

Newton's Method

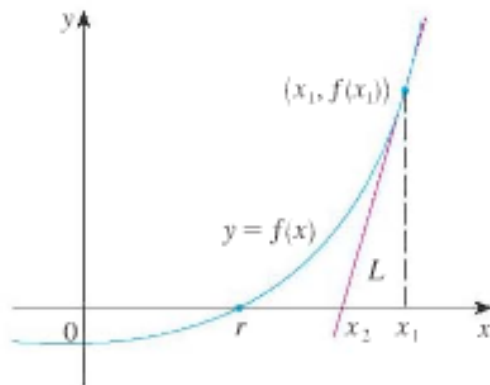
In this section we will explore a method for estimating the solutions of an equation $f(x) = 0$ by a sequence of approximations that approach the solution.

Note that for a quadratic equation $ax^2 + bx + c = 0$, we can solve for the solutions using the quadratic formula. There are formulas available to find the zeros of cubic and quartic polynomials, however they are more difficult (and tedious) to apply than the quadratic formula. Some notes on the history of the solutions have been included at the end of the lecture. **It can be shown that the derivation of such a formula is impossible for polynomials with degree 5 or higher and there is no such systematic method to find their solutions.** The methods below apply to all polynomial functions and a wide range of other functions.

Procedure for Newton's Method



To estimate the solution of an equation $f(x) = 0$, we produce a sequence of approximations that approach the solution. We find the first estimate by sketching a graph or by guessing. We choose x_1 which is close to the solution according to our estimate. The method then uses the tangent line to the curve at the point $(x_1, f(x_1))$ as an estimate of the path of the curve.



We label the point where this tangent to the graph of $f(x)$ cuts the x-axis, x_2 . Usually x_2 is a better approximation to the solution of $f(x) = 0$ than x_1 . We can find a formula for x_2 in terms of x_1 by using the equation of the tangent at $(x_1, f(x_1))$:

$$y = f(x_1) + f'(x_1)(x - x_1).$$

The x-intercept of this line, x_2 , occurs when $y = 0$ or

$$f(x_1) + f'(x_1)(x_2 - x_1) = 0 \quad \text{or} \quad -f(x_1) = f'(x_1)(x_2 - x_1)$$

This implies that

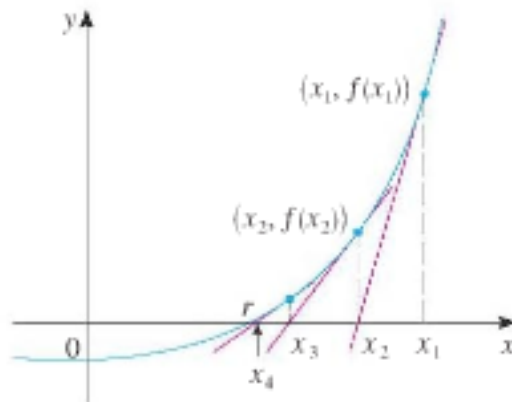
$$-\frac{f(x_1)}{f'(x_1)} = x_2 - x_1 \quad \text{or} \quad \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}.$$

If $f(x_2) \neq 0$, we can repeat the process to get a third approximation to the zero as

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In the same way we can repeat the process to get a third and a fourth approximation to the zero using the formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}.$$



Note When applying Newton's method it may happen that the sequence of approximations gets further and further away from the zero of the function (or further apart from each other) as the value of n increases. It may also happen that one of the approximations falls outside the domain of the function. In this case you should start over with a different approximation.

Example 1 Use Newton's method to find the fourth approximation, x_4 , to the root of the following equation

$$x^3 - x - 1 = 0$$

starting with $x_1 = 1$.

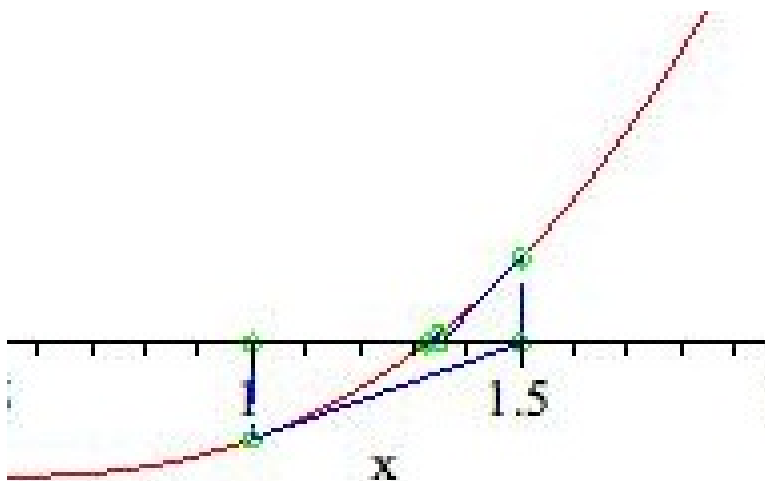
Note that if $f(x) = x^3 - x - 1$, then $f(1) = -1 < 0$ and $f(2) = 5 > 0$. Therefore by the Intermediate Value Theorem, there is a root between $x = 1$ and $x = 2$.

It is helpful to make a table for computation:

		$f(x) =$	$f'(x) =$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	1			
2				
3				

As a **rule of thumb**, when successive approximations x_n and x_{n+1} agree up to K decimal places, then our approximation is accurate up to K decimal places. For example the table below shows the approximations x_1, x_2, \dots, x_6 for the above problem. We see that x_5 and x_6 agree up to 9 decimal places. By the rule of thumb above, the solution to the equation is equal to 1.324717957 when rounded off to 9 decimal places.

		$f(x) = x^3 - x - 1$	$f'(x) = 3x^2 - 1$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	2	1.5
2	1.5	0.875	5.75	1.347826087
3	1.347826087	0.100682173	4.449905482	1.325200399
4	1.325200399	0.002058362	4.268468292	1.324718174
5	1.324718174	0.000000924	4.264634722	1.324717957
6	1.324717957	-1.8672×10^{-13}	4.264632999	1.324717957



Example 2 Use Newton's method to estimate $\sqrt{2}$ correct up to 5 decimal places.

Find an equation $f(x) = 0$ for which $\sqrt{2}$ is a solution.

		$f(x) =$	$f'(x) =$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1				
2				
3				
4				
5				

Example 3 Do the curves $y = \cos x$ and $y = 3x$ meet?

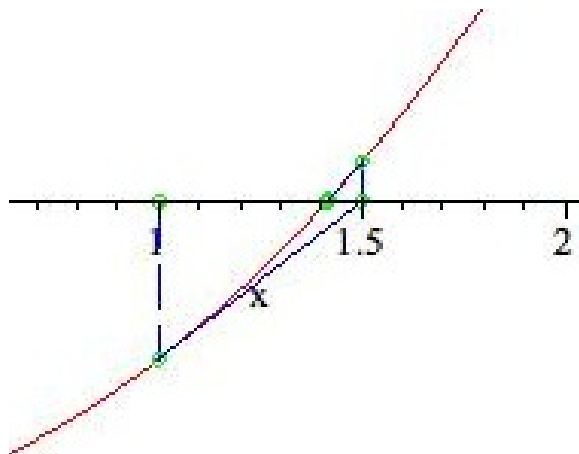
If so estimate the value of x at which they meet up to 3 decimal places.

		$f(x) =$	$f'(x) =$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1				
2				
3				
4				

We stop when successive approximations are equal up to 5 decimal places.

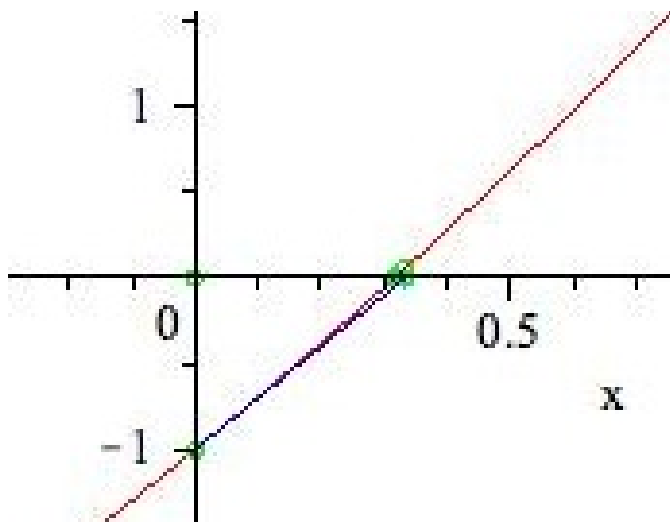
Example 2 Solution From calculator ($\sqrt{2} = 1.41421$). To use Newton's method, we look for a positive root of $f(x) = x^2 - 2$. We stop when successive approximations are equal up to 5 decimal places.

		$f(x) = x^2 - 2$	$f'(x) = 2x$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	2	1.5
2	1.5	.25	3	1.416666666
3	1.416666666	.006944444	2.833333333	1.414215686
4	1.414215686	.000006005	2.828431372	1.414213563



Example 3 Solution To use Newton's method, $f(x) = 3x - \cos x$. We use $x_1 = 0$ stop when successive approximations are equal up to 3 decimal places.

		$f(x) = 3x - \cos x$	$f'(x) = 3 + \sin x$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	0	-1	3	$1/3 = .3333333333$
2	$1/3$.05504305	3.3271946	.316789952
3	.316789952	.000129555	3.31517854	.316750829



Niccolò Fontana Tartaglia

From Wikipedia, the free encyclopedia

Niccolò Fontana Tartaglia (1499/1500, Brescia, Italy – 13 December 1557, Venice, Italy) was a mathematician, an engineer (designing fortifications), a surveyor (of topography, seeking the best means of defense or offense) and a bookkeeper from the then-Republic of Venice (now part of Italy). He published many books, including the first Italian translations of Archimedes and Euclid, and an acclaimed compilation of mathematics. Tartaglia was the first to apply mathematics to the investigation of the paths of cannonballs; his work was later validated by Galileo's studies on falling bodies.



Niccolò Fontana Tartaglia.

Niccolò Fontana was the son of Michele Fontana, a rider and deliverer. In 1505, Michele was murdered and Niccolò, his two siblings, and his mother were impoverished. Niccolò experienced further tragedy in 1512 when the French invaded Brescia during the War of the League of Cambrai. The militia of Brescia defended their city for seven days. When the French finally broke through, they took their revenge by massacring the inhabitants of Brescia. By the end of battle, over 45,000 residents were killed. During the massacre, a French soldier sliced Niccolò's jaw and palate. This made it impossible for Niccolò to speak normally, prompting the nickname "Tartaglia" (stammerer).

There is a story that Tartaglia learned only half the alphabet from a private tutor before funds ran out, and he had to learn the rest for himself. Be that as it may, he was essentially self-taught. He and his contemporaries, working outside the academies, were responsible for the spread of classic works in modern languages among the educated middle class.

His edition of Euclid in 1543, the first translation of the *Elements* into any modern European language, was especially significant. For two centuries Euclid had been taught from two Latin translations taken from an Arabic source; these contained errors in Book V, the Eudoxian theory of proportion, which rendered it unusable. Tartaglia's edition was based on Zamberti's Latin translation of an uncorrupted Greek text, and rendered Book V correctly. He also wrote the first modern and useful commentary on the theory. Later, the theory was an essential tool for Galileo, just as it had been for Archimedes.

Solution to cubic equations

Tartaglia is perhaps best known today for his conflicts with Gerolamo Cardano. Cardano nagged Tartaglia into revealing his solution to the cubic equations, by promising not to publish them. Several years later, Cardano happened to see unpublished work by Scipione del Ferro who independently came up with the same solution as Tartaglia. As the unpublished work was dated before Tartaglia's, Cardano decided his promise could be broken and included Tartaglia's solution in his next publication. Since Cardano credited his discovery, Tartaglia was extremely upset. He responded by publicly insulting Cardano.

Gerolamo Cardano

From Wikipedia, the free encyclopedia

"*Cardanus*" *redirects here*. For the lunar crater, see *Cardanus (crater)*. For the stag beetle genus, see *Cardanus (beetle)*.

Gerolamo (or **Girolamo**, or **Geronimo**) **Cardano** (French **Jérôme Cardan**; Latin **Hieronymus Cardanus**) (24 September 1501 – 21 September 1576) was an Italian Renaissance mathematician, physician, astrologer and gambler.

Contents

- 1 Early life and education
- 2 Mathematics
- 3 Family
- 4 Miscellaneous
- 5 Works
- 6 Notes
- 7 References
- 8 External links

Early life and education

He was born in Pavia, Lombardy, the illegitimate child of Fazio Cardano, a mathematically gifted lawyer, who was a friend of Leonardo da Vinci. In his autobiography, Cardano claimed that his mother had attempted to abort him. Shortly before his birth, his mother had to move from Milan to Pavia to escape the plague; her three other children died from the disease.

In 1520, he entered the University of Pavia and later in Padua studied medicine. His eccentric and confrontational style did not earn him many friends and he had a difficult time finding work after his studies had ended. In 1525, Cardano repeatedly applied to the College of Physicians in Milan, but was not admitted owing to his reputation and illegitimate birth.

Eventually, he managed to develop a considerable reputation as a physician and his services were highly valued at the courts. He was the first to describe typhoid fever.

Mathematics

Today, he is best known for his achievements in algebra. He published the solutions to the cubic and quartic equations in his 1545 book *Ars Magna*. The solution to one particular case of the cubic, $x^3 + ax = b$ (in modern notation), was communicated to him by Niccolò Fontana Tartaglia (who later claimed that Cardano had sworn not to reveal it, and engaged Cardano in a decade-long fight), and the quartic was solved by Cardano's student Lodovico Ferrari. Both were acknowledged in the foreword of the book, as well as in several places within its body. In his exposition, he acknowledged the existence of what are now called imaginary numbers, although he did not understand their properties (Mathematical field theory was developed centuries later). In *Opus novum de proportionibus* he introduced the binomial coefficients and the binomial theorem.

Girolamo Cardano



Girolamo Cardano

Born	24 September 1501 Pavia
Died	21 September 1576 (aged 74) Rome
Nationality	Italian
Fields	Mathematics Physics
Alma mater	University of Pavia
Known for	Algebra

Cardano was notoriously short of money and kept himself solvent by being



Portrait of Cardano on display at the School of Mathematics and Statistics, University of St Andrews.

Cardano was notoriously short of money and kept himself solvent by being an accomplished gambler and chess player. His book about games of chance, *Liber de ludo aleae* ("Book on Games of Chance"), written in 1526, but not published until 1663, contains the first systematic treatment of probability, as well as a section on effective cheating methods. Cardano invented several mechanical devices including the combination lock, the gimbal consisting of three concentric rings allowing a supported compass or gyroscope to rotate freely, and the Cardan shaft with universal joints, which allows the transmission of rotary motion at various angles and is used in vehicles to this day. He studied hypocycloids, published in *de proportionibus* 1570. The generating circles of these hypocycloids were later named Cardano circles or cardanic circles and were used for the construction of the first high-speed printing presses.

He made several contributions to hydrodynamics and held that perpetual motion is impossible, except in celestial bodies. He published two encyclopedias of natural science which contain a wide variety of inventions, facts, and occult superstitions. He also introduced the Cardan grille, a cryptographic tool, in 1550.

Someone also assumed to Cardano the credit for the invention of the so called *Cardano's Rings*, also called Chinese Rings, but it is very probable that they are more ancient than Cardano.

Significantly, in the history of education of the deaf, he said that deaf people were capable of using their minds, argued for the importance of teaching them, and was one of the first to state that deaf people could learn to read and write without learning how to speak first. He was familiar with a report by Rudolph Agricola about a deaf mute who had learned to write.

Family

Cardano's eldest and favorite son was executed in 1560 after he confessed to having poisoned his cuckolding wife. His other son was a gambler, who stole money from him. He allegedly cropped the ears of one of his sons. Cardano himself was accused of heresy in 1570 because he had computed and published the horoscope of Jesus in 1554. Apparently, his own son contributed to the prosecution, bribed by Tartaglia. He was arrested, had to spend several months in prison and was forced to abjure his professorship. He moved to Rome, received a lifetime annuity from Pope Gregory XIII (after first having been rejected by Pope Pius V) and finished his autobiography.