P}_D := \{P_{4t'+1}, \dots, P_{6t'}\}.$ 

Let A be the end vertices of  $\mathcal{R}$  and B be the start vertices of  $\mathcal{R}$ . Hence, A is contained in the top of the pseudo-fence and B is part of its bottom. We now take a new linkage  $\mathcal{R}' \subseteq (\mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}) \setminus \mathcal{Q}_{M_r}$  from B to A of order  $|\mathcal{R}|$  such that  $\mathcal{R}'$  is  $M_l$ -minimal. Since no internal vertex of any path in  $\mathcal{R}$  is contained in  $V(\mathcal{Q})$ , such a choice is possible. Note that  $\mathcal{R}' \cap \mathcal{Q}_{M_r} = \emptyset$ .

As the paths in  $\mathcal{P}_{M_l}$  occur in order on the paths in  $\mathcal{Q}_{M_l}$ , we can adapt the concept of rows and jumps from Definitions 6.6 and 6.7 in the obvious way. Note, though, that  $M_l$  and  $\mathcal{Q}_{M_l}$  may no longer form a pseudo-fence, as we have taken a subset  $\mathcal{P}_M$  of  $\mathcal{P}'_M$  above. However,  $M_l$  and  $\mathcal{Q}_{M_l}$  do form a segmentation, which is all we need in the following argument.

Claim 2. Let f be the function defined in Corollary 5.17. Let  $c = \binom{f(3p_1)}{3p_1} \cdot 12p_1^2$  and let  $q_1 = f(3p_1) \cdot c!$ . Suppose there is a set  $\mathcal{P}' \subseteq \mathcal{P}_{M_l}$  of c consecutive paths in  $\mathcal{P}_{M_l}$ , i.e. there is a  $2t'+1 \leq i \leq 4t'-c-1$  such that  $\mathcal{P}'$  contains the subpaths of  $P_i, \ldots, P_{i+c}$  contained in  $\mathcal{P}_{M_l}$ , and  $V(P) \cap V(\mathcal{R}') = \emptyset$  for all  $P \in \mathcal{P}'$ . Then G contains a cylindrical grid of order  $p_1$  as a butterfly minor.

Proof. By construction,  $(\mathcal{Q}_{M_l}, \mathcal{P}')$  form a  $q_1$ -segmentation of order c. Hence, by Lemma 5.19,  $\mathcal{P}' \cup \mathcal{Q}_{M_l}$  contains a  $(p_1, p_1)$ -grid formed by a subset  $\mathcal{Q}_G \subseteq \mathcal{Q}_{M_l}$  and subpaths of  $\mathcal{P}'$ .

Now let g be the function defined in Lemma 5.5 and suppose  $p_1 \geq g(p_2)$ . Then, by Lemma 5.5, this grid contains a  $(p_2, p_2)$ -fence  $\mathcal{F}$  with top and bottom being part of the top and bottom of the grid. Let A' be the top of  $\mathcal{F}$  and B' the bottom. By construction,  $\mathcal{F} \cap \mathcal{R}' = \emptyset$ . See Figure 9 b) for an illustration of this step of the proof.

We now take a linkage  $\mathcal{L}_B$  of order  $\frac{1}{2}p_2$  from B' to B, which exists by Claim 1. Let  $\mathcal{R}_B \subseteq \mathcal{R}$  be the paths in  $\mathcal{R}$  starting in the end vertices of  $\mathcal{L}_B$  and let  $A'' \subseteq A$  be the end vertices of the paths in  $\mathcal{R}_B$ . Again by Claim 1 there is a linkage  $\mathcal{L}_A$  of order  $\frac{1}{4}p_2$  from A'' to A'. Hence,  $\mathcal{L}_B \cup \mathcal{R}_B \cup \mathcal{L}_A$  form a  $\frac{1}{2}$ -integral linkage from a set  $B'' \subseteq B'$  to a set  $A''' \subseteq A'$ . Therefore, there is an integral linkage from B'' to A''' of order  $\frac{1}{8}p_2$ . Moreover, if  $p_2 \geq 8((p-1)(2p-1)+1)$ , then by Lemma 5.21 there is a subfence of  $\mathcal{F}$  of order (p-1)(2p-1)+1 with top A'' and bottom B''. Hence, by Lemma 6.3 there is a cylindrical grid of order p as a butterfly minor, as required.

Hence, we can assume that every path in  $\mathcal{R}'$  hits at least one path in every consecutive block of at least c paths in  $\mathcal{P}_{M_l}$ . As  $|\mathcal{R}'| \geq r_1 \cdot 2t'/3c$  and  $|\mathcal{P}_{M_1}| \geq p_3 = cp_4$  this implies that there is a subset  $\mathcal{R}'' \subseteq \mathcal{R}'$  of order  $r_1$  and a subset  $\mathcal{P}'_l \subseteq \mathcal{P}_{M_l}$  of order  $p_4$  such that every  $R \in \mathcal{R}''$  hits every  $P \in \mathcal{P}'_l$ .

See Figure 9 c) for an illustration of this step of the proof.

As the paths in  $\mathcal{P}_{M_l}$  occur in order on the paths in  $\mathcal{Q}_{M_l}$ , we can adapt the concept of rows and jumps from Definitions 6.6 and 6.7 to pseudo-fences in the obvious way. As  $\mathcal{R}''$  is minimal with respect to  $M_l$  (in fact to the entire pseudo-fence except  $M_r$ ) we can apply Lemma 6.8 to show that there is a subset  $\mathcal{P}' \subseteq \mathcal{P}_{M_l}$  of order  $p_2$  and  $\mathcal{R}''' \subseteq \mathcal{R}''$  of order r' such that in the pseudo-fence generated by  $\mathcal{Q}_{M_l}$  and  $\mathcal{P}'$  the linkage  $\mathcal{R}'''$  goes up row by row. Note that we take  $p_3, r_1$  that guarantee to have  $p_2 \geq t$  and  $r' \geq t$  in Lemma 6.8.

Hence, we can choose a subset  $\mathcal{P}'' \subseteq \mathcal{P}'$  of order  $p' \geq p_2/2$  such that the paths in  $\mathcal{P}''$  occur in  $\mathcal{R}'''$  in order. Hence,  $(\mathcal{P}'', \mathcal{R}''')$  form a (p', r')-web and, by Theorem 5.4 (with d=0, and  $p=\min\{p',r'\}$  implying t=16p'' in Theorem 5.4),  $\mathcal{P}'' \cup \mathcal{R}''$  contains an acyclic grid of order 16p'' and therefore, by Lemma 5.5 a fence  $\mathcal{F}$  of order  $16\hat{p}$  (where we take  $p'', \hat{p}$  as p'', p in Lemma 5.5). Note that the grid is obtained from  $\mathcal{P}''$  and  $\mathcal{R}'''$  by selecting some paths in  $\mathcal{R}'''$  and some subpaths of  $\mathcal{P}''$ . From this we get the fence by internal rerouting.

This implies that we have a linkage  $\mathcal{L}_B$  of order  $16\hat{p}$  from B (i.e., start vertices of  $\mathcal{R}$ ) to the top of the fence (consisting of initial subpaths of paths in  $\mathcal{R}''$  used to create the grid) and a linkage  $\mathcal{L}_A$  of order  $16\hat{p}$  from the bottom of the fence to A (consisting of final subpaths of the paths used to create the grid). Let A' be the end vertices of the paths in  $\mathcal{L}_A$ . By Claim 1, there exists a linkage  $\mathcal{L}_{A'}$  of order  $8\hat{p}$  form A' to the top of  $\mathcal{Q}_{M_r}$ . Let  $\mathcal{Q}'$  be the paths in  $\mathcal{Q}_{M_r}$  starting at the end vertices of the paths in  $\mathcal{L}_{A'}$  and let B'' be the end vertices of these paths. Again by Claim 1, there exists a linkage  $\mathcal{L}_{B''}$  of order  $8\hat{p}$  form B'' to the start points of the paths in  $\mathcal{L}_B$ . Then,  $\mathcal{L}_A \cup \mathcal{L}_{A'} \cup \mathcal{Q}' \cup \mathcal{L}_{B''} \cup \mathcal{L}_B$  contains a quarter-integral linkage  $\mathcal{L}'$  from the bottom of  $\mathcal{F}$  to its top (such that each path in  $\mathcal{L}'$  is internally disjoint from  $\mathcal{F}$ ) and therefore also a fully integral linkage  $\mathcal{L}$  of order  $\frac{|\mathcal{L}'|}{4} = 2\hat{p}$ . Applying Lemma 6.3 with  $\hat{p} \geq p^2$  we get a cylindrical grid of order p as a butterfly minor, as required.

**Remark 6.10.** Note that in the definition of a pseudo-fence  $(\mathcal{P}, \mathcal{Q})$  where  $\mathcal{P} := (P_1, \ldots, P_{2p})$ , we require that there is an edge from the endpoint of every  $P_i$ , for i > 1, to the start point of  $P_{i-1}$ . However, it is easily seen that the proof of the previous lemma only requires that for even i there is an edge from the end vertex of  $P_i$  to the start vertex of  $P_{i-1}$ . This observation will be useful later on.

We now prove the result mentioned above that if we have a fence with a bottomup linkage avoiding the vertical paths then we also have a cylindrical grid of large order as a butterfly minor. The proof is almost identical, but considerably simpler, than the proof of the previous lemma and we therefore refrain from repeating it here.

- **Lemma 6.11.** For every  $p \ge 1$  there is an integer t' such that if G is a digraph containing a(t,t)-fence  $W = (\mathcal{P},\mathcal{Q})$ , for some  $t \ge 3 \cdot t'$ , and a linkage  $\mathcal{R}$  of order t' from bottom of W to top of W such that no path in  $\mathcal{R}$  contains any vertex of  $V(\mathcal{Q})$ , then G contains a cylindrical grid of order p as a butterfly minor.
- 6.4. Constructing a Cylindrical Grid. In this section we complete the proof of our main result, Theorem 3.7, and thus also of Theorem 1.2. The starting point is Theorem 5.2, i.e. we assume that there are linkages  $\mathcal{P}$  of order 6p and  $\mathcal{Q}$  of order q forming a well linked fence. Let  $\mathcal{F} := (\mathcal{P}, \mathcal{Q})$  with  $\mathcal{P} := (P_1, \ldots, P_{6p})$  and  $\mathcal{Q} := (Q_1, \ldots, Q_q)$  be a (3p, q)-fence with top  $A := \{a_1, \ldots, a_q\}$  and bottom  $B := \{b_1, \ldots, b_q\}$ . Recall that we assume that  $\mathcal{P}$  and  $\mathcal{Q}$  are ordered from top to bottom and from left to right, respectively. We divide  $\mathcal{F}$  into three parts  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ , where  $\mathcal{F}_i$  is bounded by  $P_{2(i-1)p+1}, P_{2ip}$ , together with  $Q_1, Q_q$  for i = 1, 2, 3. See Figure 10.

Let  $\mathcal{R} = \{R_1, \dots, R_{\frac{q}{3}}\}$  be such that the linkage  $\mathcal{R}$  joins the last third of the bottom vertices  $(b_{\frac{2q}{3}+1}, \dots, b_q)$  to the first third of the top vertices  $(a_1, \dots, a_{\frac{q}{3}})$ . We define the following notation for the rest of this section.

- Let  $x_i$  be the last vertex of  $R_i$  in  $\mathcal{F}_3$  for  $i = 1, \ldots, \frac{q}{3}$ . Let  $X = \{x_1, \ldots, x_{\frac{q}{3}}\}$ .
- Let  $y_i$  be the first vertex of  $R_i$  in  $\mathcal{F}_1$  for  $i = 1, \ldots, \frac{q}{3}$ . Let  $Y = \{y_1, \ldots, y_{\frac{q}{3}}\}$ .
- Let  $\mathcal{R}'$  be the linkage obtained from  $\mathcal{R}$  by taking a subpath of each path in  $\mathcal{R}$  between one endpoint in X and the other endpoint in Y.
- Let  $a_i'$  be the first vertex of  $Q_i$  in  $\mathcal{F}_2$ , for  $1 \leq i \leq q$ . Let  $A' = \{a_1', \dots, a_q'\}$ .
- Let  $b_i'$  be the last vertex of  $Q_i$  in  $\mathcal{F}_2$ , for  $1 \leq i \leq q$ . Let  $B' = \{b_1', \dots, b_q'\}$ .

• Let  $Q'_i$  be the subpath of  $Q_i$  between  $a'_i$  and  $b'_i$  for i = 1, ..., q. Let  $Q' = \{Q'_1, ..., Q'_q\}$ .

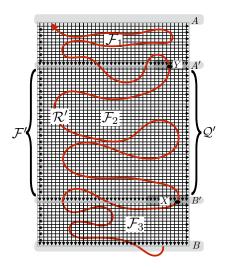


FIGURE 10. Schematic overview of the situation in Section 6.4.

Let  $\mathcal{F}' = \mathcal{Q}' \cup \mathcal{R}' \cup \{P_i : 2p+1 \leq i \leq 4p\}$ . Figure 10 illustrates the notation introduced so far. By our assumption, no vertex in  $\mathcal{F}'$  is in  $\mathcal{F}_1$  or in  $\mathcal{F}_3$ , except for the endpoints of paths in  $\mathcal{Q}' \cup \mathcal{R}'$ . We first show the following lemma.

**Lemma 6.12.** For every t, r', q' there are  $r, q, q^*$  and p, where  $q^*$  only depends on q' and t but not on r', such that if  $\mathcal{R}'$ ,  $\mathcal{Q}'$  and  $\mathcal{P}$  of order r, q and 3p, respectively, are as above then either G contains a cylindrical grid of order t as a butterfly minor or there is a linkage  $\mathcal{R}'' \subseteq \mathcal{R}'$  of order r' and a linkage  $\mathcal{Q}''$  in  $\mathcal{F}'$  of order q' such that the set of endpoints of  $\mathcal{Q}''$  is a subset of the endpoints of  $\mathcal{Q}'$  and such that every  $Q \in \mathcal{Q}''$  hits every  $R \in \mathcal{R}''$ . Furthermore, for every  $Q \in \mathcal{Q}''$  and every  $e \in E(Q) \setminus E(\mathcal{R}'')$  there are at most  $q^*$  paths from  $Q_1$  to  $Q_2$  in  $\mathcal{R}'' \cup \mathcal{Q}'' - e$ , where  $Q = Q_1 e Q_2$ .

In addition, for every  $Q \in \mathcal{Q}''$  there is an edge  $e = e(Q) \in E(Q) \setminus E(\mathcal{R}'')$  splitting Q into two subpaths l(Q), u(Q) with Q = u(Q) e l(Q) and for every  $R \in \mathcal{R}''$  there are edges  $e_1 = e_1(R), e_2 = e_2(R)$  splitting R into three subpaths  $R_1, R_2, R_3$  with  $R = R_1 e_1 R_2 e_2 R_3$  such that  $R_2$  and  $R_3$  both intersect u(Q) for all  $Q \in \mathcal{Q}''$  but not l(Q) and  $R_1$  intersects l(Q) for all  $Q \in \mathcal{Q}''$ .

*Proof.* Let g be the function implicitly defined in Lemma 6.11, i.e. let  $g: \mathbb{N} \to \mathbb{N}$  be such that if t' = g(p) then the condition of Lemma 6.11 is satisfied. Furthermore, let  $f: \mathbb{N} \to \mathbb{N}$  be the function defined in Corollary 5.17. Starting from  $\mathcal{R}', \mathcal{Q}'$  and  $\mathcal{P}$ , in the course of the proof we will take several subsets of these of decreasing order and split paths in other ways. Instead of calculating numbers directly we will state the necessary conditions on the order of these sets as we go along.

In analogy to Definition 6.6 we divide  $\mathcal{F}_2$  into rows  $\mathcal{Z}_0, \ldots, \mathcal{Z}_{k+1}$  ordered from top to bottom, each containing  $2 \cdot q_1$  paths from  $\mathcal{P}$ . Thus we require

$$(5) p \geq (k+2) \cdot 2q_1$$

$$(6) k \geq 2 \cdot q_2.$$

Recall that  $\mathcal{R}'$  consists of subpaths of paths  $R_i \in \mathcal{R}$  starting at  $x_i$  and ending at  $y_i$ . By construction, the initial subpath of  $R_i$  from its beginning to  $x_i$  may contain some vertices of  $\mathcal{F}_2$  but only in row  $\mathcal{Z}_{k+1}$ . Similarly, the final subpath of  $R_i$  from  $y_i$  to the end can contain vertices of  $\mathcal{F}_2$  but only in  $\mathcal{Z}_0$ . Hence,  $R_i \cap \bigcup_{i=1}^k \mathcal{Z}_i \subseteq \mathcal{R}'$ . Let  $\mathcal{Z} := \bigcup_{i=1}^k \mathcal{Z}_i$  and let  $\mathcal{P}_Z \subseteq \mathcal{P}$  be the set of paths in  $\mathcal{P}$  contained in  $\mathcal{Z}$  and  $\mathcal{Q}_Z$  be the maximal subpaths of paths in  $\mathcal{Q}$  which are entirely contained in  $\mathcal{Z}$ . Finally, for every  $1 \le i \le k$  let  $\mathcal{Q}_{\mathcal{Z}_i}$  be the maximal subpaths of the paths in  $\mathcal{Q}$  contained in  $\mathcal{Z}_i$ .

Recall that  $\mathcal{R}'$  consists of subpaths of  $\mathcal{R}$  and that  $\mathcal{R}$  is a bottom up linkage which, by Lemma 6.8, goes up row by row. Hence,  $\mathcal{R}'$  also goes up row by row in terms of  $\mathcal{Z}$  within  $\mathcal{F}_2$ . For the following claim we require

$$(7) q_1 \ge 2 \cdot g(t).$$

Claim 1. If there is a row  $\mathcal{Z}_i$ , for some  $1 \leq i \leq k$ , and sets  $\mathcal{Q}_1 \subseteq \mathcal{Q}$  and  $\mathcal{R}_1 \subseteq \mathcal{R}'$ , both of order  $q_1$ , such that  $V(\mathcal{Q}_1) \cap V(\mathcal{Z}_i) \cap V(\mathcal{R}_1) = \emptyset$  then G contains a cylindrical grid of order t as a butterfly minor.

Proof. Suppose  $Q_1$  and  $\mathcal{R}_1$  exist. Let  $\mathcal{P}_{Z_i}$  be the set of paths from  $\mathcal{P}$  contained in  $\mathcal{Z}_i$  and let  $Q_{Z_i}$  be the subpaths of paths in  $Q_1$  restricted to  $\mathcal{Z}_i$ . Then  $\mathcal{P}_{Z_i} \cup Q_{Z_i}$  form a fence of order  $q_1$  such that  $\mathcal{R}_1$  avoids  $Q_{Z_i}$ . Let  $A_{\mathcal{Z}_i} \subseteq A$  and  $B_{\mathcal{Z}_i} \subseteq B$  be the end points and start points of the paths in  $\mathcal{R}_1$ , respectively. Let  $A'_{\mathcal{Z}_i}$  be the set of start points and  $B'_{\mathcal{Z}_i}$  be the set of end points of  $Q_{\mathcal{Z}_i}$ . Then there is a linkage  $\mathcal{L}_A$  from  $A_{\mathcal{Z}_i}$  to  $A'_{\mathcal{Z}_i}$  of order  $q_1$  and there is a linkage of order  $q_1$  from  $B'_{\mathcal{Z}_i}$  to  $B_{\mathcal{Z}_i}$  such that these two linkages are internally disjoint from  $(\mathcal{P}_{\mathcal{Z}_i}, Q_{\mathcal{Z}_i})$ . Hence,  $\mathcal{L}_A \cup \mathcal{L}_B \cup \mathcal{R}_1$  forms a half-integral linkage from the bottom  $B'_{\mathcal{Z}_i}$  to the top  $A'_{\mathcal{Z}_i}$  of the fence  $(\mathcal{P}_{\mathcal{Z}_i}, Q_{\mathcal{Z}_i})$ . By Lemma 2.10, there also exists an integral linkage  $\mathcal{L} \subseteq \mathcal{L}_A \cup \mathcal{L}_B \cup \mathcal{R}_1$  of order  $q_1 = g(t)$  from  $g'_{\mathcal{Z}_i}$  to  $g'_{\mathcal{Z}_i}$ . Hence, by Lemma 6.11,  $g'_{\mathcal{Z}_i}$  contains a cylindrical grid of order  $g'_{\mathcal{Z}_i}$  as a butterfly minor as required.

By the previous claim, we can now assume that in each row  $\mathcal{Z}_i$  at most  $q_1 \cdot \binom{q}{q_1}$  paths in  $\mathcal{R}'$  avoid at least  $q_1$  paths in  $\mathcal{Q}'$  restricted to  $\mathcal{Z}_i$ . For otherwise, by the pigeon hole principle there would be a row  $\mathcal{Z}_i$  and a set  $\mathcal{R}_1 \subseteq \mathcal{R}'$  of order  $q_1$  such that every  $R \in \mathcal{R}_1$  avoids the same set  $\mathcal{Q}_1 \subseteq \mathcal{Q}_{\mathcal{Z}_i}$  of at least  $q_1$  paths. Hence, by the previous claim, this would imply a cylindrical grid of order t as a butterfly minor. The rest of the proof needs several steps.

Step 1. Let us now consider the rows  $\mathcal{Z}' := \{\mathcal{Z}_{2i} : 1 \leq i \leq q_2\}$ , which is possible as  $k \geq 2q_2$ . It follows that there is a set  $\mathcal{R}_2^* \subseteq \mathcal{R}'$  of order  $r_2^* := r - q_2 \cdot q_1 \cdot \binom{q}{q_1}$  such that in each  $Z \in \mathcal{Z}'$  each path of  $\mathcal{R}_2^*$  hits all but at most  $q_1$  paths in  $\mathcal{Q}'$  restricted to row Z. As we require

(8) 
$$r \ge q_2 \cdot q_1 \cdot \begin{pmatrix} q \\ q_1 \end{pmatrix} + \begin{pmatrix} q \\ q_1 \end{pmatrix}^{q_2} \cdot r_2$$

and therefore  $r_2^* \geq \binom{q}{q_1}^{q_2} \cdot r_2$ , we can find a set  $\mathcal{R}_2 \subseteq \mathcal{R}_2^*$  of order  $r_2$  such that any two paths in  $\mathcal{R}_2$  hit in each row  $Z \in \mathcal{Z}'$  exactly the same paths in  $\mathcal{Q}'$  restricted to Z. We now choose in each row  $\mathcal{Z}_{2i} \in \mathcal{Z}'$ , for  $1 \leq i \leq q_2$ , a path  $Q_{2i} \in \mathcal{Q}'$  such that every  $R \in \mathcal{R}_2$  has a non-empty intersection with  $Q_{2i}$  in  $\mathcal{Z}_{2i}$ . For all  $1 \leq i \leq q_2$  let  $Q_i^2$  be the restriction of  $Q_{2i}$  to row  $\mathcal{Z}_{2i} \in \mathcal{Z}'$  and let  $Q_2 := \{Q_i^2 : 1 \leq i \leq q_2\}$ .

As  $\mathcal{R}'$ , and hence  $\mathcal{R}_2$ , goes up row by row, it follows that all paths in  $\mathcal{R}_2$  go through the paths in  $\mathcal{Q}_2$  strictly in the same order  $Q_{q_2}^2, \ldots, Q_1^2$ . Hence  $(\mathcal{Q}_2, \mathcal{R}_2)$  form a  $q_2$ -split of  $(\mathcal{Q}', \mathcal{R}_2)$  of order  $r_2$ . As we require

$$(9) r_2 \geq f(r_3)$$

(10) 
$$q_2 \geq \binom{r_2}{q_5} (q_5)! (q_5)^2,$$

we can apply Lemma 5.18 to get a  $(q_5, r_3)$ -grid  $\mathcal{H} := (\mathcal{B}', \mathcal{R}_3)$  such that  $\mathcal{R}_3 \subseteq \mathcal{R}_2$ . Let  $\mathcal{B}$  be the set of maximal subpaths of paths in  $\mathcal{B}'$  with both endpoints on paths in  $\mathcal{R}_3$  but which are internally vertex disjoint from  $\mathcal{R}_3$ . By Lemma 5.18, every  $H \in \mathcal{B}$  is a subpath of some  $Q \in \mathcal{Q}_2$  and hence  $V(\mathcal{B}) \subseteq V(\mathcal{Q}_2)$ . Finally, for every  $B \in \mathcal{B}'$  let  $r_u(B) := \min\{l : V(B) \cap V(\mathcal{Z}_l) \neq \emptyset\}$  and let  $r_l(B)$  be the maximal number in this set. That is, the path B is formed by subpaths of paths in  $\mathcal{Q}_2$  in the rows  $\bigcup_{l=r_u(B)}^{r_l(B)} \mathcal{Z}_l$ .

Step 2. Let  $\tilde{\mathcal{Q}}_2 := \{Q \in \mathcal{Q}' : V(Q) \cap V(\mathcal{Q}_2) \neq \emptyset\}$ . Now let  $\mathcal{Q}_3 \subseteq (\mathcal{R}_3 \cup (\mathcal{Q}' \setminus \tilde{\mathcal{Q}}_2)) \cap \mathcal{Z}$  be an  $\mathcal{R}_3$ -minimal linkage of order  $q_3$ , for some value of  $q_3$  to be determined below, such that the start and end vertices of the paths in  $\mathcal{Q}_3$  are from the set of start and end vertices of the maximal subpaths of  $\mathcal{Q}'$  in  $\mathcal{Z}$ . This is possible as we require

$$(11) q' \ge q_3 + q_2.$$

We set  $q^* := q_3$ . Note that, by minimality, for every  $Q \in \mathcal{Q}_3$  and every edge  $e \in E(Q) \setminus E(\mathcal{R}_3)$ , if  $Q = Q_1 e Q_2$  then there are at most  $q^*$  paths from  $Q_1$  to  $Q_2$  in  $(\mathcal{R}_3 \cup \mathcal{Q}_3) - e$ . Note furthermore, that  $V(\mathcal{Q}_3) \cap V(\mathcal{H}) \subseteq V(\mathcal{R}_3)$ .

Let  $\mathcal{R}_3 := (R_1, \ldots, R_{r_3})$  be ordered in the order in which the paths in  $\mathcal{R}_3$  occur on the grid  $\mathcal{H}$ , from left to right, and let  $\mathcal{B}' := (B_1, \ldots, B_{q_5})$  be ordered in the order in which they appear in  $\mathcal{H}$  from top to bottom in the fence  $\mathcal{F}$ . That is, the paths in  $\mathcal{R}_3$  go through the paths in  $\mathcal{B}'$  in the order  $B_{q_5}, \ldots, B_1$ . We require that

$$(12) r_3 \geq t_w \cdot t_c,$$

$$q_5 \geq t_r \cdot t_w,$$

for some values of  $t_c, t_w, t_r$  to be determined below.

For  $1 \leq j \leq t_c$  let  $\mathcal{R}^j := \{R_{(j-1)\cdot t_w}, \dots, R_{j\cdot t_w-1}\}$ . Furthermore, for every  $1 \leq i \leq t_r$  we let  $\mathcal{R}^j_i$  be the set of subpaths of paths  $R \in \mathcal{R}^j$  starting at the first vertex of  $R \cap B_{i\cdot t_w-1}$  and ending at the last vertex of  $R \cap B_{(i-1)\cdot t_w}$ . Finally, let  $\mathcal{S}_{i,j}$  be the subgrid of  $\mathcal{H}$  induced by the paths  $\mathcal{R}^j_i$  and the minimal subpaths of the paths  $B \in \{B_{i\cdot t_w-1}, \dots, B_{(i-1)\cdot t_w}\}$  which contain all of  $V(B) \cap V(\mathcal{R}^j_i)$ . We set  $I := \{1, \dots, t_r\}$  and  $J := \{1, \dots, t_c\}$ . Finally, we set  $r_u(\mathcal{S}_{i,j}) := r_u(B_{(i-1)\cdot t_w})$  and  $r_l(\mathcal{S}_{i,j}) := r_l(B_{i\cdot t_w-1})$ . Hence, all paths B which intersect  $\mathcal{S}_{i,j}$  are contained in  $\bigcup_{r_u(\mathcal{S}_{i,j}) \leq l \leq r_l(\mathcal{S}_{i,j})} \mathcal{Z}_l$ .

For all  $i \in I$  and  $j \in J$  let  $\alpha(S_{i,j})$  be the set of paths  $Q \in \mathcal{Q}_3$  which contain a subpath  $Q^* \subseteq Q$  with first vertex in  $V(\mathcal{R}_3) \cap \bigcup_{l < r_u(\mathcal{S}_{i,j})} V(\mathcal{Z}_l)$ , last vertex in

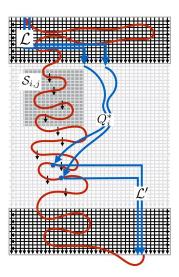


FIGURE 11. Illustration of the construction in Claim claim: R\*:3 of Lemma 6.12.

 $V(\mathcal{R}_3) \cap \bigcup_{l > r_l(\mathcal{S}_{i,j})} V(\mathcal{Z}_l)$  and internally vertex disjoint from  $V(\mathcal{R}_3)$ . We call such a subpath  $Q^*$  a jump of Q over  $\mathcal{S}_{i,j}$ .

Claim 2. If there is a pair  $i \in I, j \in J$  such that  $|\alpha(S_{i,j})| \geq t_2$  then G contains a cylindrical grid of order t as a butterfly minor.

*Proof.* First, we require that

$$(14) t_w \ge t_v' + 2 \cdot t_3.$$

See Figure 6.4 for an illustration of the following construction. Let  $\mathcal{S}_{i,j}^u$  be the subgrid of  $\mathcal{S}_{i,j}$  induced by the paths  $B_{(i-1)t_w}, \ldots, B_{(i-1)t_w+t_3}$ , let  $\mathcal{S}_{i,j}^l$  be the subgrid of  $\mathcal{S}_{i,j}$  induced by the paths  $B_{it_w-t_3-1}, \ldots, B_{it_w-1}$  and let  $\mathcal{S}_{i,j}^m$  be the remaining subgrid, i.e. the subgrid of  $\mathcal{S}_{i,j}$  induced by the paths  $B_{(i-1)t_w+t_3+1}, \ldots, B_{it_w-t_3-2}$ .

For every  $Q \in \alpha(\mathcal{S}_{i,j})$  choose a subpath  $Q^*$  as above, i.e. a path  $Q^*$  that jumps over the subgrid  $\mathcal{S}_{i,j}$ . Recall that  $\mathcal{S}_{i,j}$  is the subgrid of  $\mathcal{H}$  formed by the paths in  $\mathcal{R}_i^j$  and the minimal subpaths of the paths  $B \in \{B_{i \cdot t_w - 1}, \dots, B_{(i-1) \cdot t_w}\}$  which contain all of  $V(B) \cap V(\mathcal{R}_i^j)$ . Each  $B_l$ , for  $(i-1)t_w \leq l \leq B_{it_w - 1}$ , is constructed from subpaths of paths  $Q' \in Q'$  within distinct rows  $\mathcal{Z}_{l'} \in \mathcal{Z}'$ . As  $\mathcal{R}'$  and therefore  $\mathcal{R}_3$  is going up row by row within  $\mathcal{F}$ , and  $\mathcal{Z}'$  only contains every second row of  $\mathcal{Z}$ , there is no path from any vertex in some row  $\mathcal{Z}_{2l} \in \mathcal{Z}'$  to a vertex in row  $\mathcal{Z}_{2(l+1)}$  that does not contain a vertex of some path  $Q' \in Q'$ .

Hence, there is an initial subpath  $\hat{Q}_1$  of  $Q^*$  which has a non-empty intersection with at least  $t_3$  distinct rows in  $\mathcal{S}^u_{i,j}$  and a terminal subpath  $\hat{Q}_2$  of  $Q^*$  which has a non-empty intersection with  $t_3$  rows in  $\mathcal{S}^m_{i,j}$ . Let  $\hat{Q}$  be the remaining subpath of  $Q^*$  such that  $Q^* = \hat{Q}_1 \hat{Q} \hat{Q}_2$ .

As  $|\alpha(S_{i,j})| \geq t_2 \geq t_3$  we can choose a set  $\{Q_1, \ldots, Q_{t_3}\}$  of  $t_3$  distinct paths  $Q \in \alpha(S_{i,j})$  and  $t_3$  distinct rows  $\{Z_i : i \in I\}$  in  $S_{i,j}^u$ , for an index set I of order  $t_3$ , and a set  $T_1 := \{v_i : i \in I\}$  of vertices on these rows which are on distinct paths

 $\hat{Q}_1$ , for  $Q \in \{Q_1, \dots, Q_{t_3}\}$ . Furthermore, we can choose  $t_3$  rows  $\{\mathcal{Z}_i : i \in J\}$  in  $\mathcal{S}_{i,j}^l$ , for some index set J of order  $t_3$ , and a set  $T_2 := \{u_i : i \in J\}$  of vertices on these rows which are on distinct paths  $\hat{Q}_2$  for  $Q \in \{Q_1, \dots, Q_{t_3}\}$ .

Note that the rows  $\mathcal{Z}_i$  for  $i \in I$ , and hence the set  $T_1$ , appear in  $\mathcal{F}$  higher up than the subgrid  $\mathcal{S}^m_{i,j}$  whereas the rows  $\mathcal{Z}_j$ ,  $j \in J$ , and hence the set  $T_2$  appears below  $\mathcal{S}^m_{i,j}$  in  $\mathcal{F}$ . It is easily seen that there is a linkage  $\mathcal{L}$  of order  $t_3$  from the endpoints of  $\mathcal{R}_3$  to the set  $T_1$  which is contained entirely in rows above  $\mathcal{S}^m_{i,j}$ . Similarly, there is a linkage  $\mathcal{L}'$  of order  $t_3$  from  $T_2$  to the start vertices of paths in  $\mathcal{R}_3$  which lies entirely below  $\mathcal{S}^m_{i,j}$ , such that  $\mathcal{L} \cup \mathcal{L}' \cup \mathcal{R}_3$  contain a half-integral linkage of order  $t_3$ , and hence an integral linkage  $\mathcal{L}''$  of order  $\frac{t_3}{2}$  from the bottom of  $\mathcal{S}^m_{i,j}$  to its top. Furthermore, we can choose this linkage  $\mathcal{L}''$  so that it satisfies the condition of Lemma 6.5. As we require that

(15) 
$$\frac{1}{2}t_3 \geq 2((t-1)(2t-1)+1)$$

$$(16) t_v' \geq \frac{3}{2}t_3,$$

we can apply Lemma 6.5 to obtain a cylindrical grid of order t as a butterfly minor, as required. Note that here we use the fact that  $V(\mathcal{Q}_3) \cap V(\mathcal{H}) \subseteq V(\mathcal{R}_3)$ , so that  $\mathcal{L}''$  does not intersect  $\mathcal{S}^m_{i,j}$ .

Thus, we can now assume that  $|\alpha(S_{i,j})| \leq t_2$  for all  $i \in I$  and  $j \in J$ . In particular, this implies that every  $Q \in \mathcal{Q}_3 \setminus \alpha(S_{i,j})$  intersects a path in  $S_{i,j}$ .

As we require that

$$(17) t_r \ge \begin{pmatrix} q_3 \\ t_2 \end{pmatrix}^{t_c} \cdot t_r',$$

there is a subset  $I' \subseteq I$  of order  $t'_r$  such that  $\alpha(\mathcal{S}_{s,j}) = \alpha(\mathcal{S}_{s',j})$  for all  $s, s' \in I$  and  $j \in J$ . Furthermore, as

$$(18) t_c \ge t_c' \binom{q_3}{t_2},$$

there is a subset  $J' \subseteq J$  of order  $t'_c$  such that  $\alpha(S_{i,j}) = \alpha(S_{i,j'})$  for all  $i \in I'$  and  $i, i' \in J$ .

Now let  $Q_4 := Q_3 \setminus \alpha(S_{i,j})$  for some (and hence all)  $i \in I', j \in J'$ . Let  $q_4 := |Q_4|$ . So every  $Q \in Q_4$  has a non-empty intersection with some  $R \in \mathcal{R}_i^j$ .

For every  $Q \in \mathcal{Q}_4$  and all  $i \in I'$  let  $v_i(Q)$  be the last vertex on Q in  $V(\bigcup_{j \in J'} \mathcal{R}_i^j)$ , i.e. the last vertex of Q in the row i, when traversing Q from beginning to end. For simplicity of notation we assume that  $I' := \{1, \ldots, t_r'\}$ .

Claim 3.  $v_{i_1}(Q), \ldots, v_{i_{l'_r}}(Q)$  appear on Q in this order and furthermore, for all  $i_l \in I'$ , the subpath of Q from  $v_i$  to the end of Q has a non-empty intersection with  $V(\mathcal{R}_{i+1}^j)$ , for all  $j \in J'$ .

Proof. Otherwise there would be  $i < i' \in I'$  such that  $v_i(Q)$  appears on Q after  $v_{i'}(Q)$ . But then the subpath of Q from  $v_i(Q)$  to the end of Q would not intersect any  $\mathcal{R}_{i'}^j$ , for all  $j \in J'$ , contradicting the fact that  $Q \notin \alpha(\mathcal{S}_{i',j})$ .

Now, for all  $Q \in \mathcal{Q}_4$ ,  $i \in I'$  with i > 1 and  $j \in J'$  let

$$\beta_Q(\mathcal{S}_{i,j}) := \{ R \in \mathcal{R}^j : \begin{array}{l} Q \text{ intersects } R \cap \mathcal{R}_i^j \text{ in the subpath} \\ \text{of } Q \text{ from } v_{i-1}(Q) \text{ to the end of } Q \end{array} \}.$$

Note that  $\beta_Q(\mathcal{S}_{i,j}) \neq \emptyset$  by Claim 3. As

$$(19) q_4 \ge 2^{t_w} \cdot q_6$$

there is some  $\mathcal{R}_{i,j} \subseteq \mathcal{R}^j$  and some  $\mathcal{Q}_{i,j} \subseteq \mathcal{Q}_4$  such that  $|\mathcal{Q}_{i,j}| = q_6$  and  $\beta_Q(\mathcal{S}_{i,j}) = \mathcal{R}_{i,j}$  for all  $Q \in \mathcal{Q}_{i,j}$ . As we require that

$$(20) t_r' \geq \left( \begin{pmatrix} q_4 \\ q_6 \end{pmatrix} \cdot 2^{t_w} \right)^{t_c'} \cdot t_r''$$

$$(21) t_c' \geq \begin{pmatrix} q_4 \\ q_6 \end{pmatrix} \cdot 2^{t_w} \cdot t_c''$$

there is a set  $I'' \subseteq I'$  with  $|I''| = t''_r$  and a set  $J'' \subseteq J'$  with  $|J'| = t''_c$  such that  $\mathcal{R}_{i,j} = \mathcal{R}_{i',j}$ , for all  $i,i' \in I''$  and  $j \in J''$  and  $\mathcal{Q}_{i,j} = \mathcal{Q}_{i',j'}$  for all  $i,i' \in I$  and  $j,j' \in J$ .

We let  $\mathcal{Q}'' := \mathcal{Q}_{i,j}$  for some (and hence all)  $i \in I''$  and  $j \in J''$  and set  $\mathcal{R}'' := \bigcup_{j \in J''} \mathcal{R}_{i,j}$  for some  $i \in I''$ . We claim that if  $t''_c = r$  and  $q_6 = q'$  then  $\mathcal{R}''$  and  $\mathcal{Q}''$  constitute the second outcome of the lemma. To see this we need to define the edges  $e_1(R), e_2(R)$  and e(Q) for all  $R \in \mathcal{R}''$  and  $Q \in \mathcal{Q}''$ .

We require that  $t_r'' \geq 4$ . Hence we can choose  $i_0 < i_1 < i_2 < i_3 \in I''$ . For  $R \in \mathcal{R}''$  choose  $e_1(R) \in E(R)$  as the first edge on R after the last vertex of R in  $\bigcup_{j \in J''} \mathcal{R}_{i_3}^j$  and  $e_2(R)$  as the first edge on R after  $\bigcup_{j \in J''} \mathcal{R}_{i_2}^j$ . For every  $Q \in \mathcal{Q}''$  let e(Q) be the first edge  $e \in E(Q) \setminus E(\mathcal{R}'')$  on Q after  $v_{i_2}(Q)$  (which exists as  $\mathcal{R}_3$  only goes up). Let u(Q) and l(Q) be the subpaths of Q such that Q = u(Q) e l(Q). Then, if  $R = R_1 e_1(R) R_2 e_2(R) R_3 \in \mathcal{R}''$ , then  $R_1$  does not intersect any u(Q), for  $Q \in \mathcal{Q}''$  but  $R_1$  intersects l(Q) and each of  $R_2$  and  $R_3$  intersect u(Q). Hence, this constitutes the second outcome of the lemma.

We are now ready for the last step of the argument. Before we present the final steps of the proof we need an extra lemma.

The next lemma is a refinement of Lemma 5.13. Recall the definition of a web with linkedness c from Definition 5.11.

**Lemma 6.13.** For all  $c, x, y, p, q \ge 0$  with  $p \ge x$  there is a number q' such that if G contains a (p, q')-web  $W := (\mathcal{P}, \mathcal{Q})$  with linkedness c, then G contains a y-split  $(\mathcal{P}', \mathcal{Q}')$  of  $(\mathcal{P}, \mathcal{Q})$  of order q or an x-segmentation  $(\mathcal{P}', \mathcal{Q}')$  of  $(\mathcal{P}, \mathcal{Q})$  of order q with the following additional property.

Let  $\mathcal{P}' = (P_1, \dots, P_x)$  be ordered in the order in which the paths appear on the paths in  $\mathcal{Q}'$ . Then at most y-1 paths in  $\mathcal{P}'$  are subpaths of the same path in  $\mathcal{P}$ . Finally, for every path  $P \in \mathcal{P}$ , either  $V(P) \subseteq V(\mathcal{P}')$  or  $V(P) \cap V(\mathcal{P}') = \emptyset$ .

*Proof.* We will determine the value of q' as we go along the proof. The first step of the proof is almost identical to the proof of Lemma 5.13. Fix  $Q^* := Q$  for the rest of the proof. All applications of Corollary 5.10 will be with respect to this original linkage  $Q^*$ .

For all  $0 \le i \le xy$  we define values  $q_i$  inductively as follows. We set  $q_{xy} := (c+1) \cdot p$  and  $q_{i-1} := q_i \cdot (q_i + c) \cdot p$ . Let  $(\mathcal{P}, \mathcal{Q})$  be a  $(p, q_0)$ -web. For  $0 \le i \le xy$  we construct numbers  $x_i$  and  $y_i$  and a sequence  $\mathcal{M}_i := (\mathcal{P}^i, \mathcal{Q}^i, \mathcal{S}^i_{seq}, \mathcal{S}^i_{split})$ , where

 $\mathcal{P}^i, \mathcal{Q}^i \subseteq \mathcal{Q}^*, \mathcal{S}^i_{seg}, \mathcal{S}^i_{split}$  are linkages of order  $p, q_i, x_i$  and  $y_i$  respectively and such that  $Q^i$  is an  $x_i$ -segmentation of  $S^i_{seg}$  and  $(S_{split}, Q^i)$  is a  $y_i$ -split of  $(\mathcal{P}, \mathcal{Q})$ . Recall that in particular this means that the paths in  $S_{split}$  are the subpaths of a single path in  $\mathcal{P}$  split at edges  $e \in E(P) \setminus E(\mathcal{Q}^*)$ .

We first set  $\mathcal{M}_0 := (\mathcal{P}^0 := \mathcal{P}, \mathcal{Q}^0 := \mathcal{Q}^*, \mathcal{S}^0_{seg} := \emptyset, \mathcal{S}^0_{split} := \emptyset)$  and define  $x_0 = y_0 := 0$ . Clearly, this satisfies the conditions on  $\mathcal{M}_0$  defined above.

Now suppose that  $\mathcal{M}_i$  has already been defined for some  $i, x_i \geq 0$  and  $y_i > 0$ . If  $S_{split}^i \setminus S_{seg}^i = \emptyset$ , we first choose a set  $Q^+ \subset Q^i$  of order  $\frac{q_i}{p} = q_i \cdot (q_i + c)$  and a path  $P \in \mathcal{P}_i$  such that every path in  $\mathcal{Q}^+$  intersects P before any other path in  $\mathcal{P}_i$ . Note that this is possible by the pigeon hole principle. We set  $\mathcal{S}_{split} = \{P\}$ . Otherwise, if  $S_{split}^i \setminus S_{seg}^i \neq \emptyset$ , we set  $S_{split} := S_{split}^i$  and  $Q^+ := Q^i$ . In both cases, we set  $S_{seq} := S_{seq}^i$ .

Now, let  $P \in \mathcal{S}_{split} \setminus \mathcal{S}_{seg}$ . We apply Corollary 5.10 to  $P, Q^+$  with respect to  $Q^*$ setting  $x = q_{i+1}$  and  $y = (q_{i+1} + c)$ . If we get a  $q_{i+1}$ -segmentation  $\mathcal{Q}_1 \subseteq \mathcal{Q}^+$  of Pwith respect to  $Q^*$  we set

$$\mathcal{P}^{i+1} := \mathcal{P}, \qquad \mathcal{Q}^{i+1} := \mathcal{Q}_1, \qquad \mathcal{S}^{i+1}_{seq} := \mathcal{S}_{seg} \cup \{P\} \quad \text{ and } \quad \mathcal{S}^{i+1}_{split} := \mathcal{S}_{split}.$$

Otherwise, we get a 2-split  $((P_1, P_2), \mathcal{Q}_2)$  of order  $q_{i+1}$  where  $\mathcal{Q}_2 \subseteq \mathcal{Q}^*$ . Then we

$$\begin{split} \mathcal{P}^{i+1} &:= & (\mathcal{P} \setminus \{P\}) \cup \{P_1, P_2\}, \\ \mathcal{Q}^{i+1} &:= & \mathcal{Q}_2, \\ \mathcal{S}^{i+1}_{seg} &:= & \mathcal{S}^i_{seg} \quad \text{and} \\ (\mathcal{S}^{i+1}_{split} &:= & \mathcal{S}^i_{split} \setminus \{P\}) \cup \{P_1, P_2\}. \end{split}$$

It is easily verified that the conditions for  $\mathcal{M}^{i+1} := (\mathcal{P}^{i+1}, \mathcal{Q}^{i+1}, \mathcal{S}^{i+1}_{seg}, \mathcal{S}^{i+1}_{split})$  are maintained. In particular, the linkedness c of  $(\mathcal{P}^{i+1}, \mathcal{Q}^{i+1})$  is preserved as deleting or splitting paths cannot increase forward connectivity (in contrast to the minimality property). This concludes the construction of  $\mathcal{M}_{i+1}$ .

We stop this process as soon as for some i

- (1)  $|\mathcal{S}_{split}^{i}| \geq y$  or (2)  $|\mathcal{S}_{seg}^{i}| \geq x$  and  $|\mathcal{S}_{split}^{i} \setminus \mathcal{S}_{seg}^{i}| = 0$ .

Note that in the construction, after every y steps, either we have found a set  $S_{split}^i$  of size y or  $S_{split}^i \setminus S_{seg}^i$  has become empty at some point. More precisely, we start with a path  $P \in \mathcal{P}$  to put into  $S_{split}$ . Then in every step we try to split a path in  $\mathcal{S}_{split}$ . If this works and we find a splittable edge, we add both subpaths to  $\mathcal{S}_{split}$ . Otherwise, the path will be added to  $S_{seq}$  and then we do not try to split it again later on. Hence, continuing in this way, for the path P we started with, either it will be split y times and we stop the construction or at some point all its subpaths generated by splitting will also be contained in  $S_{seg}$ . We then stop working on Pand choose a new path  $P' \in \mathcal{P}$  on which we repeat the process.

Hence, in the construction above, in each step we either increase  $x_i$  and decrease  $|\mathcal{S}_{split}^i \setminus \mathcal{S}_{seq}^i|$  or we increase  $y_i$ . After at most  $i \leq xy$  steps, either we have constructed a set  $\mathcal{S}_{seg}^{i}$  of order x and  $\mathcal{S}_{split}^{i} \setminus \mathcal{S}_{seg}^{i} = \emptyset$  or a set  $\mathcal{S}_{split}^{i}$  of order y.

If we found a set  $S_{split}^i$  of order y then we can choose any set  $Q' \subseteq Q^i$  of order q and  $(\mathcal{S}_{split}^i, \mathcal{Q}')$  is the first outcome of the lemma.

So suppose that instead we get a set  $S_{seg} := S_{seg}^i$  of order  $y' \geq y$  such that  $S_{seg} \setminus S_{split}^i = \emptyset$ . This implies that we get a segmentation  $(S_{seg}, \mathcal{Q}^i)$  such that for every path  $P \in \mathcal{P}$ , either  $V(P) \cap V(S_{seg}) = \emptyset$  or  $V(P) \subseteq V(S_{seg})$ . Note further that as we are in the second case, no path P was split y or more times. Hence at most y-1 paths in  $S_{seg}$  belong to the same path  $P \in \mathcal{P}$ . Hence,  $(S_{seg}, \mathcal{Q}^i)$  satisfy the conditions for the second outcome of the lemma.

We are now ready to complete the proof of Theorem 6.1. Let us recall the current situation. After Lemma 6.12, we have a linkage  $\mathcal{R}''$  of order r'' and the linkage  $\mathcal{Q}''$  of order q'' as in the statement of the lemma. In particular, for every  $Q \in \mathcal{Q}''$  there is a split edge  $e(Q) \in E(Q) \setminus E(\mathcal{R}'')$  splitting Q into two subpaths l(Q) and u(Q) with Q = u(Q)e(Q)l(Q). Furthermore, for every  $R \in \mathcal{R}''$  there are distinct edges  $e_1(R), e_2(R)$  splitting R into subpaths  $l(R), u_1(R)$  and  $u_2(R)$  such that  $R = l(R)e_1(R)u_1(R)e_2(R)u_2(R)$  and

- (1) the subpath  $u_1(R)e_2(R)u_2(R)$  does not intersect l(Q) for every  $Q \in \mathcal{Q}''$
- (2)  $u_1(R)$  and  $u_2(R)$  both intersect every u(Q) for  $Q \in \mathcal{Q}''$
- (3) l(R) intersects every l(Q) for  $Q \in \mathcal{Q}''$  (but may also intersect u(Q)).

By construction, at most  $q^*$  paths in  $\mathcal{R}''$  can contain a vertex of some u(Q) before they contain a vertex of l(Q) for the same  $Q \in \mathcal{Q}''$ . We can therefore take a subset  $\mathcal{R}^* \subseteq \mathcal{R}''$  of order  $r^* \geq r'' - q^* \cdot q''$  such that, for all  $Q \in \mathcal{Q}''$ , no path in  $\mathcal{R}^*$  intersects l(Q) after it has intersected u(Q).

We now construct a cylindrical grid of order k as a butterfly minor as follows. For every  $R \in \mathcal{R}^*$  let i(R) be the last vertex of R in  $l(\mathcal{Q}^*)$ . Let M'(R) be the subpath of R of minimal length which starts at the successor of i(R) and which intersects every u(Q) for  $Q \in \mathcal{Q}''$ . Let m(R) be the last vertex of M'(R) and let S(R) be the subpath of R of minimal length which starts at the successor of m(R) and which intersects every u(Q) for  $Q \in \mathcal{Q}''$ . Such vertices i(R) and m(R) as well as the subpaths M'(R) and S(R) exist by construction, i.e. by Property 1–3 above. See Figure 12 a) for an illustration of the construction so far.

In the sequel we will impose various conditions on the size of the linkages we construct which will eventually determine the values of p and r in Theorem 6.1. We refrain from calculating these numbers precisely but rather state conditions on the size of the linkages. It is straightforward to verify that these conditions can always be satisfied.

By the pigeon hole principle and as we require

$$(22) r^* \ge r_1(q'')^2$$

there is a set  $\mathcal{R}_1 \subseteq \mathcal{R}^*$  of order  $r_1$  such that m(R) = m(R') and i(R) = i(R') for all  $R, R' \in \mathcal{R}_1$ .

For every  $R \in \mathcal{R}_1$  let  $<_R^S$  be the order on  $\mathcal{Q}''$  where  $Q <_R^S Q'$  if the first vertex S(R) has in common with Q appears on S(R) before the first vertex S(R) has in common with Q'. Again, by the pigeon hole principle and as we require

$$(23) r_1 \ge r_2 \cdot (q'')!$$

we can choose a subset  $\mathcal{R}_2 \subseteq \mathcal{R}_1$  of order  $r_2$  such that  $<_R^S = <_{R'}^S$  for all  $R, R' \in \mathcal{R}_2$ . Let  $<^S := <_R^S$  for some (and hence all)  $R \in \mathcal{R}_2$ .

Let  $Q_1, \ldots, Q_{q''}$  be the paths in  $\mathcal{Q}''$  ordered by  $<^S$  and let  $\mathcal{O} := \{Q_{q''-t}, \ldots, Q_{q''}\}$ , for some suitable constant t to be determined below. See Figure 12 b) for an illustration. In the figure, the black dots are the first vertex R has in common with

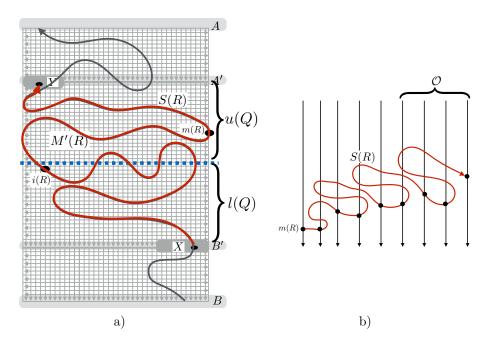


FIGURE 12. Illustration of a) the final part of the proof and b) of the construction of the set  $\mathcal{O}$ .

a path Q. The obvious but important observation used below is that the initial subpath of a path  $R \in \mathcal{R}_2$  taken of minimal length so that it intersects every  $Q \in \mathcal{Q}'' \setminus \mathcal{O}$  does not intersect any path in  $\mathcal{O}$ .

For every  $R \in \mathcal{R}_2$  and every  $v \in V(R) \cap l(Q)$  for some  $Q \in \mathcal{Q}''$  let  $<_v$  be the order on  $\mathcal{Q}''$  where  $Q <_v Q'$  if on the subpath R' of R starting at v to the end of R, the first vertex that R' has in common with u(Q) appears before the first vertex that R' has in common with u(Q'). For any such v let  $Q'_1, \ldots, Q'_{q''}$  be the paths in  $\mathcal{Q}''$  ordered with respect to  $<_v$  and define  $omit(v) := \{Q'_{q''-t}, \ldots, Q'_{q''}\}$ . A vertex  $v \in V(R) \cap l(Q)$  for some  $Q \in \mathcal{Q}''$  is good, if  $Q \not\in omit(v) \cup \mathcal{O}$ .

**Lemma 6.14.** For every  $R \in \mathcal{R}_2$  there is a path  $Q \in \mathcal{Q}''$  such that R contains a good vertex  $v(R) \in V(l(Q))$  and the subpath of R from the beginning of R to v(R) intersects l(Q) for all  $Q \in \mathcal{Q}'' \setminus (omit(v(R)) \cup \mathcal{O})$ .

*Proof.* We give a constructive proof of this lemma. For  $0 \le i \le t$ , we will construct a set  $\mathcal{O}_i \subseteq \mathcal{Q}''$ , a path  $Q_i \in \mathcal{Q}'' \setminus (\mathcal{O}_i \cup \mathcal{O})$  and a vertex  $v_i \in V(R) \cap l(Q_i)$  such that

- (1)  $\mathcal{O}_i \subseteq omit(v_i)$ ,
- (2) the subpath of R from the beginning to  $v_i$  intersects l(Q) for all  $Q \in \mathcal{Q}'' \setminus (\mathcal{O}_i \cup \mathcal{O})$  and
- (3) the subpath of R from  $v_i$  to i(R) intersects l(Q) for all  $Q \in \mathcal{O}_i$ .

Let  $\mathcal{O}_0 := \emptyset$ . Let  $Q_0 \in \mathcal{Q}'' \setminus \mathcal{O}$  be the path containing the last vertex  $v_0$  R has in common with l(Q) for any  $Q \in \mathcal{Q}'' \setminus \mathcal{O}$ . Clearly this satisfies Property 1-3 above.

So suppose  $\mathcal{O}_i, Q_i, v_i$  have already been constructed. If  $Q_i \notin omit(v_i)$  then  $v_i$  is good and we are done. Otherwise, we set  $\mathcal{O}_{i+1} := \mathcal{O}_i \cup \{Q_i\}$ . Let  $v_{i+1}$  be the last vertex that R has in common with l(Q) for any  $Q \in \mathcal{Q}'' \setminus (\mathcal{O} \cup \mathcal{O}_{i+1})$ . Let

 $Q_{i+1} \in \mathcal{Q}'' \setminus (\mathcal{O} \cup \mathcal{O}_{i+1})$  be the path containing  $v_{i+1}$ . By construction,  $Q_{i+1}$  is the last path on R before  $v_i$  such that R intersects  $l(Q_{i+1})$  and such that  $Q_{i+1} \notin \mathcal{O} \cup \mathcal{O}_{i+1}$ . We claim that  $\mathcal{O}_{i+1} \subseteq omit(v_{i+1})$ . By induction hypothesis,  $\mathcal{O}_i \subseteq omit(v_i)$ . Hence, in the order  $<_{v_i}$ , every path in  $\mathcal{O}_i$  was among the last t paths with respect to  $<_{v_i}$ . Now suppose some  $Q \in \mathcal{O}_{i+1}$  is not in  $omit(v_{i+1})$ . This means that Q is no longer among the last t paths hit by R with respect to  $<_{v_{i+1}}$ . The only reason for this to happen is that the subpath of R from  $v_{i+1}$  to  $v_i$  intersects u(Q). But, by Property 3 above, the subpath of R from  $v_i$  to i(R) intersects l(Q). But this violates the construction of  $\mathcal{R}^*$  as in  $\mathcal{R}^*$ , no path  $R' \in \mathcal{R}^*$  intersects any l(Q) after it has intersected u(Q). Hence, the subpath of R between  $v_{i+1}$  and  $v_i$  cannot intersect u(Q) and therefore  $\mathcal{O}_{i+1} \subseteq omit(v_{i+1})$  as required. The other conditions are obviously satisfied as well.

This concludes the construction of  $\mathcal{O}_i, v_i, Q_i$  for all i. By construction, in every step i in which no good vertex is found (i.e.,  $Q_i \notin omit(v_i)$ ), the set  $\mathcal{O}_i$  increases. However, as  $\mathcal{O}_i \subseteq omit(v_i)$  and  $|omit(v_i)| \leq t$  by definition, this process must terminate after at most  $j \leq t$  iterations. Hence,  $v_j$  is a good vertex.

We require

$$(24) r_2 \ge r_3 \cdot q'' \cdot \begin{pmatrix} q'' \\ t \end{pmatrix}$$

so that the previous lemma implies the next corollary.

**Corollary 6.15.** There is a set  $\mathcal{R}_3 \subseteq \mathcal{R}_2$  of order  $r_3$  such that there is a set  $\mathcal{O}_1 \subseteq \mathcal{Q}''$  of order t and a path  $Q \in \mathcal{Q}'' \setminus (\mathcal{O} \cup \mathcal{O}_1)$  such that every  $R \in \mathcal{R}_3$  contains a good vertex  $v(R) \in V(Q)$  satisfying the condition in Lemma 6.14 and  $omit(v(R)) = \mathcal{O}_1$ .

Recall from above the definition of S(R). For every  $R \in \mathcal{R}_3$  let v(R) be the good vertex as defined in the previous corollary. We define M(R) to be the subpath of R from the successor of v(R) on R to the vertex m(R). We define I(R) to be the initial subpath of R from its beginning to v(R). By construction, I(R) intersects I(Q) for all  $Q \in \mathcal{Q}'' \setminus (\mathcal{O} \cup \mathcal{O}_1)$ , where  $\mathcal{O}_1$  is as in the previous corollary. Furthermore, M(R) intersects u(Q) for all  $Q \in \mathcal{Q}''$  and so does S(R). We write  $M(\mathcal{R}_3) := \{M(R) : R \in \mathcal{R}_3\}$ .

We require that  $|\mathcal{Q}'' \setminus \mathcal{O}_1| = q'' - t \ge q_s$  and that  $r_3$  is such that

if in Lemma 6.13 we take p := q'' - t,  $q' := r_3$ , c := q'',  $y := q_s$ ,

(25)  $x := q_1$  and  $q := r_5$  then there is a  $q_s$ -split of order  $r_5$  or a  $q_1$ -segmentation of order  $r_5$ .

Applying Lemma 6.13 to  $(\mathcal{Q}'' \setminus \mathcal{O}_1, M(\mathcal{R}_3))$ , which has linkedness q'', where  $\mathcal{Q}'' \setminus \mathcal{O}_1$  takes on the role of  $\mathcal{P}$  and  $M(\mathcal{R}_3)$  plays the role of  $\mathcal{Q}$ , we either get

- (1) a  $q_s$ -split  $(Q_s, \mathcal{R}_5)$  of order  $r_5$  obtained from a single path  $Q \in \mathcal{Q}'' \setminus \mathcal{O}_1$  which is split into  $q_s$  subpaths, i.e.  $Q = Q_1 \cdot e_1 \cdot Q_2 \dots e_{q_s-1} \cdot Q_{q_s}$ , or
- (2) we obtain a  $q_1$ -segmentation  $(\mathcal{Q}_1, \mathcal{R}_5)$  of order  $r_5$  defined by a subset  $\mathcal{R}_5 \subseteq M(\mathcal{R}_3)$  of order  $r_5$  and a set  $\mathcal{Q}_1$  of order  $q_1$  of subpaths of paths in  $\mathcal{Q}'' \setminus \mathcal{O}_1$  satisfying the extra conditions of Lemma 6.13.

In the first case, we easily get a cylindrical grid of order k as a butterfly minor as follows. Let  $\mathcal{R}_4 = \{R \in \mathcal{R}^* : M(R) \in \mathcal{R}_5\}$  be a linkage of order  $r_4 := r_5$ . Hence,  $\mathcal{R}_4$  is a linkage from the bottom of the original fence  $\mathcal{F}$  to its top and  $\mathcal{R}_4$ 

and  $Q_s$  form a pseudo-fence  $\mathcal{F}_p$ . See Figure 13 for an illustration of the following construction.

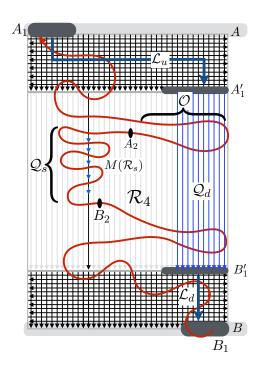


FIGURE 13. Creating a grid from a split and the resulting pseudo-fence.

We can now define paths back from the end vertices of  $\mathcal{R}_4$  to their start vertices as follows. Let  $A_1 \subseteq A$  be the endpoints of the paths in  $\mathcal{R}_4$  and let  $B_1 \subseteq B$  be the start vertices of the paths in  $\mathcal{R}_4$ . Choose a set  $\mathcal{Q}_d \subseteq \mathcal{O}_1$  of order  $r_4$ , which is possible as we require

$$(26) t \geq r_4.$$

Let  $A_1' \subseteq A'$  and  $B_1' \subseteq B'$  be the set of start and end vertices of the paths in  $\mathcal{Q}_d$ . Then there is a linkage  $\mathcal{L}_u$  of order  $r_4$  from  $A_1$  to  $A_1'$  and a linkage  $\mathcal{L}_d$  of order  $r_4$  from  $B_1'$  to  $B_1$ . Hence,  $\mathcal{L}_u \cup \mathcal{Q}_d \cup \mathcal{L}_d$  form a linkage  $\mathcal{L}$  of order  $r_4$  from  $A_1$  to  $B_1$ .

Let  $B_2$  be the start vertices of the paths in  $M(\mathcal{R}_4)$  and  $A_2$  be their end vertices. Every path  $R \in \mathcal{R}_4$  can be split into three disjoint subpaths, D(R), M(R), U(R), where D(R) is the initial component of R-M(R) and U(R) is the subpath following M(R). Then,  $\mathcal{L} \cup \bigcup \{U(R), D(R) : R \in \mathcal{R}_4\}$  forms a half-integral linkage from  $A_2$  to  $B_2$  of order  $r_4$  and hence, by Lemma 2.10, there is an integral linkage  $\mathcal{L}'$  of order  $\frac{1}{2}r_4$  from  $A_2$  to  $B_2$ .

Note that  $M(\mathcal{R}_4)$  and  $\mathcal{L}'$  are vertex disjoint (but  $\mathcal{Q}_s$  may not be disjoint from  $\mathcal{L}'$ ). Let  $f: \mathbb{N} \to \mathbb{N}$  be the function implicitly defined in Lemma 6.9, i.e. for every p if  $t' \geq f(p)$  and  $t \geq 3t' \cdot \binom{t'}{t'_2}$  then we can apply the lemma to get a cylindrical

grid of order p as a butterfly minor. As we require that

$$\frac{1}{2}r_4 \geq f(k).$$

$$(28) q_s \geq 3 \cdot \frac{1}{2} r_4 \cdot \left(\frac{\frac{1}{2} r_4}{\frac{r_4}{4}}\right)$$

we can apply Lemma 6.9 to  $\mathcal{L}'$ ,  $M(\mathcal{R}_4)$  and  $\mathcal{Q}_s$  to obtain a cylindrical grid of order k as a butterfly minor.

Let us now consider the second case above, i.e. where we obtain a  $q_1$ -segmentation  $S_1:=(\mathcal{Q}_1,\mathcal{R}_5)$  of order  $r_5$ . This case and part of the following construction is illustrated in Figure 14. We define  $\mathcal{R}_5'':=\{R\in\mathcal{R}^*:M(R)\in\mathcal{R}_5\}\subseteq\mathcal{R}^*$ . Recall that when obtaining the segmentation  $S_1$ , some paths in  $\mathcal{Q}_1$  can be obtained by splitting a single path in  $\mathcal{Q}''\setminus\mathcal{O}_1$ . However, no path in  $\mathcal{Q}''\setminus\mathcal{O}_1$  is split more than  $q_s-1$  times. We define an equivalence relation  $\sim$  on  $\mathcal{Q}_1$  by letting  $\mathcal{Q}\sim\mathcal{Q}'$  if  $\mathcal{Q}$  and  $\mathcal{Q}'$  are subpaths of the same path in  $\mathcal{Q}''$ . Note that, by Lemma 6.13, either every or no vertex of a path in  $\mathcal{Q}''$  occurs on a path in  $\mathcal{Q}_1$ . As  $M(\mathcal{R}^*)\cap l(\mathcal{Q}'')=\emptyset$ , it follows that in each equivalence class of  $\sim$  there is exactly one path containing a vertex in  $l(\mathcal{Q}'')$ . Let  $\mathcal{Q}_1'^l$  be the set of paths in  $\mathcal{Q}_1$  containing a vertex in  $l(\mathcal{Q}'')$ . Hence,  $(\mathcal{Q}_1'^l,\mathcal{R}_5)$  form a  $q_1'^l$ -segmentation of order  $r_5$  for some  $q_1'^l\geq \frac{q_1}{q_s-1}$ .

**Lemma 6.16.** For every k there are integers  $q'_l, r'_l$  such that if there is a set  $\mathcal{Q}'_1 \subseteq \mathcal{Q}'^l_1$  of order  $q'_l$  and a set  $\mathcal{R}'^l_5 \subseteq S(\mathcal{R}''_5)$  of order  $r'_l$  such that no path in  $\mathcal{R}'^l_5$  intersects any path in  $\mathcal{Q}'_1$ , then G contains a cylindrical grid of order k as a butterfly minor.

*Proof.* Let  $\mathcal{R}_5^l := \{M(R) : S(R) \in \mathcal{R}_5^{\prime l}\}$ . Let  $f : \mathbb{N} \to \mathbb{N}$  be the function defined in Corollary 5.17. We require  $r_l'$  to be big enough so that

$$(29) r_l' \ge f(r_l'').$$

As every path  $R \in \mathcal{R}_5^l$  intersects every path  $Q \in \mathcal{Q}_1'$ , we can apply Corollary 5.17 to  $(\mathcal{R}_5^l, Q)$ , for every path  $Q \in \mathcal{Q}_1'$ , to get a sequence  $\mathcal{S}(Q) := (R_1, \dots, R_{r_i''})$  of paths in  $\mathcal{R}_5^l$  such that Q contains a subpath linking  $R_i$  to  $R_{i+1}$ , for all i, which is internally vertex disjoint from  $\bigcup \{R_1, \dots, R_{r_i''}\}$ . Furthermore, this sequence can be chosen according to Condition 2 of Corollary 5.17, i.e. so that the last vertex of  $\bigcup \{R_1, \dots, R_{r_i''}\}$  on Q is on  $R_{r_i''}$ . As

$$(30) q_l' \ge \begin{pmatrix} r_l' \\ r_l'' \end{pmatrix} \cdot q_1'',$$

there is a subset  $\mathcal{Q}_1''$  of order  $q_1''$  such that  $\mathcal{S}(Q)$  is the same for all  $Q \in \mathcal{Q}_1''$ . Let  $\mathcal{S} := \mathcal{S}(Q)$  be this sequence. Hence, if we require that

$$(31) q_l'' \geq (r_l'')^2$$

we can argue as in the proof of Lemma 5.19 to show that  $\mathcal{Q}_1''$  and  $\mathcal{S}$  contain a  $(r_l'', \frac{q_1''}{r_l''})$ -grid  $(\mathcal{H}, \mathcal{V})$  where the paths in  $\mathcal{V}$  are formed from subpaths of paths in  $\mathcal{Q}_1''$  and subpaths of paths in  $\mathcal{S}$  and  $\mathcal{H} \subseteq \mathcal{S}$ . Furthermore, for every  $H \in \mathcal{H}$ , the last vertex H has in common with  $\mathcal{V}$  is on  $V_{\frac{q_1''}{r''}}$ , where  $(V_1, \ldots, V_{\frac{q_1''}{r''}})$  is the order in

which the paths in  $\mathcal{V}$  appear on the paths in  $\mathcal{H}$ . Let  $\mathcal{C} := \{S(R) : M(R) \in \mathcal{H}\}$ . By construction and the assumption of the lemma,  $\mathcal{C}$  is disjoint from  $(\mathcal{H} \cup \mathcal{V})$  except for the start vertices of the paths in  $\mathcal{C}$ , which are the end vertices of the paths in  $\mathcal{H}$ . Furthermore, every path in  $\mathcal{C}$  intersects any u(Q) for  $Q \in \mathcal{Q}_1''$ . As  $\mathcal{Q}_1''$  only contains

parts of the paths  $Q \in \mathcal{Q}_1$  which contain a vertex in l(Q) and  $\mathcal{C}$  does not intersect any path in  $\mathcal{Q}''_1$  by assumption of the lemma, every path in  $\mathcal{C}$  intersects u(Q) in the upper part of Q, i.e. in a vertex from which in Q there is a path to the subpath of Q in  $\mathcal{Q}''_1$ .

So in particular,  $\mathcal{Q}_1 \cup \mathcal{C}$  contains a half-integral linkage of order  $|\mathcal{C}|$ , and therefore an integral linkage  $\mathcal{L}$  of order  $\frac{|\mathcal{C}|}{2}$ , from the start vertices of the paths in  $\mathcal{C}$  to the top of the grid. Let  $\mathcal{H}' \subseteq \mathcal{H}$  be the paths ending in start vertices of paths in  $\mathcal{L}$  and let  $\mathcal{V}' \subseteq \mathcal{V}$  be the paths starting in end vertices of paths in  $\mathcal{L}$ . Then  $(\mathcal{H}', \mathcal{V}')$  constitute a (t''', t''')-grid  $\mathcal{G}$ , where  $t''' := \frac{|\mathcal{C}|}{2} = \frac{r''_l}{2}$ .

As we require

$$(32) r_1'' \geq 6(2(k-1)(2k-1)+1)$$

we can apply Lemma 6.5 to obtain a cylindrical grid of order k as a butterfly minor.

By the previous lemma, if

(33) 
$$r_5 \geq \binom{q_1^l}{q_1^\prime} \cdot (r_5^\prime + r_l^\prime) \text{ and }$$

$$(34) q_1^l \ge q_1'^l - q_l'$$

we can therefore assume that there is a set  $\mathcal{R}'_5 \subseteq \mathcal{R}''_5$  of order  $r'_5$  and a set  $\mathcal{Q}_1^l \subseteq \mathcal{Q}_1^{\prime l}$  of order  $q_1^l$ , such that for every  $R \in \mathcal{R}'_5$ , the subpath S(R) intersects every path in  $\mathcal{Q}_1^l$ .

We now apply the same construction to  $(\mathcal{Q}_1^l, S(\mathcal{R}_5'))$  (which again has linkedness  $q^*$ ). By Lemma 6.13 we either get

- (1) a  $q_s$ -split  $(Q_s, \mathcal{R}_7)$  of order  $r_7$  obtained from a single path  $Q \in \mathcal{Q}_1^l \setminus \mathcal{O}$  which is split into  $q_s$  subpaths, i.e.  $Q = Q_1 \cdot e_1 \cdot Q_2 \dots e_{q_s-1} \cdot Q_{q_s}$  or
- (2) we obtain a  $q_5$ -segmentation  $(\mathcal{Q}_5, \mathcal{R}_7)$  of order  $r_7$  defined by a subset  $\mathcal{R}_7 \subseteq S(\mathcal{R}_5')$  of order  $r_7$  and a set  $\mathcal{Q}_5$  of order  $q_5$  of subpaths of paths in  $\mathcal{Q}_1^l \setminus \mathcal{O}$  satisfying the extra conditions of Lemma 6.13.

In the first case, we easily get a cylindrical grid of order t as a butterfly minor as before, as follows. Let  $\mathcal{R}_8 \subseteq \{R \in \mathcal{R}^* : S(R) \in \mathcal{R}_7\}$  of order  $r_7$ . Hence,  $\mathcal{R}_8$  is a linkage from the bottom of the original fence  $\mathcal{F}$  to its top and  $\mathcal{R}_8$  and  $\mathcal{Q}_s$  form a pseudo-fence  $\mathcal{F}'_p$ .

We can now define paths back from the end vertices of  $\mathcal{R}_8$  to their start vertices as follows. Let  $A_1 \subseteq A$  be the end vertices of the paths in  $\mathcal{R}_8$  and let  $B_1 \subseteq B$  be the start vertices of the paths in  $\mathcal{R}_8$ . Choose a set  $\mathcal{Q}_d \subseteq \mathcal{O}$  of order  $r_7$ , which is possible as we require

$$(35) t \geq r_7.$$

Let  $A'_1 \subseteq A'$  and  $B'_1 \subseteq B'$  be the set of start and end vertices of the paths in  $\mathcal{Q}_d$ . Then there is a linkage  $\mathcal{L}_u$  of order  $r_7$  from  $A_1$  to  $A'_1$  and a linkage  $\mathcal{L}_d$  of order  $r_7$  from  $B'_1$  to  $B_1$ . Hence,  $\mathcal{L}_u \cup \mathcal{Q}_d \cup \mathcal{L}_d$  form a linkage  $\mathcal{L}$  of order  $r_7$  from  $A_1$  to  $B_1$ . Let  $B_2$  be the start vertices of the paths in  $S(\mathcal{R}_8)$  and  $A_2$  be their end vertices.

Every path  $R \in \mathcal{R}_8$  can be split into three disjoint subpaths, D(R), S(R), U(R), where D(R) is the initial component of R - S(R) and U(R) is the subpath following S(R). Then,  $\mathcal{L} \cup \bigcup \{U(R), D(R) : R \in \mathcal{R}_8\}$  form a half-integral linkage from  $A_2$  to

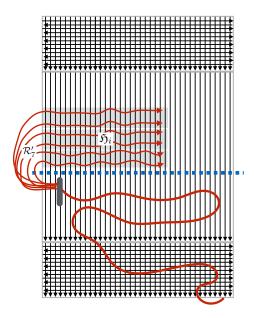


FIGURE 14. Horizontal strips in the segmentation  $S_1$ .

 $B_2$  of order  $r_7$  and hence, by Lemma 2.10, there is an integral linkage  $\mathcal{L}'$  of order  $\frac{1}{2}r_7$  from  $A_2$  to  $B_2$ . Note that  $S(\mathcal{R}_8)$  and  $\mathcal{L}''$  are vertex disjoint. Hence, if

$$(36) \frac{1}{2}r_7 \geq f(k).$$

$$q_s \geq 3 \cdot \frac{1}{2} r_7 \cdot \begin{pmatrix} \frac{1}{2} r_7 \\ \frac{r_7}{4} \end{pmatrix}$$

we can apply Lemma 6.9 to  $\mathcal{L}'' \cup L(\mathcal{R}_s)$  to obtain a cylindrical grid of order k as a butterfly minor.

So we can now assume that instead we get a  $q_5$ -segmentation  $S_2 := (\mathcal{Q}_5, \mathcal{R}_7)$  of order  $r_7$ . Let  $\mathcal{R}'_7 \subseteq \mathcal{R}_5$  be the paths R in  $\mathcal{R}_5$  which have a continuation in  $\mathcal{R}_7$ , i.e.  $\mathcal{R}'_7 := \{M(R) : R \in \mathcal{R}''_5 \text{ and } S(R) \in \mathcal{R}_7\}$ . Let  $S'_1 := (\mathcal{Q}_1^l, \mathcal{R}'_7)$  be the restriction of  $S_1$  to these paths  $\mathcal{R}'_7$ . Note that every path R in  $\mathcal{R}'_7$  ends in a vertex v such that the successor of v on R' is the start vertex of a path in  $\mathcal{R}_7$ , where  $R' \in \mathcal{R}^*$  is the path such that M(R') = R.

**Lemma 6.17.**  $(\mathcal{Q}_5, \mathcal{R}'_7 \cup \mathcal{R}_7)$  contains a cylindrical grid of order k as a butterfly minor, or a pseudo-fence  $(\mathcal{Q}'_6, \mathcal{R}_8)$  for some  $\mathcal{R}_8 \subseteq \mathcal{R}'_7 \cup \mathcal{R}_7$  of order  $r_8$  and some  $\mathcal{Q}'_6 \subseteq \mathcal{Q}_5$  of order  $q_6$ .

*Proof.* We first consider the pair  $(\mathcal{Q}_5, \mathcal{R}'_7)$ . Note that the paths in  $\mathcal{Q}_5$  are obtained from paths in  $\mathcal{Q}_1^l$  but possibly by splitting paths in  $\mathcal{Q}_1^l$ . Recall that  $(\mathcal{Q}_1^l, \mathcal{R}'_7)$  is a segmentation. It follows that  $(\mathcal{Q}_5, \mathcal{R}'_7)$  is still a segmentation but it is not necessarily true that every path in  $\mathcal{R}'_7$  hits every path in  $\mathcal{Q}_5$ . However, to obtain  $\mathcal{Q}_5$  from  $\mathcal{Q}_1^l$ , a path in  $\mathcal{Q}_1^l$  can be split at most  $q_s - 1$  times. Hence, if

$$(38) r_7 \ge (h_2 \cdot h) \cdot q_s^q$$

then there is a set  $\mathcal{Q}_6 \subseteq \mathcal{Q}_5$  of order  $q_6 \geq \frac{q_5}{q_s}$  and a set  $\tilde{\mathcal{R}}_7' \subseteq \mathcal{R}_7'$  of order  $\tilde{r_7} \geq h_2 \cdot h$  such that  $\mathcal{S}_1'' := (\mathcal{Q}_6, \tilde{\mathcal{R}}_7')$  is a segmentation and every  $R \in \tilde{\mathcal{R}}_7'$  intersects every  $Q \in \mathcal{Q}_6$ .

Let  $\tilde{\mathcal{R}}'_7 := (R_1, \dots, R_{\tilde{r}_7})$  be ordered in the order in which they appear on the paths in  $\mathcal{Q}_6$  from top to bottom. We split  $\mathcal{S}''_1$  into horizontal strips as follows. For all  $1 \le i \le h$  let  $\mathfrak{H}_i := (\mathcal{Q}_6, \mathcal{H}_i)$  where  $\mathcal{H}_i := \{R_{(i-1)\cdot h_2}, \dots, R_{i\cdot h_2-1}\}$ .

As  $r_7 \geq h_2 \cdot h$ , every  $\mathfrak{H}_i$  is itself a segmentation using  $h_2$  paths of  $\tilde{\mathcal{R}}'_7$  and the corresponding subpaths of  $\mathcal{Q}_6$ . See Figure 14 for an illustration.

For every  $\mathfrak{H}_i$  let  $\mathcal{H}_i' \subseteq \mathcal{R}_7$  be the paths in  $\mathcal{R}_7$  whose start vertex is the end vertex of a path in  $\mathcal{H}_i$ . We define  $\mathfrak{H}_i' := (\mathcal{Q}_6, \mathcal{H}_i')$ . Again, this is a segmentation. Furthermore, every horizontal path  $R \in \mathcal{H}_i$  can be continued by a path in  $\mathcal{H}_i'$ .

By construction of  $\mathcal{Q}''$ , for every  $1 \leq i \leq h$  and for every  $Q \in \mathcal{Q}_6$ , at most  $q^*$  paths in  $\mathcal{H}'_i$  can contain a vertex  $v \in V(Q)$  such that v appears on Q after the last vertex Q has in common with any path in  $\mathcal{H}_i$ .

Hence, we can take a subset  $\mathcal{H}_i'' \subseteq \mathcal{H}_i'$  of order  $h_3 := h_2 - q_6 \cdot q^*$  such that no path in  $\mathcal{H}_i''$  contains a vertex  $v \in V(Q)$ , for any  $Q \in \mathcal{Q}_6$ , which appears after the last vertex Q shares with  $\mathcal{H}_i$ . We now claim that the horizontal strips must intersect nicely as illustrated in Figure 15 b).

Claim 1. Either  $(\mathcal{Q}_5, \mathcal{R}'_7 \cup \mathcal{R}_7)$  contain a cylindrical grid of order k as a butterfly minor or for every  $1 \leq i \leq h$  there is a subset  $\hat{\mathcal{H}}_i \subseteq \mathcal{H}''_i$  of order  $h_4$  and a subset  $\mathcal{Q}'_5 \subseteq \mathcal{Q}_6$  of order  $q'_5$  such that every  $R \in \hat{\mathcal{H}}_i$  intersects every  $Q \in \mathcal{Q}'_5$  in the subpath of Q between the top path  $R_{(i-1)\cdot h_2}$  and the lowest path  $R_{i\cdot h_2-1}$  in  $\mathcal{H}_i$ .

Proof. For every  $R \in \mathcal{H}_i''$  let  $\pi_i(R)$  be the set of paths  $Q \in \mathcal{Q}_6$  such that R intersects Q only in vertices which occur on Q before the first vertex Q has in common with  $\mathcal{H}_i$ . Now suppose there are at least  $\tilde{h} \cdot \binom{q_6}{q_7}$  paths  $R \in \mathcal{H}_i''$  with  $|\pi_i(R)| \geq q_7$ . By the pigeon hole principle, there is a set  $\tilde{\mathcal{H}}_i' \subseteq \mathcal{H}_i''$  of order  $\tilde{h}$  such that  $\pi_i(R) = \pi_i(R')$  for all  $R \in \tilde{\mathcal{H}}_i'$  and  $|\pi_i(R)| \geq q_7$ . We claim that in this case we obtain a cylindrical grid of order k. The construction is illustrated in Figure 15 a).

Let  $\mathcal{V} := \pi_i(R)$  for some (and hence all)  $R \in \tilde{\mathcal{H}}'_i$ . Let  $\tilde{\mathcal{H}}_i \subseteq \mathcal{H}_i$  be the set of paths in  $\mathcal{H}_i$  ending in a start vertex of a path in  $\tilde{\mathcal{H}}'_i$ . Finally, let  $\tilde{\mathcal{Q}}$  be the set of minimal subpaths of paths  $Q \in \mathcal{V}$  containing every vertex Q has in common with  $\tilde{\mathcal{H}}_i$ . By construction, every  $Q \in \tilde{\mathcal{Q}}$  is disjoint from every  $R \in \tilde{\mathcal{H}}'_i$ . For every  $Q \in \mathcal{V}$  we can therefore take the subpath i(Q) from the beginning of Q to the predecessor of the first vertex Q has in common with  $\tilde{\mathcal{H}}_i$ . Let  $\tilde{\mathcal{Q}}' := \{i(Q) : Q \in \mathcal{V}\}$ . Then  $\tilde{\mathcal{Q}}$  and  $\tilde{\mathcal{H}}_i$  form a segmentation. By Lemma 5.19,  $(\tilde{\mathcal{Q}}, \tilde{\mathcal{H}}_i)$  contains an acyclic grid  $\mathcal{G} := (\mathcal{V}_{\mathcal{G}}, \mathcal{H}_{\mathcal{G}})$  such that the paths  $\mathcal{H}_{\mathcal{G}}$  are obtained from subpaths of  $\tilde{\mathcal{Q}}$  and  $\tilde{\mathcal{H}}_i$  preserving the end vertices of the paths in  $\tilde{\mathcal{H}}_i$ . Let  $\mathcal{V}_{\mathcal{G}} := (V_1, \dots, V_{q_7})$  be ordered in the order in which they appear on the paths in  $\mathcal{H}_{\mathcal{G}}$  and let  $\mathcal{H}_{\mathcal{G}} := (H_1, \dots, H_{h_6})$  ordered in the order in which they appear on  $\mathcal{V}_{\mathcal{G}}$ . See Figure 15 a).

Now let  $\mathcal{U}$  be the subgrid of  $\mathcal{G}$  formed by  $(\mathcal{V}_{\mathcal{U}}, \mathcal{H}_{\mathcal{U}})$  where  $\mathcal{V}_{\mathcal{U}}$  is the set of minimal subpaths of  $\mathcal{V}_{\mathcal{G}}$  to include every vertex of  $H_{\frac{1}{3}h_6}, \ldots, H_{\frac{2}{3}h_6}$  and  $\mathcal{H}_{\mathcal{U}}$  are the minimal subpaths of  $H_{\frac{1}{3}h_6}, \ldots, H_{\frac{2}{3}h_6}$  including every vertex they have in common with  $\mathcal{V}_{\mathcal{U}}$ . Then, if

(39) 
$$q_6 \ge h_6$$
,

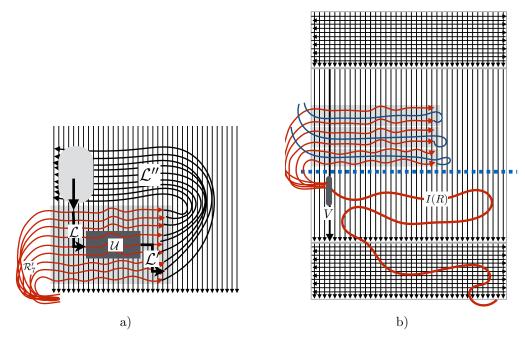


FIGURE 15. a) Creating a cylindrical grid from two disjoint horizontal strips and b) two segmentations  $S_1$  and  $S_2$  forming a pseudofence.

in  $\mathcal{G}$  there is a linkage  $\mathcal{L}$  of order  $\frac{1}{3}h_6$  from the start vertices of  $V_1, \ldots, V_{\frac{1}{3}h_6}$  to the bottom of  $\mathcal{U}$  and a linkage  $\mathcal{L}'$   $\frac{1}{3}h_6$  from the top of  $\mathcal{U}$  to the end vertices of  $H_{\frac{2}{3}h_6}, \ldots, H_{h_6}$ . Furthermore, in  $\tilde{\mathcal{H}}'_i \cup \tilde{\mathcal{Q}}'$  there is a linkage  $\mathcal{L}''$  from the end vertices of  $H_{\frac{2}{3}h_6}, \ldots, H_{h_6}$  to the start vertices of  $V_1, \ldots, V_{\frac{1}{3}h_6}$ . As  $\mathcal{L}, \mathcal{L}', \mathcal{L}''$  are pairwise disjoint except for the end vertices they have in common, they form a linkage  $\mathcal{L}'''$  from the top of  $\mathcal{U}$  to the bottom which is disjoint from  $\mathcal{U}$ . As we require that

$$(40) h_6 \ge 6((k-1)(2k-1)+1),$$

by Lemma 6.5,  $\mathcal{U} \cup \mathcal{L}'''$  form a cylindrical grid of order k as a butterfly minor.

By the previous claim, in every  $\mathcal{H}_i$  there is a path  $R \in \mathcal{H}_i$  and a path  $R' \in \mathcal{H}'_i$  such that the endpoint of R is the start vertex of R' and a set  $\gamma(R') \subseteq \mathcal{Q}_6$  of order  $q_7$  such that R' hits every path  $Q \in \mathcal{Q}'_i$  within  $\mathcal{H}_i$ . For all  $1 \leq i \leq h$  we choose such a path  $R_i$  and  $R'_i$ . Note that  $\mathcal{S}_2$  is a segmentation of  $\mathcal{Q}''$ , hence no path  $R'_i$  can intersect any  $Q \in \mathcal{Q}'_i$  at a vertex v which occurs on Q before a vertex  $v \in V(Q) \cap V(R'_i)$  for some  $v \in V(Q) \cap V(R'_i)$  for some  $v \in V(Q) \cap V(R'_i)$  for some  $v \in V(Q) \cap V(R'_i)$ 

As

$$(41) h \ge \begin{pmatrix} q'' \\ q' \end{pmatrix} \cdot r_8,$$

we can choose a set  $\mathcal{R}_8$  of paths  $R_i$  and  $R_i'$  such that  $\gamma(R_i') = \gamma(R_j')$  for all  $R_i', R_j' \in \mathcal{R}_8$ . Let  $\mathcal{Q}_6' := \gamma(R_i')$  for some (and hence all)  $R_i' \in \mathcal{R}_8$ . Hence,  $\mathcal{R}_8$  and  $\mathcal{Q}_6'$  form a pseudo-fence as required.

The current situation is illustrated in Figure 15 b). Let  $V \in \mathcal{Q}''$  be the path such that every  $R \in \mathcal{R}_3$  contains a good vertex v(R) on V. We define  $\mathcal{Q}_7 := \mathcal{Q}'_6 \cup \{V\}$ . Now,  $\mathcal{R}_8$  and  $\mathcal{Q}_7$  are no longer a pseudo-fence, but they are a pseudo-fence in restriction to  $\mathcal{Q}'_6$  and furthermore, every path  $R_i$  and  $R'_i$  in  $\mathcal{R}_8$  also intersects V.

Recall that  $\mathcal{R}_8$  is a set of paths  $R'_i \in \mathcal{R}'_7$  and  $R_i \in \mathcal{R}_7$ . Let  $(R_1, \ldots, R_{r_8})$  be an ordering of the paths  $R_i \in \mathcal{R}_8 \cap \mathcal{R}_7$  in the order in which they occur on the paths in  $\mathcal{Q}'_6$ . We require

$$(42) r_8 \ge (h_9')^2,$$

for some value of  $h'_9$  to be determined below. As in the proof of the previous lemma we define horizontal strips  $\mathcal{H}_i := \{R_{(i-1)h'_9} \cup R'_{(i-1)h'_9}, \dots, R_{ih'_9-1} \cup R'_{ih'_9-1}\}$  and  $1 \leq i \leq h'_9$ , and let  $\mathcal{V}_i := \{m_i(Q) : Q \in \mathcal{Q}'_6\}$  where  $m_i(Q)$  is the minimal subpath of Q containing every vertex of  $V(\mathcal{H}_i)$ . Recall from above that every path  $R \in \mathcal{R}^*$  is split into three distinct parts, I(R), M(R) and S(R). The subpaths M(R) and S(R) are part of the construction of  $\mathcal{R}_8$ , where the M(R) play the role of the  $R_i$  above and the S(R) play the role of  $R'_i$ . We will now use the initial subpaths, I(R). Recall further that the endpoint of each I(R) for  $R \in \mathcal{R}_3$  is on the path V.

Claim 1. There is a  $1 \leq i \leq h'_9$  such that  $I := \{I(R) : M(R) \in \mathcal{H}_i\}$  is disjoint from  $\mathcal{H}_i \cup \mathcal{V}_i$ .

Proof. Towards a contradiction, suppose the claim was false. For every  $1 \leq i \leq h'_g$  choose a path  $M(R_i) \in \mathcal{H}_i$  such that  $I(R_i)$  intersects  $\mathcal{H}_i \cup \mathcal{V}_i$ . As  $I(R_i)$  ends in V, in fact ends in l(V), and furthermore, every path  $R \in \mathcal{H}_i$  intersects u(V), this implies that there is a path  $P_i$  from u(V) to l(V) in  $\mathcal{H}_i \cup \mathcal{V}_i \cup I(R_i)$ . Note that for  $i \neq j$  the paths  $P_i$  and  $P_j$  may not be disjoint, as, e.g.,  $I(R_i)$  may intersect  $\mathcal{H}_i$  and  $\mathcal{H}_j$ .

However, the set  $\{P_i: 1 \leq i \leq h_9'\}$  forms a half-integral linkage from u(V) to l(V) and therefore, by Lemma 2.10, there also is an integral linkage of order  $\frac{1}{2}h_9'$ . See Figure 16 for an illustration.

As

$$(43) h_0' > 2q^* + 2$$

this contradicts the fact that in  $\mathcal{Q}''$  at most  $q^*$  paths can go from some u(Q) to l(Q).

Let  $i \leq h_9'$  be such that  $I := \{I(R) : M(R) \in \mathcal{H}_i\}$  is disjoint from  $\mathcal{H}_i \cup \mathcal{V}_i$ . Let  $\mathcal{H} := \{M(R) : M(R) \in \mathcal{H}_i\}$ , i.e.  $\mathcal{H}$  is the same as  $\mathcal{H}_i$  with the paths S(R) removed, which are no longer needed. We require that

$$(44) h_9' \geq f(h_{10}) \cdot q_6$$

$$q_6 \geq \binom{f(3s_1)}{3s_1},$$

where f is the function defined in Corollary 5.10. By Lemma 5.19,  $(\mathcal{H} \cup \mathcal{V}_i)$  contains a grid  $\mathcal{U} := (\hat{\mathcal{H}}, \mathcal{V}'_i)$  which can be chosen so that the start vertices of the paths M(R) are preserved. Let  $\hat{\mathcal{H}} := (H_1, \ldots, H_{h_{10}})$  be ordered in the order in which they occur on the paths in  $\mathcal{V}'_i$  and let  $\mathcal{V}'_i := (V_1, \ldots, V_{s_1})$  be ordered in the order in which the paths occur on the paths in  $\mathcal{H}$ . We now take the subgrid  $\mathcal{U}'$  induced by  $(H_{\frac{1}{3}h_{10}}, \ldots, H_{\frac{2}{3}h_{10}-1})$  and  $(V_{\frac{1}{3}s_1}, \ldots, V_{\frac{2}{3}s_1-1})$ . More precisely, for every  $H \in$ 

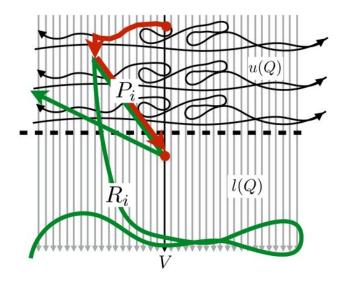


FIGURE 16. Illustration for the last claim.

 $\{H_{\frac{1}{3}h_{10}},\ldots,H_{\frac{2}{3}h_{10}-1}\}\$ let  $\rho(H)$  be the minimal subpath of H containing  $V(H)\cap V(\bigcup\{V_{\frac{1}{3}s_1},\ldots,V_{\frac{2}{3}s_1-1}\})$  and for all  $V'\in\{V_{\frac{1}{3}s_1},\ldots,V_{\frac{2}{3}s_1-1}\}\$ let  $\rho(V')$  be the minimal subpath of V' containing all of  $V(V')\cap V(\bigcup\{H_{\frac{1}{3}h_{10}},\ldots,H_{\frac{2}{3}h_{10}-1}\})$ . Then  $\mathcal{U}'$  is the grid induced by  $\{\rho(\{H_{\frac{1}{3}h_{10}}),\ldots,\rho(H_{\frac{2}{3}h_{10}-1})\}\$ and  $\{\rho(V_{\frac{1}{3}s_1}),\ldots,\rho(V_{\frac{2}{3}s_1-1})\}$ . By Lemma 5.5,  $\mathcal{U}'$  contains a fence  $\mathcal{F}'$  whose top T and bottom B are subsets of the top and bottom of  $\mathcal{U}'$ . We can now construct a linkage from B to T as follows. Let  $\mathcal{I}'$  be the set of paths I(R) with end vertex in T. By construction, every I(R) intersects every I(Q) for  $Q\in\{V_{\frac{2}{3}s_1},\ldots,V_{s_1}\}$ . Hence,  $\{V_{\frac{2}{3}s_1},\ldots,V_{s_1}\}\cup\{H_{\frac{2}{3}s_1},\ldots,H_{s_1}\}\cup\mathcal{I}'$  contains a half-integral linkage from B to T, and therefore by Lemma 2.10, also an integral linkage  $\mathcal{L}$  from B to T of order  $\frac{1}{6}s_1$ .

Finally, as

$$(46) h_{10}, s_1 \ge 6(3(k-1)(2k-1)+1)$$

Lemma 6.5 implies that  $\mathcal{U}'$  together with  $\mathcal{L}$  contains a cylindrical grid of order k as a butterfly minor. This completes the proof of Theorem 6.1 and hence the proof of Theorem 3.7 and therefore Theorem 1.2.

## 7. Conclusion

In this paper we proved the directed grid conjecture by Reed and Johnson, Robertson, Seymour and Thomas. We view this result as a first but significant step towards a more general structure theory for directed graphs based on directed tree width, similar to the grid theorem [38] for undirected graphs being the basis of more general structure theorems [40].

Our proof indeed yields the following algorithmic result, which is perhaps of independent interest.

There is a function  $f: \mathbb{N} \to \mathbb{N}$  such that given any directed graph and any fixed constant k, in polynomial time, we can obtain either (1) a cylindrical grid of order k as a butterfly minor, or

## (2) a directed tree decomposition of width at most f(k).

We also believe that this theorem will prove to be very useful for further applications of directed tree width, for instance to Erdős-Pósa type results for directed graphs. Furthermore, it is likely that the duality of directed tree width and directed grids will make it possible to develop algorithm design techniques such as bidimensionality theory or the irrelevant vertex technique for directed graphs. We are particularly optimistic that this approach will lead to some apparently new (and most likely best possible) results for the directed disjoint paths problem. We leave this to our future project.

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NATIONAL INSTITUTE OF INFORMATICS, 2-1-2 HITOTSUBASHI, CHIYODA-KU, TOKYO, JAPAN *E-mail address*: k\_keniti@nii.ac.jp

Chair for Logic and Semantics, Technical University Berlin, Sekr TEL 7-3, Ernst-Reuter Platz 7, 10587 Berlin, Germany

 $E\text{-}mail\ address: \verb| stephan.kreutzer@tu-berlin.de|\\$