Interval Vertex Deletion Admits Polynomial Kernel

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Sep 10, 2020

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Intro

Graph Modification Problem

The **Graph Modification Problem** is an intriguing type of problem. Given a class $\mathcal{F}$ of graphs, The $\mathcal{F}$-modification problem inquires

> *Given a graph G, Is it possible to add/delete at most k vertices/edges in order to make modified graph belong to $\mathcal{F}$?*

Each of them are called vertex/edges completion/deletion problems, respectively. There are a variety of special named questions about the **vertex deletion problem**.

- Feedback Vertex Set – $\mathcal{F}$ is consisted of acyclic graphs.
- Maximum Clique Problem – $\mathcal{F}$ is consisted of complete graphs.
- Minimum Vertex Cover – $\mathcal{F}$ is consisted of edgeless graphs.

...most of them looks quite hard.

Graph Modification Problem

Class $\mathcal{F}$ is called **hereditary**, if $G \in \mathcal{F}$ implies that any induced subgraph of G belongs to $\mathcal{F}$ as well.

Theorem (Lewis & Yannakakis, 1980)

- For any nontrivial hereditary graph class $\mathcal{F}$, $\mathcal{F}$-modification problem is **NP-hard**.

Known that the problem is *MaxSNP-hard*, which is out of our interest. After all, it is very hard to solve.

Graph Modification Problem

One of the canonical approach to challenge NP-hard problem is examining *FPT (Fixed-Parameter Tractibility)*.

A problem with *parameter* k is Fixed-Parameter Tractible if it is solvable in $\mathcal{O}(f(k) \cdot n^{\mathcal{O}(1)})$ for some computable function $f(k)$.

- f can be arbitrarily large such as $f(k) = k^{R(k,k^k)}$ as long as it's finite.
- Shoving all the 'hardness' of problem into k.

For example, vertex cover problem with n vertices and m edges may run in $\mathcal{O}(2^k \cdot \min(m, nk))$ time, with a pruning technique – vertex cover is FPT.

Elucidating that $\mathcal{F}$-modification problem is FPT, received attention for decades — bipartite graph, forests, chordal graphs... and finally interval graph. **(Cao & Marx, 2015)**

Current best complexity is $\mathcal{O}(6^k(n+m))$ by Cao (2016).

Kernelization – Vertex Cover Example

Look at the 'pruning scheme' for a given vertex cover instance (G, k).

1. Ignore all the isolated vertices v and solve for $(G - v, k)$.
2. If there is a vertex v with degree $\geqslant k$, include v to answer and solve for $(G - v, k - 1)$.
3. If G has more than k^2 edges and neither two rules above is not available, return `No`.

After the pruning scheme, G has at most $2k^2$ vertices, and applying the previous search tree algorithm gives the overall complexity is about $\mathcal{O}(k^2 2^k + n + m)$. Note that now we may cope with the case $k = 15$ and $n = 10^6$.

Kernelization

Kernelization means a *reduction step* from a parametrized problem instance (x, k) to an equivalent instance (x', k'), where $|x'|$ is bounded by some function $f(k')$.

- If f is $k^{\mathcal{O}(1)}$, we say the problem admits a *polynomial kernel*.

The *reduction step* need not to be polynomial, but it is common for a polynomial kernel to accept a polynomial reduction step.

In fact, kernelizability and FPT is equivalent in some sense. But FPT does not guarantee the existence of a polynomial kernel.

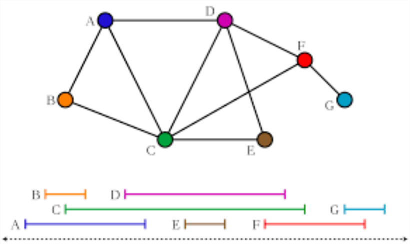
Let the story begin

Now, it's clear to say what is today's goal.

- Interval Graph Vertex Deletion (IVD) is NP-hard. (Lewis & Yannakakis, 1980)
- IVD is FPT. (Cao & Marx, SODA '14)
- IVD admits a **Polynomial Kernel. (Agrawal et al, SODA '19)**

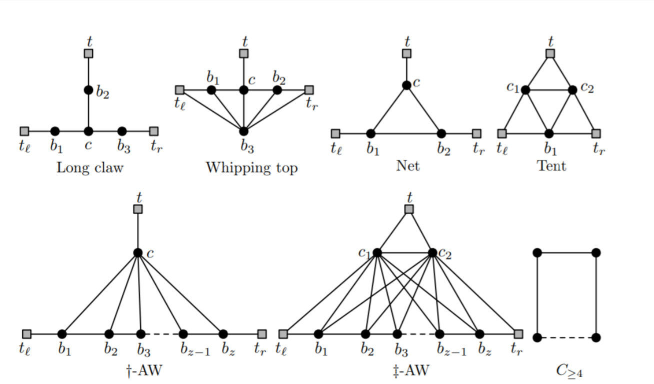
But, we didn't mention a single word about the class *Interval Graph*. We share a few slide to analyze Interval Graphs.

Interval Graph



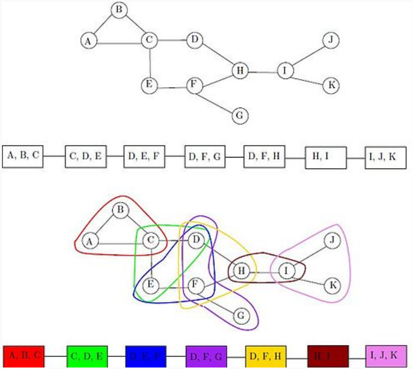
- Interval Graphs are **Chordal**. i.e. It does not have induced subgraph C_n for $n \geqslant 4$.
- Surprisingly, interval graphs are distinguishable to add a few type of forbidden induced subgraphs including the chordal condition, called *obstructions.*

Obstructions



- (t, t_l, t_r) forms a Asteroidal Triple (AT)s, two of them are connected by a path avoiding neighbor of the rest.
- t_l, t_r are called *non-shallow terminal*, t is called a *shallow terminal*.
- c_i are called 'center's, and b_i's are called 'base vertices'.
- 'Base vertices' form a chordless path.

Path Decomposition



- Another way to characterize interval graph is using the path decomoposition of graph.
- A path decomposition of G is a path equipped with some 'bags' of $V(G)$, assigning some common bags for each adjacent pair of vertices.
- Rigorous definition goes to the next slide.

Path Decomposition

A path decomposition (P, β) of G is consisted of a path P and 'bags' $\beta : V(P) \to 2^{V(G)}$ such that,

- $\bigcup_{p \in V(P)} \beta(p) = V(G)$ — bags cover the whole vertex set.
- If $uv \in E(G)$, $\exists\, q \in V(P)$ s.t. $u, v \in \beta(q)$. — adjacent vertices have some common bags.
- $P_v := \{p \in V(P) \mid v \in \beta(p)\}$ is a subpath of P — Each vertex is contained in some consecutive bags.

If $\beta(p)$ are distinct maximal cliques, (P, β) is called a *clique path.*

Theorem.

- Graph G is interval if and only if G has a clique path.

1. Computing Redundant Sets – (Chap 3)

Terminology Setup

Given an IVD instance (G, k).

- $X \subset V(G)$ is called *solution* if $G - X$ is an interval graph.
- Solution X is called a t-solution if $|X| \leqslant t$.

- For $\mathcal{W} \subseteq 2^{V(G)}$, $S \subset V(G)$ is a *hitting set* of $\mathcal{W}$ if $W \cap S \neq \emptyset \; \forall W \in \mathcal{W}$.
- $\mathcal{W}$ is called t-necessary if all t-solutions hits $\mathcal{W}$.

- Obstruction $\mathbb{O}$ is *covered* by $\mathcal{W}$, if there is $W \in \mathcal{W}$ s.t. $W \subset V(\mathbb{O})$.
- $M \subset V(G)$ is called t-redundant w.r.t $W \in \mathcal{W}$, if for every obstruction $\mathbb{O}$ either
  - $\mathbb{O}$ is covered by $\mathcal{W}$.
  - $|M \cap V(\mathbb{O})| > t$

Terminology Setup

All the other words are generic. What do you mean by *necessary* and *redundant*?

- We will eventually use the $(k+2)$-necessary family $\mathcal{W}$ and a 9-redundant solution M w.r.t. $\mathcal{W}$ such that $\mathcal{W} \subseteq 2^M$.
- Often we drop a carefully chosen vertex $v \notin M$, and claim that X, the solution of $(G-v,k)$ is a solution for (G,k).
- If it's *false*, there is an obstruction $\mathbb{O}$ such that $V(\mathbb{O}) \subset G-X$, and definitely $v \in V(\mathbb{O})$. Now we prove that $|V(\mathbb{O}) \cap M| > 9$.
- Otherwise, $\mathbb{O}$ is covered by $\mathcal{W}$. Thus there is a $Z \in \mathcal{W}$ such that $Z \subseteq V(\mathbb{O})$. Then $X \cup \{v\}$, being a $(k+1)$-solution, intersects with Z hence $X \cap Z \neq \emptyset$. This contradicts to the choice of $\mathbb{O}$.

Hence, we only need to consider the *large* obstructions *highly intersecting* with M, which leads to a structural benefit.

Main Lemma

Lemma 3.1 (Agrawal)

- Given an IVD instance (G, l). For a fixed $r \in \mathbb{N}$, it is poly-time attainable to either verify that (G, l) is `No`-instance, or compute $(\mathcal{W}, M)$ such that
  - $\mathcal{W} \subseteq 2^M$.
  - $\mathcal{W}$ is l-necessary.
  - M is r-redundant w.r.t. $\mathcal{W}$.
  - M is $(r+1)(6l)^{r+1}$-solution.

To prove Lemma 3.1, we use the following algorithm as a black-box.

Propo 3.1 (Cao)

- IVD admits a poly-time 6-approximation algorithm. We call this `ApproxIVD`(G).

Analyzing tool — copy of a graph

Define $G' := G.\text{copy}(U,t)$ for $U \subset V(G)$ and $t \in \mathbb{N}$, such that

- $V(G') = \{v^0 \mid v \in V(G)\} \cup \{v^i \mid (v,i) \in U \times [t]\}$
- $E(G') = \{u^i v^j \mid uv \in E(G), 0 \leqslant i,j \leqslant t\} \cup \{v^i v^j \mid v \in V(G), 0 \leqslant i < j \leqslant t\}$

So we make t replicas for vertices in G, and made a clique with clones. We sometimes identify G as a subgraph of G'.

Note that G' is also interval graph for an interval graph G.

In this problem, we specify $G' := G.\text{copy}(U, 6l+1)$, and analyze `ApproxIVD`$(G')$.

Let $A :=$ApproxIVD(G'). The following statement holds.

- If $|A| > 6l$, then $\{U\}$ is l-necessary.
- If $|A| \leqslant 6l$, Every obstruction $\mathbb{O}$ of G intersects with $(A \cap V(G)) \backslash U$.

Thus, we may obtain a necessary set or augment the redundancy.

Proof.

- If $|A| > 6l$, (G', l) is a No-instance. If $\{U\}$ is not necessary, there is an l-solution L s.t. $L \cap U = \emptyset$. Thus $G - L$ is an interval graph containing U. Thus $(G - L).\text{copy}(U, 6l + 1)$ equals $G' - L$ and it's an interval graph. Contradiction, since L becomes an l-solution for (G', l).
- If $|A| \leqslant 6l$, every $b \in V(\mathbb{O}) \cap U$ has a 'twin' $b^{i(b)} \in G' \backslash A$ since b has $6l + 1$ duplicates. Exchanging b and $b^{i(b)}$ gives another obstruction, hence there should be a vertex $c \in (A \cap V(G)) \backslash U$.

The algorithm `RedundantIVD`

$M_0 :=$`ApproxIVD`(G), $\mathcal{W}_0 := \emptyset$, $\mathcal{T}_0 := \{(v) \mid v \in V(M_0)\}$.
If $|V(M_0)| > 6l$ return `No`. For $i = 1, \cdots, r$,

1. $M_i := M_{i-1}$, $\mathcal{W}_i := \mathcal{W}_{i-1}$, $\mathcal{T}_i := \emptyset$.
2. For $\mathbf{v} = (v_0, \cdots, v_{i-1}) \in \mathcal{T}_{i-1}$
   2.1 $A :=$ `ApproxIVD`$(G.\text{copy}(\mathbf{v}, 6l+1))$.
   2.2 If $|A| > 6l$, $\mathcal{W}_i.\text{append}(\mathbf{v})$
   2.3 Else for each $u \in (A \cap V(G)) \backslash \mathbf{v}$, $\mathcal{T}_i.\text{append}(\mathbf{v} :: u)$, $M_i.\text{append}(u)$

return $(M_r, \mathcal{W}_r)$.

- It is clear that $\mathcal{W}_i$ is l-necessary and M_i is i-redundant solution w.r.t $\mathcal{W}_i$, and $\mathcal{W}_i \subseteq 2^{M_i}$.
- Now we analyze the size of $M_r, \mathcal{W}_r$.
  - $|\mathcal{T}_i| \leqslant (6l)^{i+1}$.
  - $|M_i| \leqslant |M_{i-1}| + 6l\,|\mathcal{T}_i| \leqslant \sum_{j=0}^{i} (6l)^{j+1} \leqslant (i+1)(6l)^{i+1}$.
  - $|\mathcal{W}_i| \leqslant |\mathcal{T}_{i-1}| \leqslant (6l)^{i+1}$.

Intermission

Remaining set

Now, we have a 9-redundant $\mathcal{O}(k^{10})$-solution M w.r.t $k+2$-necessary family $\mathcal{W}$. Additionally, we may erase all the singletons in $\mathcal{W}$ and include them in the answer.

However, we still have to bound the kernel in $G-M$. To resolve this, we introduce the concept of **graph modules**.

$C \subseteq V(G)$ is a **module** if all the vertices in C has same neighbor set outside C. (i.e. for all $v \notin C$, $N_G(v) \cap C \in \{\emptyset, C\}$)

We classify the connected components of $G-M$ into module (in G) components and non-module (in G) components. Then we need to:

- Bound the number of distinct components in kernel
- Bound the size of each components in kernel

Easy parts

- Bounding the number of non-module components is easy.
- Bounding the size of module components is easy.
- Bounding the number of module components is involved.
- Bounding the size of non-module components is *dreadful*.

Number of non-module components

Claim.

- It is possible to leave at most $(k+2)\,|M| = \mathcal{O}(k^{11})$ non-module components.

If not, there is a vertex $w \in M$ such that for at least $k+3$ non-module components $E_1, \cdots, E_{k+3}$, each E_i contains $u_i v_i \in E(G)$ such that $u_i w \in E(G)$ but $v_i w \notin E(G)$. Then for every k-subset S excluding w of G, $G - S$ contains a **Long Claw** centred at w. Thus every k-solution must contain w. So we simply pick w and remove from G until the claim hold.

Size of module components

We introduce a simple proposition beforehand.

- Given an obstruction $\mathbb{O}$ with size > 4 and a module component D, either $V(\mathbb{O}) \subseteq V(D)$ or $|V(\mathbb{O}) \cap V(D)| \leqslant 1$.

Claim

- For a minimal obstruction $\mathbb{O}$ not covered by $\mathcal{W}$ and a module component of D, $|V(\mathbb{O}) \cap V(D)| \leqslant 1$ holds.

$\mathbb{O}$ is not covered by $\mathcal{W}$ means $|M \cap V(\mathbb{O})| > 9$, thus $|V(\mathbb{O})| > 4$ holds. Also D is interval graph, hence $V(\mathbb{O}) \not\subseteq V(D)$. Thus $|V(\mathbb{O}) \cap V(D)| \leqslant 1$.

Size of module components

If there is a module component D with $|V(D)| > k+1$, we just erase an arbitrary vertex $v \in V(D)$ and solve for $(G-v, k)$.

If S is a k-solution for $(G-v,k)$, we claim that S is a k-solution for G as well. If not, take a minimal obstruction $\mathbb{O}$ of $G-S$, and v must be included.

Since $S+v$ is $(k+1)$-solution, S hits $\mathcal{W}$ by its $(k+2)$-necessity. Also, $\mathcal{W} \subset 2^M$ guarantees that S hits $\mathcal{W}$. S and $\mathbb{O}$ are disjoint, hence $\mathbb{O}$ is not covered by $\mathcal{W}$.

By the previous lemma, $\{v\} = V(\mathbb{O}) \cap V(D)$. Since D is a module component, $\mathbb{O} - v + u$ is also obstruction for all $u \in V(D)$. As $|V(D)| > k+1$, $|V(D) \backslash S \backslash v| \geqslant 1$ means there is an obstruction $\mathbb{O}' = \mathbb{O} - v + u'$ for some $u' \in V(D) \backslash S \backslash v$, which contradicts to that S is a k-solution for $G \backslash v$.

Main

Current score

- 9-redundant solution M w.r.t. $(k+2)$-necessary family $\mathcal{W}$: $\mathcal{O}(k^{10})$.
- Number of non-module components: $\mathcal{O}(k^{11})$
- Size of each module component: $k+1$
- Number of module components: $\mathcal{O}(k^{102})$.
- Size of each non-module component: ???

Main Lemma (Lemma 4.3)

Variables

- (G,k): IVD instance
- $\hat{M}$: some arbitrary vertex set. $M' := M \cup \hat{M}$, which is still 9-redundant.
- $\hat{\mathcal{C}}$: set of module compo's in $G - M'$.

In polynomial time,

We can do either

- Find an equivalent instance (G',k) such that G' is a strictly induced subgraph of G
- Find a set $B \subseteq V(\hat{\mathcal{C}})$ with $|B| \leqslant 8(k+1)^2 \left|M'\right|^{10}$ such that for all k-subset $S \subset V(G)$ which has a corresponding obstruction $\mathbb{O}_S$ such that $S \cap \mathbb{O}_S = \emptyset$ and not covered by $\mathcal{W}$, there is an obstruction $\mathbb{O}'$ such that $\mathbb{O}' \cap S = \emptyset$ and $\mathbb{O}' \cap (V(\hat{\mathcal{C}}) \setminus B) = \emptyset$.

Power of Lemma 4.3

To reduce the module component size, just put $\hat{M} = \emptyset$.

- If Lemma 4.3 gives (G', k), just solve for it.
- If Lemma 4.3 gives the set B, delete arbitrary $v \in V(\hat{\mathcal{C}}) \backslash B$ and solve $(G - v, k)$.

We prove that S, a k-solution for $(G - v, k)$ is also a k-solution for (G, k). Similarly assume the contrary, and then S hits $\mathcal{W}$ and the minimal obstruction $\mathbb{O}_S$ in $G - S$ is not covered by $\mathcal{W}$. Thus applying the Lemma 4.3, there is an obstruction $\mathbb{O}'$ with $\mathbb{O}' \cap S = \emptyset$ and $\mathbb{O}' \cap (V(\hat{\mathcal{C}}) \backslash B) = \emptyset$. Where $v \in V(\hat{\mathcal{C}}) \backslash B$, the existence of $\mathbb{O}'$ contradicts that $S + v$ is a $k + 1$-solution.

Thus, the reduction rule guarantees that $\left| V(\hat{\mathcal{C}}) \right| \leqslant 8(k+1)^2 \left| M' \right|^{10} = \mathcal{O}(k^{102})$.

So our final goal is to construct such B to prove the Lemma 4.3. we switch to the general case with $\hat{M} \neq \emptyset$ again.

A handy lemma (Lemma 4.6)

Lemma 4.6 (Agrawal)

- For $u \neq v \in V(G)$ with $uv \notin E(G)$, then either $\{u, v\} \in \mathcal{W}$ or $G[(N_G(u) \cap N_G(v)) - M']$ is a clique.

Assume $\{u, v\} \notin \mathcal{W}$ and there exists a pair of vertices $x, y \in (N_G(u) \cap N_G(v)) - M'$ such that $xy \notin E(G)$. Then $\mathbb{O} := G[\{u, v, x, y\}]$ is a hole C_4. $\mathbb{O}$ is not covered by $\mathcal{W}$ since $x, y \notin M'$, $\{u, v\} \notin \mathcal{W}$ and there are no singletons in $\mathcal{W}$. Thus $|M' \cap V(\mathbb{O})| > 9$, but $|V(\mathbb{O})| = 4$ leads to a contradiction.

Consequence of Lemma 4.6

Henceforth, for each non-adjacent u, v with $\{u, v\} \notin \mathcal{W}$, there is at most one module component $C_{uv} \in \hat{\mathcal{C}}$ such that $N_G(u) \cap N_G(v) \cap V(C_{uv}) \neq \emptyset$. Denote

$$\mathcal{C}^\star := \{C_{uv} \mid uv \notin E(G), \{u, v\} \notin \mathcal{W}, N_G(u) \cap N_G(v) \cap V(C_{uv}) \neq \emptyset\}$$

Then $|V(\mathcal{C}^\star)| \leqslant (k+1)\,|M'|^2 = \mathcal{O}(k^{21})$, which is small enough. so we just care for module components in $\mathcal{C}' := \hat{\mathcal{C}} \backslash \mathcal{C}^\star$.

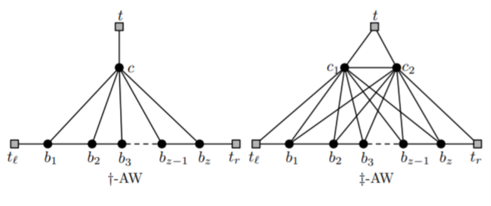
- Given a vertex $v \in V(\mathcal{C}')$ and $u \neq w \in N_G(v) \cap M'$, $\{u, w\} \in \mathcal{W}$ or $uw \in E(G)$.

Classification Lemma (Lemma 4.11)

Variables

- $C \in \mathcal{C}'$: a module component.
- $\mathbb{O}$: a minimal obstruction not covered by $\mathcal{W}$, intersecting with C.

Then $\{t\} = V(\mathbb{O}) \cap V(C)$ and $\mathbb{O}$ is an **AW**, having t as a terminal.



Since $\mathbb{O}$ is not covered, $|V(\mathbb{O})| > 9$ and thus $\mathbb{O}$ is either †-AW, ‡-AW or a hole. Since $N_G(t) \subset V(C) \cup M'$ and Lemma 4.6, for any $u, w \in N_G(t) \cap V(\mathbb{O})$ is adjacent, since $\{u, w\} \notin \mathcal{W}$. The figure implies that t must be a terminal.

- Remark that t_l, t_r are non-shallow terminals, while t is a shallow terminal.

Lemma 4.13 (Agrawal)

- There is a set $A' \subseteq V(\mathcal{C}')$ such that $|A'| \leqslant (k+1)\,|M'|^4$, all non-shallow terminals in $V(\mathcal{C}')\backslash A'$ has $k+1$ alternatives.
- Rigorously, if an obstruction $\mathbb{O}$ is not covered by $\mathcal{W}$ and $\mathbb{O}$ is an AW having a non-shallow terminal $v \in V(\mathcal{C}')\backslash A'$, there are $k+1$ elements $a \in A'$ such that $\mathbb{O} - v + a$ is still an obstruction.
- So, if S the k-solution of $G\backslash v$ is not a k-solution for (G, k)...

We won't look at the proof and the construction. It's just a case-work.

Handling shallow terminals

Lemma 4.17 (Agrawal)

- There is a set $A'' \subseteq V(\mathcal{C}')$ such that $|A''| \leqslant 4(k+1)^2\,|M'|^{10}$, all shallow terminals in $V(\mathcal{C}')\backslash(A' \cup A'')$ has $k+1$ alternatives.
- Rigorously, if an obstruction $\mathbb{O}$ is not covered by $\mathcal{W}$ and $\mathbb{O}$ is an AW having a shallow terminal $v \in V(\mathcal{C}')\backslash(A' \cup A'')$, there are $k+1$ elements $a \in A''$ such that $\mathbb{O} - v + a$ is still an obstruction.
- So, if S the k-solution of $G\backslash v$ is not a k-solution for (G, k)...

... Let's peek at this! (optional)

Observation 4.6

Let P be a chordless path. We say that P is covered by $\mathcal{W}$ if there is $W \in \mathcal{W}$ with $W \subset V(P)$, it's same as the obstructions.

- Let P be an chordless path in $G[V(G) - V(C)]$ for some $C \in \hat{\mathcal{C}}\prime$, not covered by $\mathcal{W}$ Then for all $v \in V(C)$ $|N_G(v) \cap V(P)| \leqslant 2$, and they are adjacent.
- If $u, v \in N_G(v) \cap V(P)$, since $\{u, v\} \notin \mathcal{W}$ $(u, v) \in E(G)$. Thus, u, v must be adjacent and $N_G(v) \cap V(P)$ cannot contain more than three elements to maintain chord-free property.

Handling shallow terminals

Define $M'' := M' \cup V(\mathcal{C}^\star) \cup A'$.

For all 2-subset S of M', denote $A_S := \{v \in V(\mathcal{C}')\backslash A' \mid S \subseteq N_G(v)\}$. It intuitively means the candidate of vertices can be a shallow terminal of AW, having S as center.

Denote $V(M'') = \{v_1, \cdots, v_p\}$ and $E(G[M'']) = \{e_1, \cdots, e_q\}$. We define a incidence vector $\mathrm{inc}(u) \in \mathbb{F}_2^{p+q+1}$ for all $u \in V(\mathcal{C}')\backslash A'$ as:

- $\mathrm{inc}(u)_i$ is 1 iff $v_i \in N_G(u)$. $(1 \leqslant i \leqslant p)$
- $\mathrm{inc}(u)_{p+i}$ is 1 iff both endpoints of e_i is in $N_G(u)$. $(1 \leqslant i \leqslant q)$
- $\mathrm{inc}(u)_{p+q+1} = 1$.

inc_i completely describes the possible incidence relation of u.

Basis Marking Strategy

For every 2-subset $S \subset M'$, define the **multiset** of vectors $\mathbf{V}_S := \{\mathrm{inc}(u) \mid u \in A_S\}$. Then we gradually build up the bases:

1. $\mathbf{U}_S^0 := \emptyset$.
2. For $i = 1, \cdots, k+1$, build the basis $\mathbf{B}_S^i$ for the vector subspace $\mathbf{V}_S \backslash \mathbf{U}_S^{i-1}$, and assign $\mathbf{U}_S^i = \mathbf{U}_S^{i-1} \cup \mathbf{B}_S^i$.
3. For every $\mathbf{v} \in \mathbf{U}_S^{k+1}$, choose an arbitrary vertex $u_\mathbf{v}$ such that $\mathrm{inc}(u) = \mathbf{v}$.
4. Denote $A''_S := \{u_\mathbf{v} \mid \mathbf{v} \in \mathbf{U}_S^{k+1}\}$.

Then merge all A''_S into a single set $A'' := \bigcup_S A''_S$. Note that $\left|A''_S\right| \leqslant (k+1)(p+q) \leqslant (k+1)p^2$ since A''_S is $k+1$-union of vector spaces with dim $(p+q)$. Thus $|A''| \leqslant (k+1)\,|M'|^2\,|M''|^2 \leqslant 4(k+1)^2\,|M'|^{10}$.

Nullifying path

Define a vector $\mathbf{n}^{X,Y}$, such that $\mathbf{n}_i^{X,Y} = 1$ if and only if $i \in X$ or $i - p \in Y$ or $i = p + q + 1$. i.e. $\mathbf{n}_i$ extracts the integer from vertex set X and the edge set Y.

For a path P, $\mathbf{n}^P := \mathbf{n}^{V(P) \cap M'', E(P) \cap E(G[M''])}$.

Lemma 4.16 (Agrawal)

- Let P be an induced path in $G[V(G) \setminus V(C)]$ for some $C \in \mathcal{C}'$, such that P is not covered by $\mathcal{W}$. For all $u \in V(C)$, $\mathbf{n}^P \cdot \mathrm{inc}(u) = 1 \pmod 2$ iff $N_G(u) \cap V(P) = \emptyset$.

Since the only possible case is $|N_G(u) \cap V(P)| = 0, 1, 2$, examining each case gives the proof.

Lemma 4.17 (Restated)

- Let $w \in V(\hat{\mathcal{C}}')\backslash(A' \cup A'')$, and $\mathbb{O}$ be an AW with the shallow termiinal w. Let S be the center of $\mathbb{O}$, then for all $i \in [k+1]$ there is a $v \in \mathbf{B}_S^i$ such that $\mathbb{O} - w + u_{\mathbf{v}}$ is still an obstruction.

Since $w \in A_S$ but $w \notin A''$, $\mathrm{inc}(w) \in \mathbf{V}_S$ is not spent by $\mathbf{U}_S^i$. Thus $\mathrm{inc}(w) \in \mathbf{V}_S \backslash \mathbf{U}_S^{i-1}$ for all $i \in [k+1]$.

Thus

$$\mathrm{inc}(w) = \mathrm{inc}(b_{i1}) + \cdots + \mathrm{inc}(b_{it})$$

for some $\mathrm{inc}(b_{ij}) \in \mathbf{B}_S^i$. Taking a dot product with the chordless path P of $\mathbf{O}$,

$$1 = \mathbf{n}^P \cdot \mathrm{inc}(b_{i1}) + \cdots + \mathbf{n}^P \cdot \mathrm{inc}(b_{it})$$

indicates that there is some j such that $\mathbf{n}^P \cdot \mathrm{inc}(b_{ij}) = 1$. Then $\mathbb{O} - w + b_{ij}$ is still an obstruction, and we may choose j for all $i \in [k+1]$.

Summary

Hence, all the vertices in $V(\hat{\mathcal{C}}) \setminus (V(\mathcal{C}^\star) \cup A' \cup A'')$ is a terminal of some AW, and there are $k+1$ 'replica obstructions' in $(V(\mathcal{C}^\star) \cup A' \cup A'')$.

Since $|(V(\mathcal{C}^\star) \cup A' \cup A'')| = \mathcal{O}(k^{102})$, our main Lemma 4.3 is proved.

Epilogue

Current score

- 9-redundant solution M w.r.t. $(k+2)$-necessary family $\mathcal{W}$: $\mathcal{O}(k^{10})$.
- Number of non-module components: $\mathcal{O}(k^{11})$
- Size of each module component: $k+1$
- Number of module components: $\mathcal{O}(k^{102})$.
- Size of each non-module component: ???

In current state, we may kernelize the subset with additional parameter – the vertex cover size.

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In current state, we may kernelize the subset with additional parameter – the vertex cover size.

Introducing M as a vertex cover, all the components in $G-M$ are independent set.

But we know that, the vertex cover can grow arbitrarily large. Thus for completion, we have to bound the size of each non-module component.

How to?

First, we adopt the clique-path decomposition of each non-module component. Then we may deal with each independent question:

- 9-redundant solution M w.r.t. $(k+2)$-necessary family $\mathcal{W}$: $\mathcal{O}(k^{10})$.
- Number of non-module components: $\mathcal{O}(k^{11})$
- Size of each module component: $k+1$
- Number of module components: $\mathcal{O}(k^{102})$.
- Size of each non-module component: $?_1 \times ?_2$
  - Maximum size of bags in clique-path: $?_1$ (section 5, pp.21-40)
  - Maximum clique-path length in each non-module component: $?_2$ (section 6, pp.40-62)

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  - Maximum size of bags in clique-path: $?_1$ (section 5, pp.21-40)
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$?_1 = \mathcal{O}(k^{101})$, $?_2 = \mathcal{O}(k^{1629})$. Several marking strategies are used for each cases.

Hence, we obtain size $\mathcal{O}(k^{1741})$ kernel for IVD.

Thanks!

- One simpler problem is Chordal Vertex Deletion, which has $\mathcal{O}(k^{13})$ kernel found by Agrawal et al.
- The author believes there is about $\mathcal{O}(k^{10})$ kernel and leaves it to an open problem.
- The accessible paper is quite faulty... There are some calculation errors and typos.