# A Fine-Grained Perspective on Approximating Subset Sum and Partition

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Presented by Ce Jin

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# Overview

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#### Subset Sum

Given a (multi-)set X of n positive integers and a target integer t, compute OPT :=  $\max\{\Sigma(Y): Y\subseteq X, \Sigma(Y)\leq t\}$ .



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Example:  $X = \{1, 3, 4, 4\}, t = 6$ 



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A classic NP-hard problem.



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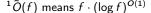
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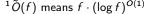
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#### Partition Problem

A special case of Subset Sum where the input set X and target integer satisfy  $t = \Sigma(X)/2$ .



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Given a (multi-)set X of n positive integers and a target integer t, compute OPT := max{ $\Sigma(Y) : Y \subseteq X, \Sigma(Y) \le t$ }.

## $(1-\varepsilon)$ -approximate Subset Sum

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Partition:  $\tilde{O}(n+1/\varepsilon^{5/3})$  (randomized) [Mucha-Węgrzycki-Włodarczyk SODA'19]

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# Main results in this paper (1)

### Conditional hardness of approximating Subset Sum

If (min, +)-convolution requires  $n^{2-o(1)}$  time, then  $(1-\varepsilon)$ -approximating Subset Sum requires  $(n+1/\varepsilon)^{2-o(1)}$  time.

(min, +)-convolution: Given a[1..n], b[1..n], compute  $c[i] = \min_{j+k=i} a[j] + b[k]$  for all i.

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(min, +)-convolution: Given a[1..n], b[1..n], compute  $c[i] = \min_{j+k=i} a[j] + b[k]$  for all i.

Previously, such a lower bound was only known for the harder problem of approximating Knapsack. [Cygan-Mucha-Węgrzycki-Włodarczyk'17, Künnemann-Paturi-Schneider'17]

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# Main results in this paper (2)

# Reduction from Subset Sum to (min, +)-convolution

If  $(\min, +)$ -convolution can be solved in T(n) time, then Subset Sum can be  $(1 - \varepsilon)$ -approximated w.h.p.<sup>a</sup> in  $\tilde{O}(n + T(1/\varepsilon))$  time.

<sup>a</sup>with high probability, i.e.,  $1-1/n^C$  for arbitrary constant C.

$$(T(n) \le n^2/2^{\Omega(\sqrt{\log n})}$$
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 $(\min, +)$ -convolution and approximating Subset Sum are *fine-grained* equivalent!

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# Main results in this paper (3)

## Better approximation algorithm for Partition

Partition can be approximated deterministically in  $\tilde{O}(n+(1/\varepsilon)^{3/2}/2^{\Omega(\sqrt{\log 1/\varepsilon})})$  time.

Improves the previous  $\tilde{O}(n+(1/\varepsilon)^{5/3})$ -time randomized algorithm.

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# Conditional lower bound

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## Knapsack

## (Exact) Knapsack

Given a set X of n items each having weight  $w_i \in \mathbb{N}^+$  and value  $v_i \in \mathbb{N}^+$ , and a capacity  $W \in \mathbb{N}^+$ , compute  $\max\{v(Y): Y \subseteq X, w(Y) \leq W\}$ , where v(Y) (and w(Y)) denote the total value (and weight) of items in Y.

We know O(nW) or  $\tilde{O}(n+W^2)$  algorithms (quadratic time).

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From (min, +)-convolution to Knapsack [Cygan et al.'17, Künnemann et al.'17]

A T(n, W)-time algorithm for Knapsack would imply a  $\tilde{O}(T(\sqrt{n}, \sqrt{n}) \cdot n)$ -time algorithm for (min, +)-convolution.



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## From Knapsack to Apx-SubsetSum (this paper)

A  $T(n, 1/\varepsilon)$ -time algorithm for approximating Subset Sum would imply a  $\tilde{O}(T(n, W) + W)$ -time algorithm for Knapsack.

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#### Decision version of Knapsack

Given a set X of n items each having weight  $w_i$  and value  $v_i$ , and a capacity W and a target total value V. Is there a  $Y \subseteq X$  such that  $w(Y) \leq W$  and  $v(Y) \geq V$ ?

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Assume all input integers are in  $\{1, 2, ..., M\}$ . Let M' = 4nM.

• Add items with  $w_i = 0$  and  $v_i = -1, -2, -4, \dots, -2^{\log M}$ . Add items with  $v_i = 0$  and  $w_i = 1, 2, 4, \dots, 2^{\log M}$ .

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- ② Define the Apx-Subset Sum instance: t := WM' V,  $\varepsilon = 1/(2W)$ , and  $x_i := w_i \cdot M' v_i$ .

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Knapsack has a solution  $\Longrightarrow$  Subset Sum has an exact solution. Knapsack has no solution  $\Longrightarrow$  Every subset sum is either > t or  $< (1 - \varepsilon)t$ .

Algorithm for Partition

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#### **Partition**

## $(1-\varepsilon)$ -Approximating Partition

Given a (multi-)set X of n positive integers, let

OPT :=  $\max\{\Sigma(Y): Y \subseteq X, \Sigma(Y) \le t\}$ , where  $t = \Sigma(X)/2$ .

Find a subset  $Y \subseteq X$  such that  $(1 - \varepsilon) \cdot \mathsf{OPT} \leq \Sigma(Y) \leq t$ .



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## Sumset

For  $A, B \subseteq \mathbb{N}$ , define

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Approximation algorithm: compute an approximate version of the sumset (which can be sparse)



# A notion of approximation

### Approximation

For  $b \in \mathbb{N}$  and  $A \subseteq \mathbb{N}$ , define

$$apx^{-}(b,A) := \max\{a \in A : a \leq b\}$$

$$apx^+(b,A) := \min\{a \in A : a \ge b\}$$

We say A  $\Delta$ -approximates B if  $A \subseteq B$  and for every  $b \in B$ ,

$$apx^+(b,A) - apx^-(b,A) \leq \Delta.$$



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This<sup>2</sup> is stronger than the usual definition of  $apx^+(b,A)-b \leq \Delta$  and  $b-apx^-(b,A) \leq \Delta$ .

#### Sumset property

If  $A_1$   $\Delta$ -approximates  $B_1$  and  $A_2$   $\Delta$ -approximates  $B_2$ , then  $A_1 + A_2$   $\Delta$ -approximates  $B_1 + B_2$ .

Notice that the approximation error  $\Delta$  doesn't blow up to  $2\Delta$ !

<sup>2</sup>Inspired by Kellerer et al.'03

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#### Transitivity

If A  $\Delta$ -approximates B, and B  $\Delta$ -approximates C, then A  $\Delta$ -approximates C.

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#### Sparsification

Given a set  $B \subseteq [t]$  <sup>a</sup>, we can compute in linear time a set A that sparsely  $\Delta$ -approximates B, where  $|A| \leq O((t/\Delta) + 1)$ .

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#### Approximate Sumset computation:

If  $A_1$  sparsely  $\Delta$ -approximates  $B_1\subseteq [t]$ ,  $A_2$  sparsely  $\Delta$ -approximates  $B_2\subseteq [t]$ , then we can compute a sparse  $\Delta$ -approximation of  $B_1+B_2$  in  $O(T_{minconv}(t/\Delta))$  time.  $(T_{minconv}(n)\leq n^2/2^{\Omega(\sqrt{\log n})}))$ 

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Goal: compute a sparse  $\Delta$ -approximation of  $\{0, x_1\} + \cdots + \{0, x_n\}$ .

Algorithm: Use approximate sumset computation to merge in a binary-tree-like fashion.

The total time in each level is at most  $O(n+(\Sigma(X)/\Delta)^2)=O(n+1/\varepsilon^2)$ 



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Issue: what if  $y \le t$  but  $\Sigma(Y) > t$  (an invalid subset)?

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Recall  $t = \Sigma(X)/2$ .

Set  $\Delta = \varepsilon \Sigma(X)/4$ .

Choose a parameter  $1 \le L \le 1/\varepsilon$ 

Divide  $\{x_1, x_2, \dots, x_n\}$  into L groups of balanced sizes.

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Take 
$$Y' = X - Y$$
 with  $\Sigma(Y') = \Sigma(X) - Y \ge \Sigma(X) - (t + \Delta) = t - \Delta$ .

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Choose  $L \approx 1/\varepsilon^{1/2}$ , the total time is  $\approx n + 1/\varepsilon^{3/2}$ .



This paper settled the complexity of approximating **Subset Sum**. There are a few other open problems.

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Thanks!

