# NUMBER THEORETIC SEQUENCES AND THE PROFINITE COMPLETION OF CLASSIFYING SPACES

by

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# Contents

## ABSTRACT

In two papers in 1979 and 1988, Lance studied the interaction between well-known number theoretic sequences and the topology of classifying spaces, building on the work of Adams, Bott, Milnor, Sullivan and others. In order to isolate the $p-$divisibility of each element, he used the $p-$localization of the classifying spaces, allowing him to study the spaces and sequences one odd prime at a time.

This dissertation seeks to further exploit the relationship between number theory and the topology of classifying spaces by moving the problem to the profinite context. In the process of making this shift, a correction to the literature needed to made, and so the tool of building a $CW-$complex that is the profinite completion was rebuilt from the beginning, and all proofs and results were re-written in the context needed for our perspective. This will add to the work done in the $p-$local setting by allowing us to more closely model the tools of algebraic number theory, and therefore exploit some of the major advances that have been made in that field. Last, the author reports on significant computational efforts which seek to find examples of specific important instances of $p-$divisibility in the number theoretic sequences which are studied in this research.

# Acknowledgements

I have always been known for, as my mother has described it, "taking the scenic route" in life. This dissertation has certainly not been an exception. There are many people who I have crossed paths with, and who have enriched my life along that journey, a few of whom I would like to acknowledge in this space.

I have had many wonderful friends in the graduate program who have helped me in ways that even they might not have realized. I would particularly like to mention Aaron Clark, Amy Kelley, Joe McCollum and Tim Lance the Younger as people who I have shared many wonderful experiences with, and I know will be a part of my life always.

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Last, I want to thank my wife Karen. Your patience and love have made this possible. You complete me. I love you, and will love you always.

I eagerly await the challenges that lie in front of me: our daughter, to be born this fall; starting my professorship at Westfield State College among a dynamic, dedicated group of mathematics faculty; and the continuation of this research program.

Marcus Jaiclin, July 2008

# Chapter 1

# Introduction

## 1.1 Historical Overview

There is a long history of connections between number theory and the topology of manifolds. For example, Kervaire and Milnor computed the size of the groups of homotopy spheres in two separate parts: first, the subgroup of spheres that form the boundary of a parallelizable manifold, and second, the quotient of the entire group of homotopy spheres by this subgroup. Strikingly, the denominator of a Bernoulli number appears in the subgroup, and the numerator of the same Bernoulli number appears in the quotient, giving the entire Bernoulli number in the order of the group of homotopy spheres. In another example, Adams showed that the Bernoulli numbers play an intrinsic role in the structure of the $J-$homomorphism, which gives information about the still-unsolved puzzle about the stable homotopy groups of the spheres.

In the 1970's and the 1980's, Lance built on the work of Milnor, Adams, Bott, Sullivan, and others to study the role of the Bernoulli numbers in the study of the classifying spaces for unitary and orthogonal bundles, $BU$ and $BO$. In order to make the problem tractable, following the work of Sullivan, the problem was moved to a

$p-$local context, which allows one to study the algebraic topology of a space one prime at a time. The work of Adams, Bott, and Sullivan show how the Bernoulli numbers, and another famous number theoretic sequence $(2^{2n-1}-1)$ can be found in the study of these two classifying spaces, and Lance used their work to carefully study their impact on the algebraic structures that are present there.

Sullivan generated the localization contruction alongside a completion construction in order to solve the Adams conjecture. His work was first detailed in a series of lectures in 1970 at MIT, the notes of which were distributed informally for many years, and finally formally published over 30 years later, after editing by Ranicki. Many of the results were also formally published in a more terse format in 1974 in a paper called "The Genetics of Homotopy Theory", where he used the localization and completion constructions to form a sort of "genetic" description of a manifold, each giving some pieces of information, that, when combined, allow one to "grow" the manifold from these pieces.

This dissertation seeks to move the work of Lance, and some of the work of those who came before him, from the $p-$local context to a $p-$adic context using the profinite completion construction. Much of modern algebraic number theory is done using the tools of the $p-$adic numbers and $p-$adic analysis, and so it is our conjecture that this shift will allow us to find deeper connections between the topology of classifying spaces (and other manifolds) and the modern developments of algebraic number theory, including Iwasawa Theory, the solution to Fermat's Last Theorem by Wiles, and possibly others.

## 1.2 Outline of the Dissertation

Chapter 2 reviews several branches of the literature. The first two sections develop tools essential to the constructions used in this paper. The next two show how localization and profinite completion can be applied to groups and rings, which gives an indication of how they will later be applied to topological spaces. Section 2.5 outlines the number theory that is found in the topology of classifying spaces, and some of what we hope to find later as this research program progresses. Section 2.6 shows how the localization construction applies to a $CW-$complex, and how that relates to the localization of a group. Section 2.7 summarizes some of the work of Lance, developing the connections between the number theory of 2.5 and the $p-$local $CW-$complexes of 2.6. Section 2.8 gives a detailed construction of one of the maps used in section 2.7 so that the reader can have an appreciation of how these connections arise.

Chapter 3 is based on the work of Sullivan in the 1970 MIT notes. In reading these notes, this author noticed an error in the notes (detailed in section 3.3), with a counterexample provided by Bousfield. We re-developed the profinite completion construction for $CW-$complexes from the beginning to be sure exactly where the error came about, and to be sure it was the only one. In the process, we clarified many of the proofs, and also generalized this construction from the special case of completing over all primes (considered by Sullivan) to a more general one where we complete over any collection of primes.

Chapter 4 applies the results of Chapter 3 to the special case of the classifying spaces $BO$ and $BU$. Some of Lance's results from section 2.7 are brought into the $p-$adic context, and more are conjectured to be present. A vision for the next steps to be taken is also presented.

Chapter 5 builds on a different branch of Lance's work which studies the computational aspects of the number theory which are present in the topology of classifying spaces. We first show how these computations give us information about these spaces, and then look at the results of the computer calculations that were performed in our study. Here, we also consider the next steps to be taken and how we will complete them.

# Chapter 2

# Literature Review

## 2.1 Eilenberg-MacLane Spaces and Postnikov Systems

This is taken directly from [DK01], pgs. 177-183, and pgs. 192-194.

**Definition 2.1.1** (Definition 7.19, pg. 177)**.** *Given an integer $n > 0$, and a group $\pi$, with $\pi$ abelian whenever $n > 1$, we define a $K(\pi, n)-$*space, or Eilenberg-MacLane Space *to be a $CW-$complex $Y$ which satisfies $\pi_k(Y) = 0$ for $k \neq n$ and $\pi_n(Y) = \pi$.*

**Theorem 2.1.2** (Theorem 7.20, pg. 178, and Corollary 7.24, pg. 182)**.** *Given any $n > 0$ and any group $\pi$, with $\pi$ abelian if $n > 1$, there exists a $K(\pi, n)-$space, which, for $n > 1$, is unique up to canonical homotopy equivalence.*

Let $[X, Y]$ denote the homotopy classes of maps from $X$ to $Y$.

**Theorem 2.1.3** (Theorem 7.22, pg. 179)**.** *There is a natural bijection: $[X, K(\pi, n)] \leftrightarrow H^n(X; \pi)$.*

**Theorem 2.1.4** (Theorem 7.40, pgs. 192-3)**.** *If $X$ is a simple space (c.f. 2.6.1, below) that is simply connected, then there exists a "tower" of spaces and maps:*

$$\begin{tikzcd}
& \vdots \arrow[d, "p_{n+1}"] \\
& X_n \arrow[d, "p_n"] \\
X \arrow[ur, "f_n"] \arrow[r, "f_{n-1}"] \arrow[ddr, "f_1"] \arrow[dddr, "f_0"'] & X_{n-1} \arrow[d, "p_{n-1}"] \\
& \vdots \arrow[d, "p_2"] \\
& X_1 \arrow[d, "p_1"] \\
& X_0
\end{tikzcd}$$

*so that each of the left-hand triangles commute, and, for each $n$, the map $p_n : X_n \to X_{n-1}$ is a fibration with fiber equal to the Eilenberg-MacLane space $K(\pi_n(X), n)$. Moreover, the spaces $X_n$ can be constructed so that $\pi_k(X_n) = 0$ for $k > n$, and $\pi_k(X_n) = \pi_k(X)$ for $n \geq k \geq 0$.*

In what follows, abbreviate $\pi_n(X)$ as $\pi_n$. For each fibration in the tower $K(\pi_n, n) \hookrightarrow X_n \to X_{n-1}$, there is a primary obstruction to finding a section; this obstuction class is called the $(n+1)^{st}$ *$k-$invariant of $X$*, and it lies in the cohomology group $H^{n+1}(X_{n-1}; \pi_n)$. Using the bijection in Theorem 2.1.3, we can see this obstruction as a map (representing a homotopy class) $X_{n-1} \to K(\pi_n, n+1)$, and so this extends the diagram representing the tower to its final version (notice that $X$ simply connected implies that $\pi_1$ is trivial, so there is no 1st $k-$invariant):

$$\begin{tikzcd}
 & \vdots \arrow[d, "p_{n+1}"] & \vdots \\
\vdots & X_n \arrow[r, "(n+1)^{\text{st}}\ k-\text{invariant}"] \arrow[d, "p_n"] & K\left(\pi_{n+1}, n+2\right) \\
X \arrow[ur, "f_n"] \arrow[r, "f_{n-1}"] \arrow[ddr, "f_1"] \arrow[dddr, "f_0"] & X_{n-1} \arrow[r, "n^{\text{th}}\ k-\text{invariant}"] \arrow[d, "p_{n-1}"] & K\left(\pi_n, n+1\right) \\
\vdots & \vdots \arrow[d, "p_2"] & \vdots \\
 & X_1 \arrow[r, "2^{\text{nd}}\ k-\text{invariant}"] \arrow[d, "p_1"] & K\left(\pi_2, 3\right) \\
 & X_0 &
\end{tikzcd} \tag{2.1.5}$$

The data of this system is summarized into a sequence of triples $\{\pi_n, p_n, k^n\}$ where:

(1). $\pi_n$ is an abelian group, with $\pi_n = \pi_n(X)$,

(2). $p_n : X_n \to X_{n-1}$ is a fibration with fiber $K\left(\pi_n, n\right)$,

(3). $X_0$ is contractible,

(4). $k^n \in H^n(X_{n-2}; \pi_{n-1})$ classifies $p_n$,

(5). the inclusion of the fiber $K\left(\pi_n, n\right) \hookrightarrow X_n$ induces the isomorphism

$$\pi_n(K\left(\pi_n, n\right)) \cong \pi_n(X_n).$$

This sequence of triples is called the *Postnikov tower* or *Postnikov decomposition* for $X$.

**Theorem 2.1.6** (Theorem 7.41, pgs. 193-4)**.** *For $X$ a simply connected simple space, the Postnikov tower determines the homotopy type of $X$; moreover, for any sequence of* *triples $\{\pi_n, p_n, k^n\}$ satisfying (1) - (5) above, there is a space $X$ with this sequence as its Postnikov tower.*

Much of this carries through without the assumption that $X$ is simply connected, though the proofs are more difficult.

This description of a $CW-$complex is dual to the description via $n-$skeleta in the following sense: Given a $CW-$complex $X$, we decompose $X$ into a sequence of $CW-$complexes $\{X^n\}$, where, for each $n$, the embedding $X^{n-1} \hookrightarrow X^n$ is a cofibration; in the Postnikov decomposition, we construct a sequence of $CW-$complexes $\{X_n\}$ where, for each $n$, the natural map $X_n \to X_{n-1}$ is a fibration. This perspective justifies the use of subscripts for the Postnikov spaces $X_n$ (as a covariant perspective) and superscripts for the skeleta $X^n$ (as a contravariant perspective).

## 2.2 Direct Limits, Inverse Limits, and other Categorical Constructions

The statements below are primarily based on those presented in Appendix 2 of [Dug66], but are modified slightly, following [Lan02], Chapter III, §10, and [Spa66], Introduction, §1. We also simultaneously consider an alternative definition, as in [AM69], Appendix 1.

First, we consider direct limits.

**Definition 2.2.1.** *We say that $\mathcal{A}$ is a* directed set *if the following hold:*

*(1). For any $\alpha \in \mathcal{A}$, $\alpha \leq \alpha$.*

*(2). If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$ for all $\alpha, \beta, \gamma \in \mathcal{A}$.*

*(3). For each $\alpha, \beta \in \mathcal{A}$, there exists $\gamma \in \mathcal{A}$ so that $\alpha \leq \gamma$ and $\beta \leq \gamma$.*

**Definition 2.2.2.** *We say that $\mathcal{A}$ is a* partially ordered set *if $\mathcal{A}$ is a directed set, and, in addition:*

*(4). If $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$.*

**Definition 2.2.3.** *A category $\mathcal{A}$ is said to* index *a category $X$ if there is a map $\varphi : \mathcal{A} \to X$; in this case, if $\varphi(\alpha) = X$, we write $X_\alpha$ to indicate the object of $X$ with its indexing element.*

**Definition 2.2.4.** *An indexing category $\mathcal{A}$ is called* filtering *if:*

*(1). For every pair of objects $\alpha, \beta \in \mathcal{A}$, there is an object $\gamma$ with morphisms $\alpha \to \gamma$ and $\beta \to \gamma$.*

*(2). If $\alpha, \beta$ are objects of $\mathcal{A}$ so that there are two morphisms $\alpha \rightrightarrows \beta$, then there is an object $\gamma$ and a morphism $\beta \to \gamma$ so that the compositions $\alpha \rightrightarrows \beta \to \gamma$ are equal.*

**Definition 2.2.5.** *Let $\mathcal{A}$ be a directed set and let $\{Y_\alpha | \alpha \in \mathcal{A}\}$ be a set of spaces indexed by $\mathcal{A}$. For each $\alpha, \beta \in \mathcal{A}$, where $\alpha \leq \beta$, assume that there exists a continuous map $\varphi_{\alpha\beta} : Y_\alpha \to Y_\beta$ such that $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}$ whenever $\alpha \leq \beta \leq \gamma$. Then the family $\{Y_\alpha; \varphi_{\alpha\beta}\}$ is called a* direct system *with spaces $Y_\alpha$ and connecting maps $\varphi_{\alpha\beta}$.*

**Definition 2.2.6.** *Given a direct system $\{Y_\alpha; \varphi_{\alpha\beta}\}$, form the space $\varinjlim_{\alpha} Y_\alpha = {}^{\bigcup Y_\alpha}/_{\sim}$, where $y_\alpha \sim y_\beta$ whenever they have a common successor in the direct system. The canonical map $\varphi_\alpha : Y_\alpha \to \varinjlim Y_\alpha$ is given by the composition of the inclusion into the infinite union, followed by projection via the equivalence relation. Then, a set $U \subset \varinjlim Y_\alpha$ is open if and only if ${\varphi_\alpha}^{-1}(U)$ is open for all $\alpha$.*

### Examples

(1). Given a $CW-$complex $X$ with $n-$skeleta $X_n$ and inclusion maps $\varphi_m \to \varphi_n$ whenever $m \leq n$, then $X = \varinjlim_{n} X_n$.

(2). The Grassmannian manifold $G_{n,k}$ is the collection of $n-$dimensional linear subspaces of $\mathbb{R}^{n+k}$, topologized as the quotient space ${}^{O(n+k)}/_{O(n)\times O(k)}$. Then, using the inclusion of $O(n+k)$ into $O(n+k+1)$ via:

$$M \mapsto \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$$

The direct limit over $k$ is the infinite Grassmannian $G_n$, or $BO(n)$ (see [MS74], pg. 63). The direct limit over $n$ of the $BO(n)$ is the space $BO$. Replacing the orthogonal groups $O(j)$ with unitary groups $U(j)$, we can construct the spaces $BU(n)$ and $BU$.

The direct limit space is always nonempty whenever any one of the $Y_\alpha$ is nonempty.

One can also define a direct limit using a filtering index category, and obtain the same description, as above.

**Definition 2.2.7.** *Let $\mathcal{A}$ be a directed set and let $\{Y_\alpha | \alpha \in \mathcal{A}\}$ be a set of spaces indexed by $\mathcal{A}$. For each $\alpha, \beta \in \mathcal{A}$, where $\alpha \leq \beta$, assume that there exists a continuous map $\varphi_{\beta\alpha} : Y_\beta \to Y_\alpha$ such that $\varphi_{\gamma\alpha} = \varphi_{\beta\alpha} \circ \varphi_{\gamma\beta}$ whenever $\alpha \leq \beta \leq \gamma$. Then the family $\{Y_\alpha; \varphi_{\beta\alpha}\}$ is called an* inverse system *with spaces $Y_\alpha$ and connecting maps $\varphi_{\alpha\beta}$.*

**Definition 2.2.8.** *Let $\{Y_\alpha; \varphi_{\beta\alpha}\}$ be an inverse system. Form the product space $\prod\limits_{\alpha\in\mathcal{A}} Y_\alpha$, and, for each $\alpha$, let $p_\alpha$ be the projection onto the factor indexed by $\alpha$. Then the* inverse limit *of the inverse system is the space $\varprojlim Y_\alpha$ of points $y \in \prod Y_\alpha$ where $p_\alpha(y) = (\varphi_{\beta\alpha} \circ p_\beta)(y)$ for all $\alpha$, $\beta$ where $\alpha \leq \beta$.*

**Examples**

(1). The Cantor ternary set is the inverse limit of the collections of points $C_n = \left\{ \sum\limits_{j=0}^{n} \frac{c_j 3^j}{3^n} \middle| c_j = 0 \text{ or } 2 \right\}$, with maps given by projection onto the first $n-1$ coordinates.

(2). Consider the tower of field extensions:

$$\cdots \to \mathbb{Q}(\zeta_{p^n}) \to \mathbb{Q}(\zeta_{p^{n-1}}) \to \cdots \to \mathbb{Q}(\zeta_{p^2}) \to \mathbb{Q}(\zeta_p) \to \mathbb{Q}$$

The Galois group of the extension $\mathbb{Q}(\zeta_{p^k}) \to \mathbb{Q}$, for $k \geq 1$, is $\left({}^{\mathbb{Z}}/_{p^k\mathbb{Z}}\right)^* \cong {}^{\mathbb{Z}}/_{(p-1)\mathbb{Z}} \oplus \left(\bigoplus_{j=2}^{k} {}^{\mathbb{Z}}/_{p\mathbb{Z}}\right)$. Maps from one stage to the next are those induced by the Galois structure of the tower; they are given by projection onto all but the last coordinate of the sum of the groups ${}^{\mathbb{Z}}/_{p\mathbb{Z}}$. Taking inverse limits of the tower and its Galois groups gives an infinite extension over $\mathbb{Q}$ with Galois group equal to the multiplicative group of units in the $p-$adic integers. (see [Iwa58]). The direct sum given provides a natural decomposition of the units of the $p-$adic integers; the component ${}^{\mathbb{Z}}/_{(p-1)\mathbb{Z}}$ is the collection of $(p-1)^{\text{st}}$ roots of unity, and each of the succesive components is the collection of units which are congruent to 1 (mod $p^{j-1}$), but not (mod $p^j$).

Similarly, one may define the inverse limit using a filtering index category and obtain the same description, as above.

There is no guarantee that an inverse limit space is nonempty, in fact, Dugundji notes that it is possible to construct an inverse system so that the inverse limit is empty, despite satisfying all of the following: each connecting map is surjective, all spaces are nonempty, and all maps $\varphi_{\alpha\alpha}$ from $Y_\alpha$ to itself are the identity.

If each $Y_\alpha$ is any of the following:

(1). Hausdorff

(2). Hausdorff and compact

(3). nonempty, Hausdorff and compact

(4). totally disconnected

then the inverse limit of the $Y_\alpha$ also satisfies that property. Notice that Hausdorff is required with compactness to ensure that the inverse limit is closed in the product space. Also notice that if the $Y_\alpha$ are all discrete, then the inverse limit is totally disconnected, but not necessarily discrete (this is the case for the $p-$adic numbers, below). A space that is totally disconnected, Hausdorff and compact is sometimes referred to as a *Stone space*. The inverse limit of finite groups, under its natural topology, is a Stone space, since the only Hausdorff topology on a finite set is the discrete topology, so this is the natural topology for a finite group.

The restriction of the projection map $p_\beta : \prod_{\alpha\in\mathcal{A}} Y_\alpha \to Y_\beta$ to the inverse limit will be denoted $\varphi_\beta$.

**Proposition 2.2.9.** *The natural topology on the inverse limit (i.e. the subspace topology of the product topology) has as a basis the collection $\{\varphi_\alpha^{-1}(U)|U$ is open in $Y_\alpha\}$.*

**Definition 2.2.10.** *Given a partially ordered set $\mathcal{A}$, a subset $\mathcal{B}$ is called* cofinal *in $\mathcal{A}$ if, for each $\alpha \in \mathcal{A}$, there is some $\beta \in \mathcal{B}$ so that $\alpha \leq \beta$.*

Alternatively, one can define:

**Definition 2.2.11.** *Let $\mathcal{A}$ be a filtering index category, $\mathcal{B}$ an index category, and let $\varphi : \mathcal{A} \to \mathcal{B}$. Then $\mathcal{B}$ is cofinal in $\mathcal{A}$ if:*

*(1). For every $\beta \in \mathcal{B}$ there is an $\alpha \in \mathcal{A}$ and a morphism in $\mathcal{B}$ so that $\beta \to \varphi(\alpha)$.*

*(2). For $\beta \in \mathcal{B}$, if there are two morphisms in $\mathcal{B}$, $\beta \rightrightarrows \varphi(\alpha)$, then there is an $\alpha' \in \mathcal{A}$ and a morphism $\alpha \to \alpha'$ so that the composed maps $\beta \rightrightarrows \varphi(\alpha) \to \varphi(\alpha')$ are equal.*

The following proposition motivates these definitions:

**Proposition 2.2.12.** *If $\mathcal{B}$ is cofinal in $\mathcal{A}$, then*

*(1).*

$$\varinjlim_{\mathcal{A}} Y_\alpha = \varinjlim_{\mathcal{B}} Y_\beta$$

*(2).*

$$\varprojlim_{\mathcal{A}} Y_\alpha = \varprojlim_{\mathcal{B}} Y_\beta$$

*In other words, in either the direct or the inverse limit construction, one can form the limit using a cofinal subset rather than the entire set without changing the resulting limit object.*

*Proof.* Refer to [Dug66], pages 424-5, and 431-2 for proofs, or [AM69], pages A.1.4-A.1.6. □

## 2.3 Localization

At the heart of this dissertation is the distinction between the localization and the completion, and so we look at both constructions in detail. Most of this follows [Sul05] quite directly, and all page numbers and theorem numbers that are not otherwise attributed are from his volume, as cited in the bibliography of this document.

In order to maintain some semblance of brevity, I have omitted all proofs unless the proof contains essential details that are referred to later in this document.

**Definition 2.3.1.** *Let $R$ be a ring. Then $S$ is a multiplicative subset of $R$ if:*

*(1).* $1 \in S$

*(2).* $0 \notin S$

*(3).* $a, b \in S$ *implies* $ab \in S$.

**Definition 2.3.2** (Definition 1.1, pg. 1). *The ring $R$ localized away from (or at) $S$, where $S$ is a multiplicative subset, is denoted $S^{-1}R$, and is defined as the collection of equivalence classes $\{{}^{r}/_{s} | r \in R, s \in S\}$ where ${}^{r}/_{s} \sim {}^{r'}/_{s'}$ iff $rs' = r's$.*

$S^{-1}R$ can be made into a ring via the operations:

- $[{}^{r}/_{s}] \cdot [{}^{r'}/_{s'}] = [{}^{rr'}/_{ss'}]$
- $[{}^{r}/_{s}] + [{}^{r'}/_{s'}] = [{}^{rs'+r's}/_{ss'}]$

The canonical homomorphism $R \to S^{-1}R$ sends $r \mapsto [{}^{r}/_{1}]$, and is referred to as the localization map.

**Example** If $p \subset R$ is a prime ideal, then $R \setminus p$ is a multiplicative subset; in this case we denote $(R \setminus p)^{-1}R$ as $R_{(p)}$. In $R_{(p)}$, every element outside $pR_{(p)}$ is invertible in $R_{(p)}$, and the localization map $R \to R_{(p)}$ takes $p$ to the unique maximal ideal of non-units in $R_{(p)}$.

We will typically use $R = \mathbb{Z}$, and $p$ as the prime ideal generated by a prime integer $p$. In this case, $\mathbb{Z}$ localized at $p$ is denoted $\mathbb{Z}_{(p)}$. We will also consider localization by more than one prime at a time; in this case let $\ell$ denote a nonempty collection of primes. Then $\mathbb{Z}_{(\ell)}$ denotes the ring formed by localizing using the multiplicative subset generated by all of the primes not in $\ell$.

**Examples** Other easy examples:

- $\mathbb{Z}_{(\text{all primes})} = \mathbb{Z}$
- $\mathbb{Z}_{(0)} = \mathbb{Z}_{(\emptyset)} = \mathbb{Q}$

If $M$ is an $R-$module, then we can define the localized $S^{-1}R-$module $S^{-1}M$ via:

$$S^{-1}M = M \otimes_R S^{-1}R$$

This can be used to localize abelian groups:

**Definition 2.3.3** (Definition 1.2, pg. 4). *If $G$ is an abelian group then the localization of $G$ with respect to a set of primes $\ell$, $G_{(\ell)}$ is the $\mathbb{Z}-$module $G \otimes \mathbb{Z}_{(\ell)}$.*

*The canonical localization homomorphism $\mathbb{Z} \to \mathbb{Z}_{(\ell)}$ induces a canonical localization homomorphism for $G$, $G \to G_{(\ell)}$.*

The multiplicative set $s$, consisting of the primes and products of primes not in $\ell$, can be partially ordered by divisibility; i.e. $a \leq a'$ whenever $a|a'$.

For $a \in s$, let $a = {p_1}^{e_1} \cdot {p_2}^{e_2} \cdot \ldots \cdot {p_n}^{e_n}$; then consider the set $a_m = \{b \in \mathbb{Z} | b = {p_1}^{f_1} \cdot {p_2}^{f_2} \cdot \ldots \cdot {p_n}^{f_n}, f_j \in \mathbb{Z}, f_j \leq e_j\}$. We can form equivalence classes $G_{(s)} = \{{}^{bg}/_{c} | b \in \mathbb{Z}, g \in G, c \in a_m,$ numerator ${}^{c}/_{b}$ in lowest terms divides $a\}$ where, as above, $[{}^{bg}/_{c}] \sim [{}^{b'g}/_{c'}]$ whenever $bc' = b'c$.

Then, we can form a directed system of groups and homomorphisms indexed by the directed set $\{s\}$ where:

$$G_{(s)} \xrightarrow{\text{multiplication by } {}^{s'}/_{s}} G_{(s')} \qquad \text{whenever } s \leq s'$$

So, for example, $G_{(6)}$ would have the fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$ as well as their integer and group element multiples, assuming $2, 3 \notin \ell$. The map $G = G_{(1)} \to G_{(2)}$ would take $1 \mapsto 2$, and so the image of $G$ in $G_{(2)}$ is $2G$.

The next proposition asserts that the direct limit of the $G_{(s)}$ can be used to define the localization of $G$ at $\ell$, and that this is consistent with the definition of the localization thinking of an abelian group as a $\mathbb{Z}-$module.

**Proposition 2.3.4** (Proposition 1.1, pg. 4).

$$\varinjlim_{s} G_{(s)} \cong G_{(\ell)} \stackrel{defn}{=} G \otimes \mathbb{Z}_{(\ell)}$$

**Lemma 2.3.5** (Lemma 1.2, pg. 4). *If $\ell$ and $\ell'$ are two sets of primes, then $\mathbb{Z}_{(\ell)} \otimes \mathbb{Z}_{(\ell')} \cong$*

$\mathbb{Z}_{(\ell\cap\ell')}$ *as rings.*

**Lemma 2.3.6** (Lemma 1.3, pg. 5)**.** *The $\mathbb{Z}-$module structure on an abelian group $G$ extends to a $\mathbb{Z}_{(\ell)}-$module structure if and only if $G$ is isomorphic to its localizations at every set of primes containing $\ell$.*

**Example**

(1).

$$\left(\mathbb{Z}/_{p^n\mathbb{Z}}\right)_{(\ell)} \overset{\text{defn}}{=} \mathbb{Z}/_{p^n\mathbb{Z}} \otimes \mathbb{Z}_{(\ell)} = \begin{cases} 0 & p \notin \ell \\ \mathbb{Z}/_{p^n\mathbb{Z}} & p \in \ell \end{cases}$$

(2).

$$\begin{pmatrix} \text{finitely generated} \\ \text{abelian group } G \end{pmatrix} \cong \underbrace{\mathbb{Z}_{(\ell)} \oplus \mathbb{Z}_{(\ell)} \oplus \cdots \oplus \mathbb{Z}_{(\ell)}}_{\text{rank}(G)\text{ factors}} \oplus\, \ell-\text{torsion } G$$

**Proposition 2.3.7** (Proposition 1.4, pg. 6)**.** *The process of localization takes exact sequences of abelian groups to exact sequences of abelian groups.*

**Corollary 2.3.8** (Corollary 1.5, pg. 6)**.** *If $0 \to A \to B \to C \to 0$ is an exact sequece of abelian groups, and two out of the three groups are $\mathbb{Z}_{(\ell)}-$modules, then so is the third.*

**Corollary 2.3.9** (Corollary 1.6, pg. 6)**.** *If, in the long exact sequence of groups:*

$$\ldots \to A_n \to B_n \to C_n \to A_{n-1} \to B_{n-1} \to \ldots$$

*two of the three sets of groups $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are $\mathbb{Z}_{(\ell)}-$modules, then so is the third.*

**Corollary 2.3.10** (Corollary 1.7, pg. 6)**.** *If $F \hookrightarrow E \to B$ is a fibration of connected spaces with abelian fundamental groups, then if two of $\pi_*(F)$, $\pi_*(E)$, and $\pi_*(B)$ are $\mathbb{Z}_{(\ell)}-$modules, then the third is also.*

**Proposition 2.3.11** (Proposition 1.8, pgs. 6-7)**.** *Let $F \hookrightarrow E \to B$ be a fibration in which $\pi_1(B)$ acts trivially on $\tilde{H}_*(F; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$ for primes $p$ not in $\ell$, and with $F$ and $B$ path connected. Then, if two of $\tilde{H}_*(F;\mathbb{Z})$, $\tilde{H}_*(E;\mathbb{Z})$, and $\tilde{H}_*(B;\mathbb{Z})$ are $\mathbb{Z}_{(\ell)}-$modules, then the third is also.*

**Definition 2.3.12.** *A square of abelian groups with maps of the form:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "l"'] & D \end{tikzcd}$$

*is called a* fibre square *if the sequence:*

$$0 \to A \xrightarrow{i \oplus j} B \oplus C \xrightarrow{l-k} D \to 0$$

*is exact.*

**Lemma 2.3.13** (Lemma 1.9, pg. 7)**.** *The direct limit of fibre squares is a fibre square, and the direct sum of fibre squares is a fibre square.*

**Proposition 2.3.14** (Proposition 1.10, pg. 7)**.** *If $G$ is an abelian group, and $\ell$ and $\ell'$ are two collections of primes so that $\ell \cap \ell' = \emptyset$ and $\ell \cup \ell' = \mathbb{Q}$, then:*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G \otimes \mathbb{Z}_{(\ell)} \arrow[d] \\ G \otimes \mathbb{Z}_{(\ell')} \arrow[r] & G \otimes \mathbb{Q} \end{tikzcd}$$

*is a fibre square.*

A generalization of this would be:

**Proposition 2.3.15** (Meta Proposition 1.12, pg. 8)**.** *Form the infinite diagram:*

$$\begin{tikzcd}
G_{(2)} \arrow[dr] & G_{(3)} \arrow[d] & G_{(5)} \arrow[dl] & \cdots & G_{(p)} \arrow[dlll] & \cdots \\
 & G_{(0)} & & & &
\end{tikzcd}$$

*Then $G$ is the fibre product of its localizations $G_{(2)}, \ldots, G_{(p)}, \ldots$ over $G_{(0)}$.*

# 2.4 Completions

## 2.4.a Profinite and Formal Completions of Groups

**Definition 2.4.1.** *Let $R$ be a ring with a unit element. Let*

$$I_1 \supset I_2 \supset I_3 \supset \ldots$$

*be a decreasing sequence of ideals of $R$, such that $\bigcap\limits_{j=1}^{\infty} I_j = \{0\}$. Then, define a metric on $R$ via:*

$$d(x, y) = \delta^{-k}, \delta > 1$$

*where $k$ is defined as the unique integer where $x - y \in I_k$, but $x - y \notin I_{k+1}$ (using the convention that $I_0 = R$). It is easy to verify that $d$ defines a distance function on $R$ in the usual sense.*

**Definition 2.4.2** (Definition 1.3, pg. 10)**.** *Given a ring $R$ with a distance function $d$ as described above, define the ring $\hat{R}_d$, the completion of $R$ with respect to $d$ using Cauchy sequences; that is, the elements of $\hat{R}_d$ are equivalence classes of sequences $\{x_n\}$ where $\lim\limits_{m,n\to\infty} d(x_n, x_m) = 0$, and we say that $\{x_n\} \sim \{y_n\}$ when $d(x_n, y_n) \to 0$. The obvious* *arithmetic operations make $\hat{R}_d$ into a ring; the metric $d$ makes $\hat{R}_d$ into a topological ring.*

There is a natural homomorphism, the completion homomomorphism, $c : R \to \hat{R}_d$ such that $r \mapsto [\{r, r, r, \ldots\}]$.

**Example** [Example 1, pg. 10] Let $R = \mathbb{Z}$, and let $I_j = p^j\mathbb{Z}$, the prinicipal ideal generated by $p^j$. The induced topology on $\mathbb{Z}$ is the $p-$adic topology, and the completion formed this way is the ring of $p-$adic integers, $\hat{\mathbb{Z}}_p$.

We note that, in the $p-$adic integers, the equation $x^{p-1} - 1$ can be factored completely; that is, all of the $(p-1)^{\text{st}}$ roots of unity are contained in the $p-$adic integers. It is also well-known that these are the only roots of unity contained in the $p-$adic integers.

**Example** [Example 2, pg. 11] Let $\ell$ be a non-empty subset of prime numbers (where $\{p_j\} = \{2, 3, 5, \ldots\}$ is the standard linear ordering of the prime numbers), and let $I_{j,\ell}$ be the principal ideal

$$I_{j,\ell} = \left( \prod_{p \in \ell, p \leq p_j} p^j \right) \mathbb{Z}$$

The topology which results from this collection of ideals using the process described above is the $\ell-$adic topology, and the completion is denoted $\hat{\mathbb{Z}}_\ell$.

If $\ell' \subset \ell$, then $I_{j,\ell} \subset I_{j,\ell'}$, and so any sequence that is Cauchy in the $\ell-$adic metric is also Cauchy in the $\ell'-$adic metric. This gives a natural map $\hat{\mathbb{Z}}_\ell \to \hat{\mathbb{Z}}_{\ell'}$.

**Proposition 2.4.3** (Proposition 1.13, pg. 11)**.** *Form the inverse system of rings $\left\{\mathbb{Z}/_{p^n\mathbb{Z}}\right\}$, where the map $\mathbb{Z}/_{p^n\mathbb{Z}} \to \mathbb{Z}/_{p^m\mathbb{Z}}$ is reduction mod $p^m$ when $n \geq m$. Then there is a natural ring isomorphism $\rho_p$:*

$$\hat{\mathbb{Z}}_p \xrightarrow{\rho_p} \varprojlim \left\{\mathbb{Z}/_{p^n\mathbb{Z}}\right\}$$

**Proposition 2.4.4.** $\varprojlim \mathbb{Z}/_{p^n\mathbb{Z}}$ *is compact, Hausdorff and totally disconnected in the* *inverse limit topology.*

**Corollary 2.4.5** (Unnumbered Corollary, pg. 12)**.** *$\hat{\mathbb{Z}}_p$ is compact, totally disconnected and Hausdorff.*

**Proposition 2.4.6** (Proposition 1.14, pg. 12)**.** *The product of the natural maps $\hat{\mathbb{Z}}_\ell \to \hat{\mathbb{Z}}_p$ yields an isomorphism of rings*

$$\hat{\mathbb{Z}}_\ell \xrightarrow{\cong} \prod_{p \in \ell} \hat{\mathbb{Z}}_p$$

Note: Unlike $\hat{\mathbb{Z}}_p$, the ring $\hat{\mathbb{Z}}_\ell$ is not an integral domain if $\ell$ contains more than one prime.

**Proposition 2.4.7** (Note, pg. 12)**.** *Both $\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_\ell$ are topologically cyclic – that is, there exists an element $r$ so that the multiples of $r$ are dense in the ring.*

**Definition 2.4.8.** *Let $G$ be any group, and let $\ell$ be a non-empty set of prime numbers. Denote by $\{H\}_\ell$ the collection of those subgroups $H$ of $G$ that are normal in $G$ and have index equal to a product of primes in $\ell$. If $H \in \{H\}_\ell$, then we will say that $H$ is* of $\ell-$index*.*

The set $\{H\}_\ell$ can be partially ordered by inclusion; that is, $H_1 \leq H_2$ whenever $H_1 \subseteq H_2$.

**Definition 2.4.9** (Definition 1.4, page 14)**.** *The $\ell-$profinite completion of $G$ is constructed via the inverse limit of the quotients via the normal subgroups of index equal to a product of primes in $\ell$:*

$$\hat{G}_\ell = \varprojlim_{H \in \{H\}_\ell} {}^{G}/_{H}$$

$\hat{G}_\ell$ is a topological group via the inverse limit topology induced by the discrete topology on the finite groups ${}^{G}/_{H}$, making it a totally disconnected, compact, Hausdorff space (that is also homeomorphic to the Cantor Set). The projections $G \to {}^{G}/_{H}$ induce a map to the inverse limit; this map is universal for maps from $G$ into a finite $\ell-$group. In other words, in the following diagram, the dotted arrow exists, and is uniquely determined by $\varphi$, whenever $L$ is a group whose order is a product of primes from $\ell$:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & \hat{G}_\ell \arrow[d, dotted] \\ & L \end{tikzcd}$$

The operation of profinite completion commutes with maps $G \to G'$, and so defines a functor from the category of groups with homomorphisms to the category of finite topological $\ell-$groups with continuous homomorphisms. The construction of the induced homomorphism can be seen as follows:

Given $f : G \to G'$, and $H'$, a subgroup of $G'$ of $\ell-$index, then let $H = f^{-1}H'$.

**Lemma 2.4.10.** *Let $H'$ be a normal subgroup of finite index in $G'$. Let $f : G \to G'$ be a homomorphism, and let $H = f^{-1}H'$. Then $[G : H]$ divides $[G' : H']$.*

It follows from the lemma that, given $F : G \to G'$, and $H'$, a subgroup of $\ell-$index, and $H = f^{-1}H'$, we have that $H$ is of $\ell-$index also. This says that the diagram:

$$\begin{tikzcd} H = f^{-1}H' \arrow[r] \arrow[d] & H' \arrow[d] \\ G \arrow[r, "f"] \arrow[d] & G' \arrow[d] \\ {}^{G}/_{H} \arrow[r, "\overline{f}"] & {}^{G'}/_{H'} \end{tikzcd}$$

induces a map of inverse systems via the map $\overline{f}$ in the diagram. We shall denote the map of inverse limits by $\hat{f} : \hat{G}_\ell \to \hat{G'}_\ell$.

**Examples**

(1). Let $G = \mathbb{Z}$, and let $\ell = \{p\}$. Then the only subgroups which are of $\ell-$ index are the subgroups $p^n\mathbb{Z}$. This says that the $p-$profinite completion of $\mathbb{Z}$ is the inverse limit $\varprojlim \mathbb{Z}/_{p^n\mathbb{Z}}$, which agrees exactly with the $p-$adic integers.

(2). Given $\ell = \{p_1, p_2, \dots\}$, a nonempty set of primes, we consider the partially ordered set of products of primes from $\ell$ via:

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_m^{\alpha_m} \leq p_1^{\beta_1} p_2^{\beta_2} \cdot \ldots \cdot p_n^{\beta_n} \qquad \text{whenever} \qquad \alpha_j \leq \beta_j \text{ for all } j \leq \max(m, n)$$

It is easy to see that the collection:

$$p_1^k p_2^k \cdot \ldots \cdot p_k^k$$

is cofinal in this partially ordered set, and so the inverse limit reduces to:

$$\begin{aligned}
\hat{\mathbb{Z}}_\ell &= \varprojlim_k \prod_{j=0}^{k} \mathbb{Z}/_{{p_j}^k\mathbb{Z}} \\
&= \prod_{p\in\ell} \varprojlim_k \mathbb{Z}/_{p^k\mathbb{Z}} \\
&= \prod_{p\in\ell} \hat{\mathbb{Z}}_p
\end{aligned}$$

as expected.

(3). Similarly, for any abelian group $G$, we have that:

$$\hat{G}_\ell \cong \prod_{p\in\ell} \hat{G}_p$$

(4). It follows that for a finitely generated abelian group $G$ with free part of rank $n$, we have that:

$$\hat{G}_\ell \cong \underbrace{\hat{\mathbb{Z}}_\ell \oplus \hat{\mathbb{Z}}_\ell \oplus \ldots \oplus \hat{\mathbb{Z}}_\ell}_{n \text{ summands}} \oplus\ \ell\text{−torsion of } G$$

However, since inverse and direct limits do not commute in general, we do *not* have a comparable result for non-finitely generated abelian groups.

(5). If, for each prime $p \in \ell$, the inverse of $p$ is in $G$, then the $\ell-$profinite completion is trivial. For example, $\widehat{(\mathbb{Q}/\mathbb{Z})}_\ell = 0$, and, for $q \neq p$, $\widehat{(\mathbb{Z}_{(p)})}_q = 0$.

(6). The $p-$profinite completion of the infinite direct sum $\bigoplus_{k=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is the infinite direct product $\prod_{k=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ since the limit process gives the elements with infinitely many nonzero elements.

From Example (4), we can see that the $\ell-$profinite completion functor is exact for finitely generated abelian groups, but, as noted in that example, this is not true in general. A counter-example that shows that this does not occur is as follows; the exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

becomes, after $\ell-$profinite completion, for any nonempty $\ell$,

$$0 \to \hat{\mathbb{Z}}_\ell \to 0 \to 0 \to 0$$

which is clearly not exact.

However, we can define a functor that is exact, and agrees with the $\ell-$profinite completion on finitely generated abelian groups, as follows:

**Definition 2.4.11** (Definition 1.5, pg. 16)**.** *The* formal $\ell-$completion *of an abelian group $G$, $\bar{G}_\ell$ is given by:*

$$\bar{G}_\ell = G \otimes \hat{\mathbb{Z}}_\ell$$

**Proposition 2.4.12.** *[Proposition 1.15, pg. 16] The functor $G \mapsto \bar{G}_\ell$ is exact. It is the unique functor which agrees with $\ell-$profinite completion on the finitely generated abelian groups, and commutes with direct limits.*

In the event that $\ell = \{\text{all primes}\}$, we omit mention of the set $\ell$, and simply call the profinite completion with respect to all primes the *profinite completion*, denoted $\hat{G}$. Similarly, the formal completion with respect to the set of all primes will be referred to simply as the *formal completion*, and is denoted $\bar{G}$. We note that $\bar{G} = G \otimes \hat{\mathbb{Z}} = G \otimes \bar{\mathbb{Z}}$.

### 2.4.b Comparing Localization and Completion

**Proposition 2.4.13.** *[Unnumbered Proposition, pg. 24] For any group $G$, there is a commutative diagram:*

$$\begin{tikzcd}
 & G_{(\ell)} \overset{defn}{=} G \otimes \mathbb{Z}_{(\ell)} = \varinjlim\limits_{\gcd(\ell,\ell')=1} \{\dots \xrightarrow{\cdot \ell'} G \xrightarrow{\cdot \ell'} G \xrightarrow{\cdot \ell'} \dots\} \arrow[d] \\
G \arrow[ur] \arrow[r] \arrow[dr] & \bar{G}_\ell \overset{defn}{=} G \otimes \hat{\mathbb{Z}}_\ell = \varinjlim\limits_{\substack{\text{finitely generated} \\ \text{subgroups } H \subset G}} \{\hat{H}_\ell\} \arrow[d] \\
 & \hat{G}_\ell \overset{defn}{=} \varprojlim\limits_{\substack{\text{subgroups of} \\ \ell-\text{index}}} \left\{ \begin{array}{c} \text{finite } \ell-\text{quotients} \\ \text{of } G \end{array} \right\}
\end{tikzcd}$$

**Corollary 2.4.14** (Unnumbered Corollary, pgs. 25-6)**.** *We have the following exact sequences, where $G$ is finitely generated, $\ell \cup \ell'$ is all primes, and $\ell \cap \ell' = \emptyset$:*

$$0 \longrightarrow \ell'-\text{torsion} \longrightarrow G \longrightarrow G_{(\ell)}$$

$$0 \longrightarrow \ell'-\text{torsion} \longrightarrow G \longrightarrow \hat{G}_\ell \cong \bar{G}_\ell$$

$$0 \longrightarrow G_{(\ell)} \longrightarrow \hat{G}_\ell \cong \bar{G}_\ell$$

*These combine in the following commutative diagram:*

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
0 \arrow[r] & \ell'\text{-}\mathit{torsion} \arrow[r] & G \arrow[r] \arrow[d] \arrow[dr] & G_{(\ell)} \arrow[d] \arrow[dl] \\
 & & \hat{G}_\ell \arrow[r, bend left] & \bar{G}_\ell \arrow[l, bend left, "\cong"]
\end{tikzcd}$$

**Corollary 2.4.15** (Included in above unnumbered corollary, pgs. 25-6)**.** *The natural map $G_{(\ell)} \to \bar{G}_\ell$ is equivalent to id$_G \otimes c : G \otimes \mathbb{Z}_{(\ell)} \to G \otimes \hat{\mathbb{Z}}_\ell$.*

**Proposition 2.4.16.** *Localization and formal completion (and therefore profinite completion of finitely generated abelian groups) commute with direct limits. Profinite completion of infinitely generated groups does not in general commute with direct limits. None of the three commute with inverse limits in general.*

## 2.4.c A Detailed Example: The $p-$adic Units

Another excellent source for this and related material is [BS66].

**Proposition 2.4.17** (Proposition 1.16, pg. 20)**.** *There is a 'canonical' splitting of the group of units in the $p-$adic integers:*

$$\hat{\mathbb{Z}}_p{}^* \cong \left(\mathbb{Z}/_{(p-1)\mathbb{Z}}\right) \oplus \hat{\mathbb{Z}}_p \qquad p > 2$$

$$\hat{\mathbb{Z}}_2{}^* \cong \left(\mathbb{Z}/_{2\mathbb{Z}}\right) \oplus \hat{\mathbb{Z}}_2 \qquad p = 2$$

*Proof.* The cases $p > 2$ and $p = 2$ are slightly different; we will consider only the case where $p$ is odd. Let $U$ be the collection of units of the form $1 + u$, where $u \equiv 0 \bmod p$. It is easy to see that the quotient ${}^{\hat{\mathbb{Z}}_p{}^*}/_U$ is equal to $\left(\mathbb{Z}/_{p\mathbb{Z}}\right)^*$ (since any unit in $\hat{\mathbb{Z}}_p$ is nonzero mod $p$, so is equal to $r(1+u)$ where $r$ is nonzero mod $p$, and $u$ is zero mod $p$). Using the isomorphism $\left(\mathbb{Z}/_{p\mathbb{Z}}\right)^* \cong \mathbb{Z}/_{(p-1)\mathbb{Z}}$, we have the short exact sequence:

$$0 \to U \to \hat{\mathbb{Z}}_p{}^* \to \mathbb{Z}/_{(p-1)\mathbb{Z}} \to 0$$

where the first map is inclusion and the second is reduction mod $p$. First, we will construct a splitting map $T : \mathbb{Z}/_{(p-1)\mathbb{Z}} \to \hat{\mathbb{Z}}_p{}^*$, which will show that $\hat{\mathbb{Z}}_p{}^* \cong U \oplus \mathbb{Z}/_{(p-1)\mathbb{Z}}$; we then show that $U \cong \hat{\mathbb{Z}}_p$, which will complete the proof. Both maps have significance in understanding the structure of the $p-$adic units, and so we will construct them in detail.

The splitting map $T : \mathbb{Z}/_{(p-1)\mathbb{Z}} \to \hat{\mathbb{Z}}_p{}^*$ will have as its image the same elements as the famous Teichmüller character, which has a significant role in the theory of cyclotomic fields. One way to see which elements are in this image is to apply the Frobenius dynamical system (i.e. iterates of the map $x \mapsto x^p$) to the units in $\hat{\mathbb{Z}}_p{}^*$, as follows.

By Fermat's Little Theorem, we have that:

$$\begin{aligned} x^p &\equiv x \pmod{p} \\ x^{p^2} &\equiv x^p \pmod{p^2} \\ &\vdots \\ x^{p^k} &\equiv x^{p^{k-1}} \pmod{p^k} \\ &\vdots \end{aligned}$$

for any $x \in \hat{\mathbb{Z}}_p{}^*$. It follows that the sum:

$$\bar{x} = x + (x^p - x) + (x^{p^2} - x^p) + \ldots + (x^{p^k} - x^{p^{k-1}}) + \ldots$$

is finite in each $\mathbb{Z}/_{p^j\mathbb{Z}}$, and therefore gives a well-defined element in the inverse limit $\hat{\mathbb{Z}}_p{}^*$.

Since $\bar{x}$ is a telescoping sum, it is easy to see that

$$\bar{x} = \lim_{k\to\infty} x^{p^k}$$

and so $\bar{x}$ is an attractor in the Frobenius dynamical system, as applied to the group of units $\hat{\mathbb{Z}}_p{}^*$. We define the map $T : \left(\mathbb{Z}/_{p\mathbb{Z}}\right)^* \to \hat{\mathbb{Z}}_p{}^*$ via $x \to \bar{x}$. In order to show that this map is well-defined, it remains to show that $\bar{x}$ depends only on the congruence class of $x$ mod $p$.

If we let $x = a + pb$, and consider $(a+pb)^{p^k}$:

$$\begin{aligned}
(a+pb)^{p^k} &= \sum_{n=0}^{p^k} \binom{p^k}{n} a^{p^k-n}(pb)^n \\
&= a^{p^k} + \sum_{n=1}^{p^k} \frac{p^k \cdot (p^k-1) \cdot \ldots \cdot (p^k-n+1)}{n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1} a^{p^k-n} p^n b^n \\
&= a^{p^k} + p^k \sum_{n=1}^{p^k} \left((p^k-1)\cdot(p^k-2)\cdot \ldots \cdot (p^k-n+1)\right) \frac{p^n}{n!} a^{p^k-n} b^n
\end{aligned}$$

Since the multiplicity of $p$ in $n!$ is always less than $n$,* $\frac{p^n}{n!}$ is a well-defined element of $\left(\mathbb{Z}/_{p\mathbb{Z}}\right)^*$, and so we can write:

$$(a+pb)^{p^k} \equiv a^{p^k} \pmod{p^k}$$

This shows that, if $x \equiv a$ mod $p$, then $\bar{x} = \bar{a}$, and so $\bar{x}$ depends only on the congruence class of $x$ mod $p$, which shows that the splitting map $T$ is well-defined.

It remains to show that $U \cong \hat{\mathbb{Z}}_p$. In fact, we will show that $U \cong p\hat{\mathbb{Z}}_p$; this will suffice

*In fact, $\nu_p(n!) = \left\lfloor \frac{n-\varphi_p(n)}{p-1} \right\rfloor$, where $\varphi_p(n) =$ the sum of the digits when $n$ is expressed in base $p$; a sketch of the proof is by induction: case $n = 1$ is obvious; for $n = k$, if $k$ is not divisible by $p$, both $n$ and $\varphi_p(n)$ increase by 1 under an increase from $k-1$ to $k$; if $k$ is divisible by $p$, $n$ will increase by 1, and $\varphi_p(n)$ will decrease by $\nu_p(k) \cdot (p-1) - 1$.

since $\hat{\mathbb{Z}}_p$ is torsion-free. We will construct maps in both directions:

$$U \underset{\exp}{\overset{\log}{\rightleftarrows}} p\hat{\mathbb{Z}}_p \subset \hat{\mathbb{Z}}_p$$

We know that an element of $U$ has the form $1+u$, where $u \equiv 0 \bmod p$. This allows us to define the standard power series $\log(1+u) \in p\hat{\mathbb{Z}}_p$:

$$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \ldots + (-1)^{k+1}\frac{u^k}{k} + \ldots$$

Since $u \equiv 0 \bmod p$ (and so we write $u = pv$), the terms in this series are clearly contained in $p\hat{\mathbb{Z}}_p$. Since $\nu_p(pv^k)$ increases much more rapidly than $\nu_p(k)$, it is easy to see that the terms go to zero as $k \to \infty$. In $p-$adic analysis, this is sufficient to show that the series converges in $p\hat{\mathbb{Z}}_p$. We show that this is an isomorphism by constructing its inverse, exp, in the obvious way.

Again, we use the familiar power series expansion for exp:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \ldots$$

We write $x = py$, and use the above expression for $\nu_p(k!)$ to see that this series is defined on $p\hat{\mathbb{Z}}_p$ and converges for all $y \in \hat{\mathbb{Z}}_p$. Since log and exp are inverses as formal power series, we can see that they are inverses where they are defined, which is exactly on the domains given above; $\log : U \to p\hat{\mathbb{Z}}_p$ and $\exp : p\hat{\mathbb{Z}}_p \to U$. $\square$

### 2.4.d Three More Examples

**Example**

If we start with any abelian group, localize at some collection of primes $\ell$, and then profinitely complete at another collection of primes $\ell'$, it is easy to see that localizing at $\ell$ kills all torsion outside of $\ell$, and then profinitely completing kills all torsion outside of $\ell'$, so the only torsion that survives is of an order contained in the intersection of the two. Also, if $p$ is a prime in $G$ which is invertible in $G$, then profinitely completing over this prime yields the trivial group, so the only free components will also be those that result from primes in the intersection of $\ell$ and $\ell'$. Hence,

$$\widehat{(G_{(\ell)})}_{\ell'} = \begin{cases} \prod_{p\in\ell\cap\ell'} \hat{G}_p & \text{if } \ell\cap\ell' \neq \emptyset \\ 0 & \text{if } \ell\cap\ell' = \emptyset \end{cases}$$

**Example**

If, instead, we localize and then *formally* complete, we get different objects. For example, if we localize the integers at zero (i.e., localize away from all primes), we get the rational numbers, and then formally completing at a prime $p$ yields $\mathbb{Q}\otimes\hat{\mathbb{Z}}_p = \hat{\mathbb{Q}}_p$, the $p-$adic number field for the prime $p$. The unit disk in $\hat{\mathbb{Q}}_p$ is the ring of $p-$adic integers.

**Example**

If we localize at zero again, but formally complete at all primes, the resulting ring is a restricted product over all $p$ of the $p-$adic numbers:

$$\mathbb{Q}\otimes\hat{\mathbb{Z}} = \mathbb{Q}\otimes \prod_{p \text{ is prime}} \hat{\mathbb{Z}}_p = \widehat{\prod_{p \text{ is prime}}} \hat{\mathbb{Q}}_p \subsetneq \prod_{p \text{ is prime}} \hat{\mathbb{Q}}_p$$

The product is restricted since the tensor product is distributive over direct sums, but not direct products. It follows that this restricted product consists of the infinite sequences of elements:

$$r = (r_2, r_3, r_5, \ldots, r_p, \ldots)$$

where all of the $r_p$ are $p-$dic numbers, and all but finitely many are $p-$adic integers. We denote this ring by $\bar{\mathbb{Q}}$.

We can embed $\mathbb{Q}$ into $\bar{\mathbb{Q}}$ via the map:

$$\frac{n}{m} \mapsto \left(\frac{n}{m}, \frac{n}{m}, \ldots, \frac{n}{m}, \ldots,\right)$$

Since $\frac{n}{m}$ is a $p-$adic integer whenever $p \nmid m$, we can see that $\frac{n}{m}$ is a $p-$adic integer in all but finitely many coordinates, and so the embedding is well-defined. If we combine this embedding with the standard embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$ to get:

$$\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}} \times \mathbb{R}$$

we obtain an embedding of $\mathbb{Q}$ as a discrete subgroup with a compact quotient ([Wei82], pg. 1).

The ring $\bar{\mathbb{Q}} \times \mathbb{R}$ is called *the ring of Adeles* for $\mathbb{Q}$. Adeles can be constructed for other number fields as well. These rings of Adeles have natural metrics, and the volumes of the compact quotients constructed as above have interesting number theoretic consequences. The units in the ring of Adeles are called *ideles*, and can be used to construct abelian extensions of a number field. For more details, see [Wei82].

### The Arithmetic Square

**Proposition 2.4.18** (Proposition 1.17, pg. 28)**.** *The following is a fibre square:*

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r, "\hat{}"] \arrow[d, "\otimes\mathbb{Q}"'] & \hat{\mathbb{Z}}_p \arrow[d, "\otimes\mathbb{Q}"] \\ \mathbb{Q} \arrow[r, "\otimes\hat{\mathbb{Z}}_p"'] & \hat{\mathbb{Q}}_p \end{tikzcd} \quad \overset{defn}{=}$$

$$\begin{tikzcd} \left\{\begin{array}{c}\textit{integers localized}\\ \textit{at } p\end{array}\right\} \arrow[r, "p-\textit{adic completion}"] \arrow[d, "\textit{localization at zero}"'] & \left\{\begin{array}{c}p-\textit{adic}\\ \textit{integers}\end{array}\right\} \arrow[d, "\textit{localization at zero}"] \\ \left\{\begin{array}{c}\textit{rational}\\ \textit{numbers}\end{array}\right\} \arrow[r, "\textit{formal completion at } p"'] & \left\{\begin{array}{c}p-\textit{adic}\\ \textit{numbers}\end{array}\right\} \end{tikzcd}$$

**Corollary 2.4.19** (Unnumbered Corollary, pg. 28)**.** *The ring of integers localized at $p$ is the fibre product of the rational numbers with the ring of $p-$adic integers over the $p-$adic numbers.*

**Proposition 2.4.20** (Proposition 1.18, pg. 29)**.** *The following is a fibre square:*

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "\otimes\mathbb{Q}"'] & \hat{\mathbb{Z}} \arrow[d, "\otimes\mathbb{Q}"] \\ \mathbb{Q} \arrow[r, "\otimes\hat{\mathbb{Z}}"'] & \bar{\mathbb{Q}} \end{tikzcd} \overset{\text{defn}}{=}$$

$$\begin{tikzcd}[column sep=huge, row sep=large] \{\textit{integers}\} \arrow[r, "\textit{profinite completion}"] \arrow[d, "\textit{localization at zero}"'] & \left\{\begin{array}{c} \textit{product over all } p \\ \textit{of the } p\textit{-adic} \\ \textit{integers} \end{array}\right\} \arrow[d, "\textit{localization at zero}"] \\ \left\{\begin{array}{c} \textit{rational} \\ \textit{numbers} \end{array}\right\} \arrow[r, "\textit{formal completion}"'] & \left\{\begin{array}{c} \textit{restricted product} \\ \textit{over all } p \textit{ of the} \\ p\textit{-adic numbers} \\ \overset{\textit{defn}}{=} \text{ ``\textit{finite Adeles}''} \end{array}\right\} \end{tikzcd}$$

**Corollary 2.4.21** (Unnumbered Corollary, pg. 30)**.** *The ring of integers is the fibre product of the rational numbers and the infinite product of the $p-$adic integers for all primes $p$, over the ring of finite Adeles.*

To generalize the most recent Proposition, for any finitely generated abelian group $G$ and for any nonempty set of primes $\ell$, we have the following fibre square:

$$\begin{tikzcd}[column sep=huge] G \otimes \mathbb{Z}_{(\ell)} = G_{(\ell)} \arrow[r, "\ell\text{-adic completion}"] \arrow[d, "\text{localization at zero}"'] & \hat{G}_\ell = G \otimes \hat{\mathbb{Z}}_\ell \arrow[d, "\text{localization at zero}"] \\ G \otimes \mathbb{Q} = G_{(0)} \arrow[r, "\text{formal completion at } \ell"'] & \left(\hat{G}_\ell\right)_{(0)} = \overline{\left(G_{(0)}\right)}_\ell = G \otimes \mathbb{Q} \otimes \hat{\mathbb{Z}}_\ell \end{tikzcd}$$

If we take $\ell$ to be {all primes}, then we see that $G \otimes \mathbb{Z}_{(\ell)} = G$, and so $G$ can be reconstructed from the natural maps from its localization at zero, $G \otimes \mathbb{Q}$, and its profinite completion over all primes into the ring $G \otimes$ (finite Adeles).

Taking $\ell = \{p\}$, we can reconstruct the localization of $G$ at $p$ from its localization at zero and its $p-$adic completion.

Going one step futher, given a suitable topological space $X$, we can form its profinite completion and its localization at zero, and these both map into an Adele space, which completely capture the space $X$:

$$\begin{tikzcd} & \hat{X} \arrow[d] \\ X_{(0)} \arrow[r] & X_A \end{tikzcd}$$

## 2.5 Iwasawa Theory and $p-$divisibility of Bernoulli Numbers

### 2.5.a Bernoulli Numbers

There is an enormous volume of literature on the Bernoulli numbers, which appear in many different areas of mathematics. We have used all of the following references: [IR90], [MS74], [BS66], and [Nie23]. The last is a delightful, extensive book which meanders through a wide range of computational results about Bernoulli numbers, and which seems to still be the most complete reference on the topic to this day.

Following [IR90], consider the following three famous problems of classical Number Theory:

(1).

$$1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$$

$$1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$

$$1^3 + 2^3 + \cdots + (n-1)^3 = \frac{n^2(n-1)^2}{4} = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$$

and so on. In considering these problems, Jacob Bernoulli highlighted the importance of the coeffecient of $n$ in each of these formulas, and so defined the sequence of rational numbers which now bear his name:

$$B_0 = 1; B_1 = -\frac{1}{2}; B_2 = \frac{1}{6}; B_3 = 0, \ldots$$

(2). The Riemann zeta function is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

In 1734, Leonhard Euler showed that $\zeta(2) = \frac{\pi^2}{6}$, and then later computed:

$$\zeta(2n) = (-1)^{n+1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}$$

or, solving for the Bernoulli Number, we see that:

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n)$$

In his proof, he computed the following power series expansion:

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

showing that $\frac{xe^x}{e^x-1}$ is an exponential generating function for the Bernoulli numbers. It follows easily from these two expressions that the odd Bernoulli numbers beyond the first are all zero (by observing that $\frac{xe^x}{e^x-1}+\frac{x}{2}$ is an even function), and the even Bernoulli Numbers are all nonzero, and, from $n=2$ forward, they alternate sign (since $\zeta(s)>0$ for $s>1$).

He also showed (using the analytic continuation of $\zeta$) that $\zeta(0)=-\frac{1}{2}$, and that, for $n>1$, $\zeta(1-n)=-\frac{B_n}{n}$.

(3). Fermat's Last Theorem states that:

$$x^n+y^n=z^n$$

has no integer solution for $n>2$. It suffices to show that this is true for $n=4$ and for $n=p$, where $p$ is an odd prime. The most naive approach to solving the Fermat Problem starts by factoring:

$$(x+y)(x+\zeta_p y)\dots(x+\zeta_p{}^{p-1}y)=z^p$$

and then observing that, if the ring $\mathbb{Z}[\zeta_p]$ has unique factorization, then each of the terms on the left side must be a $p^{\text{th}}$ power of some element of $\mathbb{Z}[\zeta_p]$ (since the right side is a $p^{\text{th}}$ power, and the factors on the left are relatively prime), and then showing that this is impossible. Of course, Kummer showed that $\mathbb{Z}[\zeta_p]$ is not a unique factorization domain for $23\leq p\leq 163$, which was extended to all $p$, $23\leq p$ by Uchida and Montgomery in 1971 [Rib80].

The ring $\mathbb{Z}[\zeta_p]$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_p)$. Two ideals $I,J$ in $\mathbb{Q}(\zeta_p)$ are said to be equivalent if there exist principal ideals $(\alpha)$ and $(\beta)$ so that $(\alpha)I=(\beta)J$. The *class number* $h_p$ for the prime $p$ is the number of equivalence classes of ideals under this relation; unique factorization is present in the integers of a cyclotomic field if and only if $h_p = 1$.

Kummer, working in the general case where unique factorization is not necessarily present, showed that, in the cyclotomic field $\mathbb{Q}(\zeta_p)$, if the $p^{\text{th}}$ power of a nonprincipal ideal is not principal, then the Fermat Problem for that prime $p$ is not solvable. He called such primes $p$ *regular*, and all others *irregular*. In an attempt to understand how many primes his solution covered, he came up with a succession of computational conditions for regularity, the two most important of which are:

**Theorem 2.5.1.** *A prime $p$ is regular if and only if $p$ does not divide the class number $h_p$.*

and

**Theorem 2.5.2.** *A prime $p$ is regular if and only if $p$ does not divide the numerator of any of the Bernoulli numbers $B_2, B_4, \ldots B_{p-3}$.*

In modern notation, it is common to take Theorem 2.5.2 as the definition.

Computational tests indicate that, in known intervals, approximately 40% of primes are irregular and 60% are regular. However, it has been proven that there are infinitely many irregular primes (by Jensen, 65 years after Kummer proved that there exists an irregular prime), but no one has proven whether or not there are infinitely many regular primes.

### 2.5.b Computational Number Theory

There is an extensive literature on the prime divisibility of Bernoulli numbers and its connection to Fermat's Theorem. The first highlight in this line of work comes from Kummer, who proved the Kummer congruences:

**Theorem 2.5.3.** *If $n$ is even, $n \geq 2$, then:*

$$\frac{B_n}{n} \equiv \frac{B_{n+p-1}}{n+p-1} \pmod{p}$$

*and, in general,*

$$(-1)^n(1-p^{n-1})\frac{B_n}{n} \equiv (-1)^m(1-p^{m-1})\frac{B_m}{m} \pmod{p^{j+1}}$$

*whenever $n \equiv m$ (mod $\varphi(p^{j+1})), n \not\equiv 0 \pmod{p-1}$.*

Next, came the theorem, proved independently by Clausen and von Staudt (c.f. [BS66], pg. 384, or [IR90], pg. 233):

**Theorem 2.5.4.** *An odd prime $p$ divides the denominator of $B_n$ ($n$ even, $n \geq 2$) exactly once if and only if $p-1$ divides $n$; otherwise $p$ does not divide the denominator of $B_n$ at all.*

Vandiver, in the first half of the 20th century, made significant case-by-case progress on the Fermat problem using computational techniques.

In pursuing these problems, the following distinction is made:

**Definition 2.5.5.** *An irregular prime $p$ is said to have* index of irregularity *equal to $i(p)$ if $p$ divides $i(p)$ different Bernoulli numbers in the collection $B_2, B_4, \ldots B_{p-3}$.*

Of all computed irregular primes (up to 12 million, see [BCE$^+$01]), the largest known index of irregularity is 7. The smallest irregular prime is 37, which divides $B_{32}$.

## 2.5.c Generalized Bernoulli Numbers, Algebraic Number Theory and Iwasawa Invariants

Here, we are following [Iwa72], [Iwa58], and [FW79].

Given a character $\chi$ on $(\mathbb{Z}/n\mathbb{Z})^*$, the number of elements in the image of $\chi$ is called the *conductor* of $\chi$, and is usually denoted $f_\chi$, or simply $f$. A character $\chi$ is called *even* if $f_\chi$ is an even number, and likewise for odd. We write $\delta_\chi = +1$ if $\chi$ is odd, and $\delta_\chi = 0$ if $\chi$ is even. It is easy to show that $\chi(-1) = \delta_\chi$ for all characters $\chi$. If $f = 1$, then $\chi$ is the trivial character $\chi_0$.

**Definition 2.5.6.** *The* generalized Bernoulli numbers associated to the character $\chi$ *are generated by the following generating function (where $f$ is the conductor of $\chi$):*

$$F_\chi(t) = \sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!}$$

Notice that $B_{n,\chi_0} = B_n$. If $\chi$ is non-trivial, then $B_{0,\chi} = 0$, $B_{n,\chi} = 0$ when $n \not\equiv \delta_\chi$, and $B_{n,\chi} \neq 0$ whenever $n \equiv \delta_\chi$.

We can use these to construct the Dirichlet $L-$functions, which are the corresponding generalization of the Riemann $\zeta-$function:

**Definition 2.5.7.** *The* Dirichlet $L-$functions *are defined via:*

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$$

Just as

$$\zeta(s) = \prod_{p \text{ is prime}} (1-p^{-s})^{-1}$$

the Dirichlet $L-$functions satisfy:

$$L(s,\chi) = \prod_{p \text{ is prime}} (1-\chi(p)p^{-s})^{-1}$$

and also $L(1-n,\chi) = -B_{n,\chi}/n$ for $n \geq 1$.

Computing $L(1,\chi)$ for $\chi$ nontrivial was a difficult problem solved in 1952 by H.

Hasse.

Iwasawa constructed a similar collection of functions defined on the $p-$adic numbers, which are called the $p-$adic $L-$functions, and are denoted $L_p(s, \chi)$.

He used these to construct the Iwasawa invariants, which I will now consider.

Recall that $h_p$ is the class number of the cyclotomic field $\mathbb{Q}(\zeta_p)$. The field $\mathbb{Q}(\zeta_p)$ contains a maximal real subfield, which is equal to $\mathbb{Q}(\zeta_p + {\zeta_p}^{-1})$. The class number of this is denoted ${h_p}^+$. Kummer showed that ${h_p}^+$ divides $h_p$; the quotient is denoted ${h_p}^-$; the factor ${h_p}^-$ is called the *first factor* of the class number $h_p$, and ${h_p}^+$ is called the *second factor*.

Kummer showed that a prime $p$ divides $h_p$ if and only if $p$ divides the first factor ${h_p}^-$. Recall that a Bernoulli number is regular if and only if it does not divide the class number $h_p$, and so therefore its first factor ${h_p}^-$.

Let $q_n = p^n$, and consider the cyclotomic fields $\mathbb{Q}(\zeta_{q_n})$. Let $h_{q_n}^-$ denote the first factor of the class number of $\mathbb{Q}(\zeta_{q_n})$, and let $p^{e_n}$ be the largest power of $p$ that divides $h_{q_n}^-$.

**Theorem 2.5.8** (Iwasawa)**.** *There exist constants $\lambda_p$, $\mu_p$, and $\nu_p$ that are independent of $n$ for sufficiently large $n$ such that:*

$$e_n = \lambda_p n + \mu_p p^n + \nu_p$$

Based on the proof, it seems very likely that this can be strengthened to $n \geq 0$ if one could prove even a weak bound on the index of irregularity of $p$ relative to the size of $p$.

This can be easily generalized to an arbitrary integer $q_n = mp^n$ with $m$ coprime to $p$, but we are particularly interested in the case $q_n = p^n$.

The constants $\lambda_p$, $\mu_p$, and $\nu_p$ are called the *Iwasawa invariants*. There are a few results known about the Iwasawa invariants:

**Theorem 2.5.9** (Iwasawa)**.** *If $p$ is regular, all of the Iwasawa invariants are zero.*

**Theorem 2.5.10** (Iwasawa)**.** *If $p$ is irregular, either $\lambda_p$ or $\mu_p$ is nonzero.*

**Theorem 2.5.11** (Ferrero and Washington)**.** $\mu_p = 0$ *for all* $p$

**Corollary 2.5.12.** $\lambda_p \neq 0$ *for all irregular primes* $p$*.*

A partial result in computing $\lambda_p$ can be found in [Was82], pg. 201.

**Theorem 2.5.13.** *Let $p$ be an irregular prime. If:*

*(1). (Kummer-Vandiver Conjecture) $p$ does not divide the second factor of the class number ${h_p}^+$.*

*(2). For any index $j$ where $p|B_n$, we have that $\frac{B_n}{n} \not\equiv \frac{B_{n+p-1}}{n+p-1} (\text{mod} p^2)$.*

*(3). For the generalized Bernoulli number associated to the Teichmüller character, $B_{1,\omega}$, we have that $B_{1,\omega} \not\equiv 0 (\text{mod} p^2)$.*

*then $\lambda_p = \nu_p = i(p)$, where $i(p)$ is the index of irregularity of $p$, so that $ord_p(h_{p^n}) = i(p)(n+1)$.*

## 2.6 Homotopy Theoretical Localization and Construction of a Local $CW-$ complex

I will only summarize the main results, generally without proof. As in the sections 2.3 and 2.4, we follow [Sul05], and all otherwise unattributed citations are from that volume.

For convenience, we work in a category which simplifies many of the arguments, the category of simple spaces:

**Definition 2.6.1.** *A topological space is called a* simple space *if it has the homotopy type of a $CW$ complex, and has an abelian fundamental group whose action on the homotopy and homology groups of the universal cover is trivial.*

We fix a set of primes $\ell$, which may include no or all primes, and work with this fixed set throughout this section.

**Definition 2.6.2** (Definition 2.1, pg. 32)**.** *We say that $X_{(\ell)}$ is a* local space *if $\pi_*(X_{(\ell)})$ is local; i.e. $\pi_*(X_{(\ell)})$ is a $\mathbb{Z}_{(\ell)}-$module. We say that a map $X \xrightarrow{\ell} X_{(\ell)}$ is a* localization *of $X$ if it is universal for maps of $X$ into local spaces; i.e. given $X$, a local space $L$, and a map $f : X \to L$ there is a unique map $f_{(\ell)} : X_{(\ell)} \to L$ which makes the following diagram commute:*

$$\begin{tikzcd} X \arrow[rr, "\ell"] \arrow[dr, "f"'] & & X_{(\ell)} \arrow[dl, dotted, "f_{(\ell)}"] \\ & L & \end{tikzcd}$$

The following theorem provides simpler criteria for characterizing localizations:

**Theorem 2.6.3** (Theorem 2.1, pg. 32)**.** *For a map $X \xrightarrow{\ell} X'$, the following are equivalent:*

*(1). $\ell$ is a localization.*

*(2). $\ell$ localizes integral homology; that is, the dotted arrow represents a map that is an isomorphism that makes the following diagram commute:*

$$\begin{tikzcd} \tilde{H}_*(X;\mathbb{Z}) \arrow[rr, "\ell"] \arrow[dr, "\otimes\mathbb{Z}_{(\ell)}"'] & & \tilde{H}_*(X';\mathbb{Z}) \arrow[dl, dotted, "\cong"] \\ & \tilde{H}_*(X;\mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} & \end{tikzcd}$$

*(3). $\ell$ localizes integral cohomology; that is, the dotted arrow represents a map that is*

*an isomorphism that makes the following diagram commute:*

$$\begin{tikzcd} \tilde{H}^*(X;\mathbb{Z}) \arrow[rr, "\ell"] \arrow[dr, "\otimes\mathbb{Z}_{(\ell)}"'] & & \tilde{H}^*(X';\mathbb{Z}) \arrow[dl, dotted, "\cong"] \\ & \tilde{H}^*(X;\mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} & \end{tikzcd}$$

*(4). $\ell$ localizes homotopy groups; that is, the dotted arrow represents a map that is an isomorphism that makes the following diagram commute (for $* > 0$):*

$$\begin{tikzcd} \pi_*(X) \arrow[rr, "\ell"] \arrow[dr, "\otimes\mathbb{Z}_{(\ell)}"'] & & \pi_*(X') \arrow[dl, dotted, "\cong"] \\ & \pi_*(X) \otimes \mathbb{Z}_{(\ell)} & \end{tikzcd}$$

We have the following two easy corollaries:

**Corollary 2.6.4** (Unnumbered Corollary, pg. 33)**.** *If $X$ is a simple space, then the following are equivalent:*

*(1). $X$ is its own localization.*

*(2). $X$ has local homology.*

*(3). $X$ has local cohomology.*

*(4). $X$ has local homotopy.*

*Proof.* Let $\ell$ be the identity map. □

**Corollary 2.6.5** (Unnumbered Corollary, pg. 33)**.** *If $f : X \to X'$ is a map of local simple spaces, then the following are equivalent:*

*(1). $f$ is a homotopy equivalence.*

*(2). $f$ induces an isomorphism on local homology.*

*(3). $f$ induces an isomorphism on local cohomology.*

*(4). $f$ induces an isomorphism on local homotopy.*

*Proof.* This follows from the Theorem above, and the theorem of Whitehead that states that a map that induces isomorphisms on homotopy groups of a $CW-$complex is a homotopy equivalence. (see, e.g., [DK01], pg. 188) $\square$

We consider now the $\ell-$localization of the $k-$sphere, which will allow us to construct the localization of a $CW-$complex.

First, let $\ell'$ be the collection of primes *not* in $\ell$, and let $\{\ell'_n\}$ be a cofinal sequence in the multiplicative set generated by these primes. Construct $S^k_{(\ell)}$ as an "infinite telescope" by taking the union of the mapping cylinders induced by the sequence of maps, where $\ell'_j$ is a map of degree $\ell'_j$ on the $k-$ sphere:

$$S^k \xrightarrow{\ell'_1} S^k \xrightarrow{\ell'_2} S^k \ldots \xrightarrow{\ell'_j} S^k \ldots$$

The inclusion $S^k \hookrightarrow S^k_{(\ell)}$ of $S^k$ as the first sphere in the "telescope" is a localization since it localizes reduced homology: In dimensions not equal to $k$, reduced homology is trivial. In dimension $k$, we have the map:

$$\mathbb{Z} \hookrightarrow \varinjlim_{\ell'_n} \mathbb{Z} = \mathbb{Z}_{(\ell)}$$

which is the usual localization map for the integers.

It follows from Theorem 2.6.3 that the inclusion also localizes homotopy groups, and is a localization map.

We now use the construction of the local sphere to construct $CW-$complexes.

**Definition 2.6.6.** *For $n > 1$, a* local $n-$cell *is formed by taking the cone over the local $(n-1)-$sphere, $S^{n-1}_{(\ell)}$, and treating this as the local $n-$disk, with $S^{n-1}_{(\ell)}$ canonically embedded in $D^n_{(\ell)}$ as the topological boundary. Since there is no local $0-$sphere, there is no local $1-$cell. The local $n-$cell will be denoted $\left(D^n_{(\ell)}, S^{n-1}_{(\ell)}\right)$ or simply $D^n_{(\ell)}$.*

**Definition 2.6.7.** *Given an attaching map $f : \partial D^n \to S^{n-1}$, we construct a* local attaching map $f_{(\ell)} : \partial D^n_{(\ell)} \to S^{n-1}_{(\ell)}$ *by applying the attaching map $f : \partial D^n \to S^{n-1}$ to each $S^{n-1}$ in the sequence defining the local $(n-1)-$sphere, and extending these maps to the mapping cylinders.*

**Definition 2.6.8** (Definition 2.2, pgs. 34-5)**.** *A* local $CW-$complex *is constructed inductively, starting from either a point, or a local $1-$sphere using the local $n-$cells defined above, and the local attaching maps, as defined above, in a manner completely analogous to the construction of a standard $CW-$complex.*

**Theorem 2.6.9** (Theorem 2.2, pg. 35)**.** *If $X$ is a $CW-$complex with at most one $0-$cell, and no $1-$cells, there is a local $CW-$complex $X_{(\ell)}$ and a "cellular" map (meaning a map that takes $n-$cells of $X$ into local $n-$cells of $X_{(\ell)}$) $X \xrightarrow{\ell} X_{(\ell)}$ such that:*

*(1). $\ell$ induces a bijection between the cells of $X$ and the local cells of $X_{(\ell)}$.*

*(2). $\ell$ localizes homology, cohomology and homotopy groups.*

**Corollary 2.6.10** (Unnumbered Corollary, pg. 35)**.** *Any simply connected simple space has a localization.*

Here, we consider a local Postnikov tower, dual to the cellular skeletal structure.

**Definition 2.6.11.** *We say that a sequence $\{\pi_n, p_n, k^n\}$ is a* local Postnikov tower for $X$ *if the $\pi_n$ are all $\mathbb{Z}_{(\ell)}-$modules, and the Postnikov tower built from $\{\pi_n, p_n, k^n\}$ converges to the local space $X$.*

**Theorem 2.6.12.** *If $X$ is any $CW-$complex with Postnikov tower $\{\pi_n, p_n, k^n\}$, then there is a local Postnikov tower $X_{(\ell)}$ and a map which preserves Postnikov towers $X \to X_{(\ell)}$ that localizes homotopy groups and $k-$invariants.*

**Corollary 2.6.13.** *Any simple space has a localization.*

The uniqueness of the Postnikov tower gives the uniqueness of the localization, so this means that localization gives a well-defined functor from the category of simple spaces with continuous maps to the category of local spaces with continuous maps.

**Proposition 2.6.14.** *So defined, the localization functor preserves fibrations and cofibrations.*

We consider a brief example to prove that the localization functor cannot extend to the entire category of homotopy classes of topological spaces in a way that preserves both fibrations and cofibrations; consider the following two sequences of maps:

$$S^1 \xrightarrow{\text{double cover}} S^1 \longrightarrow \mathbb{R}P^2$$

$$S^2 \xrightarrow{\text{double cover}} \mathbb{R}P^2 \xrightarrow{\text{natural inclusion}} \mathbb{R}P^\infty = K\left(\mathbb{Z}/_{2\mathbb{Z}}, 1\right)$$

**Proposition 2.6.15.** *The first map is a cofibration, and the second is a fibration.*

*Proof.* The first is a cofibration since the cofiber of the double cover $S^1 \to S^1$ is homotopy equivalent to the mapping cone of the double cover; it is easy to see that $\mathbb{R}P^2$ is homotopy equivalent to this mapping cone.

To see that the second is a fibration, consider the well-known fibration $\mathbb{Z}/_{2\mathbb{Z}} \hookrightarrow S^2 \to \mathbb{R}P^2$, where the second map is the double cover. The Puppe sequence for this fibration is (c.f., for example, [Hu59], pg. 152):

$$\longrightarrow \pi_2(F) \longrightarrow \pi_2(E) \longrightarrow \pi_2(B) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \longrightarrow \pi_0(F)$$

$$\longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

To the left, we have $\pi_n(\mathbb{Z}/2\mathbb{Z}) = 0$ for all $n > 1$, and so all of the maps $\pi_n(S^2) \to \pi_n(\mathbb{R}P^2)$ are isomorphisms. Now, if we shift this sequence left by 1 space, this allows us to consider the fibration where $S^2$ is the fiber, and $\mathbb{R}P^2$ is the total space, with the map connecting them being the double cover. The Puppe sequence will determine the weak homotopy type of the base:

$$\longrightarrow \pi_2(F) \longrightarrow \pi_2(E) \longrightarrow \pi_2(B) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \longrightarrow \pi_0(F)$$

$$\longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

From here, we see that the weak homotopy type of the base is exactly that of a $K\left(\mathbb{Z}/2\mathbb{Z}, 1\right)$ (and hence the homotopy type by the uniqueness of Eilenberg-MacLane spaces), since it has $\pi_1(B) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_n(B) = 0$ for $n > 1$. To see that the $K\left(\mathbb{Z}/2\mathbb{Z}, 1\right)$ has the homotopy type of $\mathbb{R}P^\infty$, consider the fibration: $\mathbb{Z}/2\mathbb{Z} \hookrightarrow S^\infty \to \mathbb{R}P^\infty$. Since $S^\infty$ is contractible, all of its homotopy groups are trivial; $\mathbb{Z}/2\mathbb{Z}$ has $\pi_0(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and all others trivial. This says that the only nontrivial portion of the Puppe sequence is the short exact sequence: $0 \to 0 \to \pi_1(\mathbb{R}P^\infty) \to \mathbb{Z}/2\mathbb{Z} \to 0$, which shows that $\mathbb{R}P^\infty$ has the homotopy type of $K\left(\mathbb{Z}/2\mathbb{Z}, 1\right)$. $\square$

Going back to the example, if we localize away from 2 (so let $\ell$ be any set of primes excluding 2), then $\mathbb{R}P^\infty_{(\ell)}$ is contractible, so, if fibrations are preserved, then $\mathbb{R}P^2_{(\ell)}$ should be equivalent to $S^2_{(\ell)}$; if cofibrations are preserved, the double cover $S^1_{(\ell)} \to S^1_{(\ell)}$ is an equivalence, so $\mathbb{R}P^2_{(\ell)}$ should be contractible. Since $S^2_{(\ell)}$ is not contractible, it is clear that the requirement of simple connectivity is essential to the localization process.

**Proposition 2.6.16.** *If $X$ is a local space, then $\pi_k(X) \cong [S^k_{(\ell)}, X]_{\text{based}}$ where $k \geq 2$.*

*Proof.* Since $X$ is a local space, we know that its local homotopy groups and its homotopy groups agree exactly. Then, by naturality of localization, we have that the following diagram commutes (where $\ell$ denotes the localization map):

$$\begin{tikzcd} S^k \arrow[r, "\ell"] \arrow[d] & S^k_{(\ell)} \arrow[d] \\ X \arrow[r, "\ell=\mathrm{id}"] & X \end{tikzcd}$$

The result follows immediately from the fact that the bottom map is the identity. ☐

## 2.7 The Spaces $BO_{(p)}$ and $BU_{(p)}$, and their self-maps

Using the direct limit, and the localization of a $CW-$complex, as detailed in the previous sections, one can construct the spaces $BO_{(p)}$ and $BU_{(p)}$. Dr. Lance studied these spaces extensively in his papers [Lan79] and [Lan88], building on the work of Adams, Bott, Milnor, Sullivan, and others. The primary goal of this dissertation is to show that much of this work can be carried over to the profinite case, showing that nothing essential is lost by this change of context. We believe that this will allow us to find additional connections among the number theory, the topology and the representation theory that are presented here.

In all of what follows, $p$ will be an odd prime.

### 2.7.a The Algebraic Topology of the spaces $BO_{(p)}$ and $BU_{(p)}$

It is well-known that the homotopy groups of the spaces $BO$ and $BU$ are given by Bott Periodicity (c.f. [Bot59]):

**Theorem 2.7.1.**

$$\pi_k(BU) = \begin{cases} \mathbb{Z} & \textit{if } k \equiv 0 \pmod 2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\pi_k(BO) = \begin{cases} \mathbb{Z} & \textit{if } k \equiv 0 \pmod 4 \\ \mathbb{Z}/_{2\mathbb{Z}} & \textit{if } k \equiv 1, 2 \pmod 8 \\ 0 & \textit{otherwise} \end{cases}$$

Localizing at an odd prime $p$ has the effect of inducing $\otimes\mathbb{Z}_{(p)}$ on homotopy groups, and so we have:

**Corollary 2.7.2.**

$$\pi_k(BU_{(p)}) = \begin{cases} \mathbb{Z}_{(p)} & \textit{if } k \equiv 0 \pmod 2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\pi_k(BO_{(p)}) = \begin{cases} \mathbb{Z}_{(p)} & \textit{if } k \equiv 0 \pmod 4 \\ 0 & \textit{otherwise} \end{cases}$$

It was shown by Adams and Peterson that there exist equivalences $BU_{(p)} \simeq BO_{(p)} \times \Omega^2 BO_{(p)}$ ([Ada69], [AP76], [Pet69]); these are equivalent as $CW-$complexes (i.e. homotopy equivalent), $H-$spaces, and infinite loop spaces. Also, since there is an equivalence $BO \simeq \Omega^4 BSp$, there is an equivalence $BO_{(p)} \simeq BSp_{(p)}$, so there is no need to consider spin bundles in the localized theory as long as one is considering bundles whose structure group is the orthogonal group.

It is also well-known (c.f. [MS74], and [Lan79], who also cites [Liu68]) that the homology coalgebra and cohomology algebra both admit operations (a product, and a coproduct, resp.) to form a self-dual bipolynomial Hopf Algebra, in the following sense:

**Theorem 2.7.3.** *$H^*(BU_{(p)})$ is a polyonomial Hopf Algebra generated by the Chern classes $\mathbb{Z}_{(p)}[c_1, c_2, \ldots]$, with product given by the cup product of cohomology classes, and coproduct given by $\mu^* c_n = \sum_{i=0}^n c_i \otimes c_{n-i}$. If we let $d_n$ be the element in $H_{2n}(BU_{(p)})$ dual to ${c_1}^n$, then the correspondence $c_n \mapsto d_n$ defines an isomorphism of Hopf algebras $H^*(BU_{(p)}) \to H_*(BU_{(p)})$.*

Within each branch of the algebraic topology, first the homotopy groups, and second the homology/cohomology Hopf algebra, there is additional substructure that can be used to further understand the action of self-maps of these spaces.

The substructure in homotopy groups is used by Adams and Peterson to construct the equivalence $BU_{(p)} \simeq BO_{(p)} \times \Omega^2 BO_{(p)}$ (see above):

**Theorem 2.7.4.** *For each odd prime $p$, there exists a space $W$ so that:*

*(1).*

$$\pi_k(W) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } k = 2j(p-1), \text{ for } j = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

*(2).*

$$BU_{(p)} \simeq W \times \Omega^2 W \times \cdots \times \Omega^{2p-4} W$$
$$BO_{(p)} \simeq W \times \Omega^4 W \times \cdots \times \Omega^{2p-6} W$$

*where the equivalence in the second part can be taken as homotopy equivalence, $H-$space equivalence, or infinite loop space equivalence.*

This space is shown to exist by considering a bordism theory with singularities, showing that it is representable in the sense of Brown (c.f. 3.1.9, below). The space

$W$ is the bottom space of the spectrum that arises as a result of this representability. The space $W$ allows us to decompose the space $BU_{(p)}$ (resp. $BO_{(p)}$) as a product, and therefore will allow us to decompose the induced maps on homotopy groups into $p-1$ (resp. $\frac{p-1}{2}$) distinct components.

In homology and cohomology, there is a more detailed structure that arises out of the duality relationship between the two, which reflects the rich combinatorial structure of the symmetric polynomials. Recall (from 2.7.3, above) that $d_n \in H_{2n}(BU_{(p)})$ is dual to ${c_1}^n \in H^{2n}(BU_{(p)})$; similarly, $c_n$ is dual to ${d_1}^n$.

Given a monomial $c_1^{\alpha_1} c_2^{\alpha_2} \dots c_n^{\alpha_n}$ (or, equivalently, a monomial in the $d_j$), we fix the following notations:

- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the multidegree of the monomial.
- We abbreviate $c^\alpha = c_1^{\alpha_1} c_2^{\alpha_2} \dots c_n^{\alpha_n}$; we write $c_\alpha$ for the dual to $d^\alpha$, so that, for example, if $e_n$ is the elementary basis vector $(0, 0, \dots, 1)$, then $c_{e_n} = c^{ne_1}$
- $w(\alpha) = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$ is the weight of monomial, which is equal to half its dimension.
- $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$
- $\{\alpha\} = \dfrac{|\alpha|!}{\alpha_1! \alpha_2! \cdots \alpha_n!}$

It is shown in Bott's half of Bott and Milnor's famous corespondence published in [BM58] that it is possible to choose natural generators $\omega_n \in \pi_{2n}(BU_{(p)})$ so that the Hurewicz map carries these generators to $(n-1)! d_{e_n} \in H_{2n}(BU_{(p)})$, and so the $d_{e_n}$ are called *primitive elements* of homology. From this, it is possible to show (c.f. [Lan79], Theorem 2.2, pg. 200, who cites [BS53] and [Liu68]):

**Theorem 2.7.5.**

$$d_{e_n} = \sum_{w(\alpha)=n} (-1)^{|\alpha|+n} n d^\alpha \frac{\{\alpha\}}{|\alpha|}$$

*and an identical formula for the $c_{e_n}$.*

We use these primitive elements to construct an important subalgebra.

Given indeterminates $t_0, t_1, \ldots$, define the $k^{\text{th}}$ Witt polynomial:

$$T_k(t) = {t_0}^{p^k} + p{t_1}^{p^{k-1}} + \cdots + p^k t_k$$

Then, one can inductively define $a_{n,k}$ via the formula:

$$d_{e_{np^k}} = T_k(a_{n,0}, a_{n,1}, \ldots, a_{n,k})$$

Notice that $a_{n,k} \in H_{2np^k}(BU_{(p)})$.

It can be shown (see [Lan79], pgs. 200-202) that the collection $\{a_{n,k} | \, n$ prime to $p$, $k \geq 0\}$ is a polynomial basis for $H_*(BU_{(p)})$, and, if we let $A_n$ denote the subalgebra generated by $a_{n,0}, a_{n,1}, \ldots$, then we have the following decomposition:

**Theorem 2.7.6** ([Lan79], Theorem 2.5, pg. 202, who cites [GM74])**.**

$$H_*(BU_{(p)}) \cong \bigotimes_{n \textit{ prime to } p} A_n$$

*and*

$$H_*(\Omega^{2k} W) \cong \bigotimes_{\substack{n \textit{ prime to } p \\ n+k \equiv 0 \ (\textit{mod } p-1)}} A_n$$

There is an identical decomposition in cohomology, where we define $a_{n,k}^*$ via an identical formula, using $c_{e_{np^k}}$ in place of $d_{e_{np^k}}$. Using this, $c_{e_{np^k}}$ is dual to $na_{n,k}$. In this decomposition, we denote the subalgebra generated by the $a_{n,k}^*$, where $n$ is prime to $p$, by ${A_n}^*$, and obtain a result identical to Theorem 2.7.6 for cohomology.

All of these results on homology and cohomology can be replicated for $BO_{(p)}$, where the Chern classes are replaced by the Pontrjagin classes, and the dimensions are multiples of 4 instead of multiples of 2.

### 2.7.b Self-Maps of $BO_{(p)}$ and $BU_{(p)}$

Fundamentally, this dissertation is about the intersection where long-unanswered questions about the $p-$divisibility of number-theoretic sequences meets the algebraic topology of classifying spaces. The intersection that we are interested in here is found in the self-maps of $BO_{(p)}$ and $BU_{(p)}$. We will look at three maps, all of which have topological significance; one of them, the Adams map, is constructed in detail in the next section of this dissertation.

First, we consider the general situation: Let $f : BU_{(p)} \to BU_{(p)}$ be a continuous function. In this section, we will also require two additional properties of $f$: that $f$ is an $H-$map, and that $f$ is a fibration. In the second case, if the given map is not a fibration, one can always apply the path space construction to ensure that it is. In this case, we write:

$$F \hookrightarrow BU_{(p)} \xrightarrow{f} BU_{(p)}$$

We consider the induced maps on homotopy groups; since, for each $k$, we have that $\pi_{2k}(BU_{(p)}) = \mathbb{Z}_{(p)}$, the induced map in dimension $2k$ is given by multiplication by a $p-$local constant, which is generally denoted by $\pi_{2k}(f)$. Then, if we let $\lambda_k = \pi_{2k}(f)$ for each $k = 1, 2, \ldots,$ we define:

**Definition 2.7.7.** *The* characteristic sequence *of a map $f : BU_{(p)} \to BU_{(p)}$ is the sequence:*

$$(\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots, )$$

*The characteristic sequence of a map $f : BO_{(p)} \to BO_{(p)}$ is the sequence:*

$$(\lambda_2, \lambda_4, \ldots, \lambda_{2k}, \ldots, )$$

*In either case, we may abbreviate by saying that $f$ has characteristic sequence $\lambda$.*

Notice that composing two self-maps induces componentwise multiplication on their characteristic sequences; this will be used below as a way to cancel factors so as to isolate number-theoretically interesting terms.

Of course, using the decompositions above, one can factor a self-map of $BU_{(p)}$ into self-maps of $BO_{(p)}$ and $\Omega^2 BO_{(p)}$, or into maps on $W$ and its loop iterations. Similarly, given a self-map on $BO_{(p)}$, one can construct a self-map on $BU_{(p)}$ by combining with the identity map on $\Omega^2 BO_{(p)}$, to generate a map with characteristic sequence $(1, \lambda_2, 1, \lambda_4, \ldots)$.

One important lemma, which tells us that it suffices to consider only the characteristic sequence is the following (whose proof is an obstruction theory argument credited to J.P. May, [May77]):

**Lemma 2.7.8.** *If two self-maps $f, g$ have the same characteristic sequence, then they are homotopic.*

Using the algebraic structure, and the topology of these classifying spaces, Lance proved the following two remarkable results ([Lan79]):

**Theorem 2.7.9.** *Let $f$ be a self-map of $BU_{(p)}$ that has characteristic sequence $\lambda$, then $\lambda_{mp^k} \equiv \lambda_{np^k}$ (mod $p^{k+1}$) whenever $m \equiv n$ (mod $p-1$).*

The second result requires the following definition:

**Definition 2.7.10.** *Given a self-map $f$ with characteristic sequence $\lambda$, the* surplus sequence *$s(f)_k$ is given by:*

$$s(f)_k = \nu_p(\lambda_k) - k$$

*where $\nu_p(n)$ is the largest power of $p$ dividing $n$, and $\nu(0) = \infty$.*

Notice that, since $p$ does not divide the denominator of any element of $\mathbb{Z}_{(p)}$, $\nu_p(n)$ is well-defined.

**Theorem 2.7.11.** *For any $n$ prime to $p$, if $j$ is minimal such that $s(f)_{np^j} \leq 0$, then $s(f)_{np^{j+k}} = -k$ for $k = 1, 2, \ldots$. In this case, we define $\delta_n = j$, and $\delta_n = \infty$ otherwise.*

In other words, given any subsequence of a characteristic sequence of the form: $(\lambda_n, \lambda_{np}, \lambda_{np^2}, \ldots, )$, if the level of $p-$divisibility at any point in the subsequence drops to or below the index in the subsequence, then the level of $p-$divisibility is frozen at that level for the remainder of the sequence; $\delta_n$ is that level of $p-$divisibility.

Lance also pursued descriptions of the fibers of self-maps; we will consider here the cohomology of $F$, with coefficients in ${}^{\mathbb{Z}}/_{p\mathbb{Z}}$, using [Lan79] and [Lan88]. We state the result first, and then consider each factor in detail to indicate its origin, and the induced maps that connect it to the cohomology of $BU_{(p)}$ (or $BO_{(p)}$, as appropriate).

**Theorem 2.7.12.** *If $f$ is a self-map of $BU_{(p)}$ (or $BO_{(p)}$, resp.), with fiber $F$ (or $G$, resp.), then there are isomorphisms of Hopf Algebras:*

$$H^*(F; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \cong \bigotimes_{n \text{ prime to } p} E\left\{\sigma a_{n,j}^* \middle| 0 \leq j \leq \delta_n\right\} \otimes \left(A_n^* // \xi^{\delta_n} A_n^* \otimes {}^{\mathbb{Z}}/_{p\mathbb{Z}}\right)$$

*and*

$$H^*(G; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \cong \bigotimes_{n \text{ prime to } p} E\left\{\sigma b_{n,j}^* \middle| 0 \leq j \leq \delta_n\right\} \otimes \left(B_n^* // \xi^{\delta_n} B_n^* \otimes {}^{\mathbb{Z}}/_{p\mathbb{Z}}\right)$$

*where $b_{n,j}$ and $B_n^*$ are defined analagously, and $\xi$ is the Frobenius map $x \mapsto x^p$.*

We consider only $BU_{(p)}$; the case for $BO_{(p)}$ is completely analagous, with dimension $2k$ replaced by $4k$.

The second factor is a truncated bipolynomial Hopf Algebra, based on $A_n^*$, as defined in 2.7.6 above, where each variable is truncated at the exponent $p^{\delta_n}$, where $\delta_n$ is defined in 2.7.10. The map $H^*(BU_{(p)};{}^{\mathbb{Z}}/_{p\mathbb{Z}}) \to H^*(F;{}^{\mathbb{Z}}/_{p\mathbb{Z}})$ is just given by the projection $A_n^* \otimes {}^{\mathbb{Z}}/_{p\mathbb{Z}} \to \left(A_n^*//\xi^{\delta_n} A_n^*\right) \otimes {}^{\mathbb{Z}}/_{p\mathbb{Z}}$.

The first factor is an exterior algebra; the generators $\sigma a_{n,j}^*$ arise via the following construction. We start with the fibration used in the construction of $BU$ as a classifying space: $U(n) \hookrightarrow EU(n) \to BU(n)$. Consider the long exact sequence of the fibration in homotopy groups:

$$\ldots \to \pi_{k+1}(EU(n)) \to \pi_{k+1}(BU(n)) \xrightarrow{\delta} \pi_k(U(n)) \to \pi_k(EU(n)) \to \ldots$$

Since $EU(n)$ is a contractible space, all of its homotopy groups are trivial, and so the map $\delta : \pi_{k+1}(BU(n)) \to \pi_k(U(n))$ is an isomorphism for all $k$. Then, we use the isomorphism $\pi_{k+1}(BU(n)) \cong \pi_k(\Omega BU(n))$, and the fact that both spaces are $CW-$complexes, to show that there is a homotopy equivalence between $U(n)$ and $\Omega BU(n)$. Hence, passing to the stable limit, the path fibration $\Omega BU \hookrightarrow PBU \to BU$ is equivalent to $U \hookrightarrow PBU \to BU$. Using the transgression map in the cohomology suspension (c.f. [DK01], pgs. 279-283), we can define a map $\sigma$ via the composition:

$$\begin{tikzcd} H^k(BU;{}^{\mathbb{Z}}/_{p\mathbb{Z}}) \arrow[r, "\pi^*"'] \arrow[rr, bend left, "\sigma"] & H^k(PBU,U;{}^{\mathbb{Z}}/_{p\mathbb{Z}}) \arrow[r, "\delta^{-1}"'] & H^{k-1}(U;{}^{\mathbb{Z}}/_{p\mathbb{Z}}) \end{tikzcd}$$

This map $\sigma$ carries the polynomial algebra $A_n^* \otimes {}^{\mathbb{Z}}/_{p\mathbb{Z}}$ with generators $a_{n,j}^* \in H^{2np^j}(BU_{(p)};{}^{\mathbb{Z}}/_{p\mathbb{Z}})$ to the exterior algebra $\sigma(A_n^*)$ with generators $\sigma a_{n,j}^* \in H^{2np^j-1}(U;{}^{\mathbb{Z}}/_{p\mathbb{Z}})$. We complete the construction by considering the following diagram, where $f$ is the self-map of $BU$, and $\tilde{f}$ is its pullback:

$$\begin{tikzcd} F \arrow[r, dotted] \arrow[d, hook] & U \arrow[d, hook] \\ BU \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & EU \arrow[d] \\ & BU \end{tikzcd}$$

The dotted map shows the canonical lift $F \to U$, and the induced map in cohomology $H^*(U; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \to H^*(F; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$ carries the element $\sigma a^*_{n,j} \in H^{2np^j-1}(U; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$ to an element $\sigma a^*_{n,j} \in H^{2np^j-1}(F; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$.

For a detailed example of this structure in a special case, see section 5.1, below.

## 2.7.c Three Examples

**Example** *The Adams Map.*

As noted above, this map and its characteristic sequence are constructed in detail in the next section of this dissertation in order to provide a sense of how these maps arise, and what is involved in the computation of their characteristic sequence. The definition and the computation are both due to J.F. Adams, in [Ada62].

The Adams map $\psi^k$, on $BU_{(p)}$ has characteristic sequence $(k, k^2, \dots)$; if $k$ is prime to $p$, then $\psi^k$ is an infinite loop map in the $p-$local setting, and it is invertible since all of the elements of its characteristic sequence are invertible. If $k$ generates the units of ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$, then the fiber of the Adams map is the image of the $J-$homomorphism. One can also see that, if $k$ generates the units of ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$, then $\psi^k - 1$ is invertible on $\Omega^{2k}W$, for $k = 1, 2, \dots, p-1$.

**Example** *The Adams-Bott Cannabalistic Class.*

Given an oriented bundle $\xi$ over $B$, one can define the class $\rho^k$ via the formula: $\rho^k(\xi) = \Phi^{-1} \circ \psi^k \circ \Phi(1)$, where $\psi^k$ is the Adams map, and $\Phi$ is the Thom isomorphism. This class induces a self-map of $BO_{(p)}$, using the methods described in section 2.8. In this case, $\rho^k$ is neither additive nor multiplicative; it is exponential, meaning it takes sums in the domain to products, so one should consider the domain space $BO_{(p)}$ to have the $H-$space structure induced by Whitney sum, and the range space should have the $H-$space structure induced by tensor product. We abbreviate this by writing $\rho^k : \left(BO_{(p)}\right)_{\oplus} \to \left(BO_{(p)}\right)_{\otimes}$. One of Adams' computations while studying the $J-$homomorphism has the following immediate consequence:

**Theorem 2.7.13** ([Ada65], pgs. 166-167, Thm. 5.18)**.** *On $\pi_{4j}(BO_{(p)})$, $\rho^k$ is given by multiplication by the constant:*

$$(-1)^{j+1}(k^{2j} - 1)\frac{B_{2j}}{4j}$$

*where $B_{2j}$ is the $2j^{th}$ Bernoulli number (see section 2.5).*

Then, applying what we saw above about the invertibility of $(\psi^k - 1)$, and Theorem 2.7.9, we can choose $k$ so that it generates the units of ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$, and so form the map $\rho^k \circ (\psi^k - 1)^{-1}$. Doing so, we obtain an extension of the well-known Kummer congruences of Bernoulli numbers:

**Theorem 2.7.14.**

$$(-1)^m \frac{B_{2m}}{m} \equiv (-1)^n \frac{B_{2n}}{n} \pmod{p^{j+1}}$$

*whenever $m \equiv n$ (mod $p^j(p-1)$), $4m \not\equiv 0$ (mod $p-1$), and $4n \not\equiv 0$ (mod $p-1$).*

For $j \geq 1$, this differs from the well-known version (c.f. 2.5.3).

Taking the opposite perspective, information about the $p-$divisibility of the Bernoulli numbers gives information about the cohomology of the fiber of the self-map $\rho^k$. Here, we take $k$ so that $k$ generates the units of ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$.

First, we need to consider the denominator of $\frac{B_{2j}}{4j}$. The full prime factorization of this denominator is given by the factorization of $4j$ multiplied by primes $q$ (at multiplicity 1) where $q-1$ divides $2j$, by 2.5.4. So, if the prime $p$, where we are localized, divides this denominator, either $p$ divides $j$, or $p-1$ divides $2j$. If $p$ divides $j$, then $p$ also divides the numerator of $B_{2j}$ (c.f. [IR90], Prop. 15.2.4), and so the coefficient in its entirety is in $\mathbb{Z}_{(p)}$. If $p-1$ divides $2j$, then $p$ also divides $(k^{2j}-1)$ exactly as many times, by Fermat's Little Theorem, and so the coefficient in its entirety is again in $\mathbb{Z}_{(p)}$. In particular, this last statement shows $\rho^k$ restricted to the factor $W$ is an equivalence, and so the fiber of this restriction $\rho^k : W \to W$ is trivial.

If $p$ is a regular prime (i.e., it does not divide the numerator of any Bernoulli number with index $2, 4, \ldots, p-3$; c.f. 2.5.2), then $\rho^k$ is an equivalence on all of $BO_{(p)}$, and so, for all regular primes $p$, the fiber of $\rho^k$ is trivial.

In the case where $p$ is irregular, an additional restriction simplifies the description of the cohomology of the fiber:

**Definition 2.7.15.** *We say that an irregular prime $p$ is a* normal prime *if, for each index $2j \in \{2, 4, \ldots, p-3\}$ where $p$ divides $B_{2j}$, we have that all of the following are nonzero mod $p^2$:*

$$\frac{B_{2j}}{2j}, \frac{B_{2pj}}{2pj} \text{ and } \frac{B_{2j}}{2j} - \frac{B_{2j+p-1}}{2j+p-1}$$

All irregular primes which have been checked thus far are normal (c.f. Chapter 5).

Let $M$ denote the fiber of $\rho^k$, so that we have $M \hookrightarrow \left(BO_{(p)}\right)_{\oplus} \to \left(BO_{(p)}\right)_{\otimes}$; then, this leads us to:

**Theorem 2.7.16** ([Lan88], Thm 3.6)**.** *If $p$ is a normal prime, where $\{n_1, n_2, \ldots, n_k\}$*

*are the indices where* $B_{2j} \equiv 0$ (*mod* $p$). *Then:*

$$H^*(M; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \cong \bigotimes_{n \in S} E\{\sigma b_{n,0}^*\} \otimes {B_n}^* // \xi {B_n}^*$$

*where* $S = \{2n \,|n$ *is prime to* $p$ *and* $n \equiv n_i$ (*mod* $p-1$) *for some* $n_i\}$.

In the case where $p$ is not normal, there are additional generators. This theorem was part of the motivation for some of the computational number theory work seen in Chapter 5.

**Example** *The Sullivan Map*. In his proof of the Adams conjecture, Sullivan defined a map $\theta^k$, which can be seen as a self-map of $BO_{(p)}$ (also exponential, as is the Adams-Bott cannabalistic class) and performed a calculation that showed:

**Theorem 2.7.17.** *On* $\pi_{4j}(BO_{(p)})$, $\theta^k$ *is given by multiplication by:*

$$(-1)^{j-1} 2^{2j} (k^{2j} - 1)(1 - 2^{2j-1}) \frac{B_{2j}}{2j}$$

In this case, we examine the map using the previous work for the Adams-Bott cannabalistic class. Recalling Lemma 2.7.8, we can construct two maps which are both homotopic to $\theta^k$ (notice that, since $\rho^k$ is exponential, we need to switch from additive notation when composing on the right to multiplicative notation on the left):

$$\rho^k \circ (2\psi^2 - \psi^4) \qquad \text{and} \qquad ((\psi^2)^2/\psi^4) \circ \rho^k$$

If we let:

(1). (as above) $M \hookrightarrow \left(BO_{(p)}\right)_{\oplus} \xrightarrow{\rho^k} \left(BO_{(p)}\right)_{\otimes}$

(2). $N \hookrightarrow \left(BO_{(p)}\right)_{\oplus} \xrightarrow{\theta^k} \left(BO_{(p)}\right)_{\otimes}$

(3). $L_{\oplus} \hookrightarrow \left(BO_{(p)}\right)_{\oplus} \xrightarrow{(2\psi^2 - \psi^4)} \left(BO_{(p)}\right)_{\oplus}$

(4). $L_\otimes \hookrightarrow \left(BO_{(p)}\right)_\otimes \overset{\left((\psi^2)^2/\psi^4\right)}{\longrightarrow} \left(BO_{(p)}\right)_\otimes$

then we have the following diagrams of fibrations:

$$\begin{tikzcd}
L_\oplus \arrow[rr, bend left=10] \arrow[dr] & & BO_\oplus \arrow[rr, bend left=10, "\theta^k \sim \rho^k \circ (2\psi^2 - \psi^4)"] \arrow[dr, "(2\psi^2-\psi^4)"] & & BO_\otimes \\
& N \arrow[ur] \arrow[dr] & & BO_\oplus \arrow[ur, "\rho^k"'] & \\
& & M \arrow[ur] & &
\end{tikzcd} \tag{2.7.18}$$

(from [Lan88], pg. 405) and

$$\begin{tikzcd}
M \arrow[rr, bend left=10] \arrow[dr] & & BO_\oplus \arrow[rr, bend left=10, "\theta^k \sim ((\psi^2)^2/\psi^4) \circ \rho^k"] \arrow[dr, "\rho^k"] & & BO_\otimes \\
& N \arrow[ur] \arrow[dr] & & BO_\otimes \arrow[ur, "((\psi^2)^2/\psi^4)"'] & \\
& & L_\otimes \arrow[ur] & &
\end{tikzcd} \tag{2.7.19}$$

(from [Lan88], pg. 406).

Since $L_\oplus$ is homotopy equivalent to $L_\otimes$, we abbreviate by calling both simply $L$. Notice that, in addition to the three fibrations in each diagram that are self-maps of $BO_{(p)}$, there is also a fourth fibration; in the first diagram we have $L \hookrightarrow N \to M$, and in the second, we have $M \hookrightarrow N \to L$. We will revisit these fibrations in Chapter 5.

## 2.8 The Adams Map: Using Representation Theory to Induce a Map on the Algebraic Topology of a Classifying Space

The connection between Algebraic Topology and Number Theory that is exploited in this dissertation comes from self-maps of the classifying spaces associated with the unitary and orthogonal groups. These self-maps arise by considering the representation theory of these matrix groups. In this section, we present a detailed construction of the induced maps on the algebraic topology of $BU$ from the Adams map, named after J.F. Adams who created the technique and used it with substantial effectiveness on several major problems in topology, most notably in the study of the $J-$homomorphism.

The reference for this work is: [Ada62]

Let $\Lambda = \mathbb{R}$ or $\mathbb{C}$ for all of the following.

First, recall some definitions:

**Definition 2.8.1.** *For any finite $CW-$complex $X$, the ring $K_\Lambda(X)$ is constructed as follows: Let $F_\Lambda(X)$ be the free abelian group with generators equal to stable isomorphism classes $\{\xi\}$ of $\Lambda-$vector bundles over $X$. Then let $T_\Lambda(X)$ be the subgroup of $F_\Lambda(X)$ generated by elements of the form $\{\xi \oplus \eta\} - \{\xi\} - \{\eta\}$, and let $K_\Lambda(X)$ be the quotient $^{F_\Lambda(X)}/_{T_\Lambda(X)}$. This is a ring under Whitney sum and tensor product of bundles.*

This ring is commonly known as $K_0(X)$, with the field $\Lambda$ supressed in the notation.

**Definition 2.8.2.** *Given a topological group $G$, then a continuous homomorphism $\alpha : G \to GL(n, \Lambda)$ is called a* representation (of $G$, over $\Lambda$).

One may then define the direct sum, tensor product, exterior product, symmetric product of representations by: $(\alpha \oplus \beta)(g)(v) = \alpha(g)(v) \oplus \beta(g)(v)$, and the others analogously.

A representation gives a vector space $V$ the structure of a (left) $G-$module. A map $\rho$ between representations $\alpha : G \to V, \beta : G \to W$ is a $G-$module homomorphism from $V$ to $W$; i.e. a map so that the following the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\alpha(g)"'] & W \arrow[d, "\beta(g)"] \\ V \arrow[r, "\rho"] & W \end{tikzcd}$$

Often, a representation will be identified by the lowest-dimensional vector space on which it can act. For example, for the trivial representation $\varepsilon : G \to GL(n, \Lambda)$ given by $\varepsilon(g) = I_n$, we can identify this by the vector space generated by the matrix [1], and so we choose (usually) to act on $GL(1, \Lambda)$. The dimension of this subspace will be called the *degree* of the representation.

**Definition 2.8.3.** *The* representation ring (of $G$, over $\Lambda$) *is constructed as follows: Let two representations be equivalent if they are different by conjugation by an element of $GL(n, \Lambda)$. Let $F'_\Lambda(G)$ be the free abelian group generated by equivalence classes of representations, and let $T'_\Lambda(G)$ be the subgroup generated by elements of the form $\{\alpha \oplus \beta\} - \{\alpha\} - \{\beta\}$. Then $K'_\Lambda(G)$ is the quotient ${}^{F'_\Lambda(G)}/_{T'_\Lambda(G)}$. Elements of $K'_\Lambda(G)$ are called* virtual representations*. This is a ring under direct sum and tensor product.*

(This construction for $K_\Lambda(X)$ and $K'_\Lambda(G)$ is a general one, due to Grothendieck.) If we define a homomorphism from $F'_\Lambda(G)$ to $\mathbb{Z}$ which assigns, to each representation, its degree, this extends to $K'_\Lambda(g)$, and the value of this map is called the *virtual degree* of a virtual representation.

**Definition 2.8.4.** *A representation $V$ is* indecomposable *if there do not exist representations $V_1, V_2$ such that $V \cong V_1 \bigoplus V_2$. $V$ is* irreducible *if it contains no proper subrepresentations.*

**Theorem 2.8.5.** *(c.f. [FH91]) Any representation can be written in a unique way (up to re-ordering) as a direct sum of irreducible representations.*

**Theorem 2.8.6.** *(c.f. [FH91]) For any representation $V$ over $\mathbb{C}$, $V$ is indecomposable if and only if it is irreducible.*

I will (following Adams) use the following conventions in this section:

- $f, g, h$ will be used for maps of $CW-$ complexes
- $\xi, \eta, \zeta$ will be vector bundles
- $\kappa, \lambda, \mu$ will be elements of $K_\Lambda(X)$
- $\alpha, \beta, \gamma$ will be representations
- $\theta, \varphi, \psi$ will be virtual representations.

The next step is to define ways to compose each of the above objects with each other. We will need compositions of the following forms, which we will write with a solid dot, to distinguish them from other products:

- If $f : X \to Y$ and $g : Y \to Z$ are maps of $CW-$complexes, then $g \cdot f$ is the composition in the usual sense.
- If $\alpha : G \to GL(n, \Lambda)$ and $\beta : GL(n, \Lambda) \to GL(n', \Lambda')$ are representations, then $\beta \cdot \alpha$ is also the composition in the usual sense.
- If $f : X \to Y$ is a map of $CW-$complexes, and $\xi$ is a bundle, then the composition $\xi \cdot f$ is the pullback bundle over $Y$ through $f$.
- If $\alpha : GL(n, \Lambda) \to GL(n', \Lambda')$ is a representation, and $\xi$ is a bundle with structure group $GL(n, \Lambda)$, then $\alpha \cdot \xi$ is the bundle formed by changing the fibers of $\xi$ to $\Lambda'$

and applying the map $\alpha$ to the coordinate transformation functions of the bundle $\xi$. Equivalently, one can let $h : E(\xi) \to BGL(n, \Lambda)$, the classifying space for $n-$bundles over $\Lambda$, and then consider $\overline{\alpha} : BGL(n, \Lambda) \to BGL(n, \Lambda')$, the induced map of classifying spaces. Then $\alpha \cdot \xi$ is the bundle which is classified by the map $\overline{\alpha} \circ h$.

The compositions with representations in the first factor are complicated by the fact that some of the dimensions must match in order for these to be defined. So, to ensure that our compositions are well-defined, we will change the first factor to a sequence of representations. Let $\Theta = (\theta_n)$, where for each $n$, $\theta_n$ is a virtual representation of $GL(n, \Lambda)$ over $\Lambda'$. We will consistently use the letters $\Theta, \Phi, \Psi$ for these sequences. Let $\pi : GL(n, \Lambda) \times GL(m, \Lambda) \to GL(n, \Lambda)$ be projection onto the first factor, and $\varpi : GL(n, \Lambda) \times GL(m, \Lambda) \to GL(m, \Lambda)$ be projection onto the second factor. Note that these are representations of $GL(n, \Lambda) \times GL(m, \Lambda)$.

**Definition 2.8.7.** *The sequence $\Theta = (\theta_n)$ is* additive *if we have $\theta_{n+m} \cdot (\pi \oplus \varpi) = (\theta_n \cdot \pi) + (\theta_m \cdot \varpi)$ for all $n, m$.*

Then, for any additive sequences of representations, to form composites $\Theta \cdot \alpha$ or $\Phi \cdot \xi$, we simply choose the appropriate degree representation from the sequence, and take the composition as defined above. Further, the composition $\Phi \cdot \Theta$ can defined by taking $(\Phi \cdot \Theta)_n = \Phi \cdot \theta_n$. We also note that additive sequences will guarantee that composition with virtual bundles (and similarly for virtual representations) is well-defined, since the following diagram will commute (as it must, since the two bundles on the left are in the same equivalence class):

$$\begin{tikzcd}
\xi \arrow[r, "\Theta"] \arrow[d, "\text{incl.}"'] & \theta_n \cdot \xi \arrow[d, "\text{incl.}"] \\
\xi \oplus \epsilon^m \arrow[r, "\Theta"] & \theta_{n+m} \cdot (\xi \oplus \epsilon^m)
\end{tikzcd}$$

**Lemma 2.8.8.** *If $\Phi$ and $\Theta$ are additive sequences, $\theta$ is a virtual representation, and $\kappa$ is an element of $K_\Lambda(X)$, then for the composites $\Phi \cdot \Theta$, $\Phi \cdot \theta$, and $\Theta \cdot \kappa$ as defined above, $\Phi \cdot \Theta$ is a well-defined additive sequence, $\Phi \cdot \theta$ is a virtual representation, and $\Theta \cdot \kappa$ is an element of $K_\Lambda(X)$, so that:*

*(1). Each composite is bilinear.*

*(2). Associativity of composites holds in each circumstance where all composites are defined above.*

*(3). For any finite complex $X$, any additive sequence $\Theta$ defines a natural group homomorphism $\Theta : K_\Lambda(X) \to K_{\Lambda'}(X)$, given by $\kappa \mapsto \theta_n \cdot \kappa$, where $\kappa$ is an $n-$bundle.*

All of these ideas can be repeated for the tensor product, which forms the product operation in both $K_\Lambda(X)$ and $K'_{\Lambda'}(G)$. This is done as follows:

**Definition 2.8.9.** *The sequence $\Theta = (\theta)_n$ is muliplicative if we have $\theta_{mn} \cdot (\pi \otimes \varpi) = (\theta_n \cdot \pi) \otimes (\theta_m \cdot \varpi)$.*

Then, we have that the compositions defined above will distribute across the tensor product, though this will require the sequence to be both additive and multiplicative. This leads us to:

**Lemma 2.8.10.** *If the sequence $\Theta$ is both additive and multiplicative, then $\Theta : K_\Lambda(X) \to K_{\Lambda'}(X)$ preserves products. Further, if $\theta_1$ has virtual degree 1, then $\Theta$ takes the unit element in $K_\Lambda(X)$ to the unit element of $K_{\Lambda'}(X)$, and so $\Theta$ defines a ring homomorphism.*

Let $V$ be a vector space of dimension $n$ over $\Lambda$, and let $\bigwedge^k(V)$ denote the $k-$fold exterior power over $V$. If $\{v_1, v_2, \ldots, v_n\}$ form an ordered basis for $V$, then the collection

$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ forms a basis for $\bigwedge^k(V)$, if one takes each sequence where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Then $\bigwedge^k(V)$ is a vector space of dimension $m = \binom{n}{k}$.

Given any element $M$ of $GL(n, \Lambda)$, we can think of it as the matrix of a linear isomorphism of $V$ with respect to the basis $\{v_1, v_2, \ldots, v_n\}$. Then, this induces a linear isomorphism of $\bigwedge^k(V)$ via: $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \mapsto Mv_{i_1} \wedge Mv_{i_2} \wedge \cdots \wedge Mv_{i_k}$, and extending by linearity and the symmetry relation $a \wedge b = -b \wedge a$. So this defines a representation $\bigwedge_\Lambda^k : GL(n, \Lambda) \to GL(m, \Lambda)$.

Let us now define two different types of symmetric functions:

- The power sum symmetric functions are of the form $\sum_{i=1}^{n} {x_i}^k$ for some fixed $k$.
- The elementary symmetric functions are of the form
  $\sigma_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$ for some fixed $k$.

It is known that, in the ring of symmetric functions in $n$ variables over any field containing $\mathbb{Q}$, the first $n$ of each of these two types of symmetric functions can be used as a polynomial basis. (In addition, the elementary symmetric polynomials can be used as a polynomial basis for the ring of symmetric functions in $n$ variables over $\mathbb{Z}$.) In particular, for each of the power sum symmetric functions, there is a polynomial (known as the Newton polynomial) in the elementary symmetric functions $Q_n^k$ such that:

$$\sum_{i=1}^{n} {x_i}^k = Q_n^k(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{2.8.11}$$

**Definition 2.8.12.** *The Adams representation $\psi_{\Lambda,n}^k : GL(n, \Lambda) \to GL(m, \Lambda)$ is the virtual representation given by $\psi_{\Lambda,n}^k = Q_n^k(\bigwedge_\Lambda^1, \bigwedge_\Lambda^2, \ldots, \bigwedge_\Lambda^n)$. Let $\Psi_\Lambda^k$ be the sequence $(\psi_{\Lambda,n}^k)$ for $k \geq 1$, let $\Psi_\Lambda^0$ be the sequence of trivial representations of degree $n$, let $\Psi_\Lambda^{-1}$ be the sequence given by $\psi_{\Lambda,n}^{-1}(M) = (M^T)^{-1}$, and lastly let $\Psi_\Lambda^{-n}$ be given by $\Psi_\Lambda^{-1} \cdot \Psi_\Lambda^n$.*

Let us consider some of the properties of $\Psi^k_\Lambda$.

**Lemma 2.8.13.** *The virtual degree of $\psi^k_{\Lambda,n}$ is $n$.*

*Proof.* By definition, $\psi^k_{\Lambda,n} = Q^k_n(\bigwedge^1_\Lambda, \bigwedge^2_\Lambda, \ldots, \bigwedge^n_\Lambda)$. Then, notice that the degree of $\bigwedge^k_\Lambda$ is $\binom{n}{k}$, which is the number of terms in the symmetric polynomial $\sigma_k$, which is the result of substituting $x_i = 1$ for all $i$ in $\sigma_k$. So, since the degree is a ring homomorphism from $K'_\Lambda(V)$ to $\mathbb{Z}$, we can find the degree of the representation $\psi^k_{\Lambda,n}$ by substituting $x_1 = 1$ for all $i$ in $Q^k_n(\sigma_1, \sigma_2, \ldots, \sigma_n)$. But this is just the power sum symmetric function $\sum_{i=1}^{n} {x_i}^k$, and plugging in $x_i = 1$ for all $i$ gives $n$ as the degree of this representation. $\square$

**Lemma 2.8.14.** *For each $k$, $\psi^k_{\Lambda,1}$ is the $k^{th}$ power of the identity representation of $GL(1, \Lambda)$.*

*Proof.* In this case, $V$ is a $1-$dimensional vector space over $\Lambda$, and so $Q^k_1$ reduces to just ${x_1}^k$. This says that the representation is given by the $1-$fold exterior power tensored with itself $k$ times, which is exactly the $k^{\text{th}}$ power of the identity representation, since the $1-$fold exterior power is the identity. For negative values of $k$, one also needs to note that this representation is invertible (it is its own inverse, in fact), and so the negative power means $k^{\text{th}}$ power of the inverse representation. $\square$

**Lemma 2.8.15.** *If $c$ is the sequence of inclusions $c_n : GL(n, \mathbb{R}) \to GL(n, \mathbb{C})$, then $\Psi^k_\mathbb{C} \cdot c = c \cdot \Psi^k_\mathbb{R}$.*

To continue, we will need the following tool from representation theory.

**Definition 2.8.16.** *Given a representation $\alpha$ of a group $G$, the* character *of $\alpha$ on an element $g \in G$ is the trace of the matrix $\alpha(g)$, and is written $\chi(\alpha)g$.*

It easy to see that $\chi(\alpha \oplus \beta) = \chi(\alpha) + \chi(\beta)$ and $\chi(\alpha \otimes \beta) = \chi(\alpha)\chi(\beta)$. Using these, one can extend the definition of the character to virtual representations. It follows from the multiplicativity of trace that the character is invariant within conjugacy classes in $G$.

**Lemma 2.8.17.** *Let $G$ be a topological group, $g \in G$, and let $\theta$ be a virtual representation of $G$ over $\Lambda$. Then $\chi(\Psi_\Lambda^k \cdot \theta)g = \chi(\theta)g^k$.*

*Proof.* We will prove it for $\Lambda = \mathbb{C}$, the case for $\Lambda = \mathbb{R}$ will follow from this and lemma 2.8.15.

For simplicity, first assume that $M$ is a diagonal matrix with non-zero complex diagonal entries $x_1, x_2, \ldots, x_n$. Then $\bigwedge_{\mathbb{C}}^r(M)$ is a diagonal matrix with entries along the diagonal of the form $x_{i_1}x_{i_2}\cdots x_{i_r}$, with $1 \le i_1 < i_2 < \cdots < i_r \le n$. The trace of this matrix is precisely the $r$th elementary symmetric polynomial.

It follows, then, that if $Q$ is any polynomial in $n$ variables, we have that $\mathrm{Tr}(Q(\bigwedge_{\mathbb{C}}^1, \bigwedge_{\mathbb{C}}^2, \ldots, \bigwedge_{\mathbb{C}}^n)M) = Q(\sigma_1, \sigma_2, \ldots, \sigma_n)$. Then, substituting $Q = Q_n^k$, with $k \ge 0$, we have that $\mathrm{Tr}(\psi_{\mathbb{C},n}^k M) = \sum\limits_{i=1}^{n} {x_i}^k = \mathrm{Tr}(M^k)$. So, the result holds if $k \ge 0$, and $\theta(g)$ is a diagonal matrix.

The result extends as follows:

If we continue to assume that $k \ge 0$, then if $\theta(g)$ is diagonalizable, it is conjugate to a diagonal matrix, and since the trace is constant on conjugacy classes, $\mathrm{Tr}(\theta(g))$ will be the same as the trace of the diagonalization of $\theta(g)$. Then, one can extend to all matrices in one of two ways: either by using continuity of the equations and denseness of diagonalizable matrices, or considering matrices which are in Jordan form, and seeing that the same results hold as did for diagonal matrices, and so extending to all matrices. Then, to extend to negative values of $k$, first extend to $k = -1$ using the definition: $\psi_{\mathbb{C},n}^{-1}(M) = (M^T)^{-1}$. Then, in order for the result to hold, we need only see that $\mathrm{Tr}\left((M^T)^{-1}\right) = \mathrm{Tr}(M^{-1})$, which follows from Cramer's Rule. Then, the result holds for all negative $k$ by combining the result for positive $k$ with the definition for $k = -1$. $\square$

**Lemma 2.8.18.** *The sequence $\Psi_\Lambda^k$ is both additive and multiplicative.*

**Lemma 2.8.19.** $\Psi_\Lambda^k \cdot \Psi_\Lambda^l = \Psi_\Lambda^{kl}$

This concludes the consideration of the Adams representation; from here we will use this to construct an operation on vector bundles, which we will use to construct an operation on characteristic classes.

By 2.8.10, we have a natural homomorphism of rings with unity: $\Psi^k_\Lambda : K_\Lambda(X) \to K_\Lambda(X)$ for any $CW-$complex $X$, and any integer $k$.

The following properties follow immediately from the corresponding properties for the representation:

**Corollary 2.8.20.** $\Psi^k_\Lambda(\Psi^l_\Lambda(K(X))) = \Psi^{kl}_\Lambda(K(X))$

**Corollary 2.8.21.** *The following diagram is commutative:*

$$\begin{tikzcd} K_{\mathbb{R}}(X) \arrow[r, "\Psi^k_{\mathbb{R}}"] \arrow[d, "c"'] & K_{\mathbb{R}}(X) \arrow[d, "c"] \\ K_{\mathbb{C}}(X) \arrow[r, "\Psi^k_{\mathbb{C}}"] & K_{\mathbb{C}}(X) \end{tikzcd}$$

**Corollary 2.8.22.** $\Psi^0_\Lambda$ *is the map which takes a $k-$dimensional bundle and assigns to it the trivial bundle of dimension $k$.*

Recall that, for any matrix in $O(n)$, $\left(M^T\right)^{-1} = M$ (since the rows form an orthonormal basis for $\mathbb{R}^n$), and similarly for any matrix in $U(n)$, $\left(M^T\right)^{-1} = \overline{M}$. So, if $\Lambda = \mathbb{R}$, then $\Psi^{-1}_{\mathbb{R}}(\xi) = \xi = \Psi^1_{\mathbb{R}}(\xi)$, and for $\Lambda = \mathbb{C}$, we have that $\Psi^{-1}_{\mathbb{C}}(\eta) = \overline{\eta} = \overline{\Psi^1_{\mathbb{C}}(\eta)}$. It follows, by definition, that $\Psi^k_{\mathbb{R}}(\xi) = \Psi^{-k}_{\mathbb{R}}(\xi)$ and that $\Psi^k_{\mathbb{C}}(\eta) = \overline{\Psi^{-k}_{\mathbb{C}}(\eta)}$. Using these, and the properties of the Adams representation, we get the following property of the Adams map:

**Corollary 2.8.23.** *If $\xi$ is a line bundle over $X$, then $\Psi^k_\Lambda(\xi) = \xi^k$.*

*Proof.* This follows from the fact (Lemma 2.8.14) that $\psi^k_{\Lambda,1}$ is the $k$th power of the identity representation of $GL(1, \Lambda)$, noting that the $k$th power is taken as tensor product for positive $k$, and taken as above for negative $k$. $\square$

In order to continue, we will need to define the Chern character of a complex bundle, denoted $ch(\omega) \in H^{\Pi}(B, \mathbb{Q})$ (see [MS74], pg. 195, exercise 16-B). We will use the following notations: $c_q(\omega) = c_q$ will denote the $q$th Chern class of $\omega$, and $ch^q(\omega)$ will denote the component of $ch(\omega)$ in $H^{2q}(B, \mathbb{Q})$. Then, for a complex $n-$bundle $\omega$:

$$ch(\omega) = n + \sum_{k=1}^{\infty} \frac{Q_k^k(c_1, c_2, \ldots, c_k)}{k!}$$

where $Q_k^k$ denotes the Newton polynomial (see eq. 2.8.11, above). These Newton polynomials can be given explicitly, by Girard's formula (see [MS74], pg. 195, exercise 16-A):

$$\frac{(-1)^k Q_k^k(\sigma_1, \sigma_2, \ldots, \sigma_k)}{k} = \sum_{i_1+2i_2+\cdots+ki_k=k} (-1)^{i_1+i_2+\cdots+i_k} \frac{(i_1+i_2+\cdots+i_k-1)!}{i_1!i_2!\cdots i_k!} \sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_k^{i_k}$$

**Lemma 2.8.24.** $ch(\omega \oplus \omega') = ch(\omega) + ch(\omega')$

**Lemma 2.8.25.** $ch(\omega \otimes \omega') = ch(\omega)ch(\omega')$

**Corollary 2.8.26.** *$ch(\omega)$ defines a natural ring homomorphism from $K_{\mathbb{C}}(X)$ to $H^*(X, \mathbb{Q})$.*

Now consider $\gamma$, the canonical line bundle over $\mathbb{C}P^\infty$. We know that $H^*(\mathbb{C}P^\infty, \mathbb{Q}) = \mathbb{Q}[c_1(\gamma)]$, and $c_2 = c_3 = \cdots = 0$. So, we will use Girard's formula to compute the Chern character (noting that $i_q = 0$ for $q \geq 2$, since these Chern classes are zero):

$$\frac{(-1)^k Q_k^k(c_1(\gamma))}{k} = \frac{(-1)^k(k-1)!}{k!}(c_1(\gamma))^k$$

which shows that $Q^k_k(c_1(\gamma)) = (c_1(\gamma))^k$, and so

$$ch(\gamma) = 1 + \sum_{k=1}^{\infty} \frac{(c_1(\gamma))^k}{k!} = \exp(c_1(\gamma))$$

Note that this holds for any complex line bundle, since the first Chern class is equal to the Euler class, which is always well-defined since every complex bundle is orientable.

Let $X$ be any finite $CW-$complex, and let $\eta$ be a complex $n-$bundle over $X$. Then let $Y$ be an $n-$fold product of $\mathbb{C}P^m$ for sufficiently large $m$, and let $Y$ have the bundle $\gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$ over it, where the $\gamma_i$ are the canonical line bundles over each of the factors of $Y$. Then, using the fact that $Y$ is the classifying space of the maximal torus (in $U(n)$) of diagonal matrices with diagonal entries of unit modulus, Borel showed that there exists an embedding $i : Y \to X$ so that $i$ commutes with the classifying maps for $\eta$ and $\gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$:

$$\begin{tikzcd}
 & BU(n) \\
Y \arrow[ur] \arrow[r, "i"] & X \arrow[u]
\end{tikzcd}$$

and, further, that $i^* : H^{2q}(X;\mathbb{Q}) \to H^{2q}(Y;\mathbb{Q})$ is injective, and the pullback bundle $i^*(\eta)$ is exactly $\gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$. (Note that, in the special case we are interested in, where $X = BU(n)$, it is well-known that the injection in cohomology takes the $j$th Chern class of $BU(n)$ to the degree $j$ elementary symmetric polynomial in $x_1, x_2, \ldots, x_n$, and that the pullback bundle is as given above since $i$ is the classifying map.)

**Theorem 2.8.27.** *Let $X$ be a finite $CW-$complex and $\eta$ a complex $n-$bundle over $X$. Then: $ch^q(\Psi^k_{\mathbb{C}}\eta) = k^q ch^q(\eta)$.*

*Proof.* Since $i^* : H^{2q}(X;\mathbb{Q}) \to H^{2q}(Y;\mathbb{Q})$ is injective, it suffices to show that $i^*(ch^q(\Psi^k_{\mathbb{C}}\eta)) = i^*(k^q ch^q(\eta))$.

Then, starting with the total Chern character, and letting $x_i$ be the first Chern class of $\gamma_i$:

$$\begin{aligned}
i^*ch(\Psi^k_{\mathbb{C}}\eta) &= ch(\Psi^k_{\mathbb{C}} i^*\eta) \\
&\qquad \text{(by naturality of } ch \text{ and } \Psi^k_{\mathbb{C}}) \\
&= ch(\Psi^k_{\mathbb{C}}(\gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n)) \\
&= ch(\Psi^k_{\mathbb{C}}\gamma_1 \oplus \Psi^k_{\mathbb{C}}\gamma_2 \oplus \cdots \oplus \Psi^k_{\mathbb{C}}\gamma_n) \\
&\qquad \text{(by additivity of } \Psi^k_{\mathbb{C}}) \\
&= ch((\gamma_1)^k + (\gamma_2)^k + \cdots + (\gamma_n)^k) \\
&\qquad \text{(by Corollary 2.8.23)} \\
&= (ch(\gamma_1))^k + (ch(\gamma_2))^k + \cdots + (ch(\gamma_n))^k \\
&\qquad \text{(by additivity and multiplicativity of Chern character)} \\
&= \exp(kx_1) + \exp(kx_2) + \cdots + \exp(kx_n) \\
&\qquad \text{(by computation above)} \\
&= \sum_{q=0}^{\infty} \frac{(kx_1)^q}{q!} + \frac{(kx_2)^q}{q!} + \cdots + \frac{(kx_n)^q}{q!} \\
&= \sum_{q=0}^{\infty} k^q \left( \frac{{x_1}^q}{q!} + \frac{{x_2}^q}{q!} + \cdots + \frac{{x_n}^q}{q!} \right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
i^*ch(\eta) &= ch(i^*\eta) \\
&= ch(\gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n)
\end{aligned}$$

$$= \exp(x_1) + \exp(x_2) + \cdots + \exp(x_n)$$

$$= \sum_{q=0}^{\infty} \frac{{x_1}^q}{q!} + \frac{{x_2}^q}{q!} + \cdots + \frac{{x_n}^q}{q!}$$

Comparing these two results in dimension $2q$ (index $q$ in the sum, since $\deg(x_i) = 2$), we have the desired result. $\square$

Now, let $X = BU(N)$ for sufficiently large $N$. By Theorem 2.8.27, the action of the Adams map $\Psi^k$ on the $q$th Chern class in $H^{2q}(BU(N); \mathbb{Q})$ is multiplication by $k^q$ (since the $q$th component of the Chern character contains a nonzero constant multiple of the Chern class). Since this is true for all sufficiently large $N$, this holds in the direct limit $BU$. This will similarly hold if we pass the the localization $BU_{(p)}$, since the factors are fractions with denominator coprime to $p$ (they are all 1). It is shown in Lance's paper ([Lan79]) that the $q$th Chern class is dual to a nonzero constant multiple of the element in the image of the Hurewicz map $h : \pi_{2q}(BU_{(p)}) \to H_{2q}(BU_{(p)}; \mathbb{Z})$. Then, since the characteristic sequence is the sequence of numbers which describe the action of the induced map on homotopy groups $(\Psi^k)_* : \pi_{2q}(BU_{(p)}) \to \pi_{2q}(BU_{(p)})$, which is $(\Psi^k)_* : \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$, this can be determined by the action on homotopy groups, the images of the generators of the homotopy groups in homology, or their duals in cohomology. We have computed the action on the elements in cohomology, so it follows that the characteristic sequence of the Adams map $\Psi^k$ is $(k, k^2, k^3, \ldots)$.

One important fact to notice in this calculation is that we only needed to know the effect of the Adams map on a *line* bundle in order to calculate its effect on *all* of the homology/cohomology/homotopy of the classifying spaces, or, as Adams generally pursued it, on the stable homotopy groups of spheres. Adams used this 'trick' to great effect here, and in his calculations of the $J-$homomorphism.

# Chapter 3

# Structure of $\hat{X}_\ell$

## 3.1 Existence of $\hat{X}_\ell$

Let $\ell$ be a nonempty set of primes such that $2 \notin \ell$. A positive integer $n$ is said to be $\ell-$divisible if $n$ is a product of primes from $\ell$. We will say that a space $L$ is homotopically $\ell-$divisible if all of its homotopy groups are finite, and have orders that are $\ell-$divisible. Notice that, by, e.g., Serre classes of abelian groups, this implies that the homology and cohomology groups of $L$ are also $\ell-$divisible whenever $L$ is simply connected ([DK01], Thm 10.5, pgs. 269-270).

We have chosen to follow Sullivan's approach to the profinite completion, as laid out in his proof of the Adams Conjecture, in [Sul05] and [Sul74]; Sullivan largely follows Michael Artin and Barry Mazur, [AM69], but with the goal in mind of representing the completion as a concrete $CW-$complex, which Artin and Mazur explicitly avoid. An alternative approach to completion can be found in the work of Bousfield and Kan (c.f. [BK72]), which has largely proved to be the dominant approach in the literature to date, though they also explicitly avoid the construction of a $CW-$complex representation. There has been a recent attempt by a French mathematician, Fabien Morel (c.f. [Mor96]) to revive Sullivan's approach, but Sullivan's approach still seems significantly in the minority.

We define a functor which takes a $CW-$complex $Y$ and assigns a topological space via the formula:

$$HL(Y) \stackrel{\text{defn}}{=} \varprojlim_{\text{finite subcomplexes } Y_\alpha} [Y_\alpha, L] \stackrel{\text{claimed}}{=} [Y, L]$$

For a proof of the claimed equality, see Lemma 3.1.3 below.

We generate the topology on $HL(Y)$ in three stages. First, consider the space Map$(Y_\alpha, L)$, with the compact-open topology, extended to make it compactly generated (see [Mun75], pgs. 285-9, and [DK01], pgs. 111-114). Then, a homotopy generates a path in this space, so the space $[Y_\alpha, L]$ is totally disconnected, since each point is a path component of Map$(Y_\alpha, L)$. The inverse limit, where the finite subcomplexes are directed by inclusion, is then a space which is totally disconnected (by comments above with definition of inverse limit) and compact (see below).

Given a $CW-$complex $X$, consider the category $\{f\}$ whose objects consist of functions $f : X \to L$, and whose morphisms $f \to f'$ are induced by maps $L' \to L$ so that the following diagram commutes, up to homotopy:

$$\begin{tikzcd} & L' \arrow[dd] \\ X \arrow[ur, "f'"] \arrow[dr, "f"'] & \\ & L \end{tikzcd}$$

In Propostions 3.1.6 and 3.1.7, below, we prove that this category is both directed and filtering. Now, we can consider the collection of pairs $L_f = (L, f)$, where $f : X \to L$, which serves to create an indexing of the spaces $L$ by the maps $f$. This allows us to consider the functor $\hat{X}_\ell$ on $CW-$complexes $Y$ given by:

$$\hat{X}_\ell(Y) = \varprojlim_{\{f\}}[Y, L_f]$$

**Theorem 3.1.1.** *There exists a unique $CW-$complex, also denoted $\hat{X}_\ell$, so that $\hat{X}_\ell(Y) = [Y, \hat{X}_\ell]$ for all $CW-$complexes $Y$.*

This will be proved in several stages.

**Lemma 3.1.2.** *If $Y_\alpha$ is a finite subcomplex of a $CW-$complex $Y$, and $L$ is a homotopically $\ell-$divisible space, then $[Y_\alpha, L]$ is a finite set.*

*Proof.* Given any single cell of $Y$, the number of homotopy classes of maps from that cell into $L$ is at most equal to the cardinality of the appropriate homotopy group of $L$, all of which are finite by assumption. Then, extending a map from a finite complex to the same complex with a single cell attached adds no more homotopy classes of maps than the cardinality of the appropriate homotopy group of $L$. So, by induction, the cardinality of $[Y_\alpha, L]$ for any finite subcomplex $Y_\alpha$ of $Y$ is always finite. $\square$

Combining this with the $p-$adic metric that is induced by the inverse limit shows that $\varprojlim_{\text{finite subcomplexes } Y_\alpha} [Y_\alpha, L]$ is compact, Hausdorff, perfect and totally disconnected, and so is a Cantor Set. The claim of perfect follows easily by looking at any cofinal sequence of subcomplexes of $Y$ (see proof of Lemma 3.1.3 below); any element in the inverse limit is the limit point of the restrictions to these subcomplexes.

**Lemma 3.1.3.**

$$[Y, L] \cong \varprojlim_{\textit{finite subcomplexes } Y_\alpha} [Y_\alpha, L]$$

*Proof.* Let $r : [Y, L] \to \varprojlim[Y_\alpha, L]$ be the map induced by restriction from $Y$ to $Y_\alpha$.

<u>$r$ is onto</u>: Let $x$ be any element of $\varprojlim[Y_\alpha, L]$. Given $\beta$, a subset of $\{\alpha\}$, let $Y_\beta = \bigcup_{\alpha \in \beta} Y_\alpha$. Then consider $\mathcal{B}$, the collection of those $\beta$ for which there is a representative

map homotopic to $x$ restricted to $Y_\beta$. If we consider the ordering of $\mathcal{B}$ by inclusion, the transitivity of homotopy implies that this a partial ordering.

If we consider a linear chain in $\{\alpha\}$, the ordering on the chain is equivalent to that induced by the number of cells in $Y_\alpha$, which is finite, so each linear chain is at most countable. Since a linear chain in $\mathcal{B}$ is equivalent to a linear chain in $\{\alpha\}$, each linear chain in $\mathcal{B}$ is also countable. Then, given a linear chain in $\mathcal{B}$, we can form the infinite mapping cylinder of the restrictions of $x$, which is an upper bound of that chain. This allows us to apply Zorn's Lemma to get a maximal element $M$ for the entire partially ordered set. The restrictions of $x$ to the mapping cylinders that are upper bounds for the linear chains combine to form a map $Y_M \to L$, which represents the homotopy class of $x$ on the subcomplex $Y_M$. However, the complex $Y_M$ must be all of $Y$, since if we choose any cell in $Y$, it is a finite subcomplex of $Y$, and so is an element of $Y_\alpha$ for some $\alpha$, and therefore is an element of some $\beta \in \mathcal{B}$ which is contained in $M$.

$r$ is injective: Let $f$ and $g$ be two maps from $Y$ to $L$ that are both mapped by $r$ to the same element of the inverse limit. Since they determine the same element of the inverse limit, they are homotopic on every finite subcomplex $Y_\alpha$ of $Y$. Then, since the homotopy groups of $L$ are finite, and $Y_\alpha$ is a finite complex, there are only finitely many homotopy classes of homotopies between $f$ and $g$, restricted to $Y_\alpha$. If we consider the subsets $\beta$ of $\{\alpha\}$ where we have a homotopy from $f$ to $g$ on $\bigcup_{\alpha\in\beta} Y_\alpha$, ordered by inclusion, transitivity of homotopy implies that this is a partial ordering. The argument is concluded in the same way as the proof that $r$ is onto, above. $\square$

**Proposition 3.1.4.** *If $L$ is homotopically $\ell-$divisible, then $HL(Y) = [Y, L]$ is a functor from the category of $CW-$complexes into the category of compact, totally disconnected Hausdorff spaces that is natural in both coordinates, contravariant in the first, and covariant in the second.*

*Proof.* Lemma 3.1.3 shows that $HL(Y)$ is a compact, totally disconnected Hausdorff space. In order to show naturality, we need to show that a continuous map $f : Y' \to Y$ induces a continuous map $f_\sharp : [Y, L] \to [Y', L]$, and that a continuous map $g : L \to L'$ induces a continuous map $g_\sharp : [Y, L] \to [Y, L']$; the second follows immediately from naturality of homotopy classes of maps.

The first is accomplished in two steps: by naturality of the space of homotopy classes of maps, $f$ induces a map $[Y, L] \to [Y', L]$. Then, if we choose $\tilde{f}$, a cellular map homotopic to $f$, this induces a map of directed inverse systems $\{Y'_\beta\} \to \{\tilde{f}(Y'_\beta)\} \subseteq \{Y_\alpha\}$, and so is a natural transformation of inverse limits. An argument similar to the one in Lemma 3.1.3 shows that $f([Y, L]) = \varprojlim_{\alpha} f([Y_\alpha, L])$. □

**Corollary 3.1.5.** *There is a one-to-one correspondence between homotopically $\ell-$ divisible spaces and contravariant functors from the category of $CW-$complexes to the category of compact, totally disconnected Hausdorff spaces:*

$$L \leftrightarrow [\ \ , L]$$

**Proposition 3.1.6.** *Given the category $\{f\}$, where given a $CW-$complex $X$ and a homotopically $\ell-$divisible space $L$, $f$ is a map $f : X \to L$, we can define an order relation so that the category is directed: given $f : X \to L$, and $f' : X \to L'$, we say that $f' \leq f$ whenever there is a morphism taking $f$ to $f'$.*

*Proof.* The first two properties of a directed set are verified trivially, using the identity map and composition of functions, respectively. The third is verified using projection from $L \times L'$:

$$\begin{tikzcd}
 & L \times L' \arrow[d] \arrow[dr] & \\
X \arrow[ur, "f \times f'"] \arrow[r] \arrow[rr, bend right] & L & L'
\end{tikzcd}$$

□

**Proposition 3.1.7.** *The category $\{f\}$ is filtering.*

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
 & L'' \arrow[d, dotted, "k"] \\
X \arrow[ur, dotted, "f''"] \arrow[r, "f'"] \arrow[dr, "f"] & L' \arrow[d, bend right, "h"'] \arrow[d, bend left, "h'"] \\
 & L
\end{tikzcd}$$

It suffices to verify that, given two objects $f : X \to L$, and $f' : X \to L'$ and two morphisms $h, h' : f \to f'$, there is a third object $f'' : X \to L''$ and a morphism $k : f' \to f''$ so that the compositions of morphisms $k \circ h, k \circ h' : f \to f''$ are equal in the category $\{f\}$ (i.e., so that $h \circ k, h' \circ k : L'' \to L$ are homotopic).

We construct a space which Sullivan calls the *coequalizer* of $h$ and $h'$. We start with the space $L'$, and to each $l' \in L'$, we attach the subset of the free path space $L^I$ consisting of all paths $\varphi$ connecting $h(l')$ to $h'(l')$. This gives the following definition:

$$C(h, h') = \left\{ l' \in L, \varphi \in L^I \,\middle|\, \varphi(0) = h(l'), \varphi(1) = h'(l') \right\}$$

Then, if we use $L'' = C(h, h')$, we can define $k(l', \varphi) = l'$. This allows us to construct the homotopy $K_{h,h'} : L'' \times I \to L$ via:

$$K_{h,h'}((l', \varphi), t) = \varphi(t)$$

Then $K|_{t=0} = h(l')$, $K|_{t=1} = h'(l')$; continuity of $K$ follows easily from continuity of $\varphi$ and from the definition of the free path space.

In the case where $h'$ is the map that takes $L'$ to a point, and $h$ is made into a fibration, then $h$ is the path space fibration, and $C(h, h')$ is its fiber. In the more general case, by a similar argument, there is an exact sequence of homotopy groups:

$$\ldots \to \pi_j(C(h, h')) \to \pi_j(L') \xrightarrow{h_* - h'_*} \pi_j(L) \to \ldots$$

and so it is clear that $C(h, h')$ has $\ell-$divisible homotopy groups, which ensures that $k$ is in the category $\{f\}$ as required. $\square$

Now we recall that $\{f\}$ was constructed as an index category where $L_f$ is the pair $(L, f)$, where $L$ is an $\ell-$divisible space, and $f : X \to L$; then we form $X_\ell(Y) = \varprojlim_{\{f\}} [Y, L_f]$. By 3.1.6 and 3.1.7 above, we now know that this index category is both filtering and directed, and so we can form the inverse limit and know that this is well-defined as a $CW-$complex for any $CW-$complex $Y$.

We need to extend this well-defined inverse limit of $CW-$complexes to a well-defined, representable inverse limit of functors, defined on the $CW-$complex $Y$, which we do in two steps, below. First, recall from the introduction that the inverse limit of nonempty, compact Hausdorff spaces is nonempty, compact and Hausdorff.

**Proposition 3.1.8.** *If we let $X_{\ell,f}$ denote the functor defined on the homotopy category of $CW-$complexes so that $X_{\ell,f}(Y) = [Y, L_f]$, then the inverse limit $\varprojlim_{\{f\}} X_{\ell,f}$ is a well-defined functor on the homotopy category of $CW-$complexes.*

*Proof.* Given a $CW-$complex $Y$ and an inverse system of functors $X_{\ell,f}$, fix the pair $L_f$. Given a map $h : L'_{f'} \to L_f$ so that $h \circ f' \sim f$, this is the same as a functor $h : f \to f'$; it induces a map $h_* : [Y, L'_{f'}] \to [Y, L_f]$, or, in other notation, $h_* : X_{\ell,f'}(Y) \to X_{\ell,f}(Y)$. In this context, let $I_f(Y)$ be the intersection of the images of all such $h_*$. The space $I_f(Y)$ is nonempty as long as $X$ and $Y$ are nonempty, and $f$ is nontrivial.

For any pair $h_1, h_2 : L' \to L$, we can construct the coequalizer (see 3.1.7, and the diagram in its proof), which yields a space $L''$, a map $k : L'' \to L'$, a map $f'' : X \to L''$, and a functor $X_{\ell,f''}$. By construction, $h_1 \circ k$ is homotopic to $h_2 \circ k$, from which it follows that $(h_1 \circ k)_* = (h_2 \circ k)_*$. It is clear that $I_{f'}(Y)$ is contained in the image of $k_*$, and so it follows that $(h_1)_*$ and $(h_2)_*$ must agree on $I_{f'}(Y)$. Since this can be done for any pair of maps $h_1$ and $h_2$, it follows that there is a unique map $\iota : I_{f'}(Y) \to X_{\ell,f}(Y)$.

Given any other object $f_2 : X \to L_2$ in $\{f\}$ we can consider the following diagram:

$$\begin{tikzcd}
 & & L' \times L_2 \arrow[dl, "p'"'] \arrow[dr, "p_2"] & \\
X \arrow[urr, bend left, "f' \times f_2"] \arrow[r, "f'"'] \arrow[rrr, bend left=10, "f_2"] \arrow[drr, bend right, "f"'] & L' \arrow[dr, "h'"'] & & L_2 \arrow[dl, "h_2"] \\
 & & L &
\end{tikzcd}$$

This diagram homotopy commutes (by directedness, 3.1.6), since each of the four routes from $X$ to $L$ other than $f$ is homotopic to $f$, as each vertical route is a functor in $\{f\}$. From this, it follows immediately that the following diagram commutes:

$$\begin{tikzcd}
 & X_{\ell,f'\times f_2}(Y) \arrow[dl, "(p')_*"'] \arrow[dr, "(p_2)_*"] & \\
X_{\ell,f'}(Y) \arrow[dr, "(h')_*"'] & & X_{\ell,f_2}(Y) \arrow[dl, "(h_2)_*"] \\
 & X_{\ell,f}(Y) &
\end{tikzcd}$$

We know that $(h')_*$ restricted to $I_{f'}(Y)$ agrees with the unique map $I_{f'}(Y) \to X_{\ell,f}(Y)$, and $I_{f'}(Y)$ is contained in the image of $X_{\ell,f'\times f_2}(Y)$, so the image of $I_{f'}(Y)$ under the unique map is contained in the image of $X_{\ell,f'\times f_2}(Y)$ under the composition $(h')_* \circ (p')_*$. By commutativity of the last diagram, the image of $I_{f'}(Y)$ is contained in the image of $I_{f_2}(Y)$ under $(h_2)_*$. Since $f_2$ was chosen arbitrarily, this says that the image of $I_{f'}(Y)$ is contained in the intersection of the images of all $h_*$ and so is contained in $I_f(Y)$.

This allows us to compute the inverse limit $\varprojlim_{\{f\}} X_{\ell,f}(Y)$ as the inverse limit $\varprojlim_{\{f\}} I_f(Y)$ where the maps between objects $I_f(Y)$ are unique. Since the $I_f(Y)$ are intersections of compact Hausdorff spaces, they are also compact Hausdorff. It follows that the inverse limit is nonempty compact and Hausdorff as well, as long as $X$ and $Y$ are nonempty. It is easy to see that the system of $I_f(Y)$ with the unique maps connecting them is directed, filtering, and cofinal in the collection of all $X_{\ell,f}(Y)$, and so yields a convergent inverse limit. □

**Proposition 3.1.9.** *The functor $X_\ell(Y)$ is representable; i.e., there is a $CW-$complex $\hat{X}_\ell$ so that $X_\ell(Y) = [Y, \hat{X}_\ell]$.*

*Proof.* It suffices to check the Brown axioms (c.f. [Hat02], Sec. 4.E., particularly Thm. 4E.2., pg. 449). These are:

(1). $X_\ell\left(\bigvee_{\alpha\in A} Y_\alpha\right) = \prod_{\alpha\in A} X_\ell(Y_\alpha)$, where $A$ is an arbitrary index set.

(2). The Mayer-Vietoris property: Let $Y = A\cup B$ and $Z = A\cap B$. If $a \in X_\ell\,(A\,|_Z\,) = b \in X_\ell\,(B\,|_Z\,)$ in $X_\ell(Z)$, then there is some $y \in X_\ell(Y)$ so that $a = y \in X_\ell\,(Y\,|_A\,)$, and $b = y \in X_\ell\,(Y\,|_B\,)$.

The first is essentially trivial to check, using basic properties of the homotopy classes of maps and the inverse limit:

$$\begin{aligned}
X_\ell\left(\bigvee_{\alpha\in A} Y_\alpha\right) &\overset{\text{defn}}{=} \varprojlim_{\{f\}} X_{\ell,f}\left(\bigvee_{\alpha\in A} Y_\alpha\right)\\
&\overset{\text{defn}}{=} \varprojlim_{\{f\}} \left[\bigvee_{\alpha\in A} Y_\alpha, L_f\right]\\
&= \varprojlim_{\{f\}} \prod_{\alpha\in A} [Y_\alpha, L_f]\\
&= \prod_{\alpha\in A} \varprojlim_{\{f\}} [Y_\alpha, L_f]
\end{aligned}$$

$$= \prod_{\alpha \in A} X_\ell(Y_\alpha)$$

To prove that $X_\ell$ satisfies the Mayer-Vietoris property, we start by observing that each $X_{\ell,f}(Y)$ is a $CW-$complex, and satisfies the Mayer-Vietoris property because the collection of homotopy classes of maps between $CW-$ complexes does. Then, if $X$ and $Y$ are nonempty, $X_{\ell,f}(Y)$ is nonempty, compact, and Hausdorff, and so $X_\ell(Y)$ is also nonempty, compact and Hausdorff. Since each $x$, $a$, $b$, and $y$ are limit points of a sequence in a compact Hausdorff space which satisfies the Mayer-Vietoris property, they will satisfy this property in the limit space as well. □

Putting 3.1.8 and 3.1.9 together with the statement that the inverse limit of nonempty compact Hausdorff spaces is again nonempty, compact and Hausdorff gives us:

**Corollary 3.1.10.** *The $CW-$complex $\hat{X}_\ell$ is nonempty whenever $X$ is nonempty.*

This concludes the proof of Theorem 3.1.1, and allows us to make the following:

**Definition 3.1.11.** *The complex $\hat{X}_\ell$ is called the* profinite completion of $X$ at $\ell$.

**Proposition 3.1.12.** *There exists a natural map $c : X \to \hat{X}_\ell$, called the* completion map.

*Proof.* Apply the functor $\hat{X}_\ell(Y)$ to the space $X$. This yields the space $\varprojlim_{\{f\}} [X, L_f]$. This, by definition, is a quotient of the product space $\prod_{\{f\}}[X, L_f]$. The space $X$ maps naturally into each component as the homotopy class of the index map $f$; combining all of these natural maps into the product and taking the quotient yields the natural completion map. □

## 3.2 General Properties of $\hat{X}_\ell$

We will consider the algebraic topology of the profinite completion of a $CW-$complex $X$, generally assuming that $X$ is simply connected and has finitely generated homotopy groups. The case where $X$ is not simply connected is much more difficult – it is certain that the completion cannot be generalized to all spaces, and the known classes of non-simply connected spaces that a completion construction can be applied to are generally quite restrictive. Bousfield and Kan, and several mathematicians who have followed in their footsteps, have expanded the collection of so-called "$R-$good" spaces – those that have a completion construction over $R$ that satisfies a specified list of properties (where completion is in the sense of Bousfield and Kan, which is a bit different than the one presented here).

**Theorem 3.2.1.** *If $X$ has all finitely generated homotopy groups, and $X$ is $(k-1)-$connected, $k>1$, then $\pi_k(\hat{X}_\ell) \cong (\widehat{\pi_k(X)})_\ell$*

*Proof.* By Definition 3.1.11, we have that $[Y, \hat{X}_\ell] = \varprojlim_{\{f\}} [Y, L_f]$ for any $CW-$complex $Y$. So, if we let $Y = S^k$ we have, by definition:

$$\pi_k(\hat{X}_\ell) = \varprojlim_{\{f\}} \pi_k(L_f) \tag{3.2.2}$$

Consider a group $\Lambda$ that can be generated as a quotient from $\pi_k(X)$. Then, using the Postnikov decomposition of $X$ we can form the map:

$$\begin{tikzcd}
X \arrow[r, "k^{\text{th}}\ k-\text{inv.}"] & K(\pi_k(X), k+1) \arrow[r, "\text{canon.}"] & \Omega K(\pi_k(X), k+1) \arrow[d, "\simeq"] & \\
 & & K(\pi_k(X), k) \arrow[r, "\text{quotient}"] & K(\Lambda, k)
\end{tikzcd}$$

If we let $f$ denote the composition, then we can see that the induced map on homotopy

$f_* : \pi_k(X) \to \Lambda$ is onto: the only step in the composition that is not obvious is the first, which is onto because $X$ is $(k-1)-$connected, and so in the notation of 2.1.4, $X_{k-1} = X$. The pair $(K(\Lambda, k), f)$ is included in the inverse limit on the right-hand side of 3.2.2, and so all $\ell-$divisible quotients of $\pi_k(X)$ are included in this inverse limit. It is easy to see that the collection of all $K(\pi, k)_f$, with $\pi$ any $\ell-$divisible group, and $f$ a map representing any homotopy class of maps from $X$, is cofinal in this inverse limit. Then, since the inverse limit is over the maps $f$, we can reduce to those $\pi$ that can be represented as quotients of $\pi_k(X)$, and still have a cofinal set (see 3.1.8), and so we have shown:

$$\pi_k(\hat{X}_\ell) \cong \varprojlim_{\{f\}} \pi_k(L_f) \cong \varprojlim_{\{f\}} \pi_k(K(\Lambda, k)_f)$$

which is exactly $(\widehat{\pi_k(X)})_\ell$. $\square$

**Corollary 3.2.3.** *If $X$ is contractible, then $\hat{X}_\ell$ is contractible also.*

**Lemma 3.2.4.** *If $X$ is a simply connected $CW-$complex, then, for each $k$:*

$$\varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \cong \begin{cases} H^k(X; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) & \text{if } p \in \ell \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* First, let $p \notin \ell$. Since $L$ is $\ell-$divisible and $p \notin \ell$, we know that $H^k(L; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$ is trivial, and so $\varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$ is trivial.

Now, let $p \in \ell$. Given the pair $L_f$, we have $f : X \to L$, which induces $f^k : H^k(L_f; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \to H^k(X; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$. Collectively, these maps induce $g : \varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \to H^k(X; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$.

To see that $g$ is surjective when $p \in \ell$, consider a cohomology class $a^k \in H^k(X; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$. By 2.1.3, $a^k$ can be realized as a map $\tilde{a}^k : X \to K\left({}^{\mathbb{Z}}/_{p\mathbb{Z}}, k\right)$, which can, in turn, be realized as a cohomology class in $H^k(L_f; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$, where $L_f$ is the pair $\left(K\left({}^{\mathbb{Z}}/_{p\mathbb{Z}}, k\right), \tilde{a}^k\right)$.

The map $g$ takes this class to $a^k$.

To see that $g$ is injective when $p \in \ell$, consider ker($g$): let $\theta^k$ represent the zero element in $H^k(X; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$. This corresponds to a null-homotopic map $\tilde{\theta}^k : X \to K\left({}^{\mathbb{Z}}/_{p\mathbb{Z}}, k\right)$. Then, if $b^k$ is any other cohomology class which gets mapped to zero in $H^k(X; {}^{\mathbb{Z}}/_{p\mathbb{Z}})$, then there is a homotopy commutative diagram (where $\beta$ corresponds to $b^k$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\tilde{\theta}^k"'] & L \arrow[d, "\beta"] \\ & K\left({}^{\mathbb{Z}}/_{p\mathbb{Z}}, k\right) \end{tikzcd}$$

Since the diagram is homotopy commutative, and $\tilde{\theta}^k$ is null-homotopic, the composition $\beta \circ f$ is null-homotopic, and so is part of directed set that converges to the zero element in the inverse limit. □

**Corollary 3.2.5.** *If $X$ is a simply connected $CW-$complex, then:*

$$\varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \cong H^k(X; {}^{\mathbb{Z}}/_{n_\ell\mathbb{Z}})$$

*where $n_\ell$ is the $\ell-$divisible part of $n$ (i.e. ${}^{\mathbb{Z}}/_{n_\ell\mathbb{Z}} \cong {}^{\mathbb{Z}}/_{n\mathbb{Z}} \otimes \hat{\mathbb{Z}}_\ell$).*

**Proposition 3.2.6.** *If $X$ is a simply connected $CW-$complex and $n$ is $\ell-$divisible, there exists an injective map $\varphi : \varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \to H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}})$.*

*Proof.* The cohomology group $H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}})$ is in one-to-one correspondence with $[\hat{X}_\ell, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]$ (by 2.1.3); these maps induce all of the possible maps:

$$\begin{tikzcd} {[Y, \hat{X}_\ell]} \arrow[r] \arrow[d, "="'] & {[Y, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]} \arrow[d, "="] \\ \varprojlim_{\{f\}}[Y, L_f] \arrow[r] & \varprojlim_{\text{finite subcomplexes } Y_\alpha} {[Y_\alpha, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]} \end{tikzcd} \tag{3.2.7}$$

For any map $f$ in the indexing set, and for any $d^k \in H^k(L_f; {}^{\mathbb{Z}}/_{n}\mathbb{Z})$, this corresponds to an element of $[L_f, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]$ (again by 2.1.3). Given an element of any of these cohomology groups, we can induce the map $\delta$ in the following diagram for any $CW-$complex $Y$, which in turn induces the map $\epsilon$:

$$\begin{tikzcd}
{[Y, L_f]} \arrow[r, "\delta"] \arrow[rr, bend left, "\epsilon"] & {[Y, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]} \arrow[r, "\text{restr.}"] & {[Y_\alpha, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]}
\end{tikzcd}$$

This map $\epsilon$ induces a map of objects in the inverse limits of the bottom row of 3.2.7, and so determine a map $[Y, \hat{X}_\ell] \to [Y, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]$, which, in turn, determines an element of $[\hat{X}_\ell, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]$, which determines a cohomology class in $H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}})$, as desired.

This map is injective because, if a cohomology class $d^k$ is mapped to the trivial element of $H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}})$, then the horizontal maps of 3.2.7 must be trivial for all $CW-$complexes $Y$, and then, choosing $Y = K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)$, we can see that $d^k$ is trivial also. $\square$

**Corollary 3.2.8.** *If $X$ is a simply connected $CW-$complex, and $n_\ell$ is the $\ell-$divisible part of $n$, these maps combine with the completion map to form the following commutative diagram:*

$$\begin{tikzcd}
& H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \arrow[dl, "c^*"'] & \\
H^k(X; {}^{\mathbb{Z}}/_{n_\ell\mathbb{Z}}) & & \varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \arrow[ul, "\varphi"'] \arrow[ll, "\cong"']
\end{tikzcd}$$

**Theorem 3.2.9.** *If $X$ is a simply connected $CW-$complex, then, for each $k$, we have that:*

$$H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \subseteq \overline{\varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{n\mathbb{Z}})}$$

*where $\overline{G}$ indicates the topological closure in the topology induced by the inverse limit.*

*Proof.* Let $b^k$ be a cohomology class in $H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}})$. Then $b^k$ corresponds to an element $\beta \in [\hat{X}_\ell, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]$. We consider the induced map in the diagram 3.2.7, using $Y = \hat{X}_\ell$:

$$\begin{array}{ccc} [\hat{X}_\ell, \hat{X}_\ell] & \xrightarrow{\beta^*} & [\hat{X}_\ell, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)] \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \varprojlim_{\{f\}}[\hat{X}_\ell, L_f] & \xrightarrow{\delta^*} & \varprojlim_{\text{finite subcx's } X_\alpha} [X_\alpha, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)] \end{array}$$

but, for each $L_f$ and $X_\alpha$, $\delta^*$ is a map induced by an element of $[L_f, K\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}, k\right)]$, which corresponds to a cohomology class $d^k \in H^k(L_f; {}^{\mathbb{Z}}/_{n\mathbb{Z}})$, so every cohomology class $b^k$ can be seen as a limit of cohomology classes $d^k$. $\square$

**Corollary 3.2.10.** *If $X$ is a simply connected $CW-$complex, then, for each $k$:*

$$H^k(X; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \cong \varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \subseteq H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \subseteq \overline{\varprojlim_{\{f\}} H^k(L_f; {}^{\mathbb{Z}}/_{n\mathbb{Z}})}$$

**Corollary 3.2.11.** *If $X$ is a simply connected $CW-$complex that has $\pi_k(X)$ finitely generated for each $k$, then, for each $k$:*

$$H^k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \cong H^k(X; {}^{\mathbb{Z}}/_{n_\ell\mathbb{Z}})$$

*and*

$$H_k(\hat{X}_\ell; {}^{\mathbb{Z}}/_{n\mathbb{Z}}) \cong H_k(X; {}^{\mathbb{Z}}/_{n_\ell\mathbb{Z}})$$

*where $n_\ell$ is the $\ell-$divisible part of $n$.*

*Proof.* The first is immediate because the closure of a discrete space is the same as the space. The second follows immediately from the first by Universal Coefficients. $\square$

Notice that, in this case with finite coeffecient systems, and finitely generated homotopy, this is the identical result as in the case of localization (c.f. 2.6.3).

**Theorem 3.2.12.** *If $X$ is a simply connected $CW-$complex with all finitely generated homotopy groups, then:*

$$\varprojlim_{\{f\}} H^k(L_f;\mathbb{Z}) \subseteq H^k(\hat{X}_\ell;\mathbb{Z}) \subseteq \overline{\varprojlim_{\{f\}} H^k(L_f;\mathbb{Z})}$$

*Proof.* We prove this using methods analogous to those in 3.2.6; first we construct an injective map $\varphi : \varprojlim_{\{f\}} H^k(L_f;\mathbb{Z}) \to H^k(\hat{X}_\ell;\mathbb{Z})$ for each $k$.

The cohomology group $H^k(\hat{X}_\ell;\mathbb{Z})$ is in one-to-one correspondence with $[\hat{X}_\ell, K(\mathbb{Z},k)]$ (by 2.1.3); these maps induce all of the possible maps:

$$\begin{tikzcd} {[Y,\hat{X}_\ell]} \arrow[r] \arrow[d, "="'] & {[Y,K(\mathbb{Z},k)]} \\ \varprojlim_{\{f\}}[Y,L_f] \arrow[ur] & \end{tikzcd} \tag{3.2.13}$$

For any map $f$ in the index set, then for any cohomology class $d^k \in H^k(L_f;\mathbb{Z})$, this corresponds to an element of $[L_f, K(\mathbb{Z},k)]$ (again by 2.1.3). Given any such cohomology class, we can induce the map $\delta$ for any $CW-$complex $Y$:

$$[Y,L_f] \xrightarrow{\delta} [Y,K(\mathbb{Z},k)]$$

This map $\delta$ induces a map from the inverse limit to $[Y,K(\mathbb{Z},k)]$ that forms the diagonal in 3.2.13.

This map is injective because, if a cohomology class $d^k$ is mapped to the trivial element of $H^k(\hat{X}_\ell;\mathbb{Z})$, then the horizontal map of 3.2.13 must be trivial for all $CW-$complexes $Y$, and then, choosing $Y = K(\mathbb{Z},k)$, we can see that $d^k$ is trivial also.

Let $b^k$ be a cohomology class in $H^k(\hat{X}_\ell;\mathbb{Z})$. Then $b^k$ corresponds to an element

$\beta \in [\hat{X}_\ell, K(\mathbb{Z}, k)]$. We consider the induced map in the diagram 3.2.13, using $Y = \hat{X}_\ell$:

$$\begin{tikzcd}
{[\hat{X}_\ell, \hat{X}_\ell]} \arrow[r, "\beta^*"] \arrow[d, "="'] & {[\hat{X}_\ell, K(\mathbb{Z}, k)]} \\
\varprojlim_{\{f\}} [\hat{X}_\ell, L_f] \arrow[ur, "\delta^*"] &
\end{tikzcd}$$

but, for each $L_f$, $\delta^*$ is a map induced by an element of $[L_f, K(\mathbb{Z}, k)]$, which corresponds to a cohomology class $d^k \in H^k(L_f; \mathbb{Z})$, so every cohomology class $b^k$ can be seen as a limit of cohomology classes $d^k$. □

**N.B.** The cohomology groups of $L_f$ may be non-trivial in dimensions where the cohomology groups of $X$ are trivial. See Section 3.3 for an example.

**Theorem 3.2.14.** *There is a map* $g : \varprojlim_{\{f\}} H^k(L_f; \mathbb{Z}) \to H^k(X; \mathbb{Z})$.

*Proof.* This map is defined the same way as in 3.2.4; given the pair $L_f$, we have $f : X \to L$, which induces $f^k : H^k(L_f; \mathbb{Z}) \to H^k(X; \mathbb{Z})$. Collectively, these maps induce $g : \varprojlim_{\{f\}} H^k(L_f; \mathbb{Z}) \to H^k(X; \mathbb{Z})$. □

In Lemma 3.2.4, where the coefficient group is finite, we were able to prove that $g$ is an isomorphism. In this case, both the proof that $g$ is injective, and that $g$ is surjective were dependent on the finiteness of the coefficient group. It is certain that $g$ is not surjective (see section 3.3 for a counter-example) – we have neither a counterexample nor a proof for injectivity; we conjecture that $g$ is always injective.

Notice that, since the $p-$adic integers are infinitely generated over $\mathbb{Z}$, we do not have a 'collapse' result for $CW-$complexes that have finitely generated homotopy analogous to 3.2.11. A detailed example of this is presented in section 3.3. This example also highlights the note given above, where we recognize that nontrivial homology and/or cohomology may exist in the limit where the homology and/or cohomology of $X$ is trivial.

**Theorem 3.2.15.** *If $X$ is a $CW-$complex with finitely generated homotopy groups, then $\hat{X}_\ell$ is a simple inverse limit:*

$$\hat{X}_\ell \simeq \varprojlim_n \{\cdots \to L_n \to L_{n-1} \to \cdots \to L_1 \to \{pt\}\}$$

*where (1) all of the $L_k$ have finite homotopy groups; (2) the $L_k$ are linearly ordered; and (3) there is a unique map connecting $L_k$ to $L_{k-1}$ for all $k \geq 1$.*

*Proof.* Given an $\ell-$divisible space $L$, we can fix a dimension $n$, then attach $k-$cells for all $k > n$ in order to kill all homotopy above dimension $n$. This space will be called the "$n-$coskeleton" of $L$, and will be denoted $L\langle n\rangle$. Then, using arguments as seen above, it is clear that

$$L \simeq \varprojlim_n L\langle n\rangle$$

and so we can see that the functor $\hat{X}_\ell$ can be constructed as an inverse limit with spaces that have only finitely many nontrivial homotopy groups, all of which are finite. It is obvious that this collection of spaces is countable; given any two such spaces, the collection of homotopy classes of maps between them is finite (see induction argument in the proof of 3.1.2, above).

This gives us an inverse system that is a countable collection with only finitely many maps between each object, and so it is possible to choose a linearly ordered cofinal subsystem indexed by the positive integers; since the subsystem is cofinal, it converges to the same limit, and we have:

$$\hat{X}_\ell \simeq \varprojlim_n \{\cdots \to L_n \to L_{n-1} \to \cdots \to L_1 \to \{pt\}\}$$

as desired. $\square$

**Theorem 3.2.16.** *If $X$ is a simply connected $CW-$complex with finitely generated homotopy groups, then:*

$$\pi_k(\hat{X}_\ell) \cong (\widehat{\pi_k(X)})_\ell$$

*Proof.* We consider a series of cases, starting with Eilenberg-MacLane spaces (including non-simply connected examples), and then extending to general $CW-$complexes.

Case 1: Let $X = S^1$.

$S^1 = K(\mathbb{Z}, 1)$. Let $L_j = \{K(G_j, 1)\}$, where each $G_j$ is the additive group of the $j^{\text{th}}$ ring used to form the ring of $\ell-$adic integers (see second Example on page 18 – the rings $^R/_{I_{j,\ell}}$). Let $f_j : S^1 \to L_j$ be the map induced by the canonical projection $\mathbb{Z} \to G_j$. Then it is clear that the inverse limit $\varprojlim_{\{f_j\}} K(G_j, 1)$ is $K\left(\hat{\mathbb{Z}}_\ell, 1\right) = (\hat{S}^1)_\ell$. It is easy to see that this generalizes to $K(G, 1)$, where $G$ is a finitely generated abelian group.

Case 2: Let $X = K(G, n)$, where $G$ is finitely generated abelian.

This follows immediately from Case 1 by induction, using the fibration $K(G, n-1) \hookrightarrow pt \to K(G, n)$.

Case 3: Let $X$ be given by a finite Postnikov system.

Proof is again by induction, this time on the number of nontrivial stages – the base case is contained in Case 2, since a Postnikov system with only one nontrivial stage is an Eilenberg-MacLane space. Assume the theorem is true for a $(n-1)-$stage Postnikov system.

Looking at diagram 2.1.5, we have fibrations $K(\pi, n) \hookrightarrow X_n \to X_{n-1}$, which, by a shift of index in the long exact sequence of the fibration, can be viewed as a fibration $X_n \hookrightarrow X_{n-1} \to K(\pi_n, n+1)$. Since the theorem is assumed true for an $(n-1)-$stage Postnikov system, we have a sequence $\{(L_j)_{f_j}\}$ so that $\varprojlim_{f_j} L_j = (\widehat{X_{n-1}})_\ell$.

Using Theorem 3.2.11, and the correspondence between cohomology and homotopy classes of maps, we can construct the following commutative diagram, where the $M_j$ are the fibers of the right-hand vertical sequence for each positive integer $r$ that is $\ell-$divisible:

$$\begin{tikzcd}
X_n \arrow[r, "f_j"] \arrow[d] & M_j \arrow[d] \\
X_{n-1} \arrow[r] \arrow[d, "k-\text{inv.}"'] & L_j \arrow[d, "k-\text{inv.}"'] \\
K(\pi_n, n+1) \arrow[r, "\text{reduction mod } r"] & K\left(\pi_n \otimes {}^{\mathbb{Z}}/_{r\mathbb{Z}}, n+1\right)
\end{tikzcd}$$

Then the $M_j$, together with the top horizontal map $f_j$ for each $j$, provide the system so that $\varprojlim_{f_j} M_j = \widehat{(X_n)_\ell}$.

Case 4: The general case.

Any simply connected $CW-$complex with finitely generated homotopy groups is the inverse limit of its Postnikov decomposition, and so this case follows immediately from Case 3. □

**Corollary 3.2.17.** *If $F \hookrightarrow E \to B$ is a fibration of simply connected $CW-$complexes with finitely generated homotopy groups, then $\hat{F}_\ell \hookrightarrow \hat{E}_\ell \to \hat{B}_\ell$ is a fibration also.*

*Proof.* We know that the homotopy groups of a fibration fall into a long exact sequence (the Puppe sequence); since profinite completion preserves exact sequences of finitely generated abelian groups (see comment on pg. 22), then, by Theorem 3.2.16, the homotopy groups of $\hat{F}_\ell \hookrightarrow \hat{E}_\ell \to \hat{B}_\ell$ also fall into a long exact sequence; by the Whitehead Theorem these have the homotopy type of a fibration. □

**Corollary 3.2.18.** *If $X$ and $Y$ are simply connected $CW-$complexes and both have finitely generated homotopy groups, then:*

$$\widehat{(X \times Y)}_\ell \simeq \hat{X}_\ell \times \hat{Y}_\ell$$

*Proof.* The two spaces have the same homotopy groups (by Example (4) on page 21), and therefore have the same homotopy type by Whitehead's Theorem. The assumption of finite generation is essential, because of the distinction between direct product and direct sum. $\square$

## 3.3 A Correction to Sullivan's Notes

At this point, it would be most helpful to be able to state a theorem analogous to the second and third parts of 2.6.3, telling us that the homology and cohomology of the completion can be found exactly by simply completing the homology and cohomology of the original complex. In fact, Sullivan does exactly this, for cohomology, stating:

> Theorem 3.9 If $X$ is simply connected and its homotopy groups are finitely generated then ...
>
> iii) $(\widehat{H^i(X;\mathbb{Z})}) \sim H^i(\hat{X};\hat{\mathbb{Z}})$
>
> *pg. 67, [Sul05]*

where, here, $\hat{X}$ indicates profinite completion over all primes.

Sullivan himself later provides a clue to a counter-example, which he attributes to Aldridge Bousfield. After personal correspondence with Dr. Bousfield, who provided most details of what follows, we provide the following counter-example:

Let $X = S^n$, with $n$ odd, $n > 1$. Then $\pi_n(S^n) \cong \mathbb{Z}$, and $\pi_k(S^n)$ is a finite abelian group for $k > n$, and trivial for $1 \leq k < n$. If we consider $\hat{S^n}$, then, by Theorem 3.2.16, we have that $\pi_n(S^n) \cong \hat{\mathbb{Z}}$, and the same statements about $k \neq n$. We can then consider the rationalization of this space, which is the same as the localization at zero. Then, by 2.6.3, we have that $\pi_n(\hat{S^n}_{(0)}) \cong \hat{\mathbb{Q}}$, and all other homotopy groups trivial, since the rationalization kills all of the torsion, showing that the space $\hat{S^n}_{(0)}$ is a $K\left(\hat{\mathbb{Q}}, n\right)$. In this case, we view $\hat{\mathbb{Q}}$ as a vector space over $\mathbb{Q}$ with uncountably many dimensions.

We may then apply Serre's computation of the homology of Eilenberg-MacLane spaces to show that the rational homology of a $K(V,n)$ (where $V$ is a vector space over $\mathbb{Q}$) is an exterior algebra with a set of generators equal to a basis for $\pi_n(K(V,n))$. Since, in the example of $K\left(\hat{\mathbb{Q}},n\right)$, this is a set of uncountable cardinality, and since the wedge product is additive in dimensions and nontrivial on distinct generators, we have that there are also uncountably many generators in all dimensions that are multiples of $n$. Since this is true of the rationalization space $\hat{S^n}_{(0)}$, then, by 2.6.3 (used backwards), the integral homology of the space $\hat{S^n}$ has uncountably many generators in all dimensions that are a multiple of $n$.

In particular, $H_{jn}(\hat{S^n};\hat{\mathbb{Z}}) \neq 0$ for $j > 1$, but $(\widehat{H_{jn}(S^n;\mathbb{Z})}) = 0$ for all $j > 1$. By Universal Coefficients, the same statement is true in cohomology.

# Chapter 4

# Structure and Self-Maps of $\hat{BO}_p$ and $\hat{BU}_p$

## 4.1 Structure of $\hat{BO}_p$ and $\hat{BU}_p$

Here, we look to apply the results of the previous chapter to the specific cases of the classifying spaces $BU$ and $BO$. Both spaces are simply connected $CW-$complexes that have all finitely generated homotopy groups.

In all of this chapter, $\ell$ is a collection of primes so that $2 \notin \ell$ and $p$ is an odd prime.

Using Theorem 3.2.17, we see that the fibrations $O \hookrightarrow EO \to BO$ and $U \hookrightarrow EU \to BU$ yield fibrations $\hat{O}_\ell \hookrightarrow \hat{EO}_\ell \to \hat{BO}_\ell$ and $\hat{U}_\ell \hookrightarrow \hat{EU}_\ell \to \hat{BU}_\ell$. Then, using 3.2.3, we have that $\hat{EO}_\ell$ and $\hat{EU}_\ell$ are both contractible, and so the Puppe sequence allows us to combine Bott Periodicity (2.7.1) with Theorem 3.2.16, yielding:

**Theorem 4.1.1.**

$$\pi_k(\hat{BO}_\ell) \cong \begin{cases} \hat{\mathbb{Z}}_\ell & \text{when } k \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_k(\hat{BU}_\ell) \cong \begin{cases} \hat{\mathbb{Z}}_\ell & \textit{when } k \equiv 0 \pmod 2 \\ 0 & \textit{otherwise} \end{cases}$$

**Theorem 4.1.2.** *The spaces $\hat{BU}_p$ and $\hat{BO}_p$, with $p$ odd, can be decomposed as a product:*

$$\hat{BU}_p \cong \hat{W}_p \times \Omega^2 \hat{W}_p \times \ldots \Omega^{2p-4} \hat{W}_p$$
$$\hat{BO}_p \cong \hat{W}_p \times \Omega^4 \hat{W}_p \times \ldots \Omega^{2p-6} \hat{W}_p$$

*where $\hat{W}_p$ satisfies:*

$$\pi_k(\hat{W}_p) \cong \begin{cases} \hat{\mathbb{Z}}_p & \textit{if } k = 2j(p-1) \textit{ for } j = 1, 2, \ldots \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* This follows immediately from 3.2.18 and 2.7.4 □

This allows us to factor maps and fibers as in the $p-$local case.

Moving to homology and cohomology, we have the following:

**Theorem 4.1.3.** *$H_*(\hat{BO}_\ell)$ and $H^*(\hat{BU}_\ell)$ are both torsion-free.*

*Proof.* First, the cohomology half of this theorem follows immediately from 3.2.11. The homology half follows from the first by Serre classes of abelian groups: if we let $\mathcal{C} = \{$ finite abelian groups $\}$, both spaces are cohomologically trivial mod $\mathcal{C}$, and so are homologically trivial also. Alternatively, both can be seen to follow by this Serre class from 4.1.1. □

**Conjecture 4.1.4.**

$$\tilde{H}^*(\hat{BU}_\ell;\mathbb{Z}) \cong \hat{\mathbb{Z}}_\ell[c_1, c_2, \dots]$$
$$\tilde{H}^*(\hat{BO}_\ell;\mathbb{Z}) \cong \hat{\mathbb{Z}}_\ell[p_1, p_2, \dots]$$

*where the nontrivial generators in the cohomology of $\hat{BU}_\ell$ are in dimensions $2k$, $k \geq 1$, and in the cohomology of $\hat{BO}_\ell$ are in dimension $4k$, $k \geq 1$. The cohomology algebra and the homology coalgebra admit, respectively, a coalgebra and an algebra structure making the cohomology and homology into a self-dual bibolynomial Hopf Algebra.*

## 4.2 Self-Maps of $\hat{BO}_\ell$ and $\hat{BU}_\ell$

**Proposition 4.2.1.** *A self-map of $BO$ or of $BU$ that induces a self-map of $BO_{(p)}$ or of $BU_{(p)}$ also induces a self-map of $\hat{BO}_p$ or of $\hat{BU}_p$ with the same characteristic sequence as in the $p-$local case.*

*Proof.* The completion map commutes with self-maps of $BO$ and $BU$ by naturality. The characteristic sequence is the same since $\mathbb{Z}_{(p)} \subset \hat{\mathbb{Z}}_p$. □

We also have the important congruence result from the $p-$local case:

**Theorem 4.2.2.** *Let $f$ be a self-map of $\hat{BU}_p$ that has characteristic sequence $\lambda$, then $\lambda_{mp^k} \equiv \lambda_{np^k}$ (mod $p^{k+1}$) whenever $m \equiv n$ (mod $p-1$).*

We also have the notion of the surplus sequence, and the associated result:

**Theorem 4.2.3.** *For any $n$ prime to $p$, if $j$ is minimal such that $s(f)_{np^j} \leq 0$, then $s(f)_{np^{j+k}} = -k$ for $k = 1, 2, \dots$. In this case, we define $\delta_n = j$, and $\delta_n = \infty$ otherwise.*

**Examples**

(1). *The Adams map.*

The Adams map $\psi^k$, on $\hat{BU}_p$ has characteristic sequence $(k, k^2, \ldots)$; if $k$ is prime to $p$, then $\psi^k$ is invertible since all of the elements of its characteristic sequence are invertible. One can also see that, if $k$ generates the units of ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$, then $\psi^k - 1$ is invertible on $\Omega^{2k}W$, for $k = 1, 2, \ldots, p-1$.

(2). *The Adams-Bott Cannabalistic Class.*

The Adams-Bott cannabalistic class, $\rho^k$, induces a self-map of $\hat{BO}_p$. Adams' computations have the immediate consequence that the characteristic sequence of $\rho^k$ is:

On $\pi_{4j}(\hat{BO}_p)$, $\rho^k$ is given by multiplication by the constant:

$$(-1)^{j+1}(k^{2j} - 1)\frac{B_{2j}}{4j}$$

where $B_{2j}$ is the $2j^{\text{th}}$ Bernoulli Number.

(3). *The Sullivan Map.*

The Sullivan map $\theta^k$ induces a self-map of $\hat{BO}_p$. Sullivan's computations have the immediate consequeces that the characteristic sequence of $\theta^k$ is:

On $\pi_{4j}(\hat{BO}_p)$, $\theta^k$ is given by multiplcation by:

$$(-1)^{j-1}2^{2j}(k^{2j} - 1)(1 - 2^{2j-1})\frac{B_{2j}}{2j}$$

We are again able to build the following two correspondences:

$$\theta^k \sim \rho^k \circ (2\psi^2 - \psi^4) \qquad \text{and} \qquad \theta^k \sim ((\psi^2)^2/\psi^4) \circ \rho^k$$

We have the following diagrams of fibrations:

$$\begin{tikzcd}
\hat{L}_p \arrow[rr, bend left=10] \arrow[dr] & & \hat{BO}_p \arrow[rr, bend left=10, "\theta^k \sim \rho^k \circ (2\psi^2 - \psi^4)"] \arrow[dr, "(2\psi^2 - \psi^4)"] & & \hat{BO}_p \\
& \hat{N}_p \arrow[ur] \arrow[dr] & & \hat{BO}_p \arrow[ur, swap, "\rho^k"] & \\
& & \hat{M}_p \arrow[ur] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat{M}_p \arrow[rr, bend left=10] \arrow[dr] & & \hat{BO}_p \arrow[rr, bend left=10, "\theta^k \sim ((\psi^2)^2/\psi^4) \circ \rho^k"] \arrow[dr, "\rho^k"] & & \hat{BO}_p \\
& \hat{N}_p \arrow[ur] \arrow[dr] & & \hat{BO}_p \arrow[ur, swap, "((\psi^2)^2/\psi^4)"] & \\
& & \hat{L}_p \arrow[ur] & &
\end{tikzcd}$$

Again, in addition to the three fibrations in each diagram that are self-maps of $\hat{BO}_p$, there is also a fourth fibration; in the first diagram we have $\hat{L}_p \hookrightarrow \hat{N}_p \to \hat{M}_p$, and in the second, we have $\hat{M}_p \hookrightarrow \hat{N}_p \to \hat{L}_p$.

It is apparent then, that virtually all, or all of the results in the $p-$local study of these classifying spaces carry into the $p-$adic context. It is our hope that this change of setting will yield additional information unavailable in the $p-$local, rational, and integer contexts.

## 4.3 Directions for Future Research

The next two steps in the research program that we intend to undertake are:

(1). Determine if the remainder of the structures and substructures present in the $p-$local case are also present in the profinite case.

(2). Determine if the presence of the $(p-1)^{\text{st}}$ roots of unity in the $p-$adic integers will allow us to extend the number theoretic results of this section in any way. In particular, we believe that it may be possible use the methods of section 2.8 to extend the computation of the Adams-Bott cannabalistic class, and Adams' work on the $J-$homomorphism to yield a topologically significant map whose characteristic sequence includes the generalized Bernoulli numbers.

If we are successful in the second of these goals, we will work to see if other tools of algebraic number theory can be found in the topology of classifying spaces. It would be interesting to see if the fundamental structures of the cyclotomic fields have analogues in the topology of classifying spaces. It is also our hope that some of the major advances of the last two decades in algebraic number theory can be brought into these difficult questions of topology, and, in so doing, shed some light on long-standing unsolved problems.

# Chapter 5

# Computational Number Theory

## 5.1 A Motivating Example

Here, we will consider specific examples of the fibrations 2.7.18 and 2.7.19, in order to demonstrate the effect of the $p-$divisibility questions that are considered in section 5.2. In this section, when we say $p \,\|\, n$, we mean that $p$ exactly divides $n$, i.e. that $p$ does divide $n$, but $p^2$ does not divide $n$.

Here, will consider the space $BO_{(631)}$, and the Sullivan map, $\theta^k$, for $k$ a generator of the units of ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$.

We have the following diagrams of fibrations:

$$\begin{tikzcd}
L_{\oplus} \arrow[rr, bend left] \arrow[dr] & & (BO_{\oplus})_{(631)} \arrow[rr, bend left, "\theta^k \sim \rho^k \circ (2\psi^2 - \psi^4)"] \arrow[dr, "(2\psi^2-\psi^4)"] & & (BO_{\otimes})_{(631)} \\
& N \arrow[ur] \arrow[dr] & & (BO_{\oplus})_{(631)} \arrow[ur, "\rho^k"'] & \\
& & M \arrow[ur] & &
\end{tikzcd} \tag{5.1.1}$$

and

$$
\begin{tikzcd}
M \arrow[rr] \arrow[dr] & & (BO_{\oplus})_{(631)} \arrow[rrr, "\theta^k \sim ((\psi^2)^2/\psi^4)\circ\rho^k"] \arrow[dr, "\rho^k"] & & & (BO_{\otimes})_{(631)} \\
& N \arrow[ur] \arrow[dr] & & (BO_{\otimes})_{(631)} \arrow[urr, "((\psi^2)^2/\psi^4)"'] & & \\
& & L_{\otimes} \arrow[ur] & & &
\end{tikzcd}
\tag{5.1.2}
$$

These give rise to long exact sequences in homotopy, stretches of which are entirely comprised of zeros. The nontrivial portions arise in the dimensions that are a multiple of 4, and where the characteristic sequence has an element divisible by 631. In order to see how this occurs, we look at the following diagram of long exact sequences (here, we abbreviate $L_{\oplus}$ and $L_{\otimes}$ both as $L$, since they are of the same homotopy type):

$$
\begin{tikzcd}
& \vdots \arrow[d, leftrightarrow] & & \\
\cdots \arrow[r] & \pi_{4j+1}(M) \arrow[r] \arrow[d, leftrightarrow] & \cdots & \\
\pi_{4j+1}(BO_{(631)}) \arrow[r] & \pi_{4j}(L) \arrow[r] \arrow[d, leftrightarrow] & \pi_{4j}(BO_{(631)}) \arrow[r, "\cdot(2(2^j)-4^j)"] & \pi_{4j}(BO_{(631)}) \\
\pi_{4j+1}(BO_{(631)}) \arrow[r] & \pi_{4j}(N) \arrow[r] \arrow[d, leftrightarrow] & \pi_{4j}(BO_{(631)}) \arrow[r, "\cdot(-1)^{j-1}2^{2j}(k^{2j}-1)(1-2^{2j-1})\frac{B_{2j}}{2j}"] & \pi_{4j}(BO_{(631)}) \\
\pi_{4j+1}(BO_{(631)}) \arrow[r] & \pi_{4j}(M) \arrow[r] \arrow[d, leftrightarrow] & \pi_{4j}(BO_{(631)}) \arrow[r, "\cdot(-1)^{j-1}(k^{2j}-1)\frac{B_{2j}}{4j}"] & \pi_{4j}(BO_{(631)}) \\
\cdots \arrow[r] & \pi_{4j-1}(L) \arrow[r] \arrow[d, leftrightarrow] & \cdots & \\
& \vdots & &
\end{tikzcd}
$$

Notice that the vertical sequence is exact in both directions; it is exact pointing down because of 5.1.1, and is exact pointing up because of 5.1.2.

Since $\pi_i(BO_{(631)}) = 0$ for $i \not\equiv 0 \pmod 4$, the long exact sequences reduce to a collection of short exact sequences with 12 zeros in between each. For each short exact sequence, we have:

$$0 \to \pi_{4j}(F) \to \pi_{4j}(BO_{(631)}) \xrightarrow{\cdot\lambda_j} \pi_{4j}(BO_{(631)}) \to 0$$

If the numerator of $\lambda_j$ is not divisible by 631, then the map given by multiplication by $\lambda_j$ is invertible, and so the group $\pi_{4j}(F)$ is trivial. If $\lambda_j$ is divisible by 631, then there is nontrivial homotopy in the fiber in dimension $4j$.

The fiber $L$ has nontrivial homotopy when $2^{2j-1} - 1$ is divisible by 631. This occurs first at $j = 23$, since $631 \,|\, 2^{45} - 1$ It occurs after this with period 45, so at $j = 23, 68, 113, \ldots$.

The fiber $M$ has nontrivial homotopy when the numerator of $\frac{B_{2j}}{2j}$ is divisible by 631. The prime 631 has index of irregularity 2, since it divides:

$$\frac{B_{80}}{80} = -\frac{4{,}603{,}784{,}299{,}479{,}457{,}646{,}935{,}574{,}969{,}019{,}046{,}849{,}794{,}257{,}872{,}751{,}288{,}919{,}656{,}867}{18{,}400{,}800}$$

and $\frac{B_{226}}{226}$. (The explicit fraction $\frac{B_{80}}{80}$ is given to illustrate the computational challenge associated with determining the $p-$divisibility of the Bernoulli numbers for larger values of $p$.)

Notice that, for $j = 113$, we have $p-$divisibility in both the fiber $L$ and the fiber $M$. This means that we have:

$$\mathbb{Z}/_{631\mathbb{Z}} \cong \pi_{23}(L) \cong \pi_{68}(L) \cong \pi_{113}(L) \cong \pi_{158}(L) \cong \ldots$$
$$\mathbb{Z}/_{631\mathbb{Z}} \cong \pi_{40}(M) \cong \pi_{113}(M) \cong \pi_{670}(M) \cong \ldots \qquad \text{(using the Kummer congruence)}$$

and so we can conclude that:

$$\mathbb{Z}/_{631\mathbb{Z}} \cong \pi_{23}(N) \cong \pi_{40}(N) \cong \pi_{68}(N) \cong \pi_{158}(N) \cong \dots$$

However, in dimension 113, we have the short exact sequence:

$$0 \to \mathbb{Z}/_{631\mathbb{Z}} \to \pi_{113}(N) \to \mathbb{Z}/_{631\mathbb{Z}} \to 0$$

Using a calculation of the Steenrod algebra in cohomology, Lance showed that the fiber of a self-map of $BO_{(p)}$ that is an $H-$map cannot have the homotopy type of a product (c.f. [Lan88], Corollary 2.10), and this can be used to show that the homotopy group $\pi_{113}(N)$ cannot be decomposed as a product, and so is equal to the cyclic group $\mathbb{Z}/_{(631)^2\mathbb{Z}} = \mathbb{Z}/_{398161\mathbb{Z}}$.

In cohomology (where we consider only $N$, the fiber of the Sullivan map), we have the result from 2.7.12 above:

**Theorem 5.1.3.** *If $f$ is a self-map of $BO_{(p)}$ with fiber $G$, then there is an isomorphism of Hopf Algebras:*

$$H^*(N; \mathbb{Z}/_{631\mathbb{Z}}) \cong \bigotimes_{n \text{ prime to } 631} E\left\{\sigma b_{n,j}^* \middle| 0 \le j \le \delta_n\right\} \otimes \left(B_n^*//\xi^{\delta_n} B_n^* \otimes \mathbb{Z}/_{631\mathbb{Z}}\right)$$

*where $\xi$ is the Frobenius map $x \mapsto x^{631}$.*

Recall that $\delta_n$ is the number of times that $p$ divides $\lambda_n$, $\sigma b_{n,j}^* \in H^{4np^j}(F; \mathbb{Z}/_{631\mathbb{Z}})$, $E\{\sigma b_{n,j}^*\}$ is the exterior algebra generated by the $\sigma b_{n,j}^*$, and $B_n^*$ is the polynomial algebra generated by the $b_{n,j}^*$.

For $n = 1, 2, \dots, 22$, $\delta_n = 0$, so both components are trival, and the cohomology groups are trivial.

For $n = 23$, we have $\delta_{23} = 1$, since $631 \,|\, 2^{45} - 1$, but 631 does not divide $\frac{B_{46}}{46}$.

This results in a single generator for the exterior algebra $\sigma b^*_{23,0} \in H^{91}(N; {}^{\mathbb{Z}}/_{631\mathbb{Z}})$ (here $91 = 4(23)(631^0) - 1$). By antisymmetry of the wedge product, this does not generate additional cohomology in higher dimensions by itself. In the polynomial component, we have $b^*_{23,0} = p_{e_{23}} \in H^{92}(N; {}^{\mathbb{Z}}/_{631\mathbb{Z}})$. To check for additional generators, we consider the generating function for the $b^*_{n,j}$; this comes from the Witt polynomial (see pg. 50) which, in this case, is: $p_{e_{23(631)}} = (b^*_{23,0})^{631} + 631(b^*_{23,1})$. In the fiber, we have a truncated polynomial algebra at exponent 631; also 631 as a coefficient is trivial because of the coefficient group. So we have only $b^*_{23,0}$ as a nontrivial polynomial generator. This results in nontrivial elements $(b^*_{23,0})^2, (b^*_{23,0})^2, \ldots, (b^*_{23,0})^{630}$, as well as $\sigma b^*_{23,0} \otimes b^*_{23,0} \in H^{183}(N; {}^{\mathbb{Z}}/_{631\mathbb{Z}})$, as well as the tensors involving the powers in the second coordinate. If we consider only the dimensions from 1 to 631, $n = 23$ contributes nontrivial generators in dimensions 91, 92, 183, 184, 275, 276, 367, 368, 459, 460, 551, and 552.

For $n = 40$, we also have $\delta_{40} = 1$, this time since $631 \left| \frac{B_{80}}{80} \right.$, but 631 does not divide $2^{79} - 1$. This will contribute nontrivial generators in 271, 272, 543, 544.

In addition, combining products of elements from the two previous cases contributes nontrivial generators in dimensions 362, 363 (two different ones), 364, 454, 455 (two also), 456, 546, 547 (two also), and 548.

For $n = 68$ we have nontrivial generators in dimensions 271, 272, 543, 544. These combine with all of the previous generators as well.

At $n = 113$ we have $\delta_{113} = 2$, since $631 \,|2^{225} - 1$ and $631 \left| \frac{B_{226}}{226} \right.$ as well. This will give the exterior algebra two generators $\sigma b^*_{113,0}$ and $\sigma b^*_{113,1}$ which reside in dimensions 451 and $71,302$, respectively. Their wedge product is also nontrivial, and yields a generator in dimension $71,753$.

## 5.2 Computational Results

The previous section makes it clear that the $p-$divisiblity of the Bernoulli numbers and the $p-$divisibility of the sequences $2^{2n-1}-1$ are essential in the complete understanding of the fiber of the Sullivan map; the $p-$divisibility of the Bernoulli numbers is also the determining factor in the structure of the fiber of the Adams-Bott cannabalistic class.

The specific $p-$divisibility questions which arise in our study of the topology of these classifying spaces are these four:

**Examples**

(1). Is there $p$, $n$ so that $p^2$ divides $\frac{B_{2n}}{2n}$, with $3 \leq n \leq p-3$?

(2). Is there $p$, $n$ so that $p^2$ divides $\frac{B_{2pn}}{2pn}$, with $3 \leq n \leq p-3$?

(3). Is there $p$, $n$ so that $p^2$ divides $\frac{B_{2n+p-1}}{2n+p-1} - \frac{B_{2n}}{2n}$, with $3 \leq n \leq p-3$?

(4). Is there $p$, $n$ so that $p$ divides both $\frac{B_{2n}}{2n}$ and $2^{2n-1}-1$, at least once, with $3 \leq n \leq p-3$?

From the previous section, it is obvious that $p = 631$ and $n = 113$ is an example of the last of these four questions. We undertook a growing computational search to find any examples of the first three, and any more examples of the last. Lance carried out computations on these questions in the 1980's, testing all primes up to 8000 using an Amiga desktop computer. The example at $p = 631$, $n = 113$ was the only example of all four phenomena that he found at that time.

Our initial computations in 2007 were carried out via a program written by Bernd Kellner, a graduate student at the University of Göttingen studying the $p-$divisibility of Bernoulli numbers, who hosts the site www.bernoulli.org.

Later computations were carried out via a C++ program written by the author, using different methodology, primarily on the University at Albany Research Information

Technology servers. The author wishes to acknowledge extensive support provided by Anne Shelton, Eric Warnke, and Brian Mascherone of the Research IT Department in porting this program to the system and generating computational results.

The program utilized four congruences; three were proved by H.S. Vandiver in the 1920's, who used them to look for counterexamples to Fermat's Last Theorem (c.f. [Rib80]). The last was proved by E. Lehmer in the late 1930's for the same reason. The first of the four congruences used is:

**Theorem 5.2.1.** *If $p$ is prime, $p \geq 7$, $k \geq 1$, and $p-1$ does not divide $2k$, then:*

$$\frac{4^{p-2k} + 3^{p-2k} - 6^{p-2k} - 1}{4k} B_{2k} \equiv \sum_{\frac{p}{6} < j < \frac{p}{4}} j^{2k-1} \pmod{p}$$

The fraction on the left is calculated first; if this is nonzero mod $p$, then the sum on the right is computed second. If the fraction is nonzero mod $p$ and the sum is zero, then the Bernoulli number must be zero mod $p$. However, if the fraction is zero mod $p$, then the test is inconclusive and another test must be attempted. In the event of an inconclusive result, the next three congruences were used; no pair $(p, n)$ was found that was inconclusive after all four of these tests. The three remaining congruences are:

**Theorem 5.2.2.** *If $p$ is prime, $p \geq 5$, $k \geq 1$, and $p-1$ does not divide $2k$, then:*

$$\frac{2^{p-2k} + 3^{p-2k} - 4^{p-2k} - 1}{4k} B_{2k} \equiv \sum_{\frac{p}{4} < j < \frac{p}{3}} j^{2k-1} \pmod{p}$$

**Theorem 5.2.3.** *If $p$ is prime, $p \geq 11$, $k \geq 1$, and $p-1$ does not divide $2k$, then:*

$$\frac{4^{p-2k} + 5^{p-2k} - 8^{p-2k} - 1}{4k} B_{2k} \equiv \sum_{\frac{p}{8} < j < \frac{p}{5}} j^{2k-1} + \sum_{\frac{3p}{8} < j < \frac{2p}{5}} j^{2k-1} \pmod{p}$$

**Theorem 5.2.4.** *If $p$ is prime, and $2k \not\equiv 2 \pmod{p} - 1$, then:*

$$2^{2k-1} p B_{2k} \equiv \sum_{j=1}^{(p-1)/2} (p-2j)^{2k} \pmod{p^3}$$

The first two are virtually identical in runtime; the third is a bit longer, since the sum involves more indices, and the last is much longer because of the need to compute at a larger modulus, and therefore at greater integer precision.

The program was run on a varying number of processors (each approximately on the order of 100,000 times as fast as a 1980's Amiga desktop), typically between 10 and 60 processors, continuously for about six months. We checked all of the above questions for $n = 3, 5, \ldots, p-3$ for all primes less than 1.84 million.

No additional examples of any of the four questions were found.

## 5.3 Directions for Future Computational Research

Obviously, the first direction for future research is to continue the existing calculation program, and extend the upper bound to higher reaches. We expect, at some point, to generate additional examples where the $p-$divisibility of $2^{2n-1} - 1$ coincides with the $p-$divisibility of $\frac{B_{2n}}{2n}$. If not, this might lead to a search for reasons why no more examples exist.

We expect to be putting this problem out onto NYSGrid shortly; this is a statewide consortium of research computing resources, primarily hosted at university centers across New York State. This will substantially extend the computing power available for further information on these questions.

We have also been extended an offer from NYStar (New York State Foundation for Science, Technology and Innovation) of computing time on Rensselaer Polytechnic

Institute's Computational Center for Nanotechnology Innovations supercomputer, at 10 hours/month. We expect this to even more substantially extend the results generated on the University at Albany's system using this much more powerful system. The CCNI supercomputer runs 32,768 IBM PowerPC processors, generating up to 91.75 teraflops of computing power (according to Top500.org, which tracks the 500 fastest supercomputers in the world).

It would be interesting also to see if additional congruences can be generated from the topology of classifying spaces, along the lines of 2.7.14. In particular, it is our hope that we will be able to use the additional structure afforded by the shift to the $p-$adic integers to generate congruences of generalized Bernoulli numbers. The Teichmüller character is a natural choice to pursue, since it has a conductor of exactly $p-1$, and reflects essential structure of the $p-$adic integers, as seen in subsection 2.4.c, and in the hoped-for computation of the Iwasawa invariant $\lambda_n$, seen in Theorem 2.5.13.

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