# COMP 382 Assignment 2 - Topic 1
# Context-Free Languages are not closed under intersection

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October 23, 2025

## 1 Introduction

In this assignment, we demonstrate that the set of context-free languages is not closed under the operation of intersection. We prove this by showing that the language $L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free, but can be expressed as the intersection of two context-free languages.

## 2 Construction of Two Context-Free Languages

### 2.1 Language L1

Let $L_1 = \{a^i b^j c^k \mid i = j, i, j, k \geq 0\}$.

**Definition 1.** *The context-free grammar $G_1$ for $L_1$ is defined as:*

$$S \to AB \tag{1}$$
$$A \to aAb \mid ab \tag{2}$$
$$B \to Bc \mid c \mid \varepsilon \tag{3}$$

The nonterminal $A$ generates strings of the form $a^n b^n$ for $n \geq 1$, and $B$ generates strings of the form $c^k$ for $k \geq 0$. Therefore, $L_1$ contains all strings where the number of $a$'s equals the number of $b$'s, followed by any number of $c$'s.

### 2.2 Language L2

Let $L_2 = \{a^i b^j c^k \mid j = k, i, j, k \geq 0\}$.

**Definition 2.** *The context-free grammar $G_2$ for $L_2$ is defined as:*

$$S \to AB \tag{4}$$
$$A \to aA \mid a \mid \varepsilon \tag{5}$$
$$B \to bBc \mid bc \tag{6}$$

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The nonterminal $A$ generates strings of the form $a^i$ for $i \geq 0$, and $B$ generates strings of the form $b^n c^n$ for $n \geq 1$. Therefore, $L_2$ contains all strings with any number of $a$'s followed by equal numbers of $b$'s and $c$'s.

# 3  Intersection of L1 and L2

**Theorem 1.** $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$

*Proof.* Let $w \in L_1 \cap L_2$. Then $w \in L_1$ and $w \in L_2$.
  Since $w \in L_1$, we have $w = a^i b^j c^k$ where $i = j$.
  Since $w \in L_2$, we have $w = a^i b^j c^k$ where $j = k$.
  Therefore, $i = j = k$, which means $w = a^n b^n c^n$ for some $n \geq 0$.
  Conversely, if $w = a^n b^n c^n$ for some $n \geq 0$, then:

- $w \in L_1$ because the number of $a$'s equals the number of $b$'s

- $w \in L_2$ because the number of $b$'s equals the number of $c$'s

  Therefore, $w \in L_1 \cap L_2$.
  Hence, $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$. $\qquad\square$

# 4  Proof that L is not Context-Free

**Theorem 2.** *The language $L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.*

*Proof.* We prove this by contradiction using the pumping lemma for context-free languages.

**Lemma 1** (Pumping Lemma for Context-Free Languages)**.** *If $L$ is a context-free language, then there exists a constant $p > 0$ such that for any string $s \in L$ with $|s| \geq p$, we can write $s = uvzxy$ where:*

(i) $|vzx| \leq p$

(ii) $|vx| \geq 1$

(iii) *For all $i \geq 0$, $uv^i zx^i y \in L$*

  Assume, for the sake of contradiction, that $L$ is context-free. Let $p$ be the pumping length guaranteed by the pumping lemma.
  Consider the string $s = a^p b^p c^p$. Since $s \in L$ and $|s| = 3p \geq p$, the pumping lemma applies.
  By the pumping lemma, we can write $s = uvzxy$ where conditions (i), (ii), and (iii) hold.
  We analyze all possible cases for the position of the substring $vzx$:
  **Case 1:** $vzx$ contains only $a$'s. Then $v = a^k$ and $x = a^m$ for some $k, m \geq 0$ with $k + m \geq 1$. Pumping up with $i = 2$, we get $uv^2 zx^2 y = a^{p+k+m} b^p c^p$. This string has more $a$'s than $b$'s or $c$'s, so it is not in $L$.

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**Case 2:** $vzx$ contains only $b$'s. Then $v = b^k$ and $x = b^m$ for some $k, m \geq 0$ with $k + m \geq 1$. Pumping up with $i = 2$, we get $uv^2zx^2y = a^p b^{p+k+m} c^p$. This string has more $b$'s than $a$'s or $c$'s, so it is not in $L$.

**Case 3:** $vzx$ contains only $c$'s. Then $v = c^k$ and $x = c^m$ for some $k, m \geq 0$ with $k + m \geq 1$. Pumping up with $i = 2$, we get $uv^2zx^2y = a^p b^p c^{p+k+m}$. This string has more $c$'s than $a$'s or $b$'s, so it is not in $L$.

**Case 4:** $vzx$ spans across $a$'s and $b$'s. Since $|vzx| \leq p$, the substring $vzx$ cannot span all three types of symbols. Pumping will create an imbalance between the number of $a$'s and $b$'s on one side, and $c$'s on the other side, so the resulting string is not in $L$.

**Case 5:** $vzx$ spans across $b$'s and $c$'s. Similar to Case 4, since $|vzx| \leq p$, pumping will create an imbalance between the number of $b$'s and $c$'s on one side, and $a$'s on the other side, so the resulting string is not in $L$.

In all cases, pumping leads to a string that is not in $L$, which contradicts condition (iii) of the pumping lemma.

Therefore, our assumption that $L$ is context-free must be false. Hence, $L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free. $\square$

# 5 Conclusion

We have successfully demonstrated that:

1. We constructed two context-free languages $L_1$ and $L_2$.

2. We showed that $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$.

3. We proved that $\{a^n b^n c^n \mid n \geq 0\}$ is not context-free using the pumping lemma.

Therefore, the set of context-free languages is not closed under the operation of intersection.

# 6 References

- Hopcroft, J. E., Motwani, R., & Ullman, J. D. (2006). *Introduction to Automata Theory, Languages, and Computation* (3rd ed.). Addison-Wesley.

- Sipser, M. (2012). *Introduction to the Theory of Computation* (3rd ed.). Cengage Learning.