1. 齐次方程组

$$\begin{cases} x_1-3x_2-5x_3=0 \\ 2x_1-7x_2-4x_3=0 \\ 4x_1-9x_2+ax_3=0 \\ 5x_1+bx_2-55x_3=0 \end{cases}$$

系数矩阵 $\begin{pmatrix} 1 & -3 & -5 \\ 2 & -7 & -4 \\ 4 & -9 & a \\ 5 & b & -55 \end{pmatrix}$

$$\begin{pmatrix} 1 & -3 & -5 \\ 2 & -7 & -4 \\ 4 & -9 & a \\ 5 & b & -55 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & -5 \\ 0 & -1 & 6 \\ 0 & 3 & a+20 \\ 0 & b+15 & -30 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & -5 \\ 0 & -1 & 6 \\ 0 & 0 & a+38 \\ 0 & 0 & 6b+60 \end{pmatrix}$$

只有零解相当于是有唯一解，也就是说方程个数（非零行数目）和未知数个数（3个）一样多

从而 $a+38\neq0$ 或 $6b+60\neq0$. 即 $a\neq-38$ 或 $b\neq-10$

有非零解相当于是有无穷多组解，也就是说非零行数目小于未知数个数

从而 $a+38=0$ 且 $6b+60=0$. 即 $a=38$ 且 $b=-10$.

2. $|A|$ 是 $n!$ 项的代数和，每一项都是 ±1. $n\geqslant2$ 时 $n!$ 是偶数，所以 $|A|$ 是偶数.

$n=3$ 的时候，$|A|$ 是6项代数和，则 $|A|\leqslant6$. 假设 $|A|=6$.

记 $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. 则 $|A|=\begin{matrix} a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31} \end{matrix}=6$

$\Rightarrow \begin{cases} a_{11}a_{22}a_{33}=a_{12}a_{23}a_{31}=a_{13}a_{21}a_{32}=1 \Rightarrow A \text{ 的9个元素乘积为 } 1\times1\times1=1 \\ a_{11}a_{23}a_{32}=a_{12}a_{21}a_{33}=a_{13}a_{22}a_{31}=-1 \Rightarrow A \text{ 的9个元素乘积为 } (-1)\times(-1)\times(-1)=-1 \end{cases}$

矛盾！故 $|A|\leqslant4$. 而 $\det\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}=4$，故 $n=3$ 时行列式最大值为4.

$n\geqslant3$ 时用归纳法. 假设已知道 n 时行列式不超过 $(n-1)!(n-1)$，则 $(n+1)$ 时：按第一行展开

$$\begin{aligned} |\det A| &= |a_{11}A_{11}+a_{12}A_{12}+\cdots+a_{1,n+1}A_{1,n+1}| \\ &\leqslant |a_{11}|\cdot|A_{11}|+|a_{12}|\cdot|A_{12}|+\cdots+|a_{1,n+1}|\cdot|A_{1,n+1}| \\ &\leqslant (n-1)!\cdot(n-1)+(n-1)!\cdot(n-1)+\cdots+(n-1)!\cdot(n-1) \\ &= (n-1)!\cdot(n-1)\cdot(n+1)=(n-1)!(n^2-1)=n!\cdot n-(n-1)!<n!\cdot n. \end{aligned}$$

即证.

3. (1) $\begin{vmatrix} a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & & \vdots \\ a_n+b_1 & a_n+b_2 & \cdots & a_n+b_n \end{vmatrix} \xlongequal{\text{每一列减去第一列}} \begin{vmatrix} a_1+b_1 & b_2-b_1 & \cdots & b_n-b_1 \\ a_2+b_1 & b_2-b_1 & \cdots & b_n-b_1 \\ \vdots & \vdots & & \vdots \\ a_n+b_1 & b_2-b_1 & \cdots & b_n-b_1 \end{vmatrix}$

由此可知，当$n\geq 3$时，行列式中一定存在两列成比例，则行列式的值为0.

当$n=1$时，所求为a_1+b_1. $n=2$时，所求为$(a_1+b_1)(a_2+b_2)-(a_1+b_2)(a_2+b_1)=(a_1-a_2)(b_2-b_1)$

(2) $n=1$时所求为$2a_{11}$. $n=2$时所求为$\begin{vmatrix} a_{12}+a_{11} & a_{11}+a_{12} \\ a_{22}+a_{21} & a_{21}+a_{22} \end{vmatrix}$. 两列成比例，值为0.

$n=3$时所求为

$$\begin{vmatrix} a_{13}+a_{11} & a_{11}+a_{12} & a_{12}+a_{13} \\ a_{23}+a_{21} & a_{21}+a_{22} & a_{22}+a_{23} \\ a_{33}+a_{31} & a_{31}+a_{32} & a_{32}+a_{33} \end{vmatrix} \xlongequal{\text{第二列加到第一列}} \begin{vmatrix} 2a_{11}+a_{12}+a_{13} & a_{11}+a_{12} & a_{12}+a_{13} \\ 2a_{21}+a_{22}+a_{23} & a_{21}+a_{22} & a_{22}+a_{23} \\ 2a_{31}+a_{32}+a_{33} & a_{31}+a_{32} & a_{32}+a_{33} \end{vmatrix}$$

$$\xlongequal{\text{第一列减第三列}} \begin{vmatrix} 2a_{11} & a_{11}+a_{12} & a_{12}+a_{13} \\ 2a_{21} & a_{21}+a_{22} & a_{22}+a_{23} \\ 2a_{31} & a_{31}+a_{32} & a_{32}+a_{33} \end{vmatrix} \xlongequal{\text{第一列}\times(-\frac{1}{2})\text{加到第二列}} \begin{vmatrix} 2a_{11} & a_{12} & a_{12}+a_{13} \\ 2a_{21} & a_{22} & a_{22}+a_{23} \\ 2a_{31} & a_{32} & a_{32}+a_{33} \end{vmatrix}$$

$$\xlongequal{\text{第三列减第二列}} \begin{vmatrix} 2a_{11} & a_{12} & a_{13} \\ 2a_{21} & a_{22} & a_{23} \\ 2a_{31} & a_{32} & a_{33} \end{vmatrix} = 2|A|. \quad A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$n=4$时所求为

$$\begin{vmatrix} a_{14}+a_{11} & a_{11}+a_{12} & a_{12}+a_{13} & a_{13}+a_{14} \\ a_{24}+a_{21} & a_{21}+a_{22} & a_{22}+a_{23} & a_{23}+a_{24} \\ a_{34}+a_{31} & a_{31}+a_{32} & a_{32}+a_{33} & a_{33}+a_{34} \\ a_{44}+a_{41} & a_{41}+a_{42} & a_{42}+a_{43} & a_{43}+a_{44} \end{vmatrix}$$

第一列$+(-1)\times$第二列$+$第三列$+(-1)\times$第四列$=0$. 所以行列式值为0.

从n比较小的例子我们能看出n的奇偶性比较重要.我们首先假设n是偶数.

$$\begin{vmatrix} a_{1n}+a_{11} & a_{11}+a_{12} & \cdots & a_{1,n-1}+a_{1n} \\ a_{2n}+a_{21} & a_{21}+a_{22} & \cdots & a_{2,n-1}+a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{nn}+a_{n1} & a_{n1}+a_{n2} & \cdots & a_{n,n-1}+a_{nn} \end{vmatrix} \xlongequal[\text{然后加到第1列}]{\text{将第k列乘以}(-1)^{k-1}} \begin{vmatrix} a_{1n}+a_{11}+\sum_{k=2}^{n}(-1)^{k-1}(a_{1,k-1}+a_{1k}) & \cdots \\ a_{2n}+a_{21}+\sum_{k=2}^{n}(-1)^{k-1}(a_{2,k-1}+a_{2k}) & \cdots \\ \vdots & \\ a_{nn}+a_{n1}+\sum_{k=2}^{n}(-1)^{k-1}(a_{n,k-1}+a_{nk}) & \cdots \end{vmatrix}$$

此时第j行第1列的元素为

$a_{jn}+a_{j1}+\sum_{k=2}^{n}(-1)^{k-1}(a_{j,k-1}+a_{jk})$. a_{jk}系数为$(-1)^{k-1}+(-1)^{k}=0$ $(2\le k\le n-1)$

a_{j1}系数为$1+(-1)=0$. a_{jn}系数为$1+(-1)^{n-1}=0$. 故第1列全是0，所求为0.

n是奇数时，我们也可以做同样的事情. 此时第j行第1列会成为$2a_{jn}$.

$$\text{所求}=2\begin{vmatrix} a_{1n} & a_{11}+a_{12} & \cdots & a_{1,n-1}+a_{1n} \\ a_{2n} & a_{21}+a_{22} & \cdots & a_{2,n-1}+a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{nn} & a_{n1}+a_{n2} & \cdots & a_{n,n-1}+a_{nn} \end{vmatrix} \xlongequal[\text{减第1列}]{\text{第n列}} 2\begin{vmatrix} a_{1n} & a_{11}+a_{12} & \cdots & a_{1,n-1} \\ a_{2n} & a_{21}+a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{nn} & a_{n1}+a_{n2} & \cdots & a_{n,n-1} \end{vmatrix}$$

$$\xlongequal[\text{如此进行下去，直到第2列减第3列}]{\text{第(n-1)列减第n列，然后第(n-2)列减第(n-1)列}} 2\begin{vmatrix} a_{1n} & a_{11} & \cdots & a_{1,n-1} \\ a_{2n} & a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{nn} & a_{n1} & \cdots & a_{n,n-1} \end{vmatrix}$$

互换两列$(n-1)$次，行列式不变（用到n是奇数）

$=2|A|$. 其中$A=(a_{ij})_{1\le i,j\le n}$.

$$\text{故所求为}\begin{cases} 2|A|. \quad A=(a_{ij})_{1\le i,j\le n} & n\text{为奇数} \\ 0 & n\text{为偶数} \end{cases}$$

(3) 从最后一列开始，每一列减去前一列，可知所求为

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 2 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 3 & 1 & 1 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ n & 0 & 0 & & 0 & 0 & 0 \end{vmatrix}$$

注意$(n,n-1,n-2,\cdots,1)$这个排列的逆序对数为$\frac{n(n-1)}{2}$

如图所示的反对角线是唯一在行列式完全展开中非零的项. 故所求为$(-1)^{\frac{n(n-1)}{2}}n$.

4. $$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_n & y \\ x_1^2 & x_2^2 & \cdots & x_n^2 & y^2 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^{n} & x_2^{n} & \cdots & x_n^{n} & y^{n} \end{vmatrix} = \prod_{k=1}^{n}(y-x_k)\cdot\prod_{1\le j<i\le n}(x_i-x_j)$$

所求的行列式是y^{n-1}的系数

故所求 $= -(x_1+\cdots+x_n)\cdot\prod_{1\le j<i\le n}(x_i-x_j)$

5. $$\begin{vmatrix} 1+x_1 & 1+x_2 & \cdots & 1+x_n \\ 1+x_1^2 & 1+x_2^2 & \cdots & 1+x_n^2 \\ \vdots & \vdots & & \vdots \\ 1+x_1^n & 1+x_2^n & \cdots & 1+x_n^n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1+x_1 & 1+x_2 & \cdots & 1+x_n \\ 0 & 1+x_1^2 & 1+x_2^2 & \cdots & 1+x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1+x_1^n & 1+x_2^n & \cdots & 1+x_n^n \end{vmatrix}$$

$$\underline{\underline{\text{每行减去第一行}}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & x_1 & x_2 & \cdots & x_n \\ -1 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix} = \begin{vmatrix} 2 & 1 & \cdots & 1 \\ 0 & x_1 & \cdots & x_n \\ 0 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ 0 & x_1^n & \cdots & x_n^n \end{vmatrix} - \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & x_1 & \cdots & x_n \\ 1 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ 1 & x_1^n & \cdots & x_n^n \end{vmatrix}$$

第一项按第一列展开，再每列提取公因子 x_i，就成为范德蒙德行列式

第一项 $= 2x_1\cdots x_n\prod_{1\le j<i\le n}(x_i-x_j)$

第二项本身就是一个范德蒙德行列式，值为 $(x_1-1)\cdots(x_n-1)\prod_{1\le j<i\le n}(x_i-x_j)$

故所求 $=\left(2x_1\cdots x_n-(x_1-1)\cdots(x_n-1)\right)\cdot\prod_{1\le j<i\le n}(x_i-x_j)$

6. 方法一：把很多的项都变成0，方法是每一列减去第一列

$$\begin{vmatrix} x_1-a_1 & x_2 & \cdots & x_n \\ x_1 & x_2-a_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n-a_n \end{vmatrix} = \begin{vmatrix} x_1-a_1 & x_2 & \cdots & x_n \\ a_1 & -a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_1 & 0 & \cdots & -a_n \end{vmatrix}$$

$$= a_1a_2\cdots a_n \begin{vmatrix} \frac{x_1}{a_1}-1 & \frac{x_2}{a_2} & \cdots & \frac{x_n}{a_n} \\ 1 & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & -1 \end{vmatrix} = a_1a_2\cdots a_n \begin{vmatrix} \left(\sum_{i=1}^n \frac{x_i}{a_i}\right)-1 & \frac{x_2}{a_2} & \cdots & \frac{x_n}{a_n} \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 \end{vmatrix}$$

$$= a_1a_2\cdots a_n\cdot\left[\left(\sum_{i=1}^n \frac{x_i}{a_i}\right)-1\right]\cdot(-1)^{n-1}$$

按第一列展开

方法二：加边法，其实本质上和方法一没有区别，只是写起来好看。

$$\begin{vmatrix} x_1-a_1 & x_2 & \cdots & x_n \\ x_1 & x_2-a_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n-a_n \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 0 & x_1-a_1 & x_2 & \cdots & x_n \\ 0 & x_1 & x_2-a_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_1 & x_2 & \cdots & x_n-a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ -1 & -a_1 & 0 & \cdots & 0 \\ -1 & 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & -a_n \end{vmatrix} = \begin{vmatrix} 1-\sum_{i=1}^n \frac{x_i}{a_i} & x_1 & x_2 & \cdots & x_n \\ 0 & -a_1 & 0 & \cdots & 0 \\ 0 & 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -a_n \end{vmatrix}$$

$$= \left(1-\sum_{i=1}^n \frac{x_i}{a_i}\right)(-1)^n a_1\cdots a_n.$$

7. 从最后一行开始，每一行减去上一行. 然后从最后一列开始，每一列减去上一列

$$\begin{vmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-3 & n-4 & \cdots & 0 & 1 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & -1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -2 & 0 \\ 1 & 0 & 0 & \cdots & 0 & -2 \end{vmatrix} \triangleq D_n.$$ 再按最后一列展开:

$$D_n = (-1)^{n+1} \begin{vmatrix} 1 & -2 & 0 & \cdots & 0 \\ 1 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{vmatrix} + (-2)\cdot D_{n-1} = (-1)^{n+1}\cdot(-1)^{n-1+1}(-2)^{n-2} + (-2)\cdot D_{n-1}$$

$$\Rightarrow D_n = (-2)^{n-2} + (-2)D_{n-1} \Rightarrow \frac{D_n}{(-2)^n} = -\frac{1}{4} + \frac{D_{n-1}}{(-2)^{n-1}} \Rightarrow \frac{D_n}{(-2)^n} = -\frac{1}{4}(n-2) + \frac{D_2}{(-2)^2} = -\frac{1}{4}n + \frac{1}{2} - \frac{1}{4}$$

$$\Rightarrow D_n = (-2)^n\cdot\left(-\frac{1}{4}\right)(n-1) = (-1)^{n+1}(n-1)2^{n-2}.$$

8. 从最后一行开始，每一行减去上一行

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & -(n-1) \\ 1 & 1 & 1 & \cdots & -(n-1) & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -(n-1) & 1 & \cdots & 1 & 1 \end{vmatrix}$$

每行和是0

把所有列加到最后一列

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & (n-1) & \frac{n(n+1)}{2} \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & -(n-1) & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -(n-1) & 1 & \cdots & 1 & 0 \end{vmatrix} = (-1)^{n+1}\frac{n(n+1)}{2}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1-n \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & \cdots & 1 & 1 \end{vmatrix}$$

然后从最后一行开始，每一行减去上一行.

$$所求 = (-1)^{n+1}\cdot\frac{n(n+1)}{2}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & -n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -n & \cdots & 0 & 0 \end{vmatrix} \overset{按第1列展开}{=} (-1)^{n+1}\cdot\frac{n(n+1)}{2}\cdot 1\cdot(-n)^{n-2}\cdot(-1)^{\frac{(n-2)(n-3)}{2}}$$

$$= (-1)^{n+1+\frac{1}{2}(n-2)(n-1)}\cdot\frac{n+1}{2}\cdot n^{n-1} = (-1)^{\frac{1}{2}n(n-1)}\cdot\frac{n+1}{2}\cdot n^{n-1}$$

9. 每一行减去第一行，然后按第一列展开

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & -1 & \cdots & -1 & -1 \\ 0 & -1 & 2 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & -1 & \cdots & -1 & n-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & \cdots & -1 & -1 \\ -1 & 2 & \cdots & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & n-1 \end{vmatrix} \overset{每一列减第一列}{=} \begin{vmatrix} 1 & -2 & \cdots & -2 & -2 \\ -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & n \end{vmatrix}$$

$$=\begin{vmatrix} \left(1-\frac{2}{3}-\frac{2}{4}-\cdots-\frac{2}{n}\right) & -2 & \cdots & -2 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n \end{vmatrix} = \frac{1}{2}n!\cdot\left(\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{n}\right)\times 2 = n!\cdot\left(1-\sum_{k=2}^{n}\frac{1}{k}\right)$$

$n=2$ 时 $\begin{vmatrix} x & a_0 \\ -1 & x+a_1 \end{vmatrix} = x^2+a_1x+a_0$.

10. 用归纳法证明所求为 $x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_0$. 假设 $(n-1)$ 时已成立. 按最后一行展开.

$$所求=(x+a_{n-1})\begin{vmatrix} x & 0 & 0 & \cdots & 0 \\ -1 & x & 0 & \cdots & 0 \\ 0 & -1 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x \end{vmatrix} + (-1)^{n-1+n}\cdot(-1)\begin{vmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ 0 & -1 & x & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-2} \end{vmatrix}$$

$$= x^{n-1}(x+a_{n-1}) + \left(x^{n-2}+(a_{n-2}-x)x^{n-2}+a_{n-3}x^{n-3}+\cdots+a_1x+a_0\right) = x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

11. 按最后一行展开

$$\begin{vmatrix} a_0+a_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_1+a_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & a_2+a_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-1}+a_n \end{vmatrix}$$ 记为 D_n

$$= (-1)^{2n-1}\cdot a_{n-1}\begin{vmatrix} a_0+a_1 & a_1 & \cdots & 0 \\ a_1 & a_1+a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-1} \end{vmatrix} + (a_{n-1}+a_n)\begin{vmatrix} a_0+a_1 & a_1 & \cdots & 0 \\ a_1 & a_1+a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-2}+a_{n-1} \end{vmatrix}$$

$= -a_{n-1}^2 D_{n-2} + (a_{n-1}+a_n)D_{n-1}$

于是 $D_n - a_nD_{n-1} = a_{n-1}(D_{n-1}-a_{n-1}D_{n-2}) = \cdots = a_{n-1}a_{n-2}\cdots a_3(D_3-a_3D_2)$

$D_3 = (a_0+a_1)(a_1+a_2)(a_2+a_3) - a_2^2(a_0+a_1) - a_1^2(a_2+a_3) = a_0a_1a_2+a_1a_2a_3+a_0a_1a_3+a_0a_2a_3$

$D_2 = (a_0+a_1)(a_1+a_2) - a_1^2 = a_1a_2+a_0a_2+a_0a_1 \Rightarrow D_3-a_3D_2 = a_0a_1a_2$

$\Rightarrow D_n - a_nD_{n-1} = a_{n-1}a_{n-2}\cdots a_0$. 从而

$$D_n = a_{n-1}a_{n-2}\cdots a_0 + a_n D_{n-1} = a_{n-1}a_{n-2}\cdots a_0 + a_n\left(a_{n-2}a_{n-3}\cdots a_0 + a_{n-1}D_{n-2}\right)$$
$$= \cdots = \sum_{k=0}^{n}\left(\prod_{j\neq k} a_j\right).$$

12. 先计算D_2.

$$D_2 = \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} \end{vmatrix} = \frac{(a_1+b_2)(a_2+b_1)-(a_1+b_1)(a_2+b_2)}{(a_1+b_1)(a_2+b_2)(a_1+b_2)(a_2+b_1)} = \frac{(a_1-a_2)(b_1-b_2)}{(a_1+b_1)(a_2+b_2)(a_1+b_2)(a_2+b_1)}$$

对D_n而言，每一列减去最后一列，提公因子后每一列再减去最后一列

$$D_n = \begin{vmatrix} \frac{1}{a_1+b_1}-\frac{1}{a_n+b_1} & \cdots & \frac{1}{a_1+b_n}-\frac{1}{a_n+b_n} \\ \frac{1}{a_2+b_1}-\frac{1}{a_n+b_1} & \cdots & \frac{1}{a_2+b_n}-\frac{1}{a_n+b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix} = \begin{vmatrix} \frac{a_n-a_1}{(a_1+b_1)(a_n+b_1)} & \cdots & \frac{a_n-a_1}{(a_1+b_n)(a_n+b_n)} \\ \frac{a_n-a_2}{(a_2+b_1)(a_n+b_1)} & \cdots & \frac{a_n-a_2}{(a_2+b_n)(a_n+b_n)} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix}$$

$$= \frac{\prod_{j=1}^{n-1}(a_n-a_j)}{\prod_{j=1}^{n}(a_n+b_j)} \begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \frac{1}{a_2+b_1} & \cdots & \frac{1}{a_2+b_n} \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{vmatrix} = \frac{\prod_{j=1}^{n-1}(a_n-a_j)}{\prod_{j=1}^{n}(a_n+b_j)} \begin{vmatrix} \frac{b_n-b_1}{(a_1+b_1)(a_1+b_n)} & \cdots & \frac{1}{a_1+b_n} \\ \frac{b_n-b_1}{(a_2+b_1)(a_2+b_n)} & \cdots & \frac{1}{a_2+b_n} \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{vmatrix}$$

$$= \frac{\prod_{j=1}^{n-1}(a_n-a_j)}{\prod_{j=1}^{n}(a_n+b_j)} \cdot \frac{\prod_{j=1}^{n-1}(b_n-b_j)}{\prod_{j=1}^{n}(a_j+b_n)} \begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \frac{1}{a_2+b_1} & \cdots & \frac{1}{a_2+b_n} \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{vmatrix}$$

$$= \frac{\prod_{j=1}^{n-1}(a_n-a_j)}{\prod_{j=1}^{n}(a_n+b_j)} \cdot \frac{\prod_{j=1}^{n-1}(b_n-b_j)}{\prod_{j=1}^{n}(a_j+b_n)} D_{n-1} \overset{\text{归纳}}{\Rightarrow} D_n = \frac{\prod_{1\le j<i\le n}(a_i-a_j)(b_i-b_j)}{\prod_{1\le i,j\le n}(a_i+b_j)}$$

13. 根据行列式的性质，这个行列式可以写成$n!$个行列式的和，唯一一个非零项是

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\le j<i\le n}(x_i-x_j).$$

14. 提取第一列的公因子 a_1 和最后一行的公因子 b_n.

$$\text{所求}=a_1b_n\begin{vmatrix} b_1 & a_1b_2 & a_1b_3 & \cdots & a_1b_n \\ b_2 & a_2b_2 & a_2b_3 & \cdots & a_2b_n \\ b_3 & a_2b_3 & a_3b_3 & \cdots & a_3b_n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_2 & a_3 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} 0 & a_1b_2-a_2b_1 & a_1b_3-a_3b_1 & \cdots & a_1b_n-a_nb_1 \\ 0 & 0 & a_2b_3-a_3b_2 & \cdots & a_2b_n-a_nb_2 \\ 0 & 0 & 0 & \cdots & a_3b_n-a_nb_3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_2 & a_3 & \cdots & a_n \end{vmatrix}$$

$$=(-1)^{n+1}\begin{vmatrix} a_1b_2-a_2b_1 & & * \\ & a_2b_3-a_3b_2 & \\ 0 & & \ddots \\ & & & a_{n-1}b_n-a_nb_{n-1} \end{vmatrix} = (-1)^{n+1}\prod_{i=1}^{n-1}(a_ib_{i+1}-a_{i+1}b_i)$$

15. 记所求为 D_n. 按第一行和最后一行展开

$$D_n=\begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & b & a & & \\ & \iddots & & & \ddots & \\ b & & & & & a \end{vmatrix} = \begin{vmatrix} a & b \\ b & a \end{vmatrix}\cdot(-1)^{1+2n+1+2n}\cdot D_{n-1} = (a^2-b^2)D_{n-1}$$

故 $D_n=(a^2-b^2)^{n-1}\cdot D_1=(a^2-b^2)^{n-1}\cdot\begin{vmatrix} a & b \\ b & a\end{vmatrix}=(a^2-b^2)^n$.

16. 设矩阵为 $A=(a_{ij})_{1\le i,j\le n}$. 它的第 k 行全是 1. 记 a_{ij} 的代数余子式是 A_{ij}, 则

$$\sum_{1\le i,j\le n}A_{ij}=\sum_{j=1}^{n}A_{kj}+\sum_{i\ne k}\sum_{j=1}^{n}A_{ij}=\sum_{j=1}^{n}a_{kj}A_{kj}+\sum_{i\ne k}\sum_{j=1}^{n}a_{kj}A_{ij}=|A|+\sum_{i\ne k}0=|A|$$

17. 记 $A=(a_{ij})_{1\le i,j\le n}$, $A(t)=(a_{ij}+t)_{1\le i,j\le n}$. 记 A 和 $A(t)$ 中 (i,j)-元代数余子式是 A_{ij} 和 $A_{ij}(t)$. 则

$$|A(t)|=|A|+\begin{vmatrix} t & a_{12} & \cdots & a_{1n} \\ t & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ t & a_{n2} & \cdots & a_{nn}\end{vmatrix}+\begin{vmatrix} a_{11} & t & \cdots & a_{1n} \\ a_{21} & t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & t & \cdots & a_{nn}\end{vmatrix}+\cdots+\begin{vmatrix} a_{11} & a_{12} & \cdots & t \\ a_{21} & a_{22} & \cdots & t \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & t\end{vmatrix}$$

$$=|A|+t\sum_{i=1}^{n}A_{i1}+t\sum_{i=1}^{n}A_{i2}+\cdots+t\sum_{i=1}^{n}A_{in}$$

$$=|A|+t\sum_{1\le i,j\le n}A_{ij}.\ \text{故}$$

$$|A(t)|-|A|=\begin{vmatrix} a_{11}+t & a_{12}+t & \cdots & a_{1n}+t \\ a_{21}+t & a_{22}+t & \cdots & a_{2n}+t \\ \vdots & \vdots & & \vdots \\ a_{n1}+t & a_{n2}+t & \cdots & a_{nn}+t \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}+t & a_{12}+t & \cdots & a_{1n}+t \\ a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2n}-a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}-a_{11} & a_{n2}-a_{12} & \cdots & a_{nn}-a_{1n} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2n}-a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}-a_{11} & a_{n2}-a_{12} & \cdots & a_{nn}-a_{1n} \end{vmatrix}$$

$$= t\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2n}-a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}-a_{11} & a_{n2}-a_{12} & \cdots & a_{nn}-a_{1n} \end{vmatrix} = t\sum_{1\le i,j\le n} A_{ij}$$

故我们得到一个恒等式：

$$\sum_{1\le i,j\le n} A_{ij} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2n}-a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}-a_{11} & a_{n2}-a_{12} & \cdots & a_{nn}-a_{1n} \end{vmatrix}$$

于是对 $A(t)$ 使用这个恒等式：

$$\sum_{1\le i,j\le n} A_{ij}(t) = \begin{vmatrix} 1 & \cdots & 1 \\ (a_{21}+t)-(a_{11}+t) & \cdots & (a_{2n}+t)-(a_{1n}+t) \\ \vdots & & \vdots \\ (a_{n1}+t)-(a_{11}+t) & \cdots & (a_{nn}+t)-(a_{1n}+t) \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 1 \\ a_{21}-a_{11} & \cdots & a_{2n}-a_{1n} \\ \vdots & & \vdots \\ a_{n1}-a_{11} & \cdots & a_{nn}-a_{1n} \end{vmatrix}$$

$$= \sum_{1\le i,j\le n} A_{ij}.$$ 得证.

18. 设多项式 $f(x)$ 的表达式为 $f(x)=c_{n-1}x^{n-1}+c_{n-2}x^{n-2}+\cdots+c_1x+c_0$

$f(a_i)=b_i$ 意味着 $a_i^{n-1}c_{n-1}+a_i^{n-2}c_{n-2}+\cdots+a_ic_1+c_0=b_i$

考虑以下 n 个方程，n 个变元的线性方程组（变元是 $c_{n-1}, c_{n-2}, \cdots, c_1, c_0$）

$$\begin{cases} a_1^{n-1}c_{n-1}+a_1^{n-2}c_{n-2}+\cdots+a_1c_1+c_0=b_1 \\ a_2^{n-1}c_{n-1}+a_2^{n-2}c_{n-2}+\cdots+a_2c_1+c_0=b_2 \\ \vdots \\ a_n^{n-1}c_{n-1}+a_n^{n-2}c_{n-2}+\cdots+a_nc_1+c_0=b_n \end{cases}$$

它的系数矩阵的行列式为

$$\begin{vmatrix} a_1^{n-1} & a_1^{n-2} & \cdots & 1 \\ a_2^{n-1} & a_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ a_n^{n-1} & a_n^{n-2} & \cdots & 1 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} \prod_{1\le j<i\le n} (a_i-a_j) \underset{\text{用到} a_1,\cdots,a_n \text{互不相同}}{\uparrow}{\neq} 0$$

由 Cramer 法则，该线性方程组存在唯一解，即满足条件的多项式存在唯一.

19. 不要被方程组吓到. 当 t 变化时，未知数的种类是不变的. 也就是说可以考虑 n^2 个方程组：

$$\begin{cases} \zeta_{j1}\zeta_{l1}=x_{1jl}\zeta_{11}+x_{2jl}\zeta_{21}+\cdots+x_{njl}\zeta_{n1} \\ \zeta_{j2}\zeta_{l2}=x_{1jl}\zeta_{12}+x_{2jl}\zeta_{22}+\cdots+x_{njl}\zeta_{n2} \\ \vdots \\ \zeta_{jn}\zeta_{ln}=x_{1jl}\zeta_{1n}+x_{2jl}\zeta_{2n}+\cdots+x_{njl}\zeta_{nn} \end{cases}$$

它的系数矩阵为

$$\begin{pmatrix} \zeta_{11} & \cdots & \zeta_{n1} \\ \vdots & & \vdots \\ \zeta_{1n} & \cdots & \zeta_{nn} \end{pmatrix}$$

. 行列式非零（记为 D）. 而每个 x_{ijl} 恰属于一个方程组.

故使用 n^2 次 Cramer 法则可知，满足条件的 x_{ijl} 存在唯一. 并且

$$x_{ijl}=\frac{1}{D}\cdot\begin{vmatrix} \zeta_{11} & \cdots & \zeta_{i-1,1} & \zeta_{j1}\zeta_{l1} & \zeta_{i+1,1} & \cdots & \zeta_{n1} \\ \zeta_{12} & \cdots & \zeta_{i-1,2} & \zeta_{j2}\zeta_{l2} & \zeta_{i+1,2} & \cdots & \zeta_{n2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \zeta_{1n} & \cdots & \zeta_{i-1,n} & \zeta_{jn}\zeta_{ln} & \zeta_{i+1,n} & \cdots & \zeta_{nn} \end{vmatrix}$$

20. 设 A 是严格主对角占优矩阵. 由 Cramer 法则可知只用证以 A 为系数矩阵的齐次线性方程组存在唯一解（没有非零解）. 假设有非零解 $(x_1,\cdots,x_n)$ 满足

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

设x_i是$x_1,\cdots,x_n$中绝对值最大的一个，则$|x_i|>0$，则

$$a_{ii}\cdot|x_i| = |a_{ii}x_i| = |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{i,i-1}x_{i-1} + a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n|$$
$$\leq \sum_{j\neq i}|a_{ij}|\cdot|x_j| \leq \sum_{j\neq i}|a_{ij}|\cdot|x_i| < a_{ii}\cdot|x_i|，矛盾.$$

故以上的齐次线性方程组只有零解，故$|A|\neq 0$.

进一步，若$A=(a_{ij})_{1\leq i,j\leq n}$严格主对角占优，则

$$A(t) = \begin{pmatrix} a_{11}+t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}+t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}+t \end{pmatrix} \quad (t\geq 0)$$

也是严格主对角占优矩阵. 记$f(t)=|A(t)|$，则$f(t)$是关于t的首一多项式函数，则f连续，且

$$\lim_{t\to+\infty} f(t) = +\infty.$$

另外由于$A(t)$严格主对角占优，故$f(t)$在$[0,+\infty)$上没有零点. 由介值原理知$f(t)>0$，$\forall t\in[0,+\infty)$.
特别地$f(0)>0$，即$|A|>0$.