Design & Analysis of Algorithms -I

Dynamic Programming

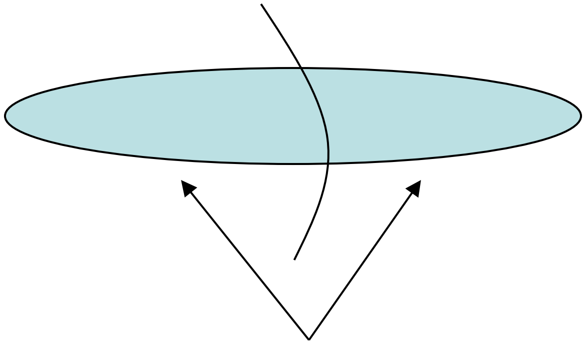
Dynamic Programming

- An algorithm design technique (like divide and conquer)
- Divide and conquer
  - Partition the problem into independent/disjoint subproblems
  - Solve the subproblems recursively
  - Combine the solutions to solve the original problem

DP - Two key ingredients

- Two key ingredients for an optimization problem to be suitable for a dynamic-programming solution:

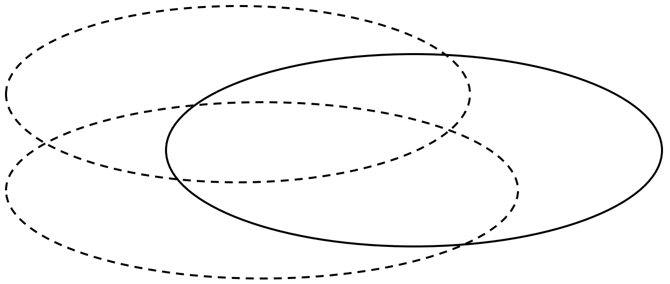
1. optimal substructures



Each substructure is optimal.

(Principle of optimality)

2. overlapping subproblems



Subproblems are dependent.

(otherwise, a divide-and-conquer approach is the choice.)

Three basic components

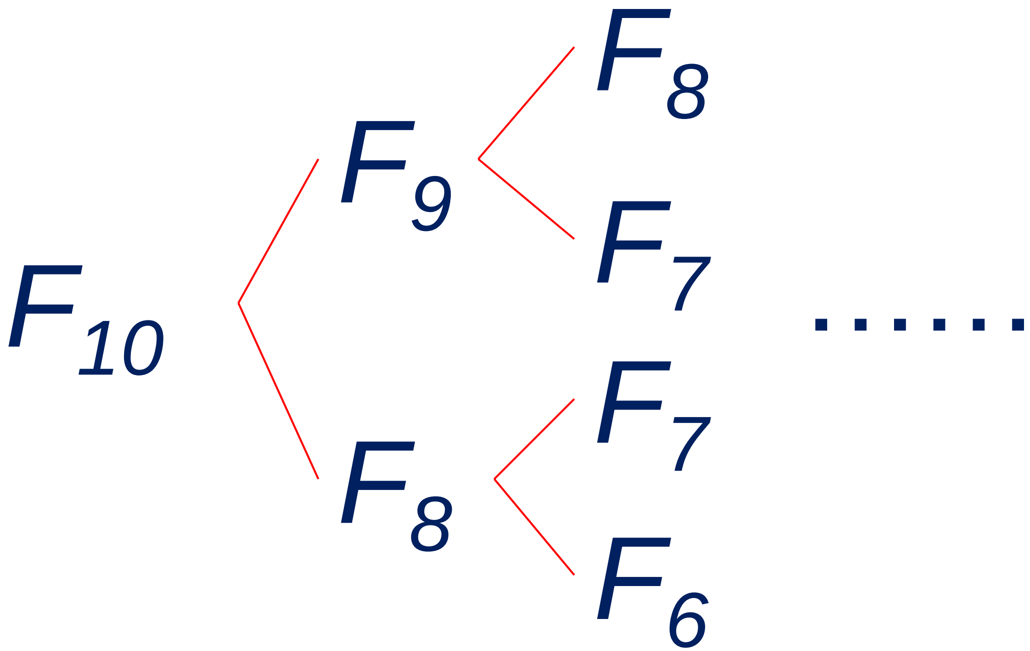
- The development of a dynamic-programming algorithm has three basic components:
  - The recurrence relation (for defining the value of an optimal solution);
  - The tabular computation (for computing the value of an optimal solution);
  - The traceback (for delivering an optimal solution).

Fibonacci numbers

The *Fibonacci numbers* are defined by the following recurrence:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1 .$$

How to compute F_{10}?



Dynamic Programming

- Applicable when subproblems are not independent
  - Subproblems share sub-subproblems

E.g.: Fibonacci numbers:

- Recurrence: $F(n) = F(n-1) + F(n-2)$
- Boundary conditions: $F(1) = 0$, $F(2) = 1$
- Compute: $F(5) = 3$, $F(3) = 1$, $F(4) = 2$

  - A divide and conquer approach would **repeatedly** solve the common subproblems
  - Dynamic programming solves every subproblem just once and stores the answer in a table

Tabular computation

- The tabular computation can avoid recompuation.

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
0	1	1	2	3	5	8	13	21	34	55

Result

The DP Methodology

We typically apply dynamic programming to ***optimization problems***. Such problems can have many possible solutions. Each solution has a value, and we wish to find a solution with the optimal (minimum or maximum) value. We call such a solution *an* optimal solution to the problem, as opposed to *the* optimal solution, since there may be several solutions that achieve the optimal value.

1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution in a bottom-up fashion
4. Construct an optimal solution from computed information

- Steps 1–3 form the basis of a dynamic-programming solution to a problem.
- For the value of an **optimal solution**, and not **the solution itself**, then we can omit step 4.
- When we do perform step 4, we sometimes maintain additional information during step 3 so that we can easily construct an optimal solution.

The Knapsack Problem

- **The 0-1 knapsack problem**
  - A thief robbing a store finds n items: the i-th item is worth v_i dollars and weights w_i pounds (v_i, w_i integers)
  - The thief can only carry W pounds in his knapsack
  - Items must be taken entirely or left behind
  - Which items should the thief take to maximize the value of his load?
- **The fractional knapsack problem**
  - Similar to above
  - The thief can take fractions of items

The 0-1 Knapsack Problem

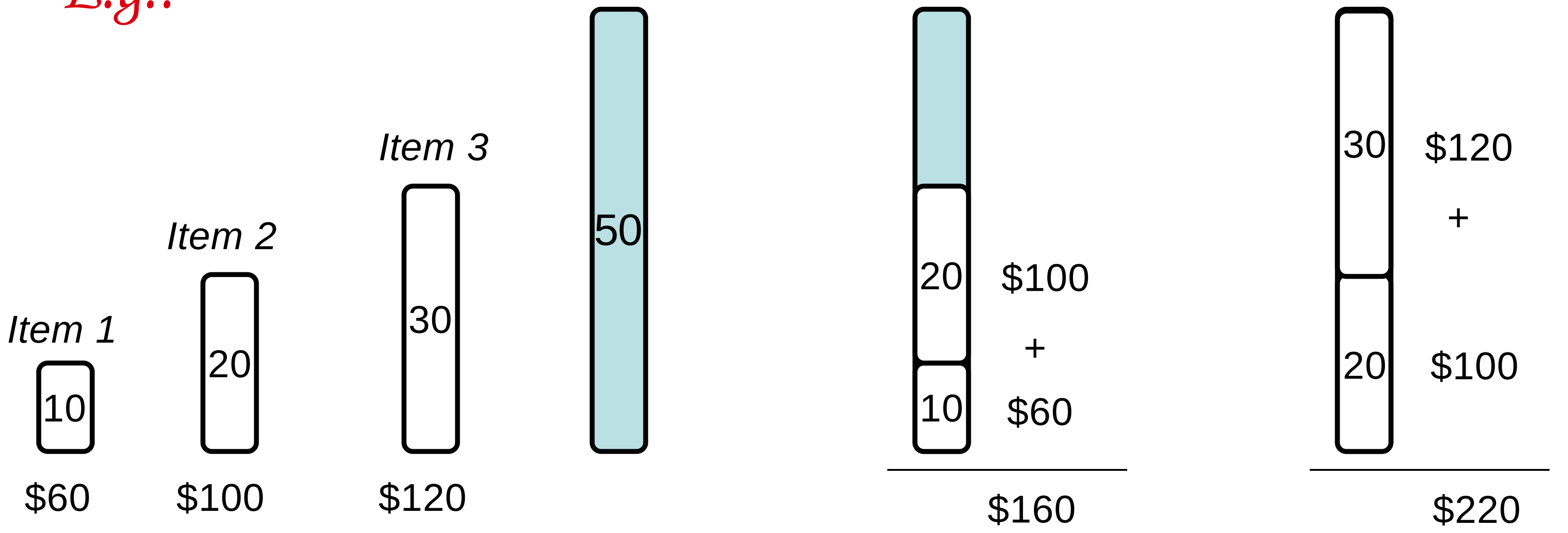
- Thief has a knapsack of capacity W
- There are n items: for i-th item value v_i and weight w_i
- Goal:
  - find x_i such that for all $x_i = \{0, 1\}$, $i = 1, 2, .., n$

    $\sum w_i x_i \leq W$ and

    $\sum x_i v_i$ is maximum

0-1 Knapsack - Greedy Strategy

- *E.g.:*



$6/pound $5/pound $4/pound

- None of the solutions involving the greedy choice (item 1) leads to an optimal solution
  - The greedy choice property does not hold

0-1 Knapsack - Dynamic Programming

- $P(i, w)$ – the maximum profit that can be obtained from items 1 to i, if the knapsack has size w
- Case 1: thief takes item i

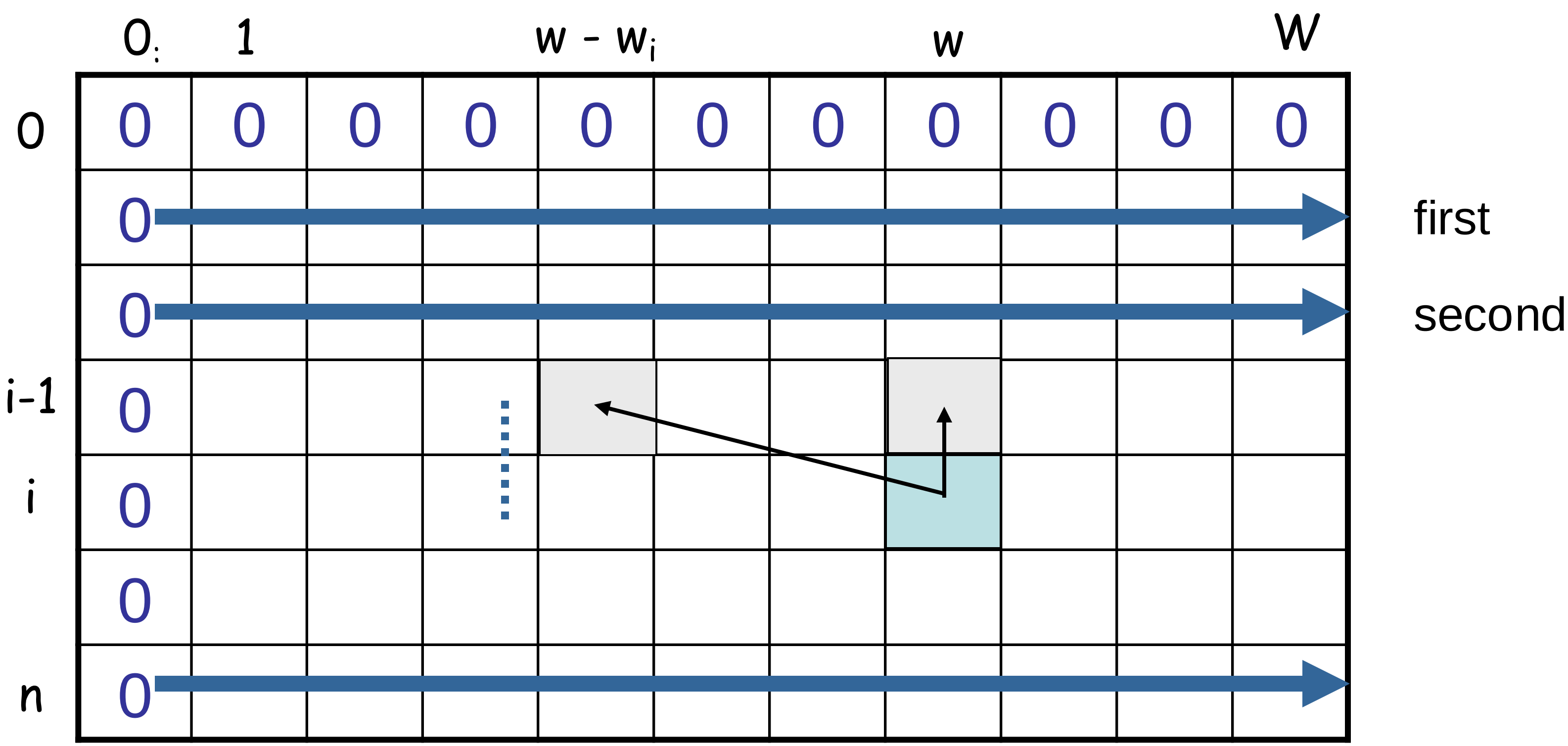
$$P(i, w) = v_i + P(i - 1, w - w_i)$$

- Case 2: thief does not take item i

$$P(i, w) = P(i - 1, w)$$

0-1 Knapsack - Dynamic Programming

$$P(i, w) = \max \{\overbrace{v_i + P(i-1, w-w_i)}^{\text{Item } i \text{ was taken}}, \overbrace{P(i-1, w)}^{\text{Item } i \text{ was not taken}} \}$$



Example:

W = 5

Item	Weight	Value
1	2	12
2	1	10
3	3	20
4	2	15

$$P(i, w) = \max\{v_i + P(i-1, w-w_i), P(i-1, w)\}$$

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	37

P(1, 1) = P(0, 1) = 0

P(1, 2) = max{12+0, 0} = 12

P(1, 3) = max{12+0, 0} = 12

P(1, 4) = max{12+0, 0} = 12

P(1, 5) = max{12+0, 0} = 12

P(2, 1)= max{10+0, 0} = 10

P(2, 2)= max{10+0, 12} = 12

P(2, 3)= max{10+12, 12} = 22

P(2, 4)= max{10+12, 12} = 22

P(2, 5)= max{10+12, 12} = 22

P(3, 1)= P(2,1) = 10

P(3, 2)= P(2,2) = 12

P(3, 3)= max{20+0, 22}=22

P(3, 4)= max{20+10,22}=30

P(3, 5)= max{20+12,22}=32

P(4, 1)= P(3,1) = 10

P(4, 2)= max{15+0, 12} = 15

P(4, 3)= max{15+10, 22}=25

P(4, 4)= max{15+12, 30}=30

P(4, 5)= max{15+22, 32}=37

Reconstructing the Optimal Solution

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	37

- Item 4
- Item 2
- Item 1

- Start at $P(n, W)$
- When you go left-up $\Rightarrow$ item i has been taken
- When you go straight up $\Rightarrow$ item i has not been taken

Overlapping Subproblems

$P(i, w) = \max \{v_i + P(i - 1, w-w_i), P(i - 1, w) \}$

	0	1				w					W
0	0	0	0	0	0	0	0	0	0	0	0
	0										
	0										
i-1	0										
i	0										
	0										
n	0										

E.g.: all the subproblems shown in grey may depend on $P(i\text{-}1, w)$

Longest Common Subsequence (LCS)

- Application: comparison of two DNA strings
- Ex: X= {A B C B D A B }, Y= {B D C A B A}
- Longest Common Subsequence:
- X = A **B** **C** **B** D **A** B
- Y = **B** D **C** A **B** **A**
- Brute force algorithm would compare each subsequence of X with the symbols in Y

Longest Common Subsequence

- Given two sequences

  $X = \langle x_1, x_2, \ldots, x_m \rangle$

  $Y = \langle y_1, y_2, \ldots, y_n \rangle$

  find a maximum length common subsequence (LCS) of X and Y

- *E.g.:*

  $X = \langle A, B, C, B, D, A, B \rangle$

- Subsequences of X:
  - A subset of elements in the sequence taken in order

  $\langle A, B, D \rangle$, $\langle B, C, D, B \rangle$, etc.

Example

$X = \langle A, B, C, B, D, A, B \rangle$ $\quad$ $X = \langle A, B, C, B, D, A, B \rangle$

$Y = \langle B, D, C, A, B, A \rangle$ $\quad$ $Y = \langle B, D, C, A, B, A \rangle$

- $\langle B, C, B, A \rangle$ and $\langle B, D, A, B \rangle$ are longest common subsequences of X and Y (length = 4)
- $\langle B, C, A \rangle$, however is not a LCS of X and Y

Brute-Force Solution

- For every subsequence of X, check whether it's a subsequence of Y
- There are 2^m subsequences of X to check
- Each subsequence takes $\Theta(n)$ time to check
  - scan Y for first letter, from there scan for second, and so on
- Running time: $\Theta(n2^m)$

LCS Algorithm

$$X = \langle x_1, x_2, \ldots, x_m \rangle \text{ and } Y = \langle y_1, y_2, \ldots, y_n \rangle.$$

sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$, we define the ith ***prefix*** of X, for $i = 0, 1, \ldots, m$, as $X_i = \langle x_1, x_2, \ldots, x_i \rangle$. For example, if $X = \langle A, B, C, B, D, A, B \rangle$, then $X_4 = \langle A, B, C, B \rangle$ and X_0 is the empty sequence.

- Define *c[i,j]* to be the length of LCS of X_i and Y_j
- Then the length of LCS of X and Y will be *c[m,n]*

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 , \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j , \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j . \end{cases}$$

LCS-LENGTH(X, Y)

$X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n$

```
m = X.length
n = Y.length
let b[1..m, 1..n] and c[0..m, 0..n] be new tables
for i = 1 to m
    c[i, 0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i - 1, j - 1] + 1
            b[i, j] = "↖"
        elseif c[i - 1, j] ≥ c[i, j - 1]
            c[i, j] = c[i - 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j - 1]
            b[i, j] = "←"
return c and b
```

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j, \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j. \end{cases}$$

LCS recursive solution

$$c[i,j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max(c[i, j-1], c[i-1, j]) & \text{otherwise} \end{cases}$$

- We start with $i = j = 0$ (empty substrings of x and y)
- Since X_0 and Y_0 are empty strings, their LCS is always empty (i.e. $c[0,0] = 0$)
- LCS of empty string and any other string is empty, so for every i and j: $c[0, j] = c[i,0] = 0$

LCS recursive solution

$$c[i,j] = \begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

- When we calculate *c[i,j]*, we consider two cases:
- **First case:** *x[i]=y[j]*:
  - one more symbol in strings X and Y matches, so the length of LCS X_i and Y_j equals to the length of LCS of smaller strings X_{i-1} and Y_{i-1} , plus 1

LCS recursive solution

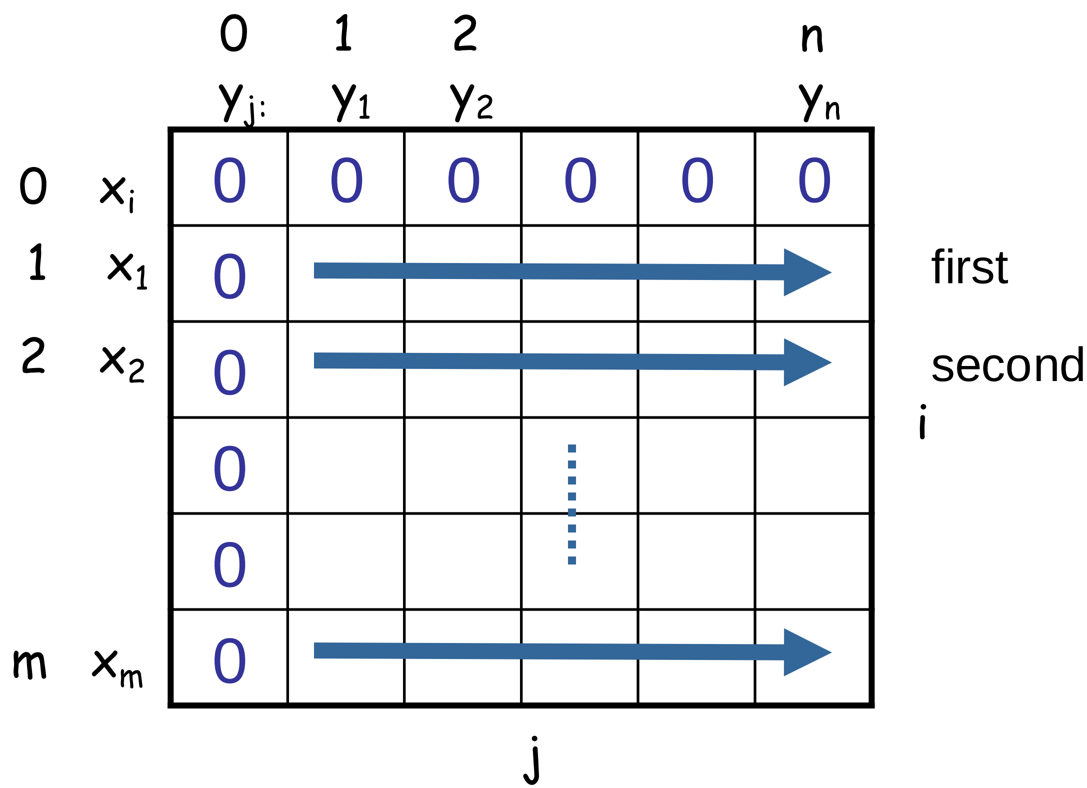
$$c[i,j] = \begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1], c[i-1,j]) & \text{otherwise} \end{cases}$$

- **Second case:** *x[i] != y[j]*
  - As symbols don't match, our solution is not improved, and the length of LCS(X_i , Y_j) is the same as before (i.e. maximum of LCS(X_i, Y_{j-1}) and LCS(X_{i-1},Y_j)

Why not just take the length of LCS(X_{i-1}, Y_{j-1}) ?

3. Computing the Length of the LCS

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$



Additional Information

$$c[i, j] = \begin{cases} 0 & \text{if } i,j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

b & c:

		0	1	2	3		n
		y_j:	A	C	D		F
0	x_i	0	0	0	0	0	0
1	A	0					
2	B	0			c[i-1,j]		
3	C	0		↖ c[i,j-1]	↑		
		0					
m	D	0					

i

j

A matrix b[i, j]:

- For a subproblem [i, j] it tells us what choice was made to obtain the optimal value
- If $x_i = y_j$

  b[i, j] = "↖"
- Else, if

  $c[i - 1, j] \geq c[i, j-1]$

  b[i, j] = " ↑ "

  else

  b[i, j] = " ← "

LCS-LENGTH(X, Y, m, n)

```
for i ← 1 to m
    do c[i, 0] ← 0
for j ← 0 to n
    do c[0, j] ← 0
for i ← 1 to m
    do for j ← 1 to n
        do if x_i = y_j
            then c[i, j] ← c[i - 1, j - 1] + 1
                b[i, j ] ← "↖"
            else if c[i - 1, j] ≥ c[i, j - 1]
                then c[i, j] ← c[i - 1, j]
                    b[i, j] ← "↑"
                else c[i, j] ← c[i, j - 1]
                    b[i, j] ← "←"
return c and b
```

The length of the LCS if one of the sequences is empty is zero

Case 1: $x_i = y_j$

Case 2: $x_i \neq y_j$

Running time: $\Theta(mn)$

Example

$X = \langle A, B, C, B, D, A \rangle$

$Y = \langle B, D, C, A, B, A \rangle$

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

If $x_i = y_j$

$b[i, j] = \text{“}\nwarrow\text{”}$

Else if

$c[i - 1, j] \geq c[i, j-1]$

$b[i, j] = \text{“}\uparrow\text{”}$

else

$b[i, j] = \text{“}\leftarrow\text{”}$

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

4. Constructing a LCS

- Start at b[m, n] and follow the arrows
- When we encounter a "↖" in b[i, j] $\Rightarrow x_i = y_j$ is an element of the LCS

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2	B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3	C	0	↑ 1	↑ 1	↖ 2	←2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	←3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

PRINT-LCS(b, X, i, j)

Running time: $\Theta(m + n)$

1. **if** $i = 0$ or $j = 0$
2. **then return**
3. **if** $b[i, j] =$ "$\nwarrow$"
4. **then** PRINT-LCS(b, X, $i - 1$, $j - 1$)
5. print x_i
6. **elseif** $b[i, j] =$ "$\uparrow$"
7. **then** PRINT-LCS(b, X, $i - 1$, j)
8. **else** PRINT-LCS(b, X, i, $j - 1$)

Initial call: PRINT-LCS(b, X, *length*[X], *length*[Y])

Improving the Code

- If we only need the length of the LCS
  - LCS-LENGTH works only on two rows of c at a time
    - The row being computed and the previous row
  - We can reduce the asymptotic space requirements by storing only these two rows

LCS Algorithm Running Time

- LCS algorithm calculates the values of each entry of the array c[m,n]
- So what is the running time?

O(m*n)

since each c[i,j] is calculated in constant time, and there are m*n elements in the array