

# Lecture 3

## Solution of Nonlinear Equations



**( Root Finding Problems )**

**Textbook: Chapter 5**  
**Page 123-133**

# Root Finding Problems

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Many problems in Science and Engineering are expressed as:

Given a continuous function  $f(x)$ ,  
find the value  $r$  such that  $f(r) = 0$

These problems are called root finding problems.

# Roots of Equations

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A number  $r$  that satisfies an equation is called a root of the equation.

The equation:  $x^4 - 3x^3 - 7x^2 + 15x = -18$

has four roots:  $-2, 3, 3, \text{and } -1$ .

i.e.,  $x^4 - 3x^3 - 7x^2 + 15x + 18 = (x + 2)(x - 3)^2(x + 1)$

The equation has two simple roots ( $-1$  and  $-2$ ) and a repeated root ( $3$ ) with multiplicity = 2.

# Zeros of a Function

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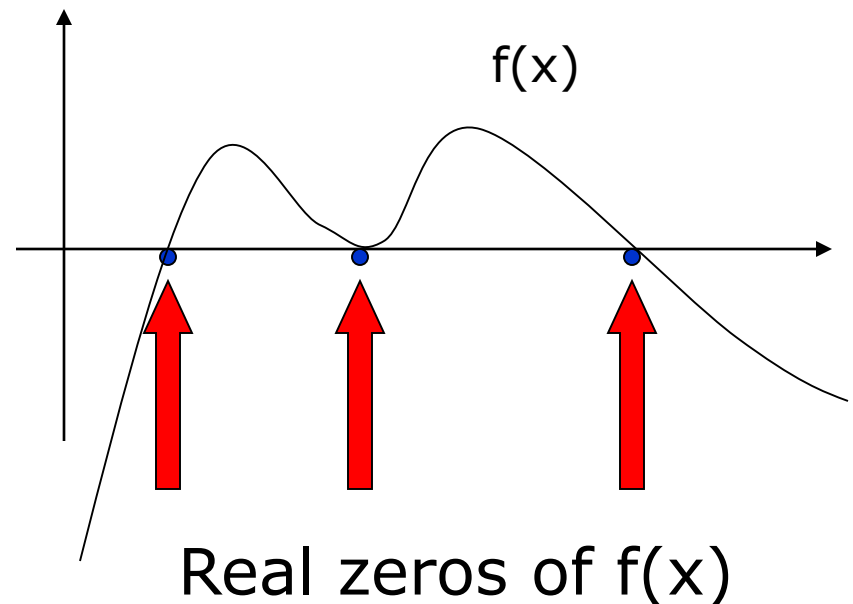
Let  $f(x)$  be a real-valued function of a real variable. Any number  $r$  for which  $f(r)=0$  is called a zero of the function.

*Examples:*

*2 and 3 are zeros of the function  $f(x) = (x-2)(x-3)$ .*

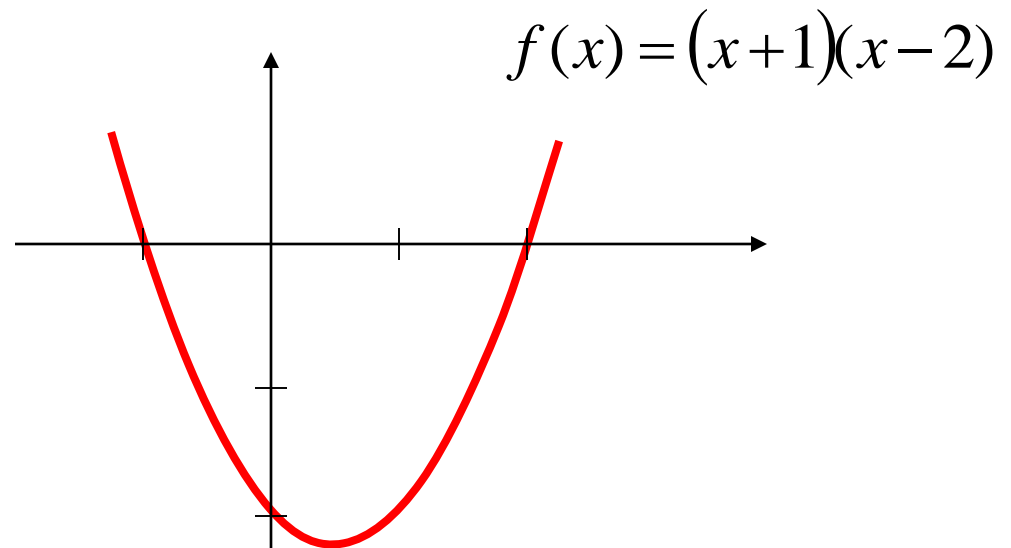
# Graphical Interpretation of Zeros

- The real zeros of a function  $f(x)$  are the values of  $x$  at which the graph of the function crosses (or touches) the x-axis.



# Simple Zeros

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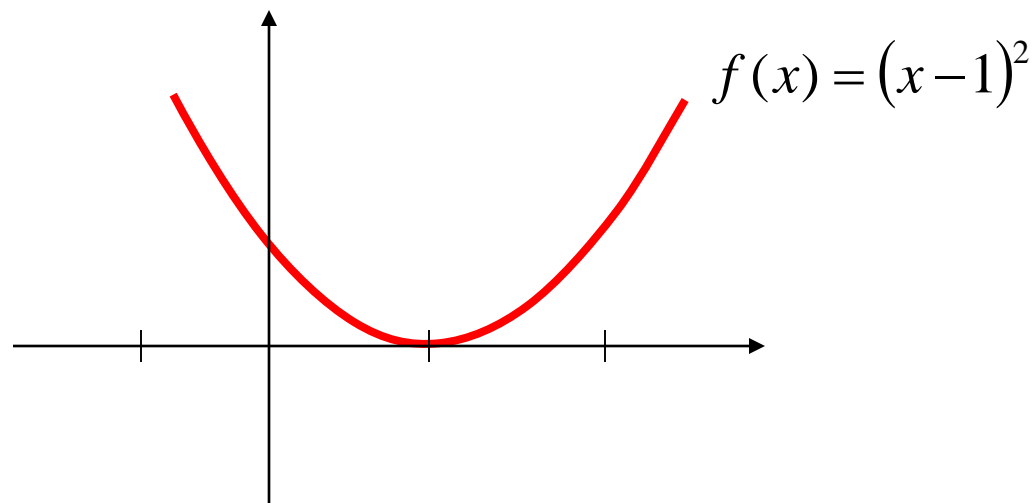


$$f(x) = (x+1)(x-2) = x^2 - x - 2$$

has two simple zeros (one at  $x = 2$  and one at  $x = -1$ )

# Multiple Zeros

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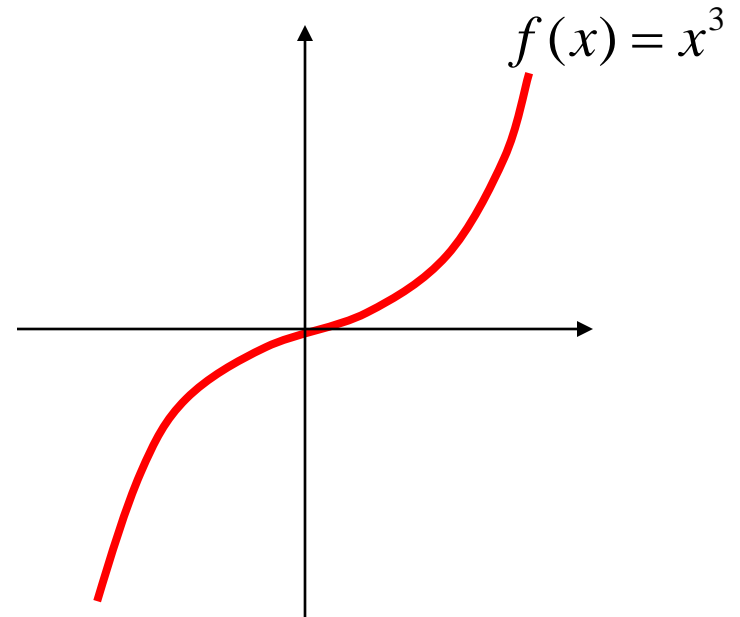


$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

has double zeros (zero with multiplicity = 2) at  $x = 1$

# Multiple Zeros

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$$f(x) = x^3$$

has a zero with multiplicity = 3 at  $x = 0$



# Facts

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- Any  $n^{\text{th}}$  order polynomial has exactly  $n$  zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If a function has a zero at  $x=r$  with multiplicity  $m$  then the function and its first  $(m-1)$  derivatives are zero at  $x=r$  and the  $m^{\text{th}}$  derivative at  $r$  is not zero.

# Roots of Equations & Zeros of Function

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Given the equation :

$$x^4 - 3x^3 - 7x^2 + 15x = -18$$

Move all terms to one side of the equation :

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$$

Define  $f(x)$  as :

$$f(x) = x^4 - 3x^3 - 7x^2 + 15x + 18$$

The zeros of  $f(x)$  are the same as the roots of the equation  $f(x) = 0$   
(Which are  $-2$ ,  $3$ ,  $3$ , and  $-1$ )

# Solution Methods

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Several ways to solve nonlinear equations are possible:

- Analytical Solutions
  - Possible for special equations only
- Graphical Solutions
  - Useful for providing initial guesses for other methods
- Numerical Solutions
  - Open methods
  - Bracketing methods

# Analytical Methods

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Analytical Solutions are available for special equations only.

Analytical solution of :  $ax^2 + bx + c = 0$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for:  $x - e^{-x} = 0$

# Graphical Methods

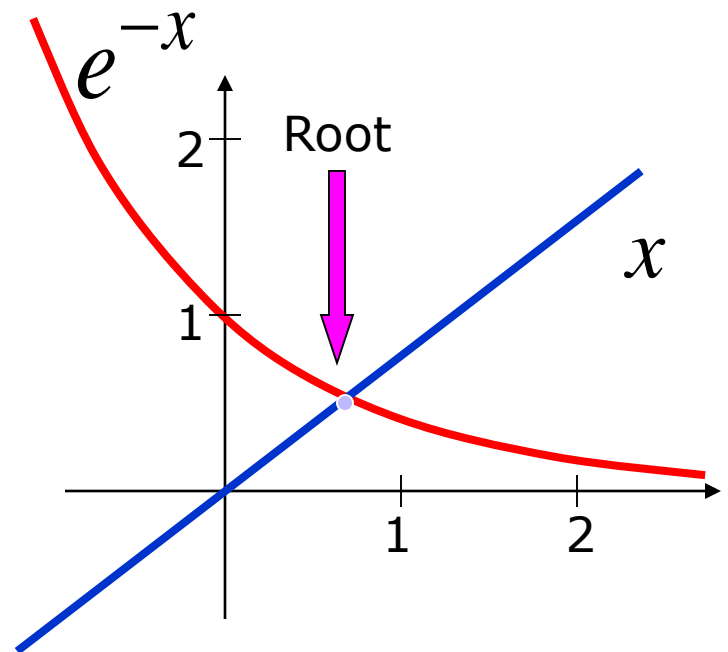
- Graphical methods are useful to provide an initial guess to be used by other methods.

*Solve*

$$x = e^{-x}$$

*The root  $\in [0,1]$*

*root  $\approx 0.6$*



# Bracketing Methods

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- In bracketing methods, the method starts with an interval that contains the root and a procedure is used to obtain a smaller interval containing the root.
- Examples of bracketing methods:
  - Bisection method
  - False position method

# Open Methods

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- ❑ In the open methods, the method starts with one or more initial guess points. In each iteration, a new guess of the root is obtained.
- ❑ Open methods are usually more efficient than bracketing methods.
- ❑ They may not converge to a root.

# Bisection Method



- The Bisection Algorithm
- Convergence Analysis of Bisection Method
- Examples



# Introduction

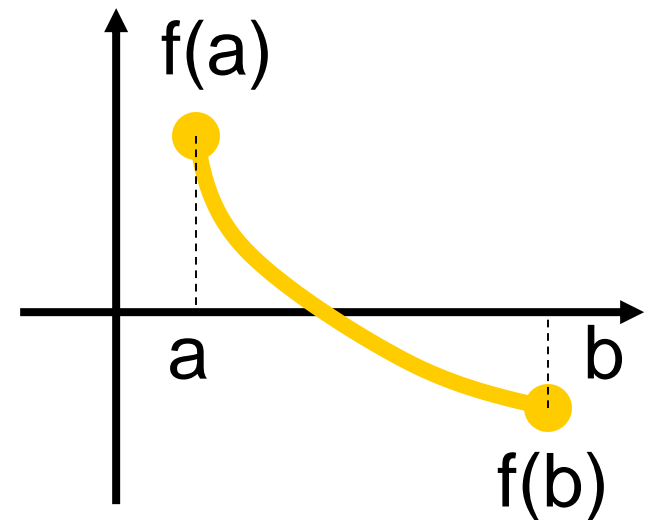
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- ❑ The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- ❑ It is also called **interval halving** method.
- ❑ To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- ❑ The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- ❑ The procedure is repeated until the desired interval size is obtained.

# Intermediate Value Theorem

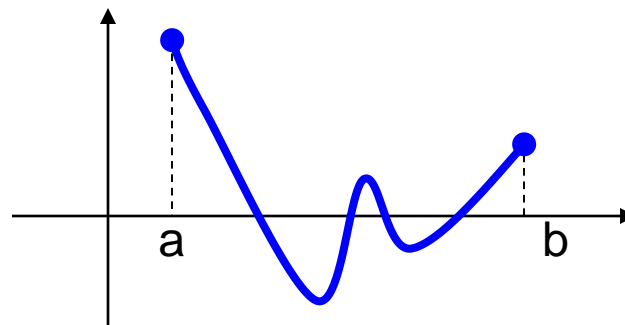
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- Let  $f(x)$  be defined on the interval  $[a,b]$ .
- Intermediate value theorem:  
if a function is continuous and  $f(a)$  and  $f(b)$  have different signs then the function has at least one zero in the interval  $[a,b]$ .



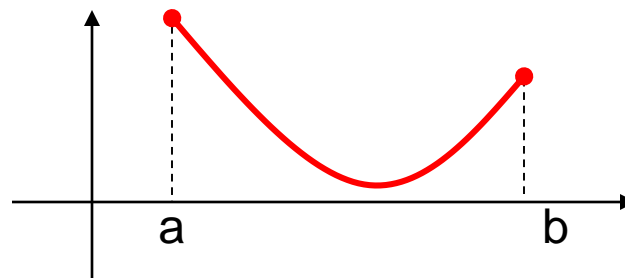
# Examples

- If  $f(a)$  and  $f(b)$  have the same sign, the function may have an even number of real zeros or no real zeros in the interval  $[a, b]$ .



The function has four real zeros

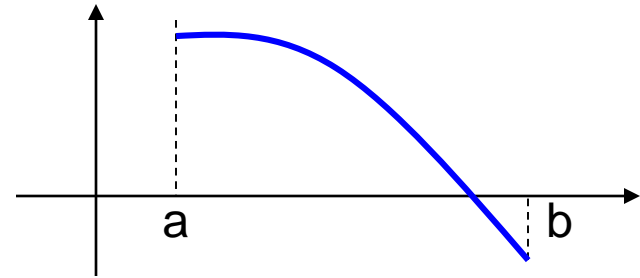
- Bisection method can not be used in these cases.



The function has no real zeros

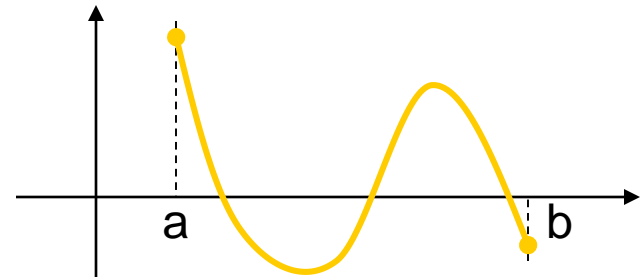
# Two More Examples

- If  $f(a)$  and  $f(b)$  have different signs, the function has at least one real zero.



The function has one real zero

- Bisection method can be used to find one of the zeros.



The function has three real zeros

# Bisection Method

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- If the function is continuous on  $[a,b]$  and  $f(a)$  and  $f(b)$  have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.
- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

# Bisection Method

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## Assumptions:

Given an interval  $[a,b]$

$f(x)$  is continuous on  $[a,b]$

$f(a)$  and  $f(b)$  have opposite signs.

These assumptions ensure the existence of at least one zero in the interval  $[a,b]$  and the bisection method can be used to obtain a smaller interval that contains the zero.

# Bisection Algorithm

## Assumptions:

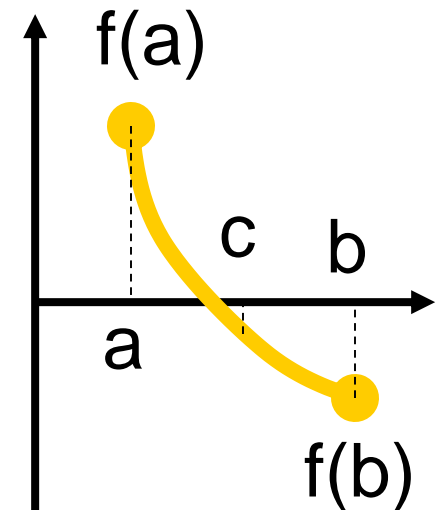
- $f(x)$  is continuous on  $[a, b]$
- $f(a) f(b) < 0$

## Algorithm:

### Loop

1. Compute the mid point  $c = (a+b)/2$
2. Evaluate  $f(c)$
3. If  $f(a) f(c) < 0$  then new interval  $[a, c]$   
If  $f(a) f(c) > 0$  then new interval  $[c, b]$

### End loop



# Example

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Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$  in the interval  $[0,2]$ ?

**Answer:**

$f(x)$  is continuous on  $[0,2]$

and  $f(0) * f(2) = (1)(3) = 3 > 0$

$\Rightarrow$  Assumptions are not satisfied

$\Rightarrow$  Bisection method can not be used



# Example

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Can you use Bisection method to find a zero of :  
 $f(x) = x^3 - 3x + 1$  in the interval  $[0,1]$ ?

## **Answer:**

$f(x)$  is continuous on  $[0,1]$

and  $f(0) * f(1) = (1)(-1) = -1 < 0$

$\Rightarrow$  Assumptions are satisfied

$\Rightarrow$  Bisection method can be used

# Best Estimate and Error Level

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Bisection method obtains an interval that is guaranteed to contain a zero of the function.

## Questions:

- What is the best estimate of the zero of  $f(x)$ ?
- What is the error level in the obtained estimate?

# Best Estimate and Error Level

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The best estimate of the zero of the function  $f(\mathbf{x})$  after the first iteration of the Bisection method is the mid point of the initial interval:

$$\text{Estimate of the zero: } r = \frac{b + a}{2}$$

$$\text{Error} \leq \frac{b - a}{2}$$

# Stopping Criteria

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Two common stopping criteria

1. Stop after a fixed number of iterations
2. Stop when the absolute error is less than a specified value

How are these criteria related?

# Stopping Criteria

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- $c_n$  : is the midpoint of the interval at the  $n^{\text{th}}$  iteration  
(  $c_n$  is usually used as the estimate of the root).  
 $r$  : is the zero of the function.

After  $n$  iterations:

$$|error| = |r - c_n| \leq E_a^n = \frac{b - a}{2^n} = \frac{\Delta x^0}{2^n}$$

# Convergence Analysis

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*Given  $f(x)$ ,  $a$ ,  $b$ , and  $\varepsilon$*

How many iterations are needed such that:  $|x - r| \leq \varepsilon$   
where  $r$  is the zero of  $f(x)$  and  $x$  is the  
bisection estimate (i.e.,  $x = c_k$ )?

$$n \geq \frac{\log(b - a) - \log(\varepsilon)}{\log(2)}$$

# Convergence Analysis – Alternative Form

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$$n \geq \frac{\log(b - a) - \log(\varepsilon)}{\log(2)}$$

$$n \geq \log_2 \left( \frac{\text{width of initial interval}}{\text{desired error}} \right) = \log_2 \left( \frac{b - a}{\varepsilon} \right)$$

# Example

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$$a = 6, \quad b = 7, \quad \varepsilon = 0.0005$$

How many iterations are needed such that:  $|x - r| \leq \varepsilon$ ?

$$n \geq \frac{\log(b - a) - \log(\varepsilon)}{\log(2)} = \frac{\log(1) - \log(0.0005)}{\log(2)} = 10.9658$$

$$\Rightarrow n \geq 11$$



# Example

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- Use Bisection method to find a root of the equation  $x = \cos(x)$  with absolute error  $< 0.02$  (assume the initial interval  $[0.5, 0.9]$ )

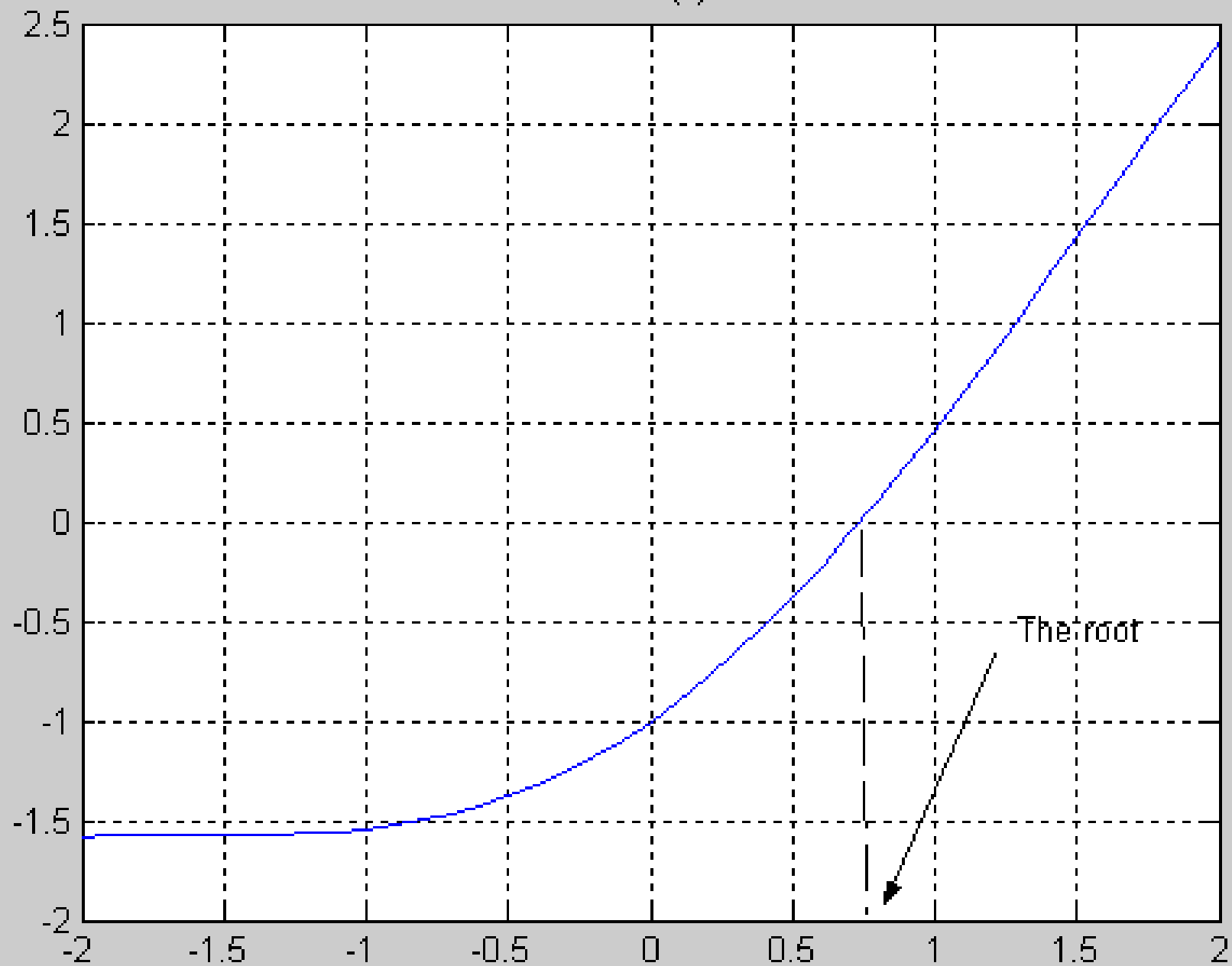
Question 1: What is  $f(x)$  ?

Question 2: Are the assumptions satisfied ?

Question 3: How many iterations are needed ?

Question 4: How to compute the new estimate ?

$$x - \cos(x)$$



# Bisection Method

## Initial Interval

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$$f(a) = -0.3776$$

$$f(b) = 0.2784$$

Error < 0.2



$$a = 0.5$$

$$c = 0.7$$

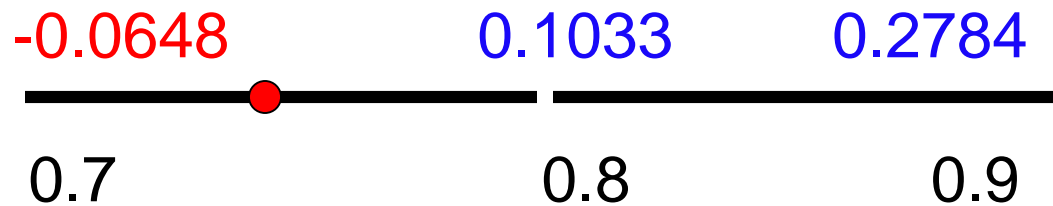
$$b = 0.9$$

# Bisection Method

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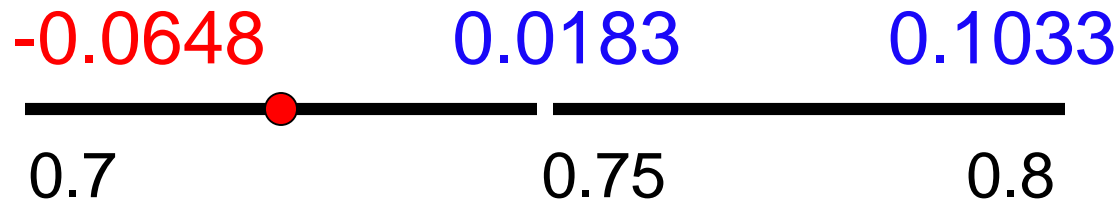
Error < 0.1



Error < 0.05

# Bisection Method

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Error < 0.025



Error < .0125

# Summary

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- Initial interval containing the root:  
[0.5,0.9]
  
- After 5 iterations:
  - Interval containing the root: [0.725, 0.75]
  - Best estimate of the root is 0.7375
  - $| \text{Error} | < 0.0125$

# A Matlab Program of Bisection Method

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```
a=.5; b=.9;  
u=a-cos(a);  
v=b-cos(b);  
for i=1:5  
    c=(a+b)/2  
    fc=c-cos(c)  
    if u*fc<0  
        b=c ; v=fc;  
    else  
        a=c; u=fc;  
    end  
end
```

```
c =  
    0.7000  
fc =  
   -0.0648  
c =  
    0.8000  
fc =  
    0.1033  
c =  
    0.7500  
fc =  
    0.0183  
c =  
    0.7250  
fc =  
   -0.0235
```

# Example

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Find the root of:

$$f(x) = x^3 - 3x + 1 \text{ in the interval: } [0,1]$$

\*  $f(x)$  is continuous

\*  $f(0) = 1, f(1) = -1 \Rightarrow f(a) f(b) < 0$

$\Rightarrow$  Bisection method can be used to find the root



# Example

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Iteration	a	b	$c = \frac{(a+b)}{2}$	f(c)	$\frac{(b-a)}{2}$
1	0	1	0.5	-0.375	0.5
2	0	0.5	0.25	0.266	0.25
3	0.25	0.5	.375	-7.23E-3	0.125
4	0.25	0.375	0.3125	9.30E-2	0.0625
5	0.3125	0.375	0.34375	9.37E-3	0.03125

# Bisection Method

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## Advantages

- ❑ Simple and easy to implement
- ❑ One function evaluation per iteration
- ❑ The size of the interval containing the zero is reduced by 50% after each iteration
- ❑ The number of iterations can be determined a priori
- ❑ No knowledge of the derivative is needed
- ❑ The function does not have to be differentiable

## Disadvantage

- ❑ Slow to converge
- ❑ Good intermediate approximations may be discarded