AINTA CUTE

ethereum

Title of paper

- Michal Kotrbčík, Martin Škoviera – “Simple Greedy 2-Approximation Algorithm for the Maximum Genus of a Graph”

- Slide by Changki Yun

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PART 1

"So… what is genus?"

So, what is “genus”?

- “Genus” is an integer indicates the number of holes on a surface.

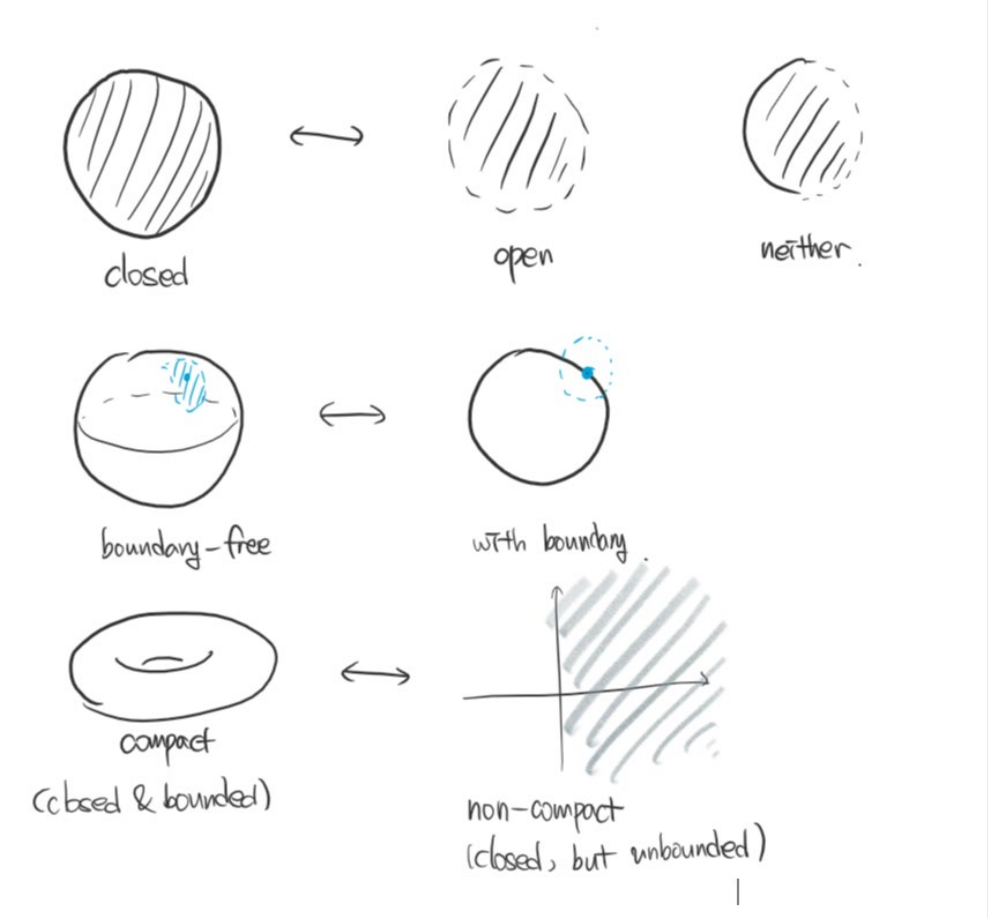
- Spheres have genus 0, Donuts have genus 1, and Pretzels have genus 3.

OK, then a “surface” is…

- Surface is a *topological space*, and each point has a *neighbor homeomorphic to* an open disk.
- In this case, we only deal with *compact* and *boundaryless* surfaces.

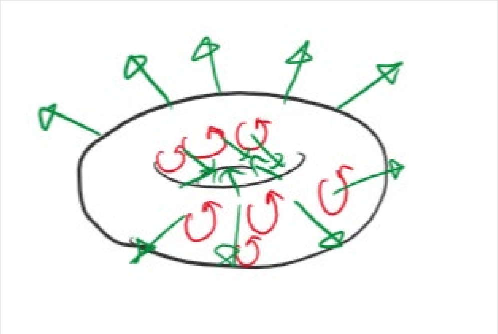
- To explain the terms *written in italic*, we need a whole-semester course on topology.
- So we substitute these stuffs (and other things involves knowledge on topology) with some graphic illustration, and we “believe” that we are dealing with just “well-behaving” surfaces.

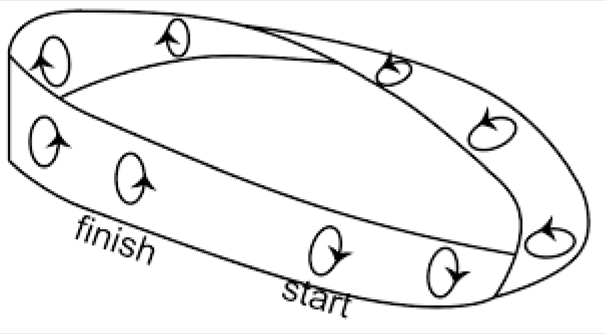
A few examples of surfaces



Orientability of surface

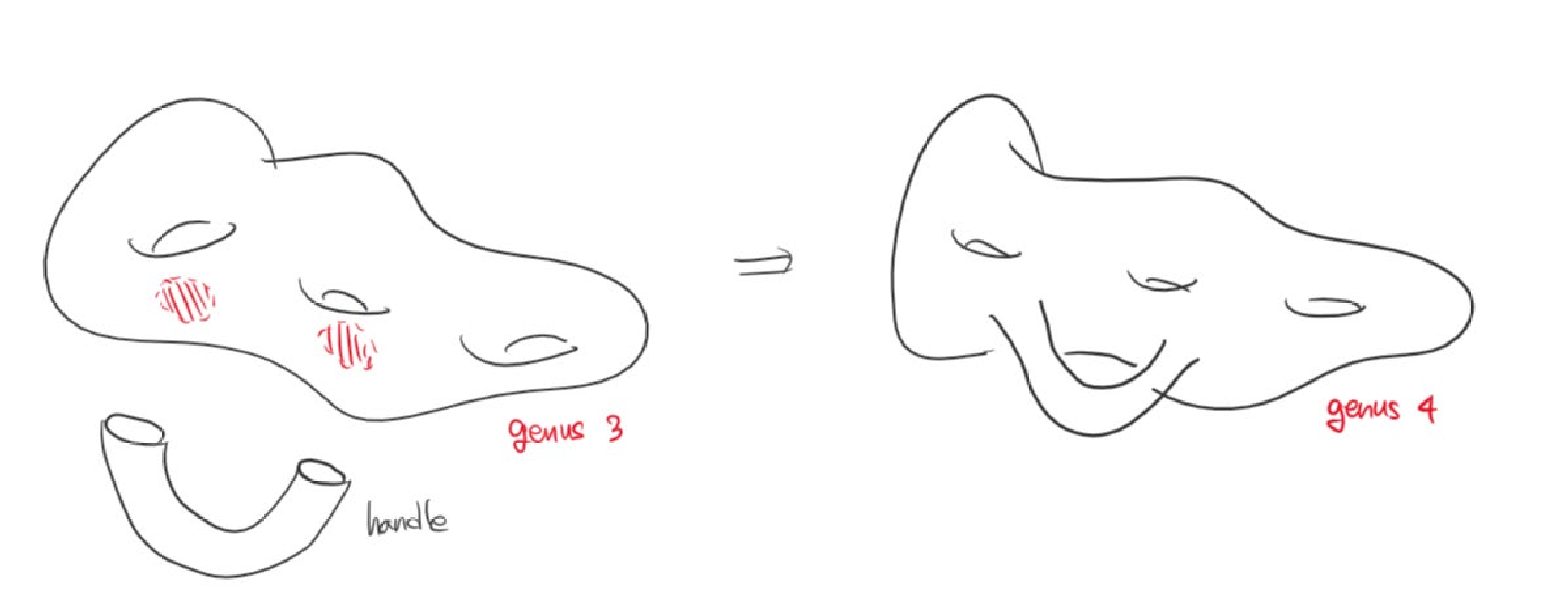
- We got something to talk about on orientability.
- A *smooth surface S* is *orientable* if there is a *smooth normal field* …
- Simply, just think if we consistently "raise our thumb" consistently. Illustration goes.
- Today's main ingredients are orientable surfaces.





Gluing surfaces

- We may artificially “generate” a new surface by gluing a “handle” on it, raising a genus by one.



Classifying surfaces

- Classification Theorem on 2-manifolds:
  - Any *compact, boundaryless, orientable* genus g-surfaces are *homeomorphic* to S_g.
  - S_g : A surface acquired by attaching g handles on a surface.

- Thus the term "genus g surface" is unified into a representative S_g, at least in this presentation.

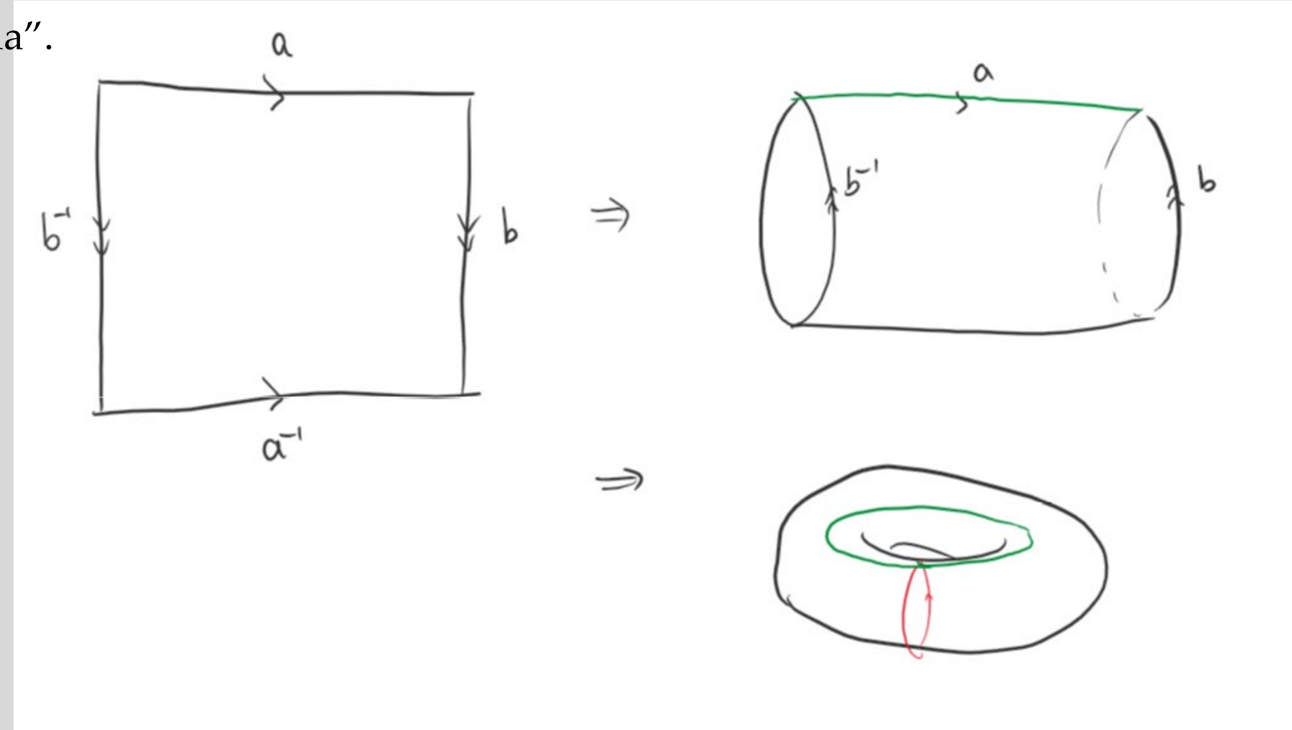
- Wait, so is the "genus" merely means the number of *holes* on it…?

In-place definition of genus

- Genus of a surface is the maximum number of *simple closed curve* does not disconnect the surface.
  - Cf) Jordan curve theorem: A single simple closed curve separates $\mathbb{R}^2$; thus $\mathbb{R}^2$ has genus 0.

Alternative way to generate a surface: plane schema

- ‘Folding’ this “paper” $aba^{-1}b^{-1}$ by matching x and x^{-1}, we obtain a torus.
- Any surfaces could be built from this “plane schema”.



- A standard script to generate a genus-g surface is
  - Repeating $(aba^{-1}b^{-1})$ for g times:

Alternative way to generate a surface: plane schema

- There are some complicated examples:

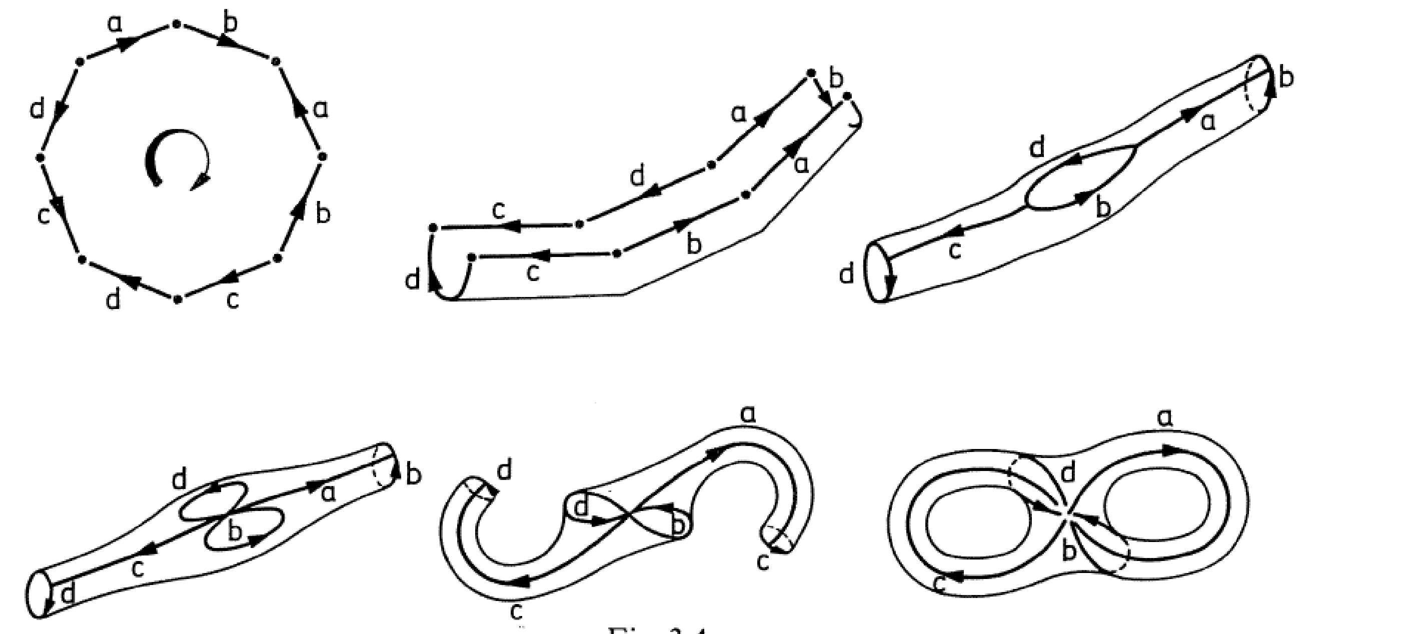


Fig. 3.4

Alternative way to generate a surface: plane schema

- There are some complicated examples:

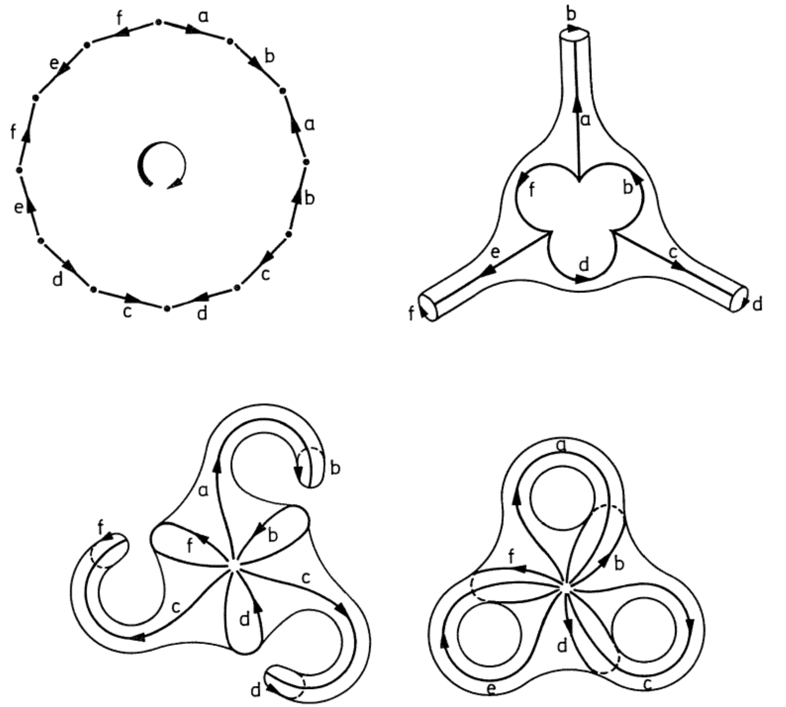
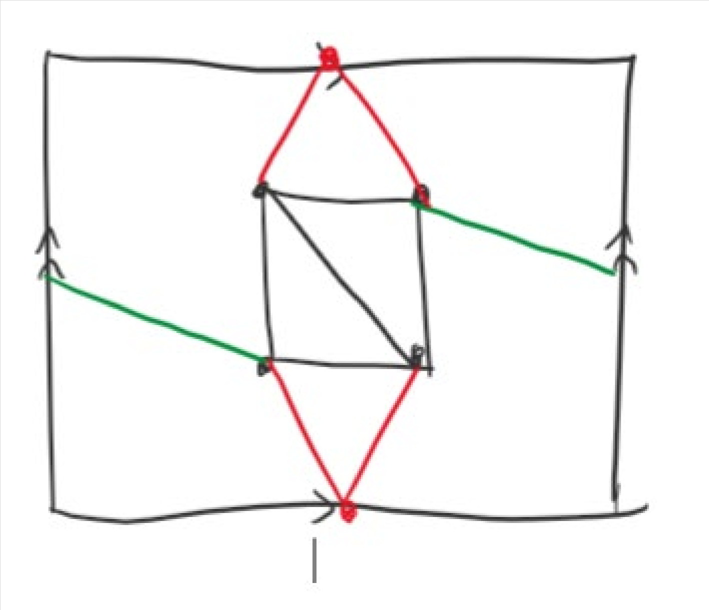


Fig. 3.5

Alternative way to generate a surface: plane schema

- Thus, we replace many drawings on a surface by its plane schema.

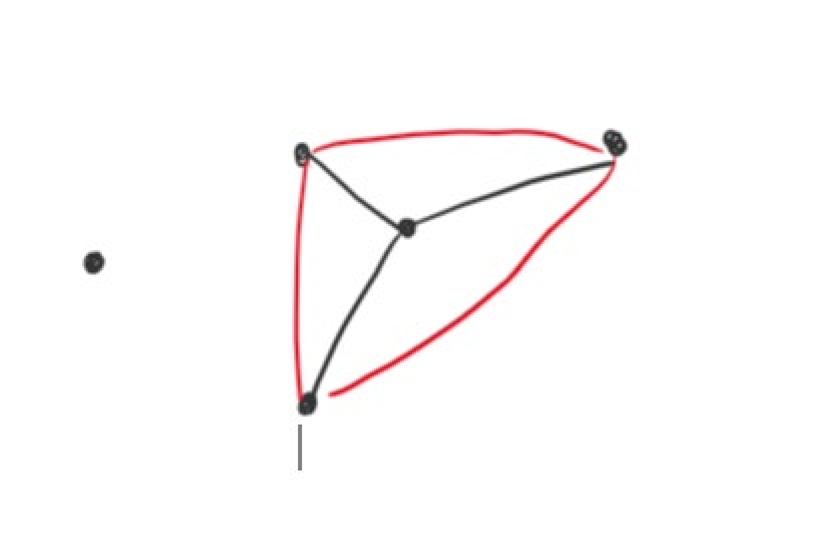
- E.g.) Drawing of K_5 on the torus



Wait, so we're on topology 'till the end?

Obviously not. We will talk about the genus of a **graph**!

Think why we cannot draw K_5 on the plane: the red arc separates two vertices on the plane.

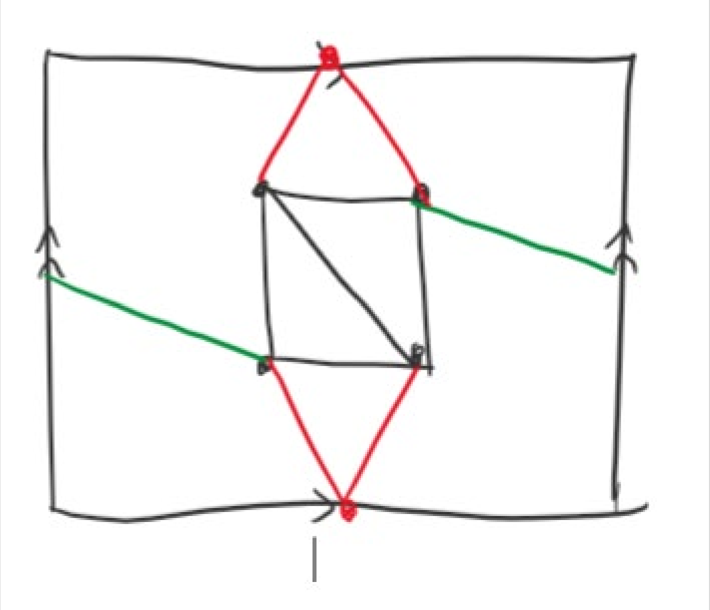


Graph genus problem

However, if we raise the genus of a surface, the embedding is achievable:

Therefore, we may attain some information about graph connection profile by its topological property.

On the next section, we will review some classic result on graph genus theory.



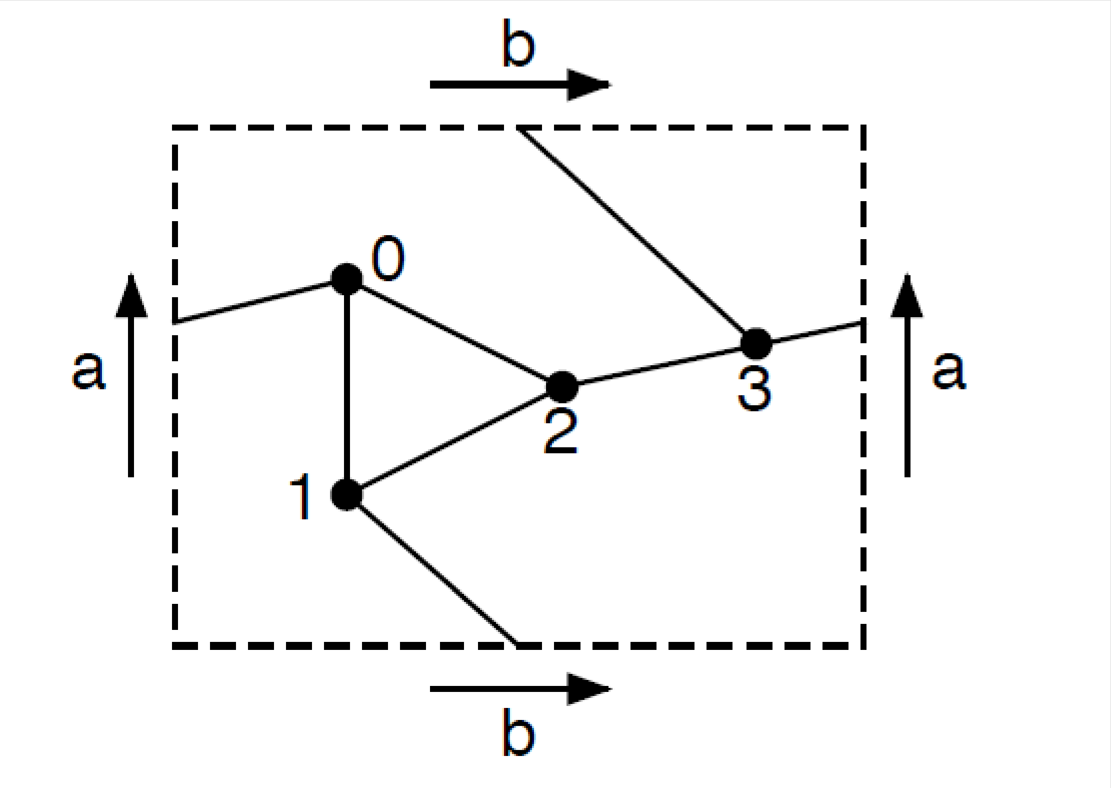
PART 2

Preceding works on graph genus

Graph embedding on a surface

A graph embedding (Drawing without edge-conflict) is called 2-cell embedding (or cellular embedding),

If every faces are homeomorphic to a disk.



- $\gamma(G)$ denotes the *genus* (or the minimum genus) of G,
  is minimum g such that embedding of G on S_g is possible.

- $\gamma_M(G)$ denotes the *maximum genus*
  is maximum g such that
  cellular-embedding of G on S_g is possible.

Graph embedding on a surface

Thm (Duke, 1962): If G has cellular embeddings on both S_l and S_r, G is 2-cellular embeddable on S_x for all $l \leq x \leq r$.

Hence, the extremal genus γ, γ_M are of the interest. Today, we will consider the maximum one.

Euler formula

Thm (Euler-Poincare): For a 2-cellular embedding of G on S_g,

$$V - E + F = 2 - 2g.$$

Here we construct an upper bound for $\gamma_M(G)$.

- Define $\beta(G) = E - V + 1$ to be the **cycle rank** or **betti number** of G.
- Then $g = \frac{1}{2}(\beta(G) + 1 - F)$. Considering $F \geq 1$, $g \leq \lfloor\frac{\beta(G)}{2}\rfloor$ holds.

- We say G is upper-embeddable (abbreviated to u.e.), if $\gamma_M(G) = \lfloor\frac{\beta(G)}{2}\rfloor$.
  - It seems intuitive, as a pair of cycles "chops" the surface into disk-like faces.
  - For the equality condition, we know that u.e. graphs has single-face (or 2-faces) embedding on the maximum genus, depending on the parity of $\beta(G)$.

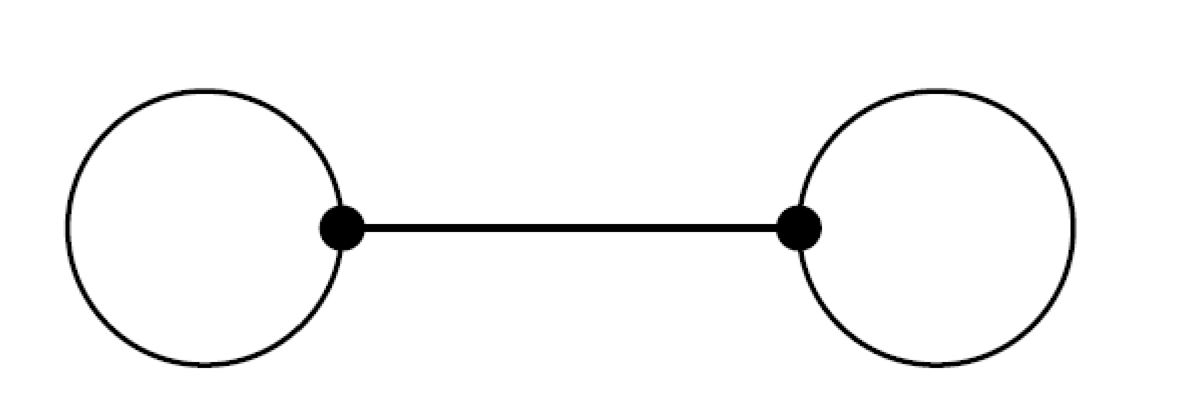
Some surveys

Theorem (Nordhaus, Jungerman): K_m, $K_{m,n}$ are u.e.

Theorem (TBA): Every 4-edge-connected graphs are u.e.

The simplest non-u.e. graph examples are ‘dumbbell graphs’.

Theorem. There is a general construction of non-u.e. 3-edge connected graphs.



PART 3

Truly “simple” and “greedy” “2”-approximation algorithm.

So, is it truly "simple"?

- Obtaining 'minimum genus' $\gamma(G)$ is NP-hard. (Thomassen, 1989)
  - There is $O(g^{256} \log^{189} n)$ – approximation algorithm in non-orientable case (Kawarabayashi, 2015)

- Unlike the 'minimum' case, maximum genus problem is solvable in polynomial time, in *deterministic way*.
  - Reducing into cographic matroid parity problem in $O(m \cdot n \cdot \Delta \cdot \log^6 m)$ time (Furst and Gross, 1988)

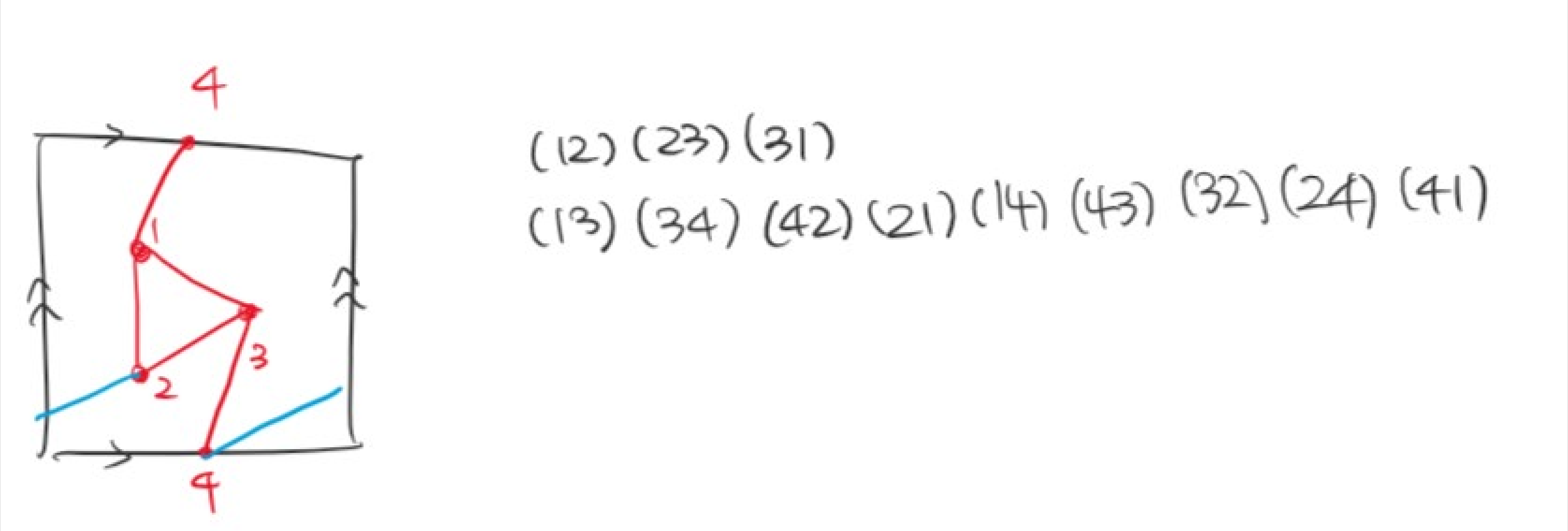
  - May convert to "optimum matching forest" problem which runs within $O(m^6)$ time (Gabow, 1985)

- There is an $O(m \log n)$ 4-approximation algorithm for $\gamma_M(G)$, requiring modification of graph and doubly-linked face tracer

- Regarding this in total, the method deserves 'simplicity'.

Building block

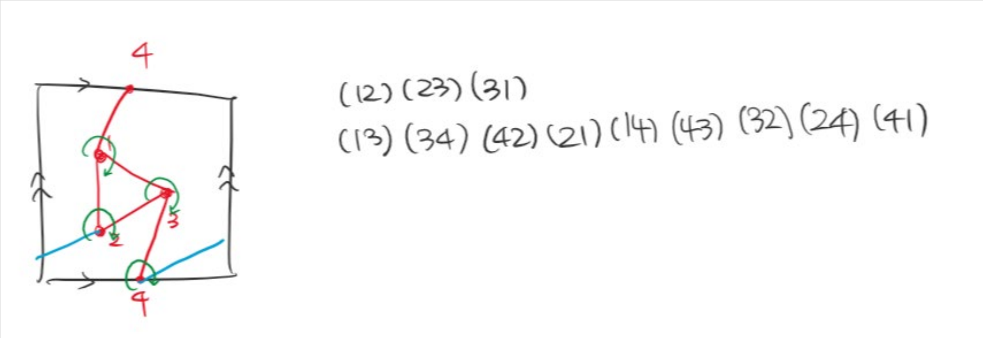
- Heffter-Edmonds Principle: Embedding of a graph makes one-to-one correspondence to its rotation system.



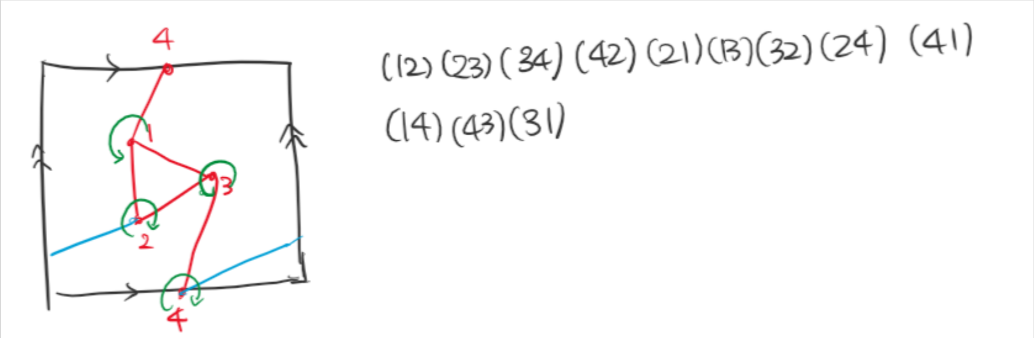
Embedding of K_4 on a torus

Building block

- Heffter-Edmonds Principle: Embedding of a graph makes one-to-one correspondence to its rotation system.



Embedding of K_4 on a torus



Same embedding with a weird rotation

Ringeisen’s Edge addition lemma

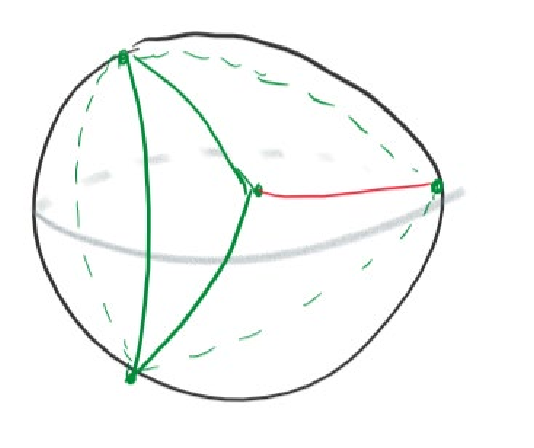
- G is a connected graph. (any graph is connected throughout this presentation)
- e is an edge not in $E(G)$, but incident to vertices in G. i.e. We will add e on G.

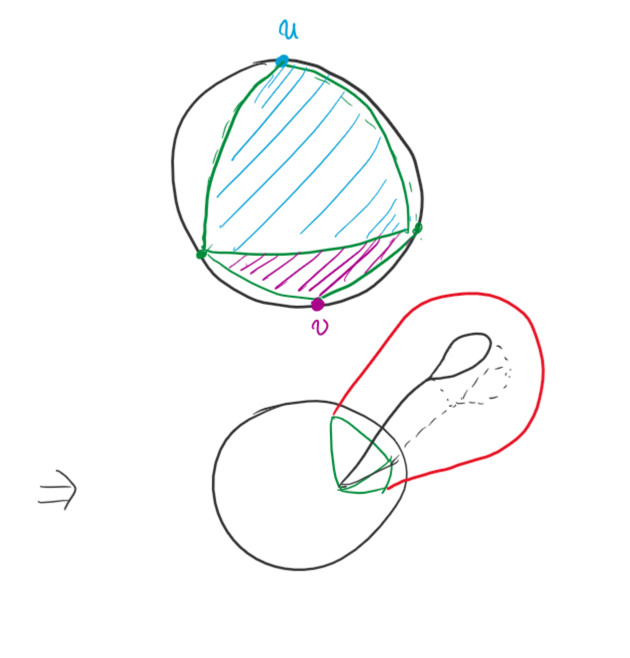
- Suppose $e = uv$ and u belongs to the face f_u and v belongs to the face f_v.
  - v or e ‘belongs to’ f, if it locates on the boundary of f. Thus f_u and f_v could be selected among a few faces.

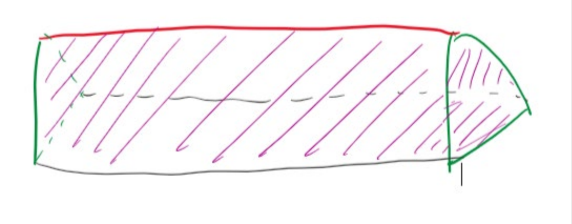
- If $f_u = f_v$, $G + e$ accepts an embedding with one additional face than G, with the same genus.
- If $f_u \neq f_v$, $G + e$ accepts an embedding with the number of face decreased by one, with the genus increased by one. (desired direction)

Ringeisen's Edge addition lemma (Cont'd)

- If $f_u = f_v$, $G + e$ accepts an embedding with one additional face than G, with the same genus.
- If $f_u \neq f_v$, $G + e$ accepts an embedding with the number of face decreased by one, with the genus increased by one. (desired direction)
- *Heffter-Edmonds principle seems to give combinatorial proof.*







Theorem (Jungerman)

- e, f are adjacent edges not in G. If G has an embedding with a single face, so does $G \cup \{e, f\}$.
  - 'Embedding with a single face' implies that $v - e + 1 = 2 - 2g$, hence $\beta(G) = 2g$.
  - Thus the case works for even betti number, and we know that G is *upper embeddable.*
  - Such embedding of $G \cup \{e, f\}$ implies that its embedding would raise genus by one compared to G.

Proof) Suppose $e = uv, f = uw$. All vertices of G belongs to a single face F.

Applying 1st case of Ringeisen, we split F by F_1 and F_2 to obtain an embedding of $G + e$.

Then apply 2nd case of Ringeisen, we merge F_1 and F_2 giving additional genus. ■

Theorem (Xuong)

- For a spanning tree T of G, Let $\xi(G,T)$ denote the Xuong deficiency; which means # of odd-edge components in $G\ -T$.

- Then, $\gamma_M(G) = \frac{\left(\beta(G) - \min_T \xi(G,T)\right)}{2}$.
- This powerful theorem states the maximum genus of G is a mere combinatorial quantity of G.

- For instance, any 4-edge-connected graph is upper embeddable.
  - Since 4-edge connected graph has 2 edge-disjoint spanning trees, say T, T', $G - T$ has only one component.
  - Thus $\xi(G,T) \leq 1$, implying that $\gamma_M(G) = \lfloor\frac{\beta(G)}{2}\rfloor$ in some detail.
  - Hence, it's enough to figure out $\gamma_M(G)$ for not 4-edge connected case.

Theorem (Xuong)

- For a spanning tree T of G, Let $\xi(G,T)$ denote the Xuong deficiency; which means # of odd-edge components in $G - T$.
- Then, $\gamma_M(G) = \frac{\left(\beta(G) - \min_T \xi(G,T)\right)}{2}$.

We prove a lemma beforehand.

Lemma) Even-sized edge set of a connected graph G can be partited into disjoint pairs of adjacent edges.

Proof) Reduce to Maximum Matching of the line graph $L(G)$. Tutte's theorem kills it.

Theorem (Xuong)

$\gamma_M(G) = \frac{\left(\beta(G) - \min_T \xi(G,T)\right)}{2}.$

Lemma) Even-sized edge set of a connected graph G can be partited into disjoint pairs of adjacent edges.

Pick a spanning tree T and exhaust the co-tree $G - T$ with disjoint pairs of adjacent edges. Note that there are exactly $(\beta - \xi)/2$ such pairs.

Draw T on the sphere, and apply the scheme in Jungerman's theorem to raise genus.

Adding remaining ξ edges (according to Ringeisen's lemma) does not "decrease" the genus, hence the genus of embedding is at least $(\beta - \xi)/2$.

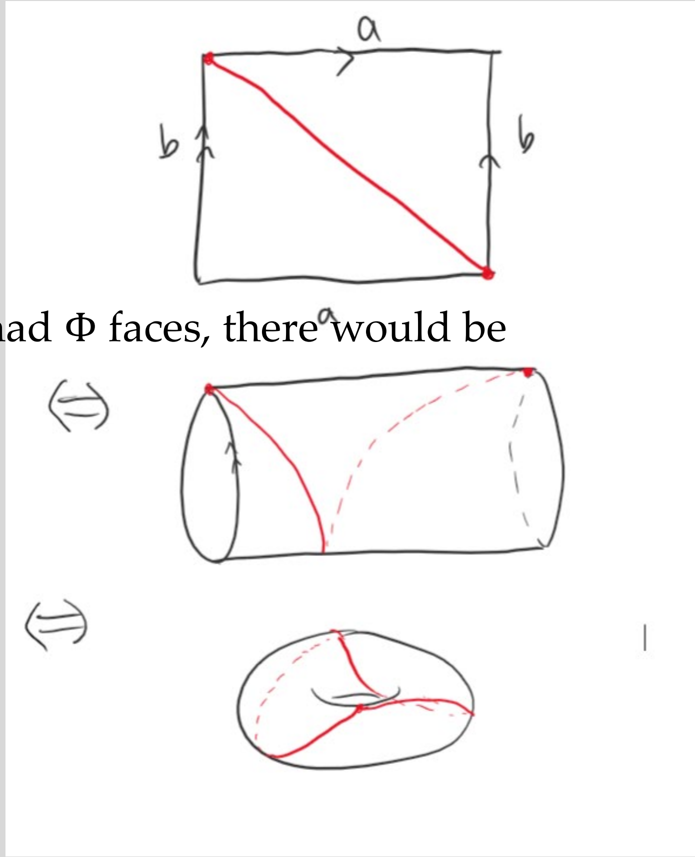
Theorem (Xuong)

- $\gamma_M(G) = \frac{\left(\beta(G) - \min_T \xi(G,T)\right)}{2}$.

Proof of Xuong's theorem) Now we move to the counterpart, $\gamma \leq (\beta - \xi)/2$.

- Consider an embedding of G and its 'plane representation'. If the embedding had Φ faces, there would be $\Phi - 1$ "chords" on plane representation.

- If we remove all such chords $e_1, e_2, \cdots e_{\Phi-1}$, remaining graph G' would be u.e.
- Hence, we have a spanning tree T such that all components of $G' - T$ are even.
- If we add all the chords, # of odd components in $G - T$ is at most $\Phi - 1$.
- Therefore $\xi(G,T) \leq \Phi - 1 = \beta - 2\gamma$, givin' the desired formula.



Theorem (Xuong)

Corollary (Khomenko): $\gamma_M(G)$ is equal to # of (disjoint) packs of adjacent edges, whose removal leave the graph connected.

Now, we completely transformed the maximum genus into combinatorial problem.

Matroid parity problem is so-called "polymatroid matching", which has sense of analogy with the problem above.

To avoid this difficulty, we provide the following algorithm:

"Merely select-and-remove a pair of adjacent edges, until the remaining graph is disconnected"

Such 'simple-greedy'. Now, does it really '2-approximable'?

The algorithm

This lemma is immediate from Ringeisen's edge-addition lemma.

"If $G - e$ is connected, $\gamma_M(G) - 1 \leq \gamma_M(G - e) \leq \gamma_M(G)$"

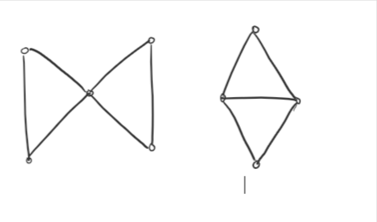
Thus if we repeatedly expel "adjacent edges" from G, $\gamma_M(G)$ would decrease by at most two.

Now here's a question: "What if we could expel adjacent edges from H with $\gamma_M(H) = 0$?"

Theorem (Nordhaus)

TFAE:

1) $\gamma_M(G) = 0$.
2) G does not contain such topological minor graph.
3) Any cycle of G does not contain a common vertex.
4) There are no adjacent edges e, e' such that $G - \{e, e'\}$ is connected.



We omit the proof here, but now we know that our algorithm is well-established!

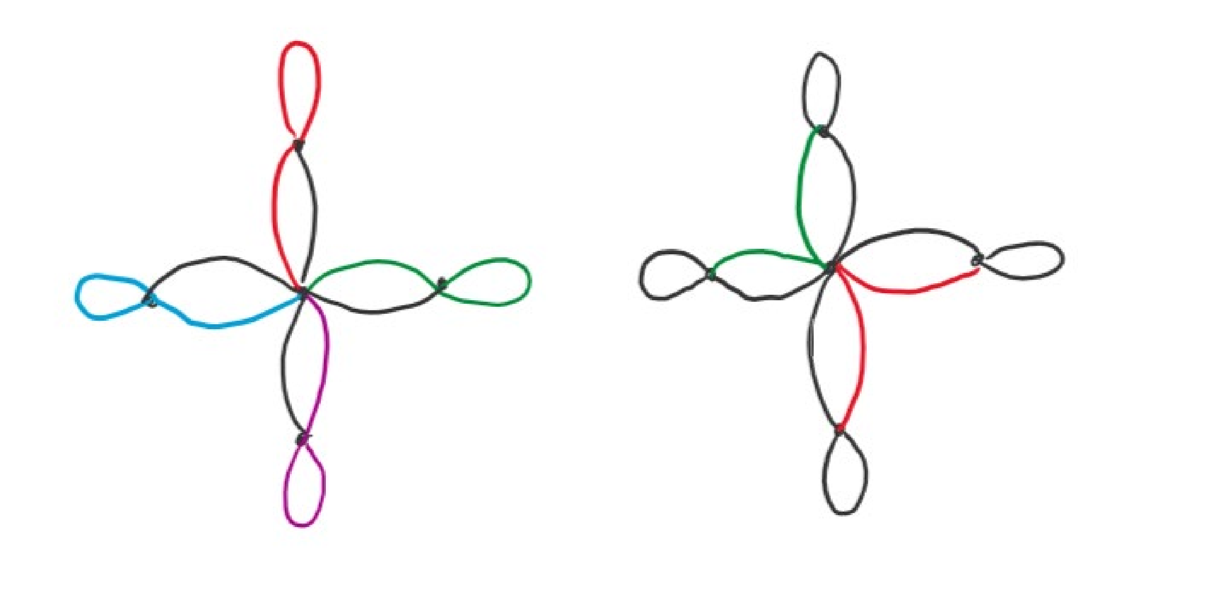
The algorithm - Runtime analysis

- Thus, this corollary guarantees 2-approximability;
  - Any “maximal” set of disjoint-adjacent edges in G has at least $\gamma_M(G)/2$ pairs of disj.adj. edges.

- There’s no need to fetch a complicated analysis for runtime:
  - $O\big((\tau+\rho)\sum\deg^2\big)$, where τ, ρ are overheads for testing connectivity / updating graph.
  - Using the best known dynamic connectivity algorithm, we obtain $\tau+\rho = O(\frac{\log^2 n}{\log\log n})$.
  - As $\sum\deg^2 v = O(\frac{m^2}{n})$, we get that the total complexity equals $O(\frac{m^2}{n}\cdot\frac{\log^2 n}{\log\log n})$. Which works well for sparse graph.

The algorithm – Worst case

- The approximation ratio is tight: The authors propose that it would be challenging to improve.



Further talk

- For the non-oritentable case, even the Euler formula is revised: $V - E + F = 2 - \gamma$.
- Thus we should fix many theoretical basis that we discussed.
- But it seems freer to discuss non-orientable imbedding, according to achievement from researchers.