The Directed Grid Theorem

Aeren

0. Notations

- Undirected and directed graphs will be denoted as graphs and digraphs, respectively.

- Every graph and digraph are finite and simple unless stated otherwise.

- For a graph(or a digraph) G, $V(G)$ denotes its vertex set and $E(G)$ its edge set.

- For an edge $e = (u, v) \in E(G)$ in a digraph G, u and v are **tail** and **head** of e respectively. e is said to be **incident to** u and v.

- For an edge $e = (u, v) \in E(G)$ and paths P_1 ending at u and P_2 in a digraph G, starting at v, $P_1 e P_2$ denotes the combined path.

- $N(u) = \{v: \{u, v\} \in E(G)\}$ for a graph G, and $N_{in}(u) = \{v: (v, u) \in E(G)\}$, $N_{out}(u) = \{u: (u, v) \in E(G)\}$ for a digraph G.

- An **abstract graph** in a digraph G is the graph obtained by ignoring the direction of edges.

- For an graph G, $G^{\leftrightarrow}$ denotes the biorientation of G, and $\bar{G}$ its completement.

- K_n is the unique (up to isomorphism) complete graph with n vertices.

- For a set S and an integer k, $[S]^{=k}$ is the set of subsets of S of size k. $[S]^{\leq k}, [S]^{<k}$, $[S]^{>k}$, $[S]^{\geq k}$ are defined similarly.

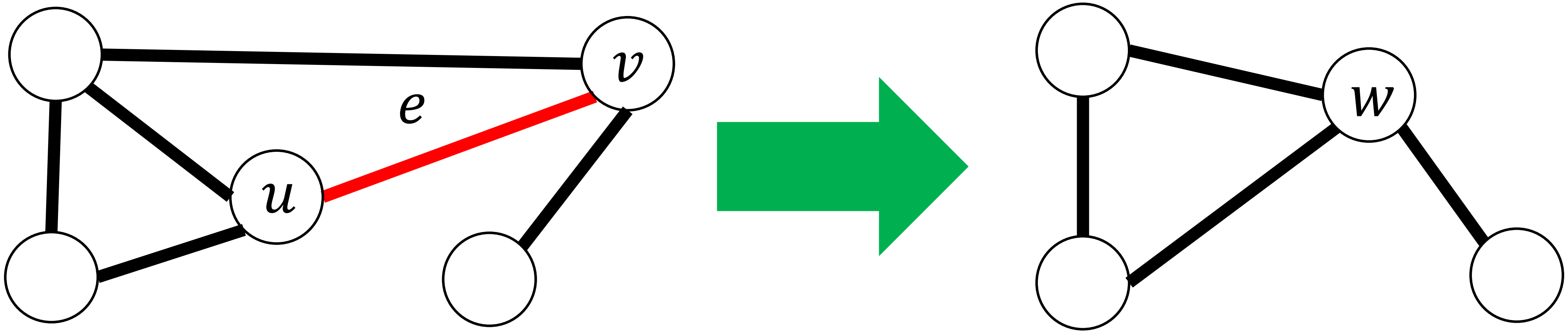
- For a graph(or a digraph) G and $S \subseteq V(G)$, $[S]$ denotes the induced subgraph.

1. Prologue

DEFINITION 1.1

A **minor** of a graph G is a graph obtained by applying the following operations zero or more times.

1. Delete an edge (Set $E(G) \leftarrow E(G) \backslash \{e\}$ for some $e \in E(G)$)
2. Delete an isolated vertex (Set $V(G) \leftarrow V(G) \backslash \{u\}$ for some $u \in V(G)$ of degree 0)
3. Contract an edge (Choose an $e = \{u, v\} \in E(G)$ and set $V(G) \leftarrow (V(G) \backslash \{u, v\}) \cup \{w\}$ and $E(G) \leftarrow (E(G) \backslash \{e\}) \cup \left\{\{w, i\}: i \in \left(N(u) \cup N(v)\right) \backslash \{u, v\}\right\}$ for a new vertex w)

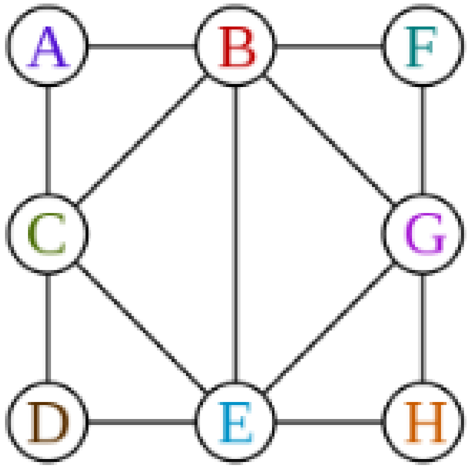


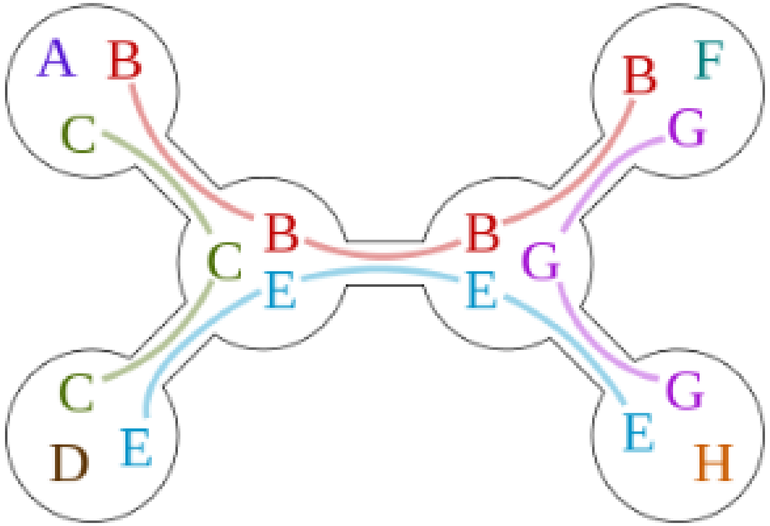
1. Prologue

DEFINITION 1.2

A **tree decomposition** of a graph G is a tree T with nodes $X_1, \ldots, X_n \subseteq V(G)$ such that

1. $\bigcup_{i=1}^{n} X_n = V(G)$ and $X_i \neq \emptyset$ for all $1 \leq i \leq n$,
2. for all $u \in V(G)$, the subgraph of T induced by $\{X\_i : u \in X\_i\}$ is connected, and
3. for all $\{u, v\} \in E(G)$, there exists an index i such that $\{u, v\} \subseteq X_i$.



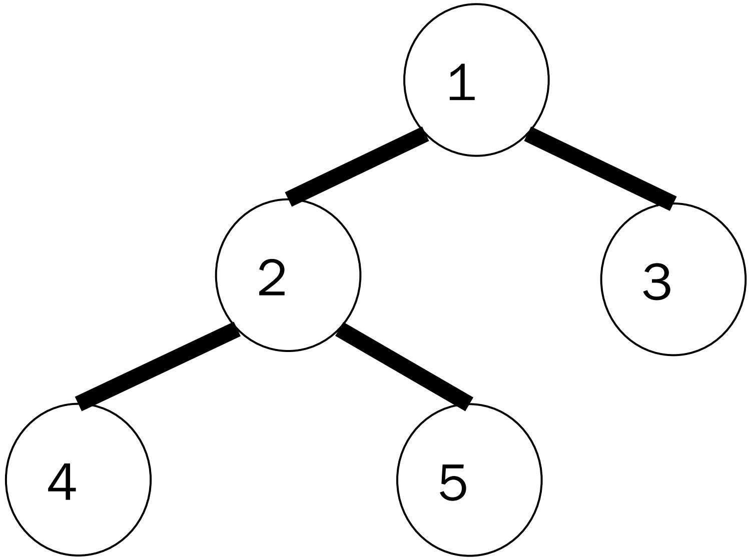


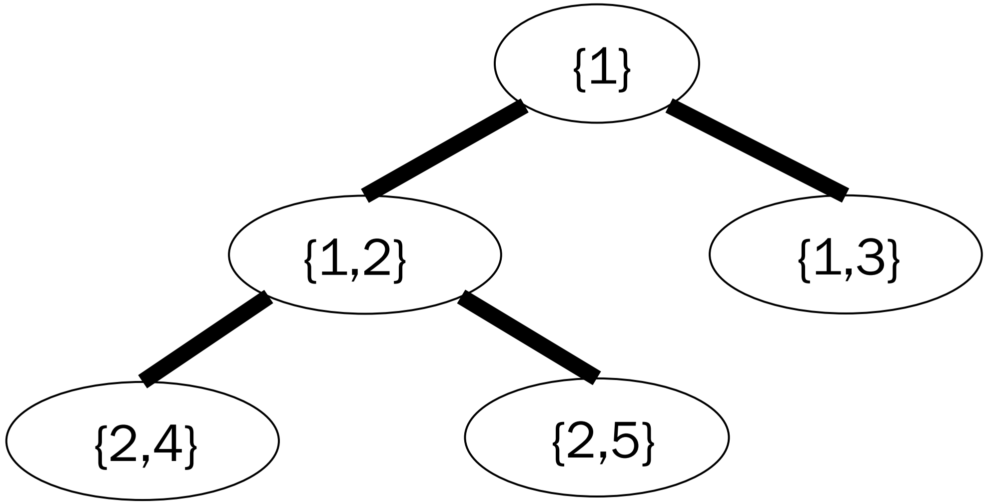
1. Prologue

DEFINITION 1.3

The **width** of a tree decomposition is $\max|X_i| - 1$.

The **tree width** $tw(G)$ of a graph G is the minimum width among all possible decompositions of G.



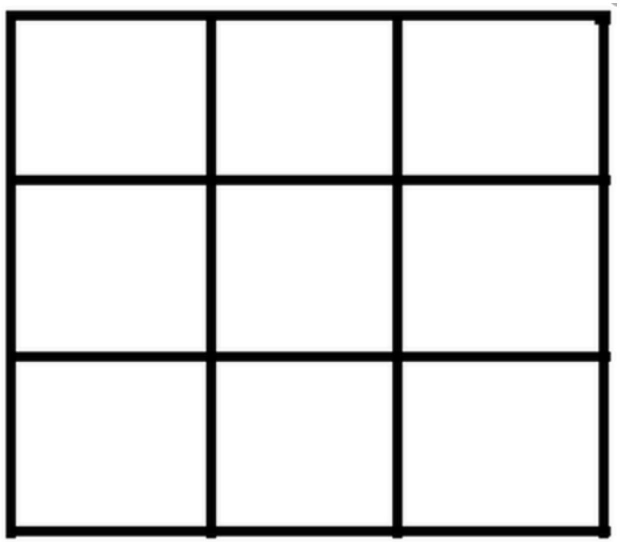


1. Prologue

DEFINITION 1.4

The **grid** of order $n \geq 1$ is a graph G with

1. $V(G) = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i,j \leq n\}$
2. $E(G) = \left\{\{(i,j),(k,l)\} \in [\mathbb{Z}^2]^{=2} : 1 \leq i,j,k,l \leq n, |i-k| + |j-l| = 1\right\}$



A grid of order 4

1. Prologue

THEOREM 1.5 (Robertson and Seymour, The Grid Theorem)
There is a function $f: \mathbb{N} \to \mathbb{N}$ such that every graph of tree width at least $f(k)$ contains a grid of order k as a minor.

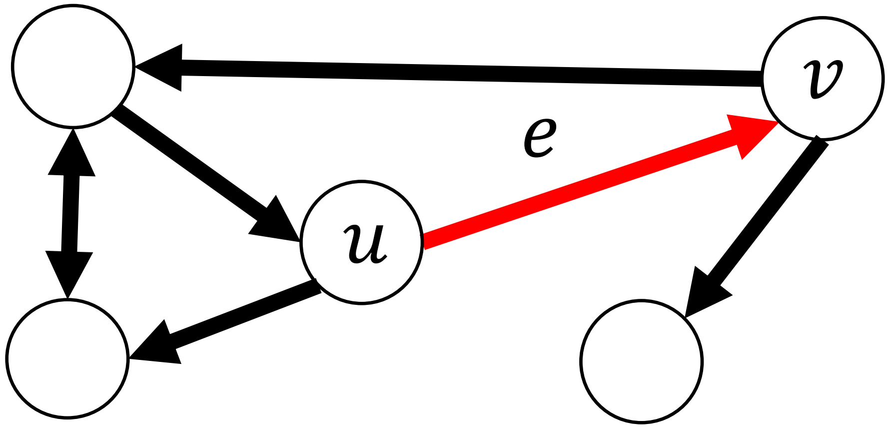
“The grid theorem is important both for structural graph theory as well as for algorithmic applications. For instance, algorithmically it is the basis of an algorithm design principle called bidimensionality theory, which has been used to obtain many approximation algorithms, PTASs, subexponential algorithms and fixed-parameter algorithms on graph classes excluding a fixed minor. These include feedback vertex set, vertex cover, minimum maximal matching, face cover, a series of vertex-removal parameters, dominating set, edge dominating set, R-dominating set, connected dominating set, connected edge dominating set, connected R-dominating set and unweighted TSP tour.”

1. Prologue

DEFINITION 1.6

Let G be a digraph. An edge $e = (u, v) \in E(G)$ is **butterfly-contractible** if either

1. e is the only outgoing edge of u or
2. e is the only incoming edge of v

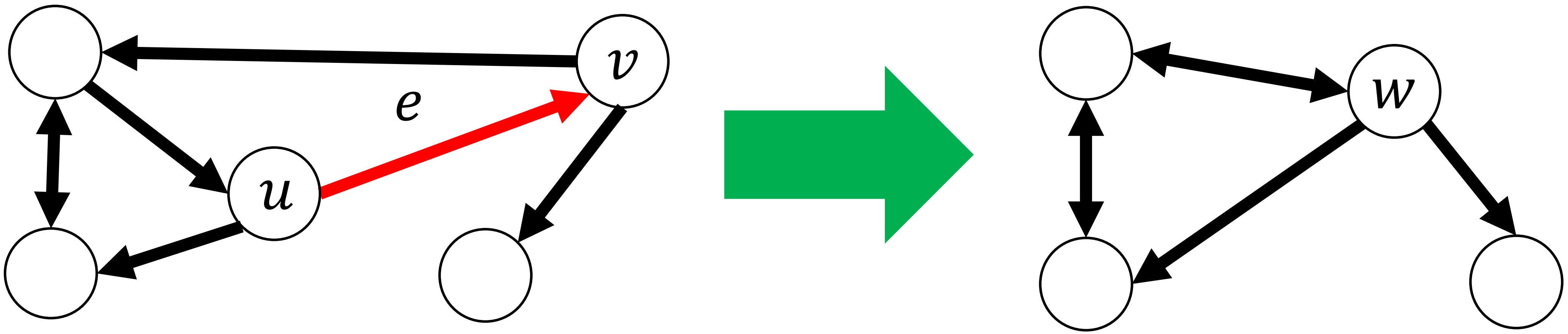


1. Prologue

DEFINITION 1.7

A **butterfly-contraction** of a digraph G with respect to a butterfly-contractible edge $e = (u, v) \in E(G)$ is the digraph G' with

1. $V(G') = (V(G) \setminus \{u, v\}) \cup \{w\}$ for a new vertex w and
2. $E(G') = \{(i, j) \in E(G): \{i, j\} \cap \{u, v\} = \emptyset\}$
   $\cup \{(i, w): i \in N_{in}(u) \cup N_{in}(v)\} \cup \{(w, i): i \in N_{out}(u) \cup N_{out}(v)\}$.



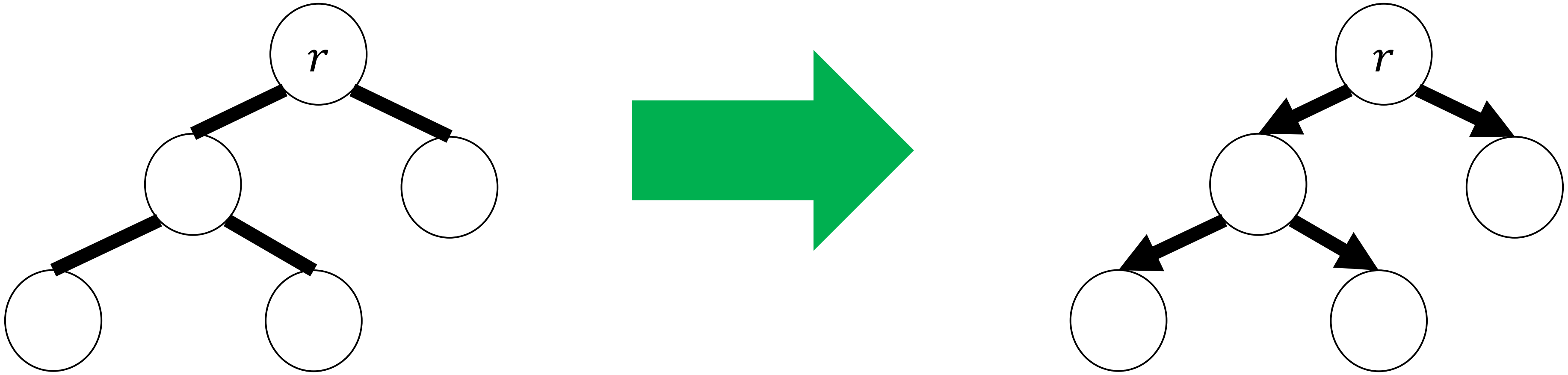
DEFINITION 1.8

A **butterfly-minor** of a digraph G is a digraph obtained from a subdigraph of G by zero or more butterfly contractions.

1. Prologue

DEFINITION 1.9

An **arborescence** is a digraph R such that there is a vertex $r \in V(R)$, called a **root** of R, with the property that for all $u \in V(R)$, there is a unique path from r to u.
We write $u < v$ for $u, v \in V(R)$ if v is reachable from u.
We also write $e < w$ for $e = (u, v) \in E(R)$ and $w \in V(R)$ if $v = w$ or $v < w$.



1. Prologue

DEFINITION 1.10

Let G be a digraph and $Z \subseteq V(G)$. A set $S \subseteq V(G)$ is Z-**normal** if $S \cap Z = \emptyset$ and there is no path P such that

1. P does not intersect with Z,
2. both endpoints of P are in S, and
3. P intersects with $V(G) \backslash (Z \cup S)$.

THEOREM 1.11

A set S is Z-normal if and only if the strongly connected components of $G - Z$ can be numbered $C_1, \ldots, C_n$ so that

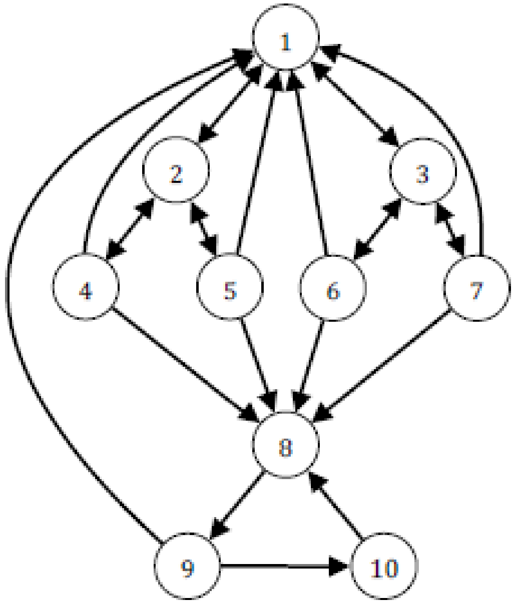
1. C_i are topologically sorted and
2. either $S = \emptyset$ or $S = C_i \cup \cdots \cup C_j$ for some $1 \leq i \leq j \leq n$.

1. Prologue

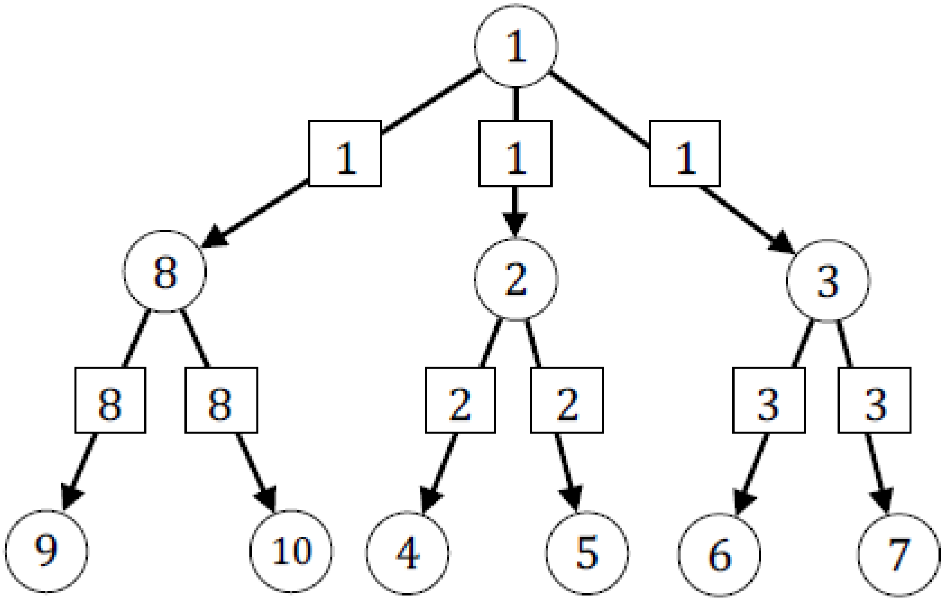
DEFINITION 1.12

A **directed tree decomposition** of a digraph G is a triple (R, W, X) where R is an arborescence, $W: V(R) \to [V(G)]^{\geq 1}$, and $X: E(R) \to [V(G)]^{\geq 0}$ such that

1. the sets $W(u)$ form a partition of $V(G)$ and
2. for $e \in E(R)$, $\bigcup(W(w): w \in V(R), w > e)$ is X_e-normal.



$a)$ A digraph G.



$b)$ a directed tree-decomposition of G.

1. Prologue

DEFINITION 1.13

The **width** of a directed tree decomposition is $\max_{u \in V(R)} |W(u) \cup \bigcup_e X(e)| - 1$ where e is taken over all edges incident to u.

The **directed tree width** $dtw(G)$ of a digraph G is the minimum width among all possible decompositions of G.

We call $W(u)$ the **bag** of $u \in V(R)$, $X(e)$ the **guard** of $e \in E(R)$.

1. Prologue

DEFINITION 1.14

A **cylindrical grid** of order $k \geq 1$ is a digraph G_k consisting of k pairwise disjoint directed cycles $C_1, \dots, C_k$, together with a set of $2k$ pairwise vertex-disjoint paths $P_1, \dots, P_{2k}$ such that

1. each path P_i has exactly one vertex in common with each C_j,
2. the paths appears in each C_i in order of increasing indices, and
3. for odd i, the cycles appears in each P_i in order of increasing indices, and for even i, in order of decreasing indices.

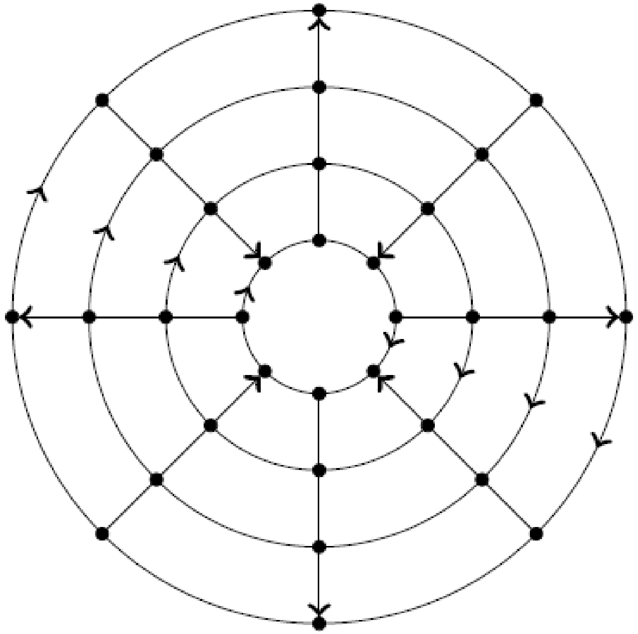


Figure 1: Cyclindrical grid G_4.

1. Prologue

CONJECTURE 1.15 (Johnson, Robertson, Seymour and Thomas)
There is a function $f: \mathbb{N} \to \mathbb{N}$ such that every digraph of directed tree width at least $f(k)$ contains a cylindrical grid of order k as a butterfly minor.

Overview of The Proof

Directed Tree Decomposition

Part 3

Bramble

Part 4

Web

Part 5

Fence

Part 6

Cylindrical Grid

2. Preliminaries

DEFINITION 2.1

Let $\mathcal{P}$ and $\mathcal{Q}$ be sets of pairwise disjoint paths in a digraph G. **The intersection graph** $\mathcal{I}(\mathcal{P}, \mathcal{Q})$ of $\mathcal{P}$ and $\mathcal{Q}$ is the bipartite graph with vertex set $\mathcal{P} \cup \mathcal{Q}$ and there's an edge between $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ if and only if $P \cap Q \neq \emptyset$.

2. Preliminaries

LEMMA 2.2

Let G be a bipartite graph with bipartition (S, T) with $|S| = s, |\mathrm{T}| = \mathrm{t}$, and let $s' \leq s, t' \leq t$ be positive integers. If $|E(G)| \leq (s - s')(t - t')/t'$, there exists $S' \subseteq S, T' \subseteq T$ such that $|S'| = s', |T'| = t'$ and $S' \cup T'$ is independent in G.

PROOF

Sort the elements in T in increasing order by their degree and let T' be the set of the first t' vertices.

Their degree must be equal or lesser than $(s - s')/t'$ otherwise the number of edges generated by the remaining $t - t'$ vertices exceeds $(s - s')(t - t')/t'$

There are at most$(s - s')/t' \times t' = s - s'$ edges incident to T', and setting S' to be the set of s' vertices in S not linked to T' completes the proof.

2. Preliminaries

THEOREM 2.3 (Ramsey)

For all integer $n \geq 1$, there exists an (minimum) integer $R(n)$ such that for every subgraph of $K_{R(n)}$, contains either K_n or $\overline{K_n}$ as a subgraph.

THEOREM 2.4

For all integer $n \geq 1$, there exists an (minimum) integer $R(n)$ such that for every edge-coloring function $c: E\left(K_{R(n)}\right) \to \{0,1\}$ of $K_{R(n)}$, there exists a monochromatic clique.

2. Preliminaries

LEMMA 2.5

There is a function $f_{clique}\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all $n, k \geq 0$, if $G := K^{\leftrightarrow}_{f_{clique}(n,k)}$ and $\gamma\colon E(G) \to [V(G)]^{\leq k}$ such that $\gamma(e) \cap e = \emptyset$ for all $e \in E(G)$, then there is $H \cong K^{\leftrightarrow}_n \subseteq G$ such that $\gamma(e) \cap V(H) = \emptyset$.

PROOF

Let $f_{clique}(n, 0) = n$ and for $k > 0$, let $f_{clique}(n, k) = R\left(\max\left(f_{clique}(n, k-1), (2k+1) \cdot f_{clique}(n-1, k)\right)\right) + 1$. We prove by the induction on k.

For $k = 0$, the statement is trivial.

So let $k > 0$. Choose a vertex $u \in V(G)$ and let $G_u = [V(G) \backslash \{u\}]$. Let $l = \max\left(f_{clique}(n, k-1), (2k+1) \cdot f_{clique}(n-1, k)\right)$. As $|V(G_u)| = R(l)$, by theorem 2.4, there's either a set $X \subseteq V(G)$ of order l such that for each edge $e \in E(G_u)$, $u \in \gamma(e)$, or a set $X \subseteq V(G)$ order l such that $u \notin \gamma(e)$.

In the first case, since $l \geq f_{clique}(n, k-1)$, the induction hypotheses allows us to choose $X' = [X]^{=n}$ such that $\gamma(e) \cap V([X']) = \emptyset$ for all $e \in E([X'])$.

2. Preliminaries

LEMMA 2.5

There is a function $f_{clique}\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all $n, k \geq 0$, if $G := K^{\leftrightarrow}_{f_{clique}(n,k)}$ and $\gamma\colon E(G) \to [V(G)]^{\leq k}$ such that $\gamma(e) \cap e = \emptyset$ for all $e \in E(G)$, then there is $H \cong K^{\leftrightarrow}_{n} \subseteq G$ such that $\gamma(e) \cap V(H) = \emptyset$.

PROOF

In the second case, we construct the set X' as follows. Initially, set $X' = \emptyset$. In each step we preserve the invariance: for each $v \in X'$, $\gamma\big((u,v)\big) \cap X' = \gamma\big((u,v)\big) \cap X' = \emptyset$. It certainly holds in the beginning. While $X \neq \emptyset$, choose $v \in X$, add it to X', and remove $\{v\} \cup \gamma\big((u,v)\big) \cup \gamma\big((v,u)\big)$ from X. Since in each step, we remove $2k + 1$ elements from X at max, $|X'| \geq f_{clique}(n-1, k)$,

By the induction hypothesis, $[X']$ contains a subgraph H' isomorphic to $K^{\leftrightarrow}_{n-1}$ such that $\gamma(e) \cap V(H') = \emptyset$ for all $e \in E(H')$, and by the construction, $H = [V(H') \cup \{u\}]$ is the subgraph isomorphic to $K^{\leftrightarrow}_{n}$ such that $\gamma(e) \cap V(H) = \emptyset$ for all $e \in E(H)$.

2. Preliminaries

LEMMA 2.6

For all $n, k \geq 0$, if $G \stackrel{\text{def}}{=} K_{n(k+1)}^{\leftrightarrow}$ and $\gamma: V(G) \to [V(G)]^{\leq k}$ such that $u \notin \gamma(u)$ for all $u \in V(G)$, then there is $H \cong K_n^{\leftrightarrow} \subseteq G$ such that $\gamma(u) \cap V(H) = \emptyset$ for all $u \in V(H)$.

PROOF

Construct H greedily. In each step, choose a vertex u from G, add it to H, and delete $\{u\} \cup \gamma(u)$ from G.

2. Preliminaries

THEOREM 2.7 (Erdős and Szekeres)

Let $s, t \geq 1, n = (s-1)(t-1)+1$, and $a_1, \dots, a_n$ be distinct reals. Then either

1. there exists $1 \leq i_1 < \cdots < i_s \leq n$ such that $a_{i_1} < \cdots < a_{i_s}$ or
2. there exists $1 \leq j_1 < \cdots < j_t \leq n$ such that $a_{j_1} > \cdots > a_{j_t}$.

2. Preliminaries

DEFINITION 2.8

A **linkage** $\mathcal{P}$ in a directed graph G is a set of pairwise vertex-disjoint paths. For $S, T \subseteq V(G)$, an $\boldsymbol{S-T}$ **path** is a path from a vertex in S to a vertex in T. An $\boldsymbol{S-T}$ **linkage** is a linkage where each path is an $S-T$ path. The **order** of the linkage, denoted by $|\mathcal{P}|$, is the number of paths.

We sometimes identify $\mathcal{P}$ as the underlying digraph, so $V(\mathcal{P}) = \bigcup_{P\in\mathcal{P}} V(P)$ and $E(\mathcal{P}) = \bigcup_{P\in\mathcal{P}} E(P)$.

DEFINITION 2.9

Let G be a digraph. $A \subseteq V(G)$ is **well-linked** if for all $X, Y \subseteq A$ with $|X| = |Y|$, there is an $X-Y$ linkage of order $|X|$.

DEFINITION 2.10

A **separation** of a digraph G is a pair (A, B) of subsets of $V(G)$ such that $V(G) = A \cup B$ and there is no edge in G from $A\backslash B$ to $B\backslash A$.The **order** of the separation is $|A \cap B|$.

THEOREM 2.11 **(Menger)**

Let G be a digraph, $S, T \subseteq V(G)$, and $k \geq 0$ an integer. Then exactly one of the following holds:

1. there is an $S-T$ linkage of order k or
2. There is a separation (X, Y) of order less than k with $S \subseteq X$ and $T \subseteq Y$.

2. Preliminaries

DEFINITION 2.12

Let G be a digraph and $A, B \subseteq V(G)$. A **half-integral $\boldsymbol{A - B}$ linkage of** order k is a set $\mathcal{P}$ of $A - B$ paths such that no vertex is contained in more than 2 paths in $\mathcal{P}$.

LEMMA 2.13

Let G be a digraph and $A, B, C \subseteq V(G)$.
1. If G contains a half-integral linkage $A - B$ of order k, then G contains an $A - B$ linkage of order $k/2$.
2. If $|B| = k$ and G contains an $A - B$ linkage $\mathcal{L}$ of order k and $B - C$ linkage $\mathcal{L}'$ of order k, then G contains an $A - C$ linkage of order $k/2$.

PROOF

For the first part, suppose G does not contain an $A - B$ linkage of order $k/2$. Then, by the Menger's theorem(2.11), there is a separation (X, Y) of order less than $k/2$ such that $A \subseteq X$ and $B \subseteq Y$. This is a contradiction since each vertices in $A \cap B \subseteq X \cap Y$ can only be used twice.

The second part follows from the first part as $\mathcal{L}$ and $\mathcal{L}'$ can be combined onto a single half-integral $A - B$ linkage of order k.

2. Preliminaries

DEFINITION 2.14

Let G be a digraph, $S, T \subseteq V(G)$, and H a subdigraph of G. For $k \geq 1$, an $S - T$ linkage $\mathcal{L}$ of order k is **minimal with respect to $\boldsymbol{H}$** or **$\boldsymbol{H}$-minimal** if for all edges $e \in E(\mathcal{L}) \setminus E(H)$, there is no $S - T$ linkage $\mathcal{M}$ of order k with $E(\mathcal{M}) \subseteq \left(E(\mathcal{L}) \cup E(H)\right) \setminus \{e\}$.

LEMMA 2.15

Let G be a digraph, $S, T \subseteq V(G)$, $\mathcal{L}'$ a linkage in G, and $\mathcal{L}$ a $\mathcal{L}'$-minimal $S - T$ linkage. Then $\mathcal{L}$ is $\mathcal{M}'$-minimal for every $\mathcal{M}' \subseteq \mathcal{L}'$.

LEMMA 2.16

Let G be a digraph, $\mathcal{P}$ a subdigraph, $\mathcal{R}$ a $\mathcal{P}$-minimal linkage between two sets $A, B \subseteq V(G)$, and $R \in \mathcal{R}$ a path. If $e \in E(R) \setminus E(\mathcal{P})$, and R_1 and R_2 are two connected components of $R - e$ such that the tail of e lies in R_1, then there are at most $r \stackrel{\text{def}}{=} |R|$ paths from R_1 to R_2.

2. Preliminaries

DEFINITION 2.17

An **elementary cylindrical wall** W_k of order k is the digraph obtained from G_k by replacing every vertex u of degree 4 with two new vertices u_{in} and u_{out} connected by an edge (u_{in}, u_{out}) such that u_{in} has the same in-neighbors as u and u_{out} has the same out-neighbors as u.

DEFINITION 2.18

A **cylindrical wall** of order k is a subdivision of W_k.

Overview of The Proof

Directed Tree Decomposition

Part 3 ↓

Bramble

Part 4 ↓

Web

Part 5 ↓

Fence

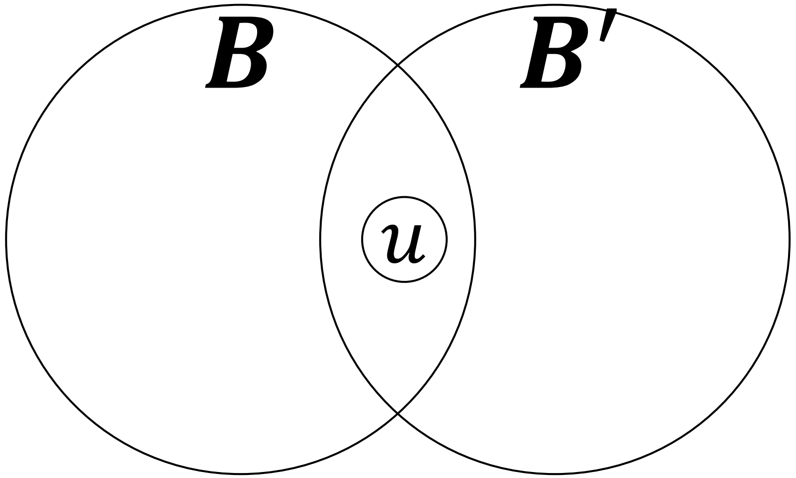
Part 6 ↓

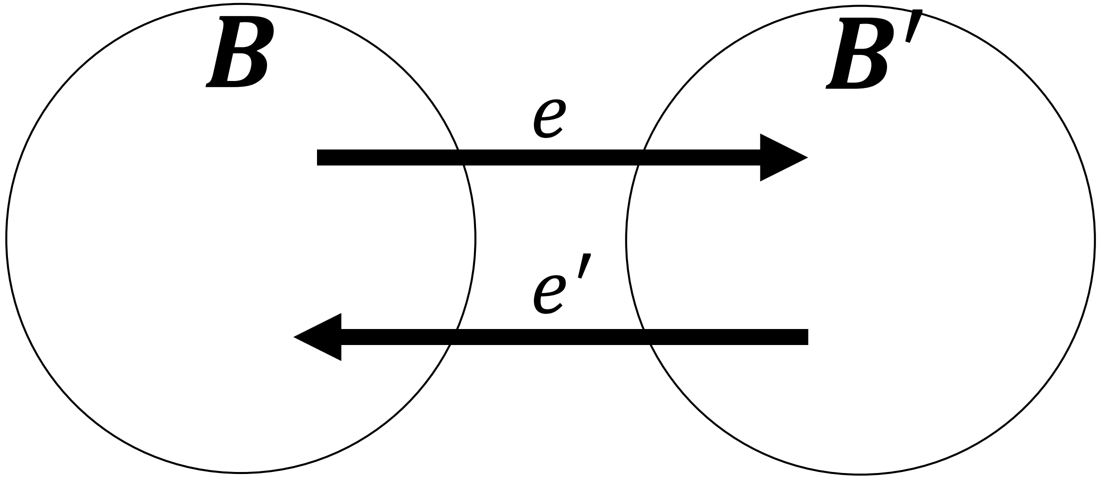
Cylindrical Grid

3. Directed Tree Decomposition → Bramble

DEFINITION 3.1

A **bramble** $\mathcal{B}$ in a digraph G is a set of strongly connected subgraphs B of G such that for all $B, B' \in \mathcal{B}$, either $V(B) \cap V(B') \neq \emptyset$ or there are $e, e' \in E(G)$ such that e links B to B' and e' links B' to B.





DEFINITION 3.2

A **cover** of a bramble $\mathcal{B}$ in a digraph G is a set $C \subseteq V(G)$ such that $V(B) \cap X \neq \emptyset$ for all $B \in \mathcal{B}$.

The **order** of $\mathcal{B}$ is the minimum size of a cover of $\mathcal{B}$.

The **bramble number** $bn(G)$ of G is the maximum order of a bramble in G.

3. Directed Tree Decomposition → Bramble

LEMMA 3.3

There are constants c, c' such that for all digraphs G', $bn(G) \leq c \cdot dtw(G) \leq c' \cdot bn(G)$.

OUTLINE OF THE PROOF

Can be proved by converting brambles into havens and back using (3.2) here.

Overview of The Proof

Directed Tree Decomposition

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4. Bramble → Web

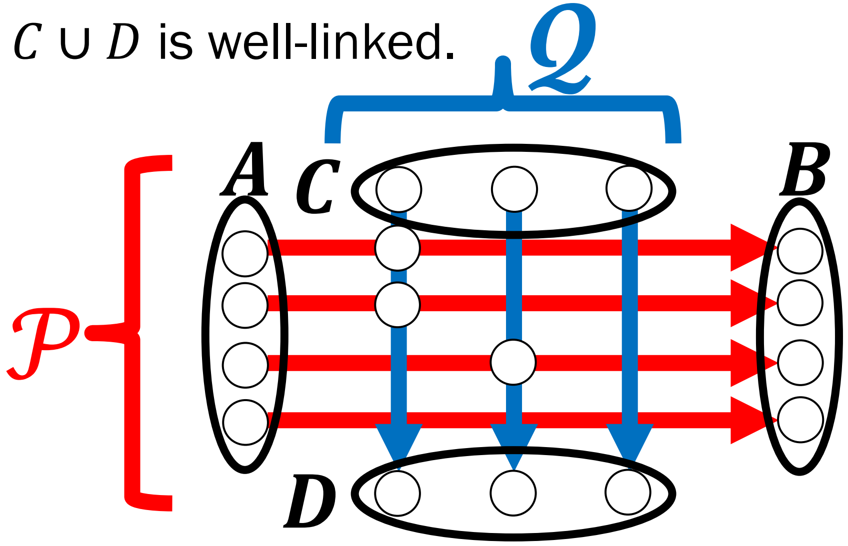
DEFINITION 4.1

Let $p, q, d \geq 0$ integers. A $(\boldsymbol{p}, \boldsymbol{q})$**-web** with avoidance d in a digraph G is a pair $(\mathcal{P}, \mathcal{Q})$ of $A - B$ linkage and $C - D$ linkage in G of order p and q, respectively, such that

1. each path $Q \in \mathcal{Q}$ intersects with all but at most p/d paths in $\mathcal{P}$ if $d \neq 0$, and with all if $d = 0$.
2. $\mathcal{P}$ is $\mathcal{Q}$-minimal.

The set $C \cap V(\mathcal{Q})$ is called the **top** of the web, denoted $top\big((\mathcal{P}, \mathcal{Q})\big)$, and $D \cap V(\mathcal{Q})$ is the **bottom**, denoted $bot\big((\mathcal{P}, \mathcal{Q})\big)$.

The web is **well-linked** if $C \cup D$ is well-linked.



$(4,3)$-web of avoidance 2

4. Bramble → Web

THEOREM 4.2 (Main Objective)

For every $k, p, l, c \geq 1$, there's an integer l' such that the following holds. Let G be a digraph with $bn(\mathrm{G}) \geq l'$. Then G contains either

1. a cylindrical grid of order k as a butterfly minor, or
2. a well-linked $(p', l \cdot p')$-web with avoidance c for some $p' \geq p$

4. Bramble → Web

LEMMA 4.3

Let $\mathcal{B}$ be a bramble in a digraph G. Then there's a path P intersecting with every $B \in \mathcal{B}$.

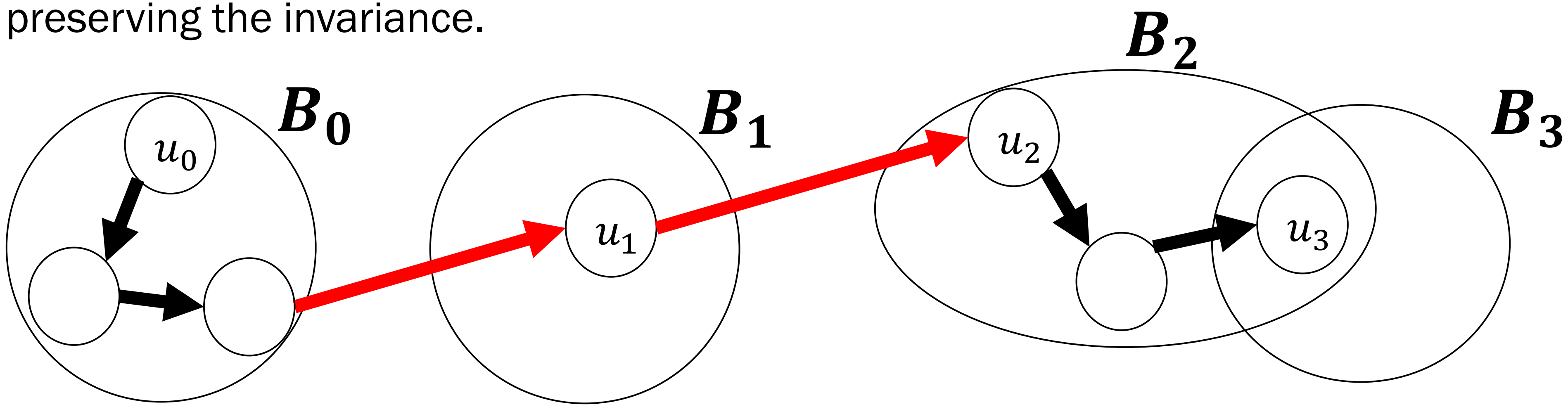
PROOF

We'll inductively construct the path P.

Choose a vertex $u_0 \in V(G)$ such that $u_0 \in B_0$ for some $B_0 \in \mathcal{B}$ and set $P = (u_0)$

We maintain the following invariance throughout the construction: there is a bramble element $B \in \mathcal{B}$ such that $B \cap V(P) = \{(\text{the last vertex of } P)\}$. It clearly holds initially.

While there's an element $B \in \mathcal{B}$ with $B \cap V(P) = \emptyset$, repeat the following: let u be the last vertex of P and $B' \in \mathcal{B}$ such that $B' \cap V(P) = \{u\}$. By the definition, there's a path P' from u to B using only edges between $B \cup B'$. Choose it so that only its endpoint is contained in B. Now we can attach P' onto P while preserving the invariance.



4. Bramble → Web

LEMMA 4.3

Let $\mathcal{B}$ be a bramble of order $k(k+2)$ in a digraph G and P a path intersecting with every $B \in \mathcal{B}$. Then there's a well-linked set $A \subseteq V(P)$ of order k.

PROOF

We first construct a sequence of subpaths $P_1, \dots, P_{2k}$ of P as follows.

Let P_1 be the minimal initial subpath of P such that $\mathcal{B}_1 \stackrel{\text{def}}{=} \{B \in \mathcal{B}: V(B) \cap V(P_1) \neq \emptyset\}$ is a bramble of order $\left\lfloor \frac{k+1}{2} \right\rfloor$. Now suppose $P_1, \dots, P_i$ and $\mathcal{B}_1, \dots, \mathcal{B}_i$ have already been constructed where $i < 2k$. Let u be the last vertex of P_i and v it's successor in P. We define P_{i+1} to be the minimal subpath of P starting at v such that $\mathcal{B}_{i+1} = \left\{B \in \mathcal{B}: B \cap \bigcup_{l \leq i} V(P_l) = \emptyset \ and \ B \cap V(P_{i+1}) \neq \emptyset\right\}$ is of order $\left\lfloor \frac{k+1}{2} \right\rfloor$.

Let a_i be the first vertex of P_{2i} and $A \stackrel{\text{def}}{=} \{a_1, \dots, a_k\}$. We'll show that A is well-linked.

Let $X, Y \subseteq A$ with $|X| = |Y|$. W.l.o.g. we may assume that $V(X) \cap V(Y) = \emptyset$. By Menger's theorem(2.11), it's enough to show that for all set $S \subseteq V(G)$ of order less than r, there's an $X - Y$ path in $G \backslash S$.

4. Bramble → Web

LEMMA 4.4

Let $\mathcal{B}$ be a bramble of order $k(k+2)$ in a digraph G and $P = P(\mathcal{B})$ a path intersecting with every $B \in \mathcal{B}$. Then there's a well-linked set $A \subseteq V(P)$ of order k.

PROOF

Let $X = \{a_{j_1}, \dots, a_{j_r}\}$ and $Y = \{a_{i_1}, \dots, a_{i_r}\}$. As $|S| < r$,

4. Bramble → Web

For the rest of the section 3, fix a digraph G with $bn(G) > 2kh(2kh + 2)$ for some $k, h \geq 1$, a bramble $\mathcal{B}$ of maximum order, a path P intersecting with every $B \in \mathcal{B}$, and a well-linked set $A \subseteq V(P)$ with $|A| = 2kh$.

Split P into subpaths $P_1, \ldots, P_h$ such that P_i is the maximum initial subpath of $P \setminus \bigcup_{j<i} P_j$ containing $2k$ elements of A.

Let $A_i = V(P_i) \cap A$. We split each A_i into A_i^{in} and A_i^{out} of order k such that each vertex in A_i^{in} appears before A_i^{out} in P.

Let $L_{i,j}$ be an $A_i^{out} - A_j^{in}$ linkage of order k (which is guaranteed to exist by the well-linkedness of A).

4. Bramble → Web

DEFINITION 4.5

Let G be a digraph and $l, p \geq 1$. An ***l*-linked path system** of order p is a tuple of sequences $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ where

1. $\mathcal{P} = (P_i)_{1 \leq i \leq p}$ is a sequence of pairwise vertex-disjoint paths in G,
2. $\mathcal{A} = \left(\left(A_i^{in}, A_i^{out}\right)\right)_{1 \leq i \leq p}$ such that A_i^{in}, A_i^{out} are subsets of $V(P_i)$, of order l, all vertices of A_i^{in} occurs before A_i^{out} in P_i, the first vertex of P_i belongs to A_i^{in} and the last to A_i^{out}, and $A = \bigcup\left(A_i^{in} \cup A_i^{out}\right)$ is a well-linked set in G, and
3. $\mathcal{L} = \left(L_{i,j}\right)_{1 \leq i \neq j \leq p}$ such that $L_{i,j}$ is a linkage of order l in G from A_i^{out} to A_j^{in} for all $i \neq j$.

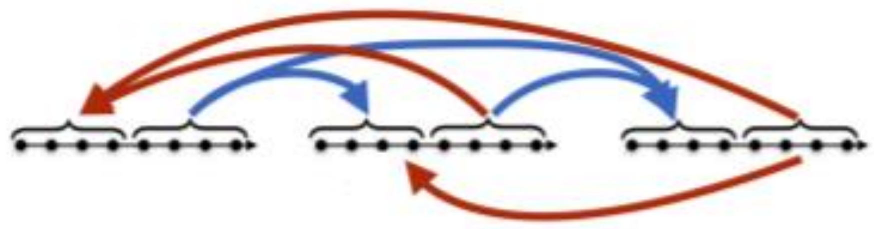


Figure 3: A 4-linked path system of order 3.

A system $\mathcal{S}$ is **clean** if for all $i \neq j$ and $Q \in L_{i,j}$, $Q \cap P_k = \emptyset$ for all $k \neq i, j$.

4. Bramble → Web

LEMMA 4.6

Let G be a digraph and $l, p \geq 1$. There is a function $f_1 \colon \mathbb{N}^2 \to \mathbb{N}$ such that if G contains a bramble of order $f_1(l, p)$ then it contains an l-linked path system of order p.

PROOF

Follows from lemma 3.4.

LEMMA 4.7

Let G be a digraph. There's a function $f_3 \colon \mathbb{N}^4 \to \mathbb{N}$ such that for all integers $l, p, k, c \geq 1$, if G contains a bramble of order $f_3(l, p, k, c)$, then G contains a clean l-linked path system of order p or a well-linked $(p', k \cdot p')$-web with avoidance c.

4. Bramble → Web

LEMMA 4.8

For every $k, p, l, c \geq 1$, there's an integer l' such that the following holds. Let $\mathcal{S}$ be a clean l'-linked path system of order k. Then either G contains either

1. a cylindrical wall of order k or
2. a well-linked $(p', l \cdot p')$-web with avoidance c.

PROOF

Let $K = k \cdot (k-1)$. We define a function $f: \{1,2,\ldots,k\} \to \mathbb{N}$ with $f(t) = (cKl)^{K-t+1}p$ and set $l' = f(1)$.

For al t, we define $g(t) = \frac{f(t)}{k \cdot l}$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ be a clean l'-linked path system of order k, where $\mathcal{P} = (P_1, \ldots, P_k)$, $\mathcal{L} = \left(L_{i,j}\right)_{1 \leq i \neq j \leq k}$ and $\mathcal{A} = \left(A_i^{in}, A_i^{out}\right)_{1 \leq i \leq k}$.

Fix an order of the pairs $\{(i,j): 1 \leq i \neq j \leq k\}$. This yields the bijection σ between $\{(i,j): 1 \leq i \neq j \leq k\}$ and $\{1,2,\ldots,k\}$.

4. Bramble → Web

LEMMA 4.8

For every $k, p, l, c \geq 1$, there's an integer l' such that the following holds. Let $\mathcal{S}$ be a clean l'-linked path system of order k. Then either G contains either

1. a cylindrical wall of order k or
2. a well-linked $(p', l \cdot p')$-web with avoidance c.

PROOF

We'll inductively construct linkages $L_{i,j}^r$ where $r \leq K$ such that

1. for all $i \neq j$ such that $\sigma(i,j) < r$, $L_{i,j}^r$ contains a single path P from A_i^{out} to A_j^{in} and P does not share an internal vertex with any path in some $L_{s,t}^r$ with $\{s,t\} \neq \{i,j\}$,
2. for $i \neq j$ with $\sigma(i,j) = r$, we have $\left|L_{i,j}^r\right| = f(r)$
3. for all $q > r$, we have $\left|L_{\sigma^{-1}(q)}^r\right| = g(r) = \frac{f(r)}{K \cdot l}$ and $L_{\sigma^{-1}(q)}^r$ is $L_{\sigma^{-1}(r)}^r$-minimal.

For $r = 1$, we choose a linkage $L_{\sigma^{-1}(1)}^1$ satisfying the second condition, and other linkages $L_{\sigma^{-1}(q)}^1$ satisfying the third condition for $q > 1$.

4. Bramble → Web

LEMMA 4.8

For every $k, p, l, c \geq 1$, there's an integer l' such that the following holds. Let $\mathcal{S}$ be a clean l'-linked path system of order k. Then either G contains either

1. a cylindrical wall of order k or
2. a well-linked $(p', l \cdot p')$-web with avoidance c.

PROOF

Now suppose the linkages have already been defined for $r < K$. Let $(i, j) = \sigma^{-1}(r)$.

Suppose for all path $P \in L_{i,j}^r$, there are i', j' with $\sigma(i', j') > r$ such that P intersects more than $\left(1 - \frac{1}{c}\right) g(r)$ paths in $L_{i',j'}^r$.

As $\left|L_{i,j}^r\right| = f(r) = g(r) \cdot K \cdot l$, by the pigeon hole principle, there is a $q > r$ such that at least $\frac{f(r)}{K} = g(r) \cdot l$ paths in $L_{i,j}^r$ intersects all but at most $\frac{g(r)}{c}$ paths in $L_{\sigma^{-1}(q)}^r$.

Let $\mathcal{Q} \subseteq L_{i,j}^r$ be the set of such paths. As a result, $\left(L_{\sigma^{-1}(q)}^r, \mathcal{Q}\right)$ forms a $\left(g(r), \frac{f(r)}{K}\right)$-web with avoidance c.

As $\frac{f(r)}{K}$=$g(r) \cdot l$ and the endpoints of the paths in $L_{i,j}^r$ are in the well-linked set $A_s^{out} \cup A_t^{in} \subseteq A$ where $(s, t) = \sigma^{-1}(q)$, this yields the second possible outcome of the lemma.

4. Bramble → Web

LEMMA 4.8

For every $k, p, l, c \geq 1$, there's an integer l' such that the following holds. Let $\mathcal{S}$ be a clean l'-linked path system of order k. Then either G contains either

1. a cylindrical wall of order k or
2. a well-linked $(p', l \cdot p')$-web with avoidance c.

PROOF

Now assume that that is never the case throughout the construction.

So there's a path $P \in L_{i,j}^r$ which, for all $q > r$, is disjoint to at least $\frac{g(r)}{c}$ paths in $L_{\sigma^{-1}(q)}^r$.

Define $L_{\sigma^{-1}(q)}^{r+1} = L_{\sigma^{-1}(q)}^r$ for all $q < r$, $L_{\sigma^{-1}(r)}^{r+1} = \{P\}$.

Let $(s,t) = \sigma^{-1}(r+1)$ and $L_{s,t}^{r+1}$ be an $A_s^{out} - A_t^{in}$ linkage of order $\frac{g(r)}{c}$ such that no path in $L_{s,t}^{r+1}$ intersects P other than its endpoints. Such a linkage exists by the choice of P.

For each $q > r+1$ and $(s', t') = \sigma^{-1}(q)$, choose an $A_{s'}^{out} - A_{t'}^{in}$ linkage $L_{s',t'}^{r+1}$ of order $g(r+1) = \frac{g(r)}{c \cdot K \cdot l}$ such that every path in it has no inner vertex in P and which is $L_{s,t}^{r+1}$-minimal.

4. Bramble → Web

LEMMA 4.8

For every $k, p, l, c \geq 1$, there's an integer l' such that the following holds. Let $\mathcal{S}$ be a clean l'-linked path system of order k. Then either G contains either

1. a cylindrical wall of order k or
2. a well-linked $(p', l \cdot p')$-web with avoidance c.

PROOF

After eventually reaching $r = K$, we have paths $P_1, \dots, P_k$ and between any two P_i, P_j with $i < j$, a path $L_{i,j}^r$. Furthermore, for all $(i,j) \neq (i',j')$ the linkages are pairwise vertex-disjoint except possibly at their endpoints. Now $\bigcup_{1 \leq i \leq k} P_i \cup \bigcup_{1 \leq i \neq j \leq k} L'_{i,j}$ contains a cylindrical wall of order k.

4. Bramble → Web

LEMMA 4.9

Let p', q', d be integers and $p \geq \frac{d}{d-1} p'$ and $q \geq q' \binom{p}{\frac{p}{d}}$. If a digraph G contains a (p, q)-web $(\mathcal{P}, \mathcal{Q})$ with avoidance d, then it contains a (p', q')-web with avoidance 0.

Overview of The Proof

Directed Tree Decomposition

Part 3

Bramble

Part 4

Web

Part 5

Fence

Part 6

Cylindrical Grid

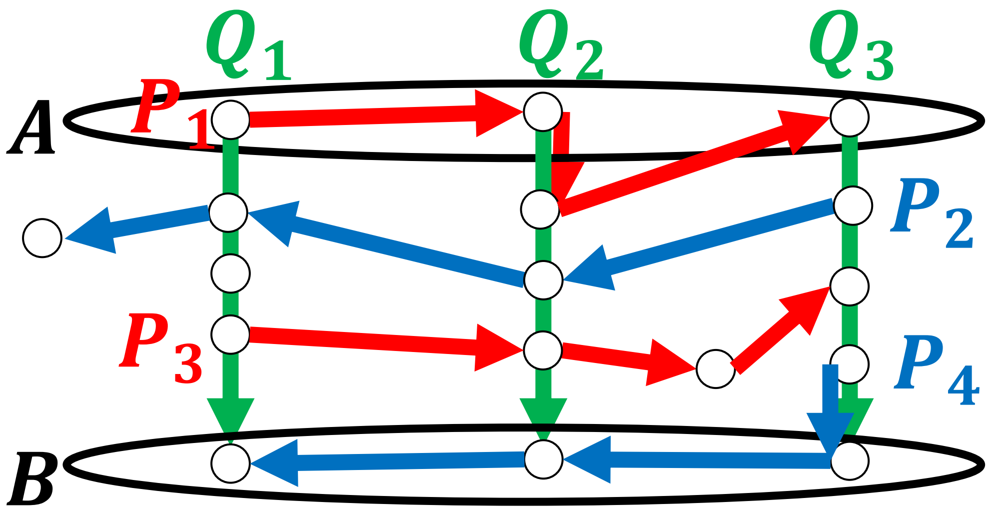
5. Web → Fence

DEFINITION 5.1

Let p, q be integers. A $(\boldsymbol{p}, \boldsymbol{q})$**-fence** in a digraph G is a sequence $\mathcal{F} = (P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ such that

1. $P_1, \dots, P_{2p}$ are pairwise disjoint paths of G and $\{Q_1, \dots, Q_q\}$ is an $A - B$ linkage for two sets $A, B \subseteq V(G)$. We call A the top, and B the bottom, and denote each by $top(\mathcal{F})$ and $bot(\mathcal{F})$, respectively,
2. $P_i \cap Q_j$ is a path for all i and j,
3. For $1 \leq j \leq q$, the paths $P_1 \cap Q_j, \dots, P_{2p} \cap Q_j$ appears in this order on Q_j, and the first vertex of Q_j belongs to P_1 and the last to P_{2p}, and
4. For odd i, the paths $P_i \cap Q_1, \dots, P_i \cap Q_q$ appears in this order on P_i, and in order of $P_i \cap Q_q, \dots, P_i \cap Q_1$ for even i

A fence is **well-linked** if $A \cup B$ is well-linked.



$(2, 3)$-fence

5. Web → Fence

THEOREM 5.2 (Main Objective)

For every $p, q \geq 1$, there's an integer p' such that any digraph G containing a well-linked (p', p')-web contains a well-linked (p, q)-fence.

5. Web → Fence

DEFINITION 5.3

An **acyclic** $(\boldsymbol{p}, \boldsymbol{q})$**-grid** is a (p, q)-web $\left(\mathcal{P} = \{P_1, \dots, P_p\}, \mathcal{Q} = \{Q_1, \dots, Q_q\}\right)$ with avoidance 0 such that

1. $P_i \cap Q_j$ is a path R_{ij} for all i, j,
2. for all i, the paths $R_{i1}, \dots, R_{iq}$ are in order in P_i, and
3. for all j, the paths $R_{1j}, \dots, R_{pj}$ are in order in Q_j.

The **top** and **bottom** of the acyclic grid is those of the underlying web.

5. Web → Fence

LEMMA 5.4

For every integer $p \geq 1$, there is an integer $p' \geq 1$ such that every digraph with an acyclic (p', p')-grid has a (p, q)-fence such that the top and bottom of the fence are subsets of the top and bottom of the acyclic grid.

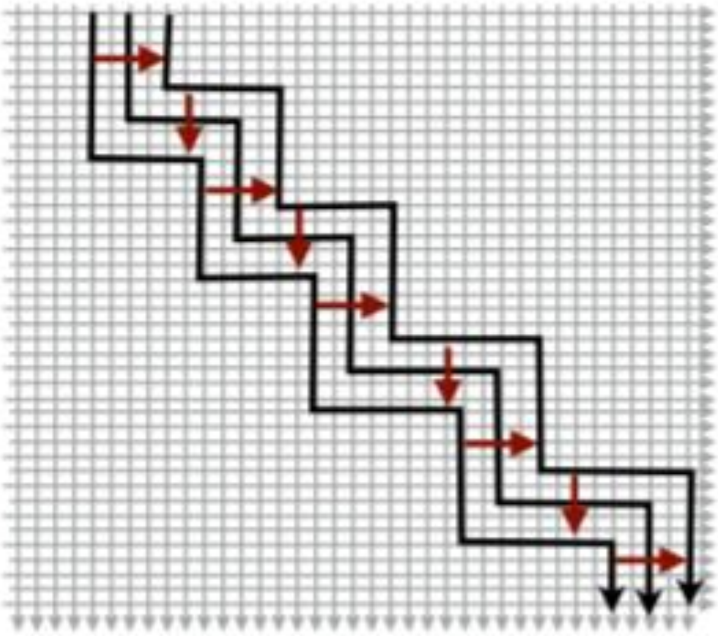


Figure 4: Constructing a Fence in a Grid.

5. Web → Fence

THEOREM 5.5

For all integers $t, d \geq 1$, there is an integer p such that every digraph G containing a well-linked (p, p)-web $(\mathcal{P}, \mathcal{Q})$ with avoidance d contains a well-linked acyclic (t, t)-grid.

5. Web → Fence

DEFINITION 5.6

Let $\mathcal{Q}^*$ be a linkage, $\mathcal{Q} \subseteq \mathcal{Q}^*$ a sublinkage of order q, and P a path intersecting every paths in $\mathcal{Q}$.

1. Let $r \geq 0$. An edge $e \in E(P) \backslash E(\mathcal{Q}^*)$ is **$\boldsymbol{r}$-splittable** with respect to $\mathcal{Q}$ and $\mathcal{Q}^*$ if there is a set $\mathcal{Q}' \subseteq \mathcal{Q}$ of order r such that for all $Q \in \mathcal{Q}'$, Q intersects with both P_1 and P_2 where P_1 and P_2 are paths with $P = P_1 e P_2$.
2. A subset $\mathcal{Q}' \subseteq \mathcal{Q}$ of order q' is a **segmentation** of P with respect to $\mathcal{Q}^*$ if there are edges $e_1, \ldots, e_{q'-1} \in E(P) \backslash E(\mathcal{Q}^*)$ with $P = P_1 e_1 P_2 \ldots e_{q'-1} P_{q'}$ for suitable subpaths $P_1, \ldots, P_{q'}$ such that $\mathcal{Q}'$ can be ordered as $(Q_1, \ldots, Q_{q'})$ such that $V(Q_i) \cap V(P) \subseteq V(P_i)$.

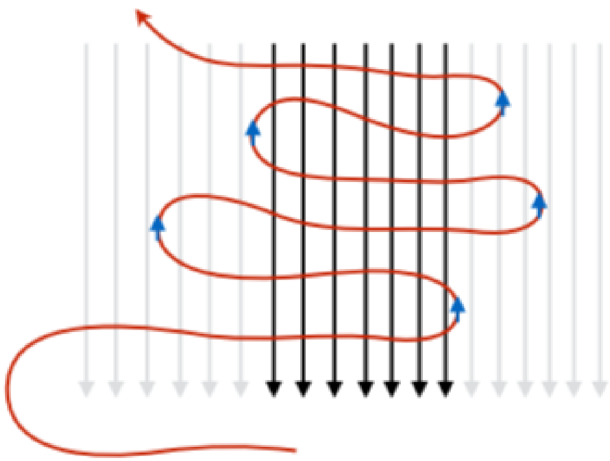
5. Web → Fence

DEFINITION 5.7

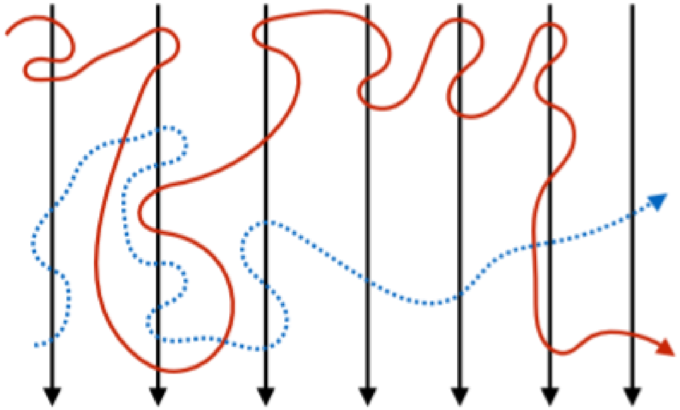
Let $\mathcal{P}$ and $\mathcal{Q}^*$ be linkages, $\mathcal{Q} \subseteq \mathcal{Q}^*$ a sublinkage of order q, and $r \geq 0$.

1. An ***r*-split** of $(\mathcal{P}, \mathcal{Q})$ of order q' with respect to $\mathcal{Q}^*$ is a pair $(\mathcal{P}', \mathcal{Q}')$ of linkages of order r and q with $\mathcal{Q}' \subseteq \mathcal{Q}$ such that $\mathcal{P}'$ can be ordered $(P_1, \dots, P_r)$ in a way that there is a path $P \in \mathcal{P}$ and edges $e_1, \dots, e_{r-1} \in E(P) \backslash E(\mathcal{Q}^*)$ with $P = P_1 e_1 P_2 \dots e_{r-1} P_r$ and every $Q \in \mathcal{Q}'$ can be segmented into subpaths $Q_1, \dots, Q_r$ such that $Q = Q_1 e_1' Q_2 \dots e_{r-1}' Q_r$ and $V(Q) \cap V(P_i) \subseteq V(Q_i)$.
2. An ***r*-segmentation** of order q' with respect to $\mathcal{Q}^*$ is a pair $(\mathcal{P}', \mathcal{Q}')$, where $\mathcal{P}'$ is a linkage of order r, $\mathcal{Q}' \subseteq \mathcal{Q}$ is a linkage of order q' such that $\mathcal{Q}'$ is a segmentation of every path P_i into segments $P_1^i e_1 P_2^i \dots e_{q'-1} P_{q'}^i$, and for all $Q \in \mathcal{Q}'$ and all $i \neq j$, if Q intersects P_i in segment P_l^i for some l, then Q intersects P_j in segment P_l^j.

An r-split $(\mathcal{P}, \mathcal{Q})$ and r-segmentation $(\mathcal{P}, \mathcal{Q})$ are **well-linked** if the set of start and endpoints of the paths in $\mathcal{Q}$ is a well-linked set.



6-split of order 7



2-segmentation of order 7

5. Web → Fence

REMARK 5.8

When $\mathcal{P}$ only contains a path P, we allow writing the r-split $(\mathcal{P}, \mathcal{Q})$ as $(P, \mathcal{Q})$.
Furthermore, as the order r-split is important, we'll often write $\left((P_1, \ldots, P_r), \mathcal{Q}'\right)$.

5. Web → Fence

LEMMA 5.9

Let $r, s \geq 0$. Let $\mathcal{Q}^*$ be a linkage and let $\mathcal{Q} \subseteq \mathcal{Q}^*$ be a sublinkage of order q. Let P be a path intersecting every path in $\mathcal{Q}$. If $q \geq r \cdot s$ then P contains an r-splittable edge with respect to $\mathcal{Q}$ and $\mathcal{Q}^*$ or there is an s-segmentation $\mathcal{Q}' \subseteq \mathcal{Q}$ with respect to $\mathcal{Q}^*$.

PROOF

It's enough to show for $q = r \cdot s$. Let $\mathcal{Q} = (Q_1, \ldots, Q_{r \cdot s})$. For $1 \leq j \leq r \cdot s$, let F_j be the minimal subpath of P that includes $V\left(P \cap Q_j\right)$.

If some edge $e \in E(P) \setminus E(\mathcal{Q}^*)$ belongs to F_j for at least r values of j, then e is r-splittable.

Otherwise, every edge in $E(P) \setminus E(\mathcal{Q}^*)$ occurs in F_j fewer than r values of j, and therefore, there are s values of j, say $j_1, \ldots, j_s$ such that $F_{j_1}, \ldots, F_{j_s}$ are pairwise disjoint. Thus $Q_{j_1}, \ldots, Q_{j_s}$ is an s-segmentation of P.

5. Web → Fence

COROLLARY 5.10

Let G be a digraph, $\mathcal{Q}^*$ a linkage in G, $\mathcal{Q} \subseteq \mathcal{Q}^*$ a sublinkage of order q, P a path in G intersecting every path in $\mathcal{Q}$, and $c \geq 0$ such that for all edge $e \in E(P) \backslash E(\mathcal{Q}^*)$, there are no c pairwise vertex-disjoint paths in G from P_1 to P_2 where $P = P_1 e P_2$. Then for all $x, y \geq 0$, if $q \geq (x + c)y$,

1. there is a y-segmentation $\mathcal{Q}' \subseteq \mathcal{Q}$ of P with respect to $\mathcal{Q}^*$ or
2. a 2-split $\big((P_1, P_2), \mathcal{Q}\big)$ of $(P, \mathcal{Q})$ order x with respect to $\mathcal{Q}^*$

PROOF

By lemma 5.9, there is a y-segmentation of P or an $(x + c)$-splittable edge $e \in E(P) \backslash E(\mathcal{Q}^*)$. In the second case, let $P = P_1 e P_2$ and $\mathcal{Q}' \subseteq \mathcal{Q}$ of order $(x + c)$ witnessing that e is $(x + c)$-splittable.

As there are less than c disjoint paths from P_1 to P_2, at most c of the paths in $\mathcal{Q}'$ intersects P_1 before P_2.

Therefore, there's a set $\mathcal{Q}'' \subseteq \mathcal{Q}'$ of order x such that for all $Q \in \mathcal{Q}''$, the last vertex of $V(P_2 \cap Q)$ occurs before the first vertex of $V(P_1 \cap Q)$. Hence, $\big((P_1, P_2), \mathcal{Q}\big)$ is a 2-split of $(P, \mathcal{Q})$.

5. Web → Fence

DEFINITION 5.11

Let $p, q, c \geq 0$ be integers and $\mathcal{Q}^*$ a linkage. A **$(\boldsymbol{p}, \boldsymbol{q})$-web with linkedness $\boldsymbol{c}$ with respect to $\boldsymbol{\mathcal{Q}}^*$** in a digraph G consists of an $A - B$ linkage $\mathcal{P} = \{P_1, .., P_p\}$ and $C - D$ linkage $\mathcal{Q} = \{Q_1, .., Q_q\}$ such that

1. for all i, Q_i intersects every path $P \in \mathcal{P}$ and
2. for all $P \in \mathcal{P}$ and every $e \in E(P) \setminus E(\mathcal{Q}^*)$, there are at most c disjoint paths from P_1 to P_2 where P_1 and P_2 are subpaths of P such that $P = P_1 e P_2$.

The set $C \cap V(\mathcal{Q})$ is called the top of the web, denoted $top\big((\mathcal{P}, \mathcal{Q})\big)$, and $D \cap V(\mathcal{Q})$ is the bottom $bot\big((\mathcal{P}, \mathcal{Q})\big)$. The web $(\mathcal{P}, \mathcal{Q})$ is **well-linked** if $C \cup D$ is well-linked.

5. Web → Fence

LEMMA 5.12

For all $p, q, r, s, c \geq 0$, and all $x, y \geq 0$ with $p \geq x$, there is a number $q' = \left(pq(q+c)\right)^{2^{(x-1)y+1}}$ such that if G contains a (p, q')-web $\mathcal{W} = (\mathcal{P}, \mathcal{Q})$ with linkedness c, then G either contains

1. a y-split $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}, \mathcal{Q})$ of order q,
2. or an x-segmentation $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}, \mathcal{Q})$ of order q.

Furthermore, if $\mathcal{W}$ is well-linked, then so is $(\mathcal{P}', \mathcal{Q}')$ (in both cases)

5. Web → Fence

DEFINITION 5.13

A $(\boldsymbol{p}, \boldsymbol{q})$-**pseudo** fence is a pair $\left(\mathcal{P} = \left(P_1, \dots, P_{2p}\right), \mathcal{Q}\right)$ of pairwise disjoint paths, where $|\mathcal{Q}| = q$, such that each $Q \in \mathcal{Q}$ can be divided into segments $Q_1, \dots, Q_{2p}$ occurring in this order in Q such that for all i, P_i intersects with each $Q \in \mathcal{Q}$ in Q_i and nowhere else. Furthermore, for all $1 \leq i \leq p$, there is an edge e_i connecting the endpoint of P_{2i} to the startpoint of P_{2i-1}.

REMAKR 5.14

Let $(\mathcal{P}', \mathcal{Q}')$ be a y-split of order q of some pair $(\mathcal{P}, \mathcal{Q})$ of linkages. Then $(\mathcal{P}', \mathcal{Q}')$ form a (y, q)-pseudo fence.

5. Web → Fence

LEMMA 5.15

There's a function $f\colon \mathbb{N} \to \mathbb{N}$ with the following property. Let $\mathcal{P}$ be a linkage and R a path intersecting every path in $\mathcal{P}$. For all $k \geq 0$, if $|\mathcal{P}| \geq f(k)$ then there's a sequence $(P_1, \dots, P_k)$ of distinct paths $P_1, \dots, P_k \in \mathcal{P}$ such that for all $1 \leq i < k$, R contains a subpath R_i from a vertex in P_i to a vertex in P_{i+1} which is internally vertex disjoint from $\bigcup_{1 \leq i \leq k} P_i$.

Furthermore, the sequence can be chosen either in a way that

1. the first vertex of $\bigcup_{1 \leq i \leq k} P_i$ on R is contained in $V(P_1)$ or
2. such that the last vertex of $\bigcup_{1 \leq i \leq k} P_i$ on R is contained in $V(P_k)$.

5. Web → Fence

LEMMA 5.16 (Split Case of Lemma 5.12)

Let f be the function on lemma 5.15. For all $k \geq 0$, $q \geq f(k)$, and $p \geq \binom{q}{k} k!\, k^2$, if G contains a p-split $\left(\mathcal{S}_{split}, \mathcal{Q}\right)$ of order q, then G contains an acyclic (k,k)-grid $(\mathcal{P}', \mathcal{Q}')$. Furthermore, $\mathcal{Q}' \subseteq \mathcal{Q}$ and for every path $P' \in \mathcal{P}'$, every subpath S of P' with both endpoints on a path in $\mathcal{Q}'$ but internally vertex disjoint from $\mathcal{Q}'$ is also a subpath of a path in $\mathcal{P}$. Finally, if $\left(\mathcal{S}_{split}, \mathcal{Q}\right)$ is well-linked, so is $(\mathcal{P}', \mathcal{Q}')$.

LEMMA 5.17 (Segmentation Case of Lemma 5.12)

Let f be the function defined on lemma 5.15. Let t be an integer and $q \geq \binom{f(3t)}{3t} \cdot 12t^2$ and $r \geq f(3t) \cdot q!$. If G contains an r-segmentation $\left(\mathcal{S}_{seg}, \mathcal{Q}\right)$ of order q, then G contains an acyclic (t,t)-grid $W' = (\mathcal{P}', \mathcal{Q}')$ such that $\mathcal{P}' \subseteq \mathcal{S}_{seg}$. Furthermore, if the set of start and end vertices of $\mathcal{Q}$ is well-linked, then so is W'. More precisely, the set of start and end vertices of $\mathcal{Q}'$ are subsets of the start and end vertices of $\mathcal{Q}$. Finally, the grid $(\mathcal{P}', \mathcal{Q}')$ can be chosen so that one (but not both) of the following properties is satisfied. Let $\mathcal{P}' = (P_1, \dots, P_t)$ be an ordering of $\mathcal{P}'$ in order in which they occur on the paths $\mathcal{Q}'$ of the grid.

1. For every $Q \in \mathcal{Q}'$, the first path $P \in \mathcal{P}$ hit by Q is P_1.
2. For every $Q \in \mathcal{Q}'$, the last path $P \in \mathcal{P}$ hit by Q is P_t.

5. Web → Fence

THEOREM 5.5

For all integers $t, d \geq 1$, there is an integer p such that every digraph G containing a well-linked (p, p)-web $(\mathcal{P}, \mathcal{Q})$ with avoidance d contains a well-linked acyclic (t, t)-grid.

PROOF

Let t, d be integers and let a (p, p)-web of avoidance d in a digraph G be given.

By the lemma 4.9, it contains a (p_1, q_1)-web with avoidance 0 as long as $p \geq \max\left(\frac{d}{d-1} p_1, q_1 \binom{p}{\frac{p}{d}}\right)$.
As noted before, any such (p_1, q_1)-web is a (p_1, q_1)-web with linkedness p_1.

By lemma 5.12, if $p_1 = p_2$, $q_1 \geq \left(p_2 q_2 (q_2 + p_1)\right)^{2^{(x-1)y+!}}$, $x = p_2$, and $y = f(3t)q2!$, then G contains a p_2-split of order q_2 or a $f(3t)q_2!$-segmentation of order q_2.

5. Web → Fence

THEOREM 5.5

For all integers $t, d \geq 1$, there is an integer p such that every digraph G containing a well-linked (p,p)-web $(\mathcal{P}, \mathcal{Q})$ with avoidance d contains a well-linked acyclic (t,t)-grid.

PROOF

In the first case, if $q_2 \geq f(t)$ and $p_2 \geq \binom{q_2}{t} t!\, t^2$, where f is the function on lemma 5.15, then lemma 5.16 implies that G contains an acyclic well-linked (t,t)-grid.

In the second case, if $q_2 \geq \binom{f(3t)}{3t} 12t^2$, then lemma 5.17 implies that G contains an acyclic well-linked (t,t)-grid.

Clearly, for any $t \geq 0$, we can choose the numbers p, p_1, p_2, x, y so that all the inequalities above are satisfied, which concludes the proof.

Overview of The Proof

Directed Tree Decomposition

Part 3

Bramble

Part 4

Web

Part 5

Fence

Part 6

Cylindrical Grid

6. Fence → Cylindrical Grid

DEFINITION 6.1

Let $(\mathcal{P}, \mathcal{Q})$ be a fence. A **$(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}})$-bottom-up linkage** is a linkage $\mathcal{R}$ from $bot(\mathcal{P}, \mathcal{Q})$ to $top(\mathcal{P}, \mathcal{Q})$. It is called **minimal $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}})$-bottom-up linkage** if $\mathcal{R}$ is $(\mathcal{P}, \mathcal{Q})$-minimal.

6. Fence → Cylindrical Grid

THEOREM 6.2 (Main Objective)

Let G be a digraph. For every $k \geq 1$, there are integers $p, r \geq 1$ such that if G contains a (p, p)-fence $\mathcal{F}$ and an $\mathcal{F}$-bottom-up linkage $\mathcal{R}$ of order r, then G contains a cylindrical grid of order k as a butterfly minor.

OUTLINE OF THE PROOF

We follow the paths in $\mathcal{R}$ from the bottom of $\mathcal{F}$ to its top and somewhere along the way we'll find a cylindrical grid, either because

1. $\mathcal{R}$ avoids a sufficiently large subfence,
2. it contains subpaths that "jump" over large fractions of the fence, or
3. $\mathcal{R}$ and $\mathcal{Q}$ intersect in a way that they generate a cylindrical grid locally.

Overview of The Proof

Directed Tree Decomposition

Part 3 ↓

Bramble

Part 4 ↓

Web

Part 5 ↓

Fence

Part 6 ↓

Cylindrical Grid

7. Epilogue

THEOREM 7.1

There is a function $f: \mathbb{N} \to \mathbb{N}$ such that every digraph of directed tree width at least $f(k)$ contains a cylindrical grid of order k as a butterfly minor.

THEOREM 7.2

There is a function $f: \mathbb{N} \to \mathbb{N}$ such that for any digraph G and a fixed constant k, in polynomial time, we can obtain either

1. a cylindrical grid of order k as a butterfly minor, or
2. a directed tree decomposition of width at most $f(k)$.

PROOF

First conclusion follows from the proof, and the second conclusion follows from the result in this which states that there's a polynomial time algorithm to to construct a directed tree decomposition of a given graph G with width $3l$ if G has directed tree width at most l.