Stochastic Processes

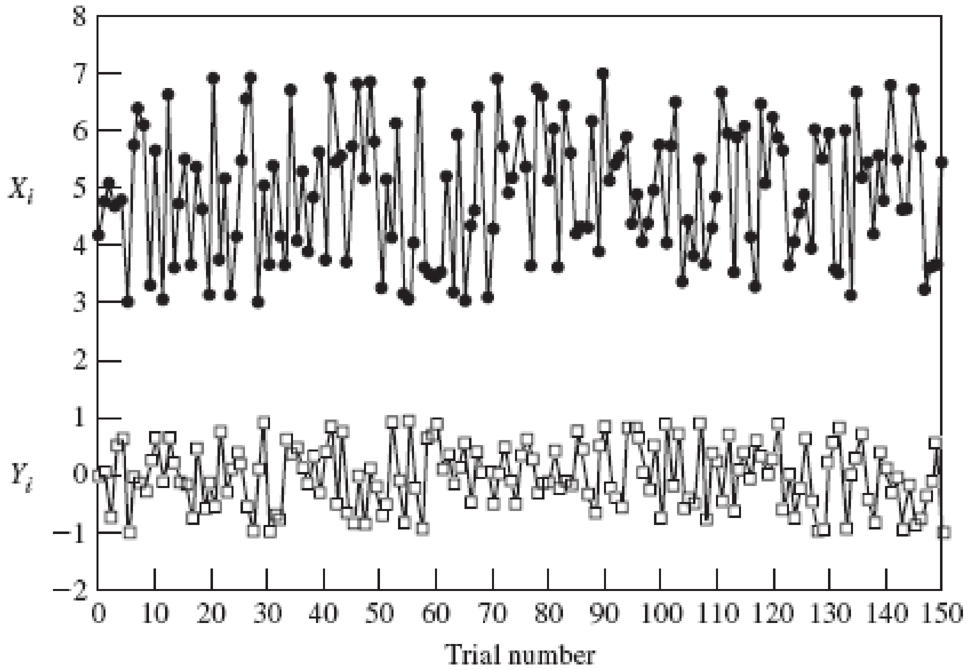
CSE-5605

Dr. Safdar Nawaz Khan Marwat
DCSE, UET Peshawar

Lecture 3

Expected Value of Discrete RV

- Entire pmf required for completely describing RV behavior
- In some cases, interest in parameters summarizing pmf



- Expected value or mean of discrete RV X defined by

$$m_X = E[X] = \sum_{x \in S_X} x p_X(x)$$

Expected Value of Discrete RV (cont.)

- The "expected value" does not mean expected outcome
- $E[X]$ not necessarily an outcome
  - E.g. the expected value of Bernoulli RV is p
  - But outcomes are always 0 or 1
- $E[X]$ corresponds to "average of X"
  - In large number of observations of X

Expected Value of Functions of RV

- Function $g(X)$ of RV X can be denoted by Z
  - Expected value of Z would be

$$E[Z] = E[g(X)] = \sum_{k} g(x_k) p_X(x_k)$$

- Or simply multiply each value of Z with its probability and add products for each k
  - For more than one value of X mapped to one value of Z

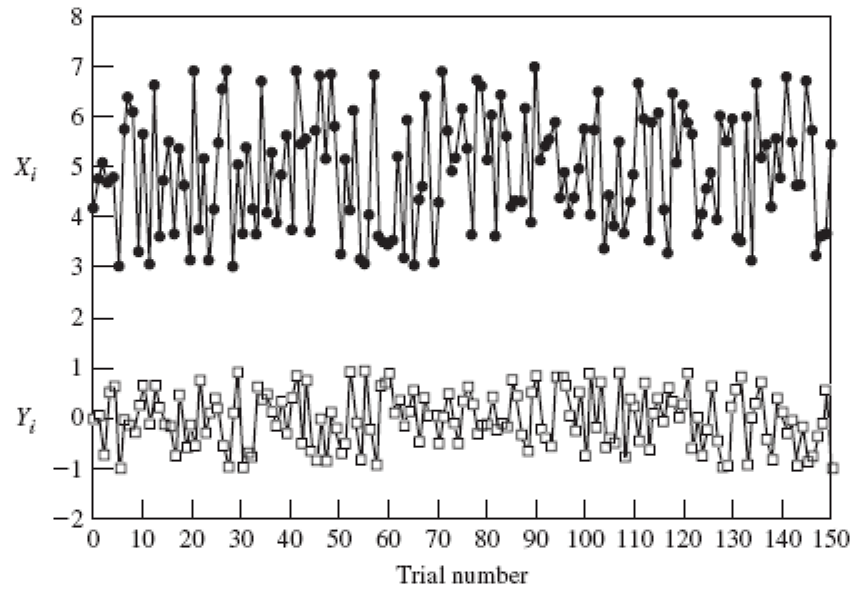
$$E[Z] = \sum_{k} g(x_k) p_X(x_k) = \sum_{j} z_j p_Z(z_j)$$

- Property (see others in Garcia):

$$E[ag(X) + c] = aE[g(X)] + c$$

Variance of Discrete RV

- Expected value provides limited information
- Interest also in the variation about expected value $X - E[X]$
- Squaring the variations gives positive values $(X - E[X])^2$
- **Variance** defined as the expected value of this square



$$\sigma^2{}_X = VAR[X] = E[(X - E[X])^2]$$

Variance of Discrete RV (cont.)

$$\sigma^2{}_X = \sum_{x \in S_X} (x - E[X])^2 p_X(x) = \sum_{k=1}^{\infty} (x_k - E[X])^2 p_X(x_k)$$

- The square root of variance is **standard deviation**

$$\sigma_X = STD[X] = \sqrt{VAR[X]}$$

- Variance also expressed as

$$E[(X - E[X])^2] = E[X^2 - 2E[X]X + E^2[X]]$$

$$= E[X^2] - 2E[X]E[X] + E^2[X] = E[X^2] - E^2[X]$$

- $E[X^2]$ is 2^{nd} moment of X, similarly $E[X^n]$ the n^{th} moment

Important Discrete RVs

- Bernoulli Random Variable

- $S_X = \{0, 1\}$
- $p_0 = q = 1 - p, p_1 = p$
- $E[X] = p, VAR[X] = pq$

- Binomial Random Variable

- $S_X = \{0, 1, 2, \ldots, n\}$
- $P_k = C^n_k \, p^k \, q^{n-k}$
- $E[X] = np, VAR[X] = npq$

Important Discrete RVs (cont.)

- Geometric Random Variable

- $S_X = \{1, 2, 3, \ldots\}$
- $P_k = q^{k-1}p$
- $E[X] = 1/p$, $VAR[X] = q/p^2$

- Uniform Random Variable

- $S_X = \{1, 2, 3, \ldots, L\}$
- $P_k = 1/L$
- $E[X] = (L+1)/2$, $VAR[X] = (L^2-1)/12$

Important Discrete RVs (cont.)

- Poisson Random Variable

- $S_X = \{0, 1, 2, \ldots\}$
- $P_k = (\alpha^k/k!)e^{-\alpha}$
- $E[X] = \alpha$, $VAR[X] = \alpha$

- Counting number of occurrences of an event in a time period
- Arises in situations where events occur at random
  - E.g. counts of emissions from radioactive substances
- Here, α is number of arrivals in time interval of length t
  - α is unitless and given as $\alpha = \lambda t$
  - λ is arrival rate with unit of jobs/time (e.g. packets/sec)

Examples

- The number N of queries arriving in t seconds at a call centre is a Poisson random variable with $\alpha = \lambda t$ where λ is the average arrival rate in queries/second. Assume that the arrival rate is 4 queries per minute. Find the probability of the following events:
  - 4 queries in 10 seconds
  - More than 4 queries in 10 seconds
  - Less than or equal to 5 queries in 2 minutes

Examples (cont.)

- The number N of packet arrivals in t seconds at a multiplexer is a Poisson random variable with $\alpha = \lambda t$ where λ is the average arrival rate in packets/second. Find the probability that there are no packet arrivals in t seconds.