Stochastic Processes

CSE-5605

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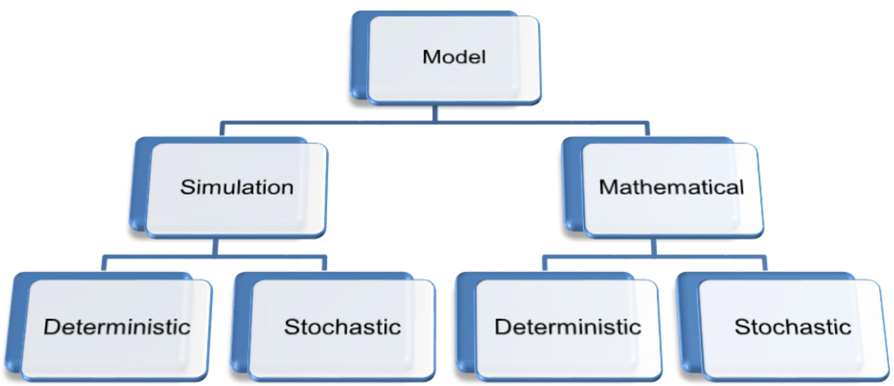
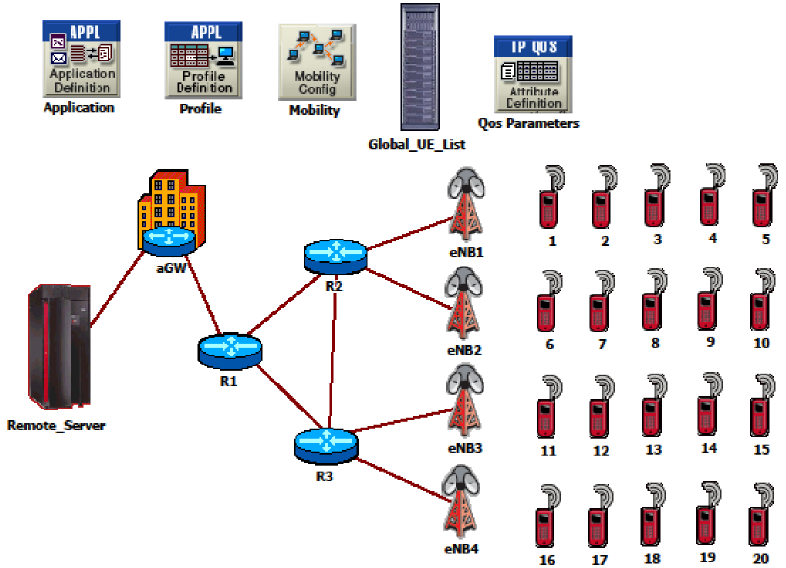
Lecture 1

Model of a Physical System

- **Model**: Approximate representation of physical situation
  - **Mathematical model**: Set of assumptions about how system works
    - **Deterministic model**: Offers repeatability of results, (e.g. Ohm's Laws)
    - **Stochastic model**: Characterizes randomness and uncertainty
  - **Simulation model**: Imitation of real system
    - **Deterministic model**: No random component involved, (e.g. chemical reaction)
    - **Stochastic model**: Must have random input component





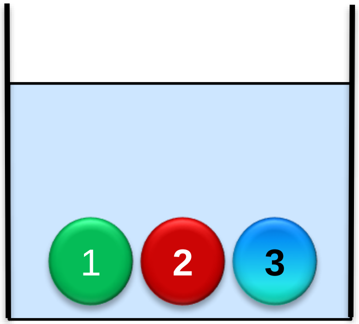
Source: S. N. K. Marwat, PhD Thesis, *University of Bremen, Germany*

Random Experiment

- **Random Experiment**: The result varies in random manner
- **Sample Space**: Set of all possible experiment results
- **Outcome**: A single element of sample space
- **Event**: A subset of sample space

- Example: An urn containing three balls, one is drawn
  - How probable it is that a ball withdrawn at random is labeled '1'?
  - Can you quantify this 'chance'?
  - Everyone of you should be able to write the sample space for this experiment!

$$S = \{ \qquad \qquad \}$$

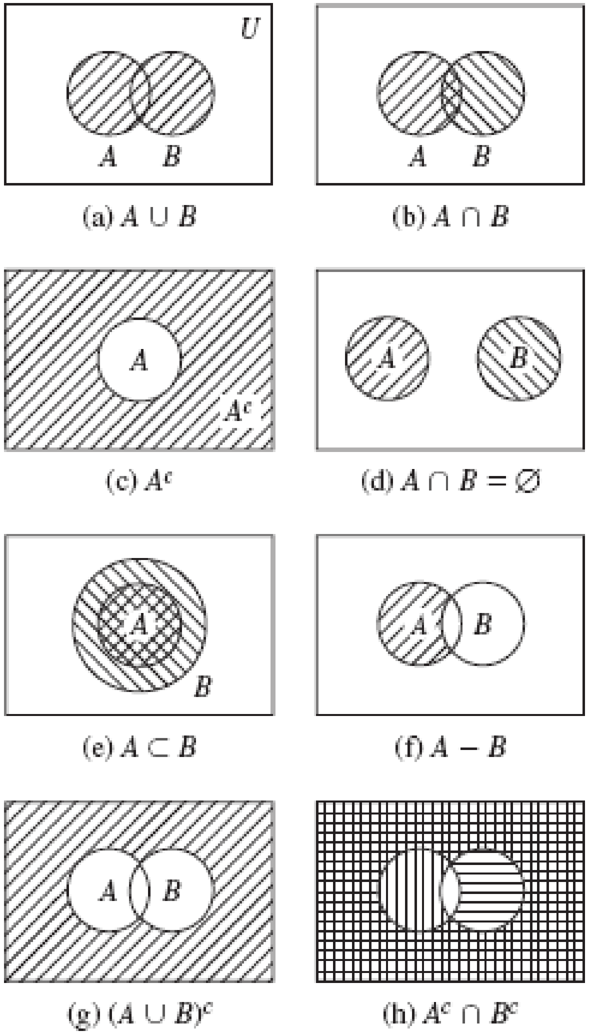


Set Theory

- Representation of events by sets
- Capital letters for names S, A, B, …
- Small letters for elements a, b, x, y, …
- Venn diagram illustrates sets and their interrelationship

Source: C. Görg, *Communication Networks II*, University of Bremen, Germany

Set Theory (cont.)



Source: A. Leon-Garcia; *Probability, Statistics, and Random Processes For Electrical Engineering*, 3rd Edition. Prentice Hall, 2008

Axioms of Probability

- **Axiom**: Universally accepted principle or rule

$$\boxed{0 \le P[A] \le 1}$$

$$\boxed{P[S] = 1}$$

- If $A \cap B = \varnothing$

$$\boxed{P[A \cup B] = P[A] + P[B]}$$

- If $A_i \cap A_j = \varnothing$ for all $i \neq j$

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k]$$

Corollaries

$$\boxed{P[A^c] = 1 - P[A]}$$

$$\boxed{P[\emptyset] = 0}$$

$$\boxed{P[A \cup B] = P[A] + P[B] - P[A \cap B]}$$

$$\boxed{\begin{aligned} P[A \cup B \cup C] = {} & P[A] + P[B] + P[C] - P[A \cap B] \\ & - P[B \cap C] - P[A \cap C] + P[A \cap B \cap C] \end{aligned}}$$

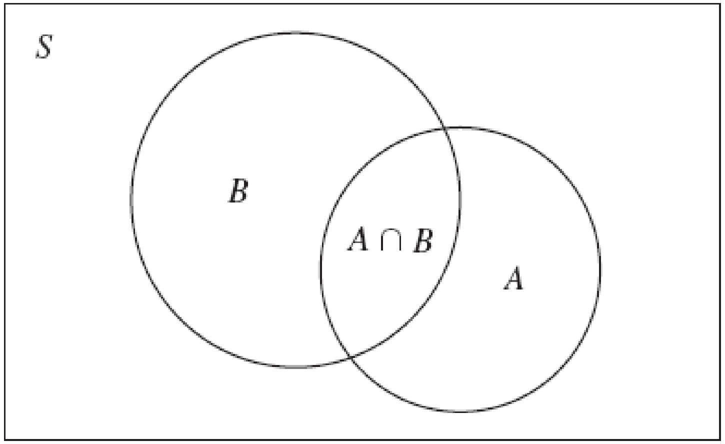
Conditional Probability

- Are two events A and B related?
- Is the occurrence of one event effecting the likelihood of other?
- **Conditional probability** of event A given that B has occurred

$$\boxed{P[A \mid B] = \frac{P[A \cap B]}{P[B]}}$$

- Knowledge of occurrence of event B reduces sample space
  - Occurrence of event A now depends on $A \cap B$
  - If $A \cap B = \emptyset$, occurrence of A is ruled out, given B occurred

Conditional Probability (cont.)



- If $A = B$, then due to the reduced sample space

$$\boxed{P[B \mid B] = 1}$$

- Similarly

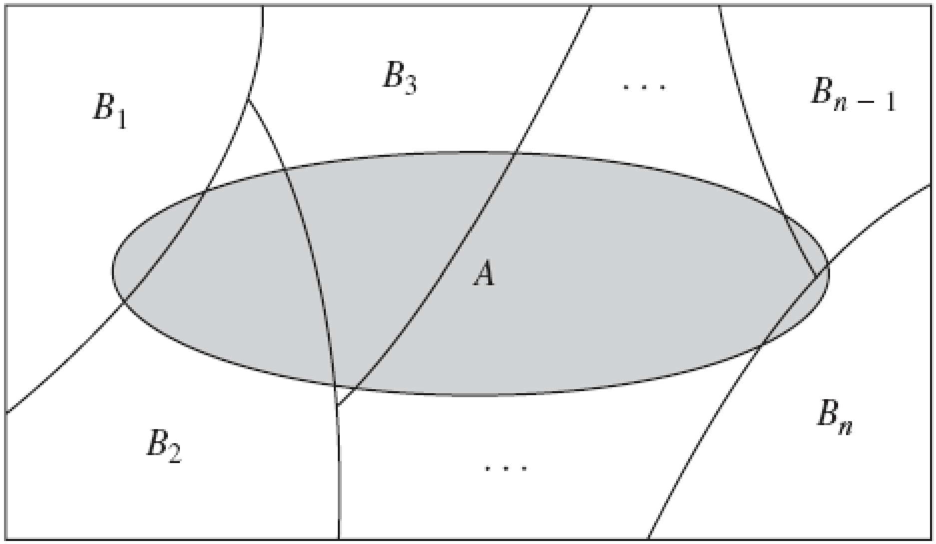
$$\boxed{P[A \cap B] = P[A \mid B]P[B]}$$

$$\boxed{P[A \cap B] = P[B \mid A]P[A]}$$

Theorem on Total Probability

- Let $B_1, B_2, \ldots, B_n$ be mutually exclusive events
- Union of all these events is S (partitions of S)
- Let A be union of events

$$\boxed{A = A \cap S = A \cap (B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n)}$$



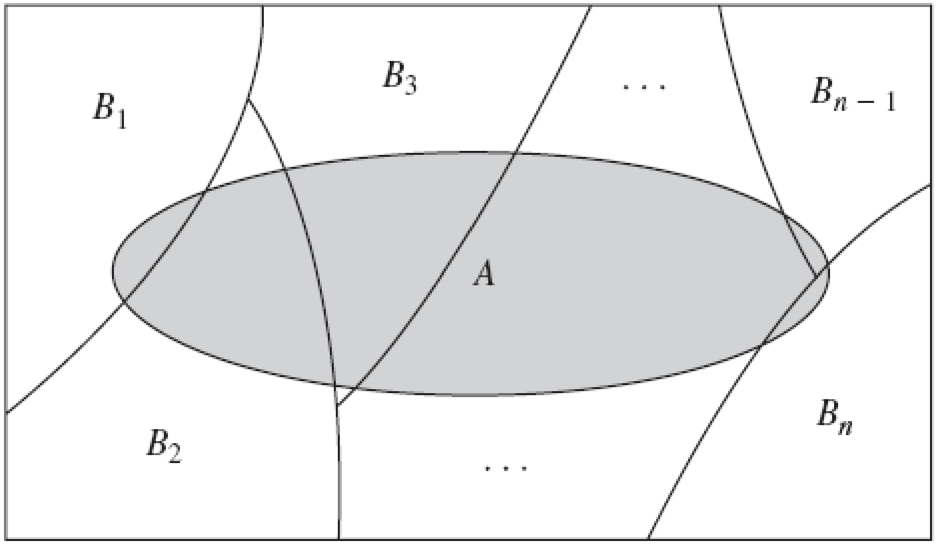
Theorem on Total Probability (cont.)

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)$$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots + P[A \cap B_n]$$

$$P[A] = P[A \mid B_1]P[B_1] + P[A \mid B_2]P[B_2] + \ldots + P[A \mid B_n]P[B_n]$$

$$P[A] = \sum_{k=1}^{n} P[A \mid B_k]P[B_k]$$



Bayes' Rule

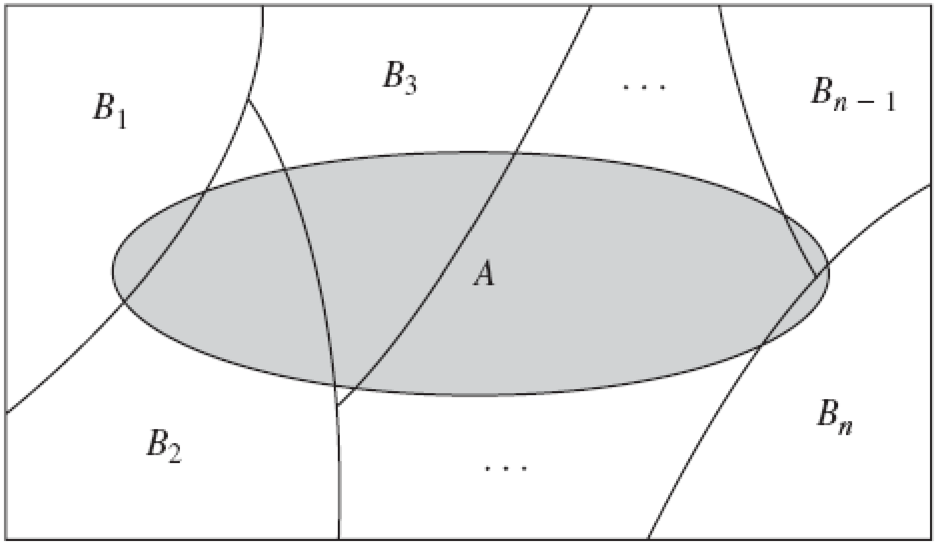
- Let $B_1, B_2, \ldots, B_n$ be mutually exclusive events
- If event A occurs, what is probability of B_j?

$$P[B_j \mid A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P[A \mid B_j]P[B_j]}{P[A]} = \frac{P[A \mid B_j]P[B_j]}{\sum_{k=1}^{n} P[A \mid B_k]P[B_k]}$$

- Simple case of Bayes' Rule is

$$P[B \mid A] = \frac{P[A \mid B]P[B]}{P[A]}$$

- Bayes' Rule
  - Method of calculating $P[B|A]$
  - Provided $P[A|B]$ available



Independence of Events

- An event occurs
  - But probability of a second event remains unchanged
  - Then the second event is independent of first
- Occurrence of one has no impact on probability of other

$$\boxed{P[A] = P[A \mid B] = \frac{P[A \cap B]}{P[B]}}$$

- What if $P[B] = 0$?

$$\boxed{P[A]P[B] = P[A \cap B]}$$

- Event A and B are said to be independent