COMP 382 Assignment 2 - Topic 1
Context-Free Languages are not closed under intersection

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1 Introduction

In this assignment, we demonstrate that the set of context-free languages is not closed under the operation of intersection. We prove this by showing that the language $L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free, but can be expressed as the intersection of two context-free languages.

2 Construction of Two Context-Free Languages

2.1 Language L1

Let $L_1 = \{a^i b^j c^k \mid i = j, i, j, k \geq 0\}$.

Definition 1. *The context-free grammar G_1 for L_1 is defined as:*

$$S \to AB \tag{1}$$
$$A \to aAb \mid ab \tag{2}$$
$$B \to Bc \mid c \mid \varepsilon \tag{3}$$

The nonterminal A generates strings of the form $a^n b^n$ for $n \geq 1$, and B generates strings of the form c^k for $k \geq 0$. Therefore, L_1 contains all strings where the number of a's equals the number of b's, followed by any number of c's.

2.2 Language L2

Let $L_2 = \{a^i b^j c^k \mid j = k, i, j, k \geq 0\}$.

Definition 2. *The context-free grammar G_2 for L_2 is defined as:*

$$S \to AB \tag{4}$$
$$A \to aA \mid a \mid \varepsilon \tag{5}$$
$$B \to bBc \mid bc \tag{6}$$

The nonterminal A generates strings of the form a^i for $i \geq 0$, and B generates strings of the form $b^n c^n$ for $n \geq 1$. Therefore, L_2 contains all strings with any number of a's followed by equal numbers of b's and c's.

3 Intersection of L1 and L2

Theorem 1. $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$

Proof. Let $w \in L_1 \cap L_2$. Then $w \in L_1$ and $w \in L_2$.
Since $w \in L_1$, we have $w = a^i b^j c^k$ where $i = j$.
Since $w \in L_2$, we have $w = a^i b^j c^k$ where $j = k$.
Therefore, $i = j = k$, which means $w = a^n b^n c^n$ for some $n \geq 0$.
Conversely, if $w = a^n b^n c^n$ for some $n \geq 0$, then:

- $w \in L_1$ because the number of a's equals the number of b's
- $w \in L_2$ because the number of b's equals the number of c's

Therefore, $w \in L_1 \cap L_2$.
Hence, $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$. □

4 Proof that L is not Context-Free

Theorem 2. *The language $L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.*

Proof. We prove this by contradiction using the pumping lemma for context-free languages.

Lemma 1 (Pumping Lemma for Context-Free Languages)**.** *If L is a context-free language, then there exists a constant $p > 0$ such that for any string $s \in L$ with $|s| \geq p$, we can write $s = uvwxy$ where:*

(i) $|vwx| \leq p$

(ii) $|vx| \geq 1$

(iii) For all $i \geq 0$, $uv^i wx^i y \in L$

Assume, for the sake of contradiction, that L is context-free. Let p be the pumping length guaranteed by the pumping lemma.

Consider the string $s = a^p b^p c^p$. Since $s \in L$ and $|s| = 3p \geq p$, the pumping lemma applies.

By the pumping lemma, we can write $s = uvwxy$ where conditions (i), (ii), and (iii) hold.

We analyze all possible cases for the position of the substring vwx:

Case 1: vwx contains only a's. Then $v = a^k$ and $x = a^m$ for some $k, m \geq 0$ with $k + m \geq 1$. Pumping up with $i = 2$, we get $uv^2wx^2y = a^{p+k+m} b^p c^p$. This string has more a's than b's or c's, so it is not in L.

Case 2: vwx contains only b's. Then $v = b^k$ and $x = b^m$ for some $k, m \geq 0$ with $k + m \geq 1$. Pumping up with $i = 2$, we get $uv^2wx^2y = a^p b^{p+k+m} c^p$. This string has more b's than a's or c's, so it is not in L.

Case 3: vwx contains only c's. Then $v = c^k$ and $x = c^m$ for some $k, m \geq 0$ with $k + m \geq 1$. Pumping up with $i = 2$, we get $uv^2wx^2y = a^p b^p c^{p+k+m}$. This string has more c's than a's or b's, so it is not in L.

Case 4: vwx spans across a's and b's. Since $|vwx| \leq p$, the substring vwx cannot span all three types of symbols. Pumping will create an imbalance between the number of a's and b's on one side, and c's on the other side, so the resulting string is not in L.

Case 5: vwx spans across b's and c's. Similar to Case 4, since $|vwx| \leq p$, pumping will create an imbalance between the number of b's and c's on one side, and a's on the other side, so the resulting string is not in L.

In all cases, pumping leads to a string that is not in L, which contradicts condition (iii) of the pumping lemma.

Therefore, our assumption that L is context-free must be false. Hence, $L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free. $\square$

5 Conclusion

We have successfully demonstrated that:

1. We constructed two context-free languages L_1 and L_2.

2. We showed that $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$.

3. We proved that $\{a^n b^n c^n \mid n \geq 0\}$ is not context-free using the pumping lemma.

Therefore, the set of context-free languages is not closed under the operation of intersection.

6 References

- Hopcroft, J. E., Motwani, R., & Ullman, J. D. (2006). *Introduction to Automata Theory, Languages, and Computation* (3rd ed.). Addison-Wesley.

- Sipser, M. (2012). *Introduction to the Theory of Computation* (3rd ed.). Cengage Learning.