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Deterministic models for rumor transmission☆

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Abstract

In this paper, we consider deterministic models for the transmission of a rumor. First, we investigate the age-independent case and introduce four models, which are classified according to whether the population is closed or not and whether the rumor is constant or variable. After formulating the models as finite-dimensional ODE systems, we show that the solutions converge to an equilibrium as $t \to \infty$. Next, we investigate a model for the transmission of a constant rumor in an age-structured population with age-dependent transmission coefficients. We formulate the model as an abstract Cauchy problem on an infinite-dimensional Banach space and show the existence and uniqueness of solutions. Then, under some appropriate assumptions, we examine the existence of its nontrivial equilibria and the stability of its trivial equilibrium. We show that the spectral radius $R_0 := r(\tilde{T})$ for some positive operator $\tilde{T}$ is the threshold. We also show sufficient conditions for the local stability of the nontrivial equilibria. Finally, we show that the model is uniformly strongly persistent if $R_0 > 1$.

Keywords: Rumor transmission; Threshold condition; Age-structured population; Rumor-free equilibrium; Rumor-endemic equilibrium; Global stability; Local stability; Uniform strong persistence

1. Introduction

In this paper we apply models similar to those used in epidemiology to the "transmission of a rumor," which is the social phenomenon that a remark spreads on a large scale in a short time through chain of communication.

Rumor transmission is an example of social contagion processes. Pioneering contributions to their modelling, based on epidemiological models, date back to [23,36–38]. In those days both deterministic models and stochastic models were used, and the former were so simple that they were solved analytically and regarded as the first approximation of the latter. Nearly a decade later, Daley and Kendall [9] explained the importance of dealing with stochastic rumor models rather than deterministic ones, henceforth stochastic models have been actively studied (see, for example, [4,7,42,49,34,35,24,11] for survey). The basic rumor transmission model which they used is called *Daley–Kendall model* after [9], and the simplified basic model is called *Maki–Thompson model* after [25]. We also refer the reader to [1,8] for details.

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Recently Pearce [32] and Gani [12] analyzed the probability generating functions in the stochastic rumor models by means of block-matrix methodology. In addition, Dickinson and Pearce [10] studied stochastic models for more general transient processes including epidemics.

Independently of this series of studies, deterministic models for rumor transmission have been studied sporadically. For example, Castillo-Chávez and Song [5] proposed the transmission models for a fanatic behavior based on the models for sexually transmitted diseases, and analyzed them qualitatively and numerically. Bettencourt et al. [2] is another recent example, which deals with the spread of ideas.

Now, some rumors alternate propagation with cessation, momentarily modified in some cases. We can take "the rumor of Orléans" [27] for instance. We shall call such rumors *recursive rumors* for the meantime. We could attribute their occurrence to the distance in space and time. One mechanism might be as follows: the rumor spread locally in a region spills over out of the region where it has never been spread with movement of people and information. Another mechanism might be as follows: since the power of fending off the rumor is weak, it survives in secret after its cessation. After a while, more and more people are unfamiliar with the rumor due to immigration from other areas, birth and the modification of the rumor. This might result in its repetition.

Noymer [29] proposed age-structured transmission models of "urban legends", which we could identify with rumor. In his models new people are constantly supplied through birth. Analyzing numerically, he found that the system seemed to converge to the steady state through damped oscillation but neither to behave periodically nor to show undamped oscillation. Note that he constructed his models based on the model for measles and hence ignored the law of mass action in the removal mechanism considered in the stochastic rumor models such as the Daley–Kendall model. Both age-structure and the law of mass action in the removal mechanism are considered to be important for rumor transmission.

In this paper, we propose and mathematically analyze deterministic models for rumor transmission. In Sections 2 and 3, we examine age-independent rumor transmission models, which are extentions of the deterministic Daley–Kendall model. In the last sections, we introduce an age-structured rumor transmission model. We owe the argument there to [17]. We first establish the well-posedness of the time evolution problem. Next, introducing a positive operator $\tilde{T}$, we show that the system has at least one nontrivial equilibrium if and only if the spectral radius $r(\tilde{T})$ is larger than 1. We examine the asymptotical stability of the equilibria. We show that the model is uniformly strongly persistent if $R_0 > 1$. Finally we briefly discuss open problems and possible extensions of the basic model.

2. Age-independent models for the transmission of a constant rumor

Let $N(t)$ denote the total population at time t. We divide the population into three classes: the susceptible class, the spreader class and the stifler class, each of which we call *rumor-class*. Each population at time t is denoted by $X(t)$, $Y(t)$, $Z(t)$, respectively. Those who belong to the susceptible class, whom we call *susceptibles*, do not know about the rumor. Those who belong to the spreader class, whom we call *spreaders*, know about the rumor and spread it actively. Those who belong to the stifler class, whom we call *stiflers*, know about the rumor and do not spread it. By definition, we have

$$X(t) + Y(t) + Z(t) = N(t).$$

We assume that no transition of rumor-class happens unless a spreader contacts someone, since the two people who are not spreaders do not talk about the rumor. That is, it is spreaders that are involved in the transition of rumor-class.

When a spreader contacts a susceptible, the spreader transmits the rumor at a constant frequency and the susceptible gets to know about it. Then the susceptible does not always become a spreader, but may doubt its credibility and consequently becomes a stifler. And so, we assume that $\alpha X(t)Y(t)\Delta t/N(t)$ susceptibles change their rumor-class and become spreaders at a constant rate $\theta \in (0, 1]$ during the small interval $(t, t + \Delta t)$, where α is a positive constant number representing the product of the contact frequency and the probability of transmitting the rumor.

When two spreaders contact, both of them transmit the rumor at a constant frequency. Hearing it again and again, the spreader gets bored, gradually loses interest in it, and consequently becomes a stifler. And so, we assume that $\beta Y(t)^2\Delta t/N(t)$ spreaders become stiflers during the small interval $(t, t + \Delta t)$, where β is a positive constant number.

When a spreader contacts a stifler, the spreader transmits the rumor at a constant frequency, and after hearing it, the stifler tries to remove it, because the stifler shows no interest in it or denies it. As a result, the spreader becomes a stifler.

And so, we assume that $\gamma Y(t)Z(t)\Delta t/N(t)$ spreaders become stiflers during the small interval $(t, t+\Delta t)$, where γ is a positive constant number.

For the meantime we assume that the rumor is "constant", that is, the same remark is transmitted at all times. Then stiflers do not change their rumor-class.

First, we consider the transmission of a constant rumor in a closed population, where people are neither born nor died. This assumption may be valid in the situation where the rumor spreads explosively in a short time and soon goes out. Then the dynamics of the population is governed by the following system:

$$\dot{X}(t) = -\alpha X(t)\frac{Y(t)}{N(t)},$$

$$\dot{Y}(t) = \alpha\theta X(t)\frac{Y(t)}{N(t)} - \beta Y(t)\frac{Y(t)}{N(t)} - \gamma Y(t)\frac{Z(t)}{N(t)},$$

$$\dot{Z}(t) = \alpha(1-\theta)X(t)\frac{Y(t)}{N(t)} + \beta Y(t)\frac{Y(t)}{N(t)} + \gamma Y(t)\frac{Z(t)}{N(t)}, \tag{2.1}$$

where $\dot{}$ denotes the differentiation with respect to t. Now $N(t)$ is independent of time and can be denoted $N_0(>0)$.

We introduce new variables x, y, z by

$$x(t) := \frac{X(t)}{N(t)}, \quad y(t) := \frac{Y(t)}{N(t)}, \quad z(t) := \frac{Z(t)}{N(t)}.$$

Then we obtain the new system for x, y, z:

$$\dot{x}(t) = -\alpha x(t)y(t),$$

$$\dot{y}(t) = \alpha\theta x(t)y(t) - \beta y(t)^2 - \gamma y(t)z(t),$$

$$\dot{z}(t) = \alpha(1-\theta)x(t)y(t) + \beta y(t)^2 + \gamma y(t)z(t).$$

Scaling time by setting $\tau := \alpha t$, we have

$$x' = -xy, \tag{2.2a}$$

$$y' = y(\theta x - by - cz), \tag{2.2b}$$

$$z' = y\{(1-\theta)x + by + cz\}, \tag{2.2c}$$

where $'$ denotes the differentiation with respect to τ and $b := \beta/\alpha$, $c := \gamma/\alpha$ are positive constants.

Let us consider the scaled system (2.2a)–(2.2c). We rewrite τ as t. We define $\Omega \subset \mathbb{R}^3$ by

$$\Omega := \{(x, y, z) \in \mathbb{R}^3_+ \mid x + y + z = 1\}. \tag{2.3}$$

It is easy to show that system (2.2a)–(2.2c) has a unique solution on $(-\infty, \infty)$ in Ω, for any initial data in Ω. Note that $x(t) = 0$ for all $t \in \mathbb{R}$ if $x(0) = 0$ and that $y(t) = 0$ for all $t \in \mathbb{R}$ if $y(0) = 0$.

In the case $x(0) > 0$ and $y(0) > 0$, we have $x(t) > 0$ and $y(t) > 0$ for all $t \in \mathbb{R}$. Hence, for all $t \in \mathbb{R}$, $x'(t) < 0$ and so $x(t)$ is strictly decreasing. Since the set $\{x(t) \mid t \in \mathbb{R}\}$ is bounded, the limits

$$x(\infty) := \lim_{t\to\infty} x(t), \quad x(-\infty) := \lim_{t\to-\infty} x(t)$$

exist and satisfy $0 \leqq x(\infty) < x(-\infty) \leqq 1$. At the same time, we find $z'(t) > 0$ for all $t \in \mathbb{R}$ and a similar discussion yields that the limits $z(\infty)$, $z(-\infty)$ exist and satisfy $0 \leqq z(-\infty) < z(\infty) \leqq 1$. Hence, as $t \to \pm\infty$, $y(t)$ converges. The limits $(x(\pm\infty), y(\pm\infty), z(\pm\infty))$ are the equilibria of the system in Ω, i.e., equal to $(t, 0, 1-t)$ $(0 \leqq t \leqq 1)$. Hence we have $y(\pm\infty) = 0$. In particular, the rumor goes out eventually.

Let

$$\Omega_1 := \{(x, y, z) \in \Omega \mid \theta x - by - cz < 0\},$$

$$\Omega_2 := \{(x, y, z) \in \Omega \mid \theta x - by - cz > 0\}.$$

In each domain $y(\theta x - by - cz)$ does not change its sign, so we find that the point $(x(t), y(t), z(t)) \in \Omega$ moves from Ω_2 into Ω_1 within a finite time, i.e., there exists some $T \in \mathbb{R}$ such that $(x(t), y(t), z(t)) \in \Omega_2$ whenever $t < T$ and $(x(t), y(t), z(t)) \in \Omega_1$ whenever $t > T$. Therefore, $y(t)$ takes its maximum at $t = T$.

Let

$$R_0 := \frac{c+\theta}{c} x(0) = \left(1 + \frac{\alpha\theta}{\gamma}\right) \frac{X(0)}{N_0}.$$

When $y(0) = Y(0)/N_0$ is sufficiently close to 0, $R_0 > 1$ means $x(0) > c/(c+\theta)$, so we see that $y(t)$ takes its maximum at $t = T > 0$, i.e., the rumor spreads and goes out just once. In contrast, $R_0 \leqq 1$ means $x(0) \leqq c/(c+\theta)$, so we see that $y(t)$ is strictly decreasing, i.e., the rumor does not spread. From this point of view, it can be safely said that R_0 is the threshold of this system.

Next, let us see the transmission of a constant rumor in a population with constant immigration and emigration. Let B be the sum of the population birth rate and the immigration rate, and μ the sum of the population death rate and the emigration rate. We assume that B, μ are positive constants, that the newcomers are all susceptibles, and that death and emigration are independent of rumor-class. These assumptions can be formulated as follows:

$$\begin{aligned}
\dot{X}(t) &= B - \alpha X(t) \frac{Y(t)}{N(t)} - \mu X(t),\\
\dot{Y}(t) &= \alpha\theta X(t) \frac{Y(t)}{N(t)} - \beta Y(t) \frac{Y(t)}{N(t)} - \gamma Y(t) \frac{Z(t)}{N(t)} - \mu Y(t),\\
\dot{Z}(t) &= \alpha(1-\theta) X(t) \frac{Y(t)}{N(t)} + \beta Y(t) \frac{Y(t)}{N(t)} + \gamma Y(t) \frac{Z(t)}{N(t)} - \mu Z(t).
\end{aligned} \tag{2.4}$$

Adding these equations, we see that the total population $N(t)$ satisfies

$$\dot{N}(t) = B - \mu N(t).$$

$N(t)$ converges to $N_0 := B/\mu$ as $t \to \infty$.

First, let us consider the limit system

$$\begin{aligned}
\dot{X}(t) &= \mu N_0 - \alpha X(t) \frac{Y(t)}{N_0} - \mu X(t),\\
\dot{Y}(t) &= \alpha\theta X(t) \frac{Y(t)}{N_0} - \beta Y(t) \frac{Y(t)}{N_0} - \gamma Y(t) \frac{Z(t)}{N_0} - \mu Y(t).
\end{aligned}$$

Scaled in the same way as (2.1), this system takes the form

$$\begin{aligned}
x'(t) &= d\{1 - x(t)\} - x(t)y(t),\\
y'(t) &= y(t)\{\theta x(t) - by(t) - c(1 - x(t) - y(t)) - d\},
\end{aligned} \tag{2.5}$$

where $d := \mu/\alpha$ is a positive constant and the scaled time τ is again called t for simplicity. It is easy to show that, for any initial data in Ω, there exists a unique solution x, y of system (2.5) on $[0, \infty)$ in Ω.

Let

$$f(x, y) := d(1-x) - xy, \quad g(x, y) := y\{\theta x - by - c(1-x-y) - d\},$$

$$\mathbf{F}(x, y) := {}^{\mathrm{t}}(f(x, y), g(x, y)),$$

where $^{\mathrm{t}}$ denotes the transpose of the vector. Let us explore the equilibria in Ω, i.e., the point $(x, y) \in \Omega$ satisfying $f(x, y) = g(x, y) = 0$.

From the latter equation $g(x, y) = 0$, we get $y = 0$ or $\theta x - by - c(1-x-y) - d = 0$. If we substitute $y = 0$ into $f(x, y) = 0$, we obtain $x = 1$. The boundary equilibrium

$$\mathbf{x}^\circ = (x^\circ, y^\circ) := (1, 0)$$

is always in Ω regardless of the parameters b, c, d, θ. This equilibrium represents the situation that no one knows about the rumor. In this context, one could also call it *rumor-free equilibrium* (RFE). The Jacobian matrix at RFE is given by

$$D\mathbf{F}(\mathbf{x}^\circ) = \begin{pmatrix} -d & -1 \\ 0 & -d+\theta \end{pmatrix}$$

and its eigenvalues are $-d(<0),\ -d+\theta$.

$(1, 0)$ is an eigenvector corresponding to $-d$. Now suppose $y(0) = 0$, then the solution is given by

$$x(t) = 1 - \mathrm{e}^{-dt}(1 - x(0)), \quad y(t) = 0.$$

Hence the segment $I_0 := [0, 1] \times \{0\}$ included in x-axis is positively invariant, and for any initial data in I_0, $(x(t), y(t))$ converges to $\mathbf{x}^\circ$ as $t \to \infty$. That is, I_0 is the intersection of Ω and the stable manifold of $\mathbf{x}^\circ$.

Let $\Omega' := \Omega \backslash I_0$, then we find that $0 < d < \theta$ implies that $\mathbf{x}^\circ$ is a saddle and any point sufficiently close to $\mathbf{x}^\circ$ in Ω' does not close to $\mathbf{x}^\circ$ eventually, so $\mathbf{x}^\circ$ is asymptotically unstable in Ω'. And we know that $d > \theta$ implies that $\mathbf{x}^\circ$ is a sink and so locally asymptotically stable in Ω.

Then, let us examine the existence of equilibria satisfying $y \neq 0$.

Suppose $(x, y) \in \Omega'$. Since $f(0, y) = d \neq 0$, any equilibrium $(x, y) \in \Omega'$, if it exists, satisfies $x \neq 0$. So, we can assume that $x > 0,\ y > 0,\ x + y \leqq 1$.

In the case $d \geqq 1$, taking into consideration the conditions $0 < x < 1 \leqq d$ and $1 - x \geqq y$, we have

$$\frac{(1-x)d}{x} > y \iff f(x, y) > 0,$$

from which it follows that no equilibrium exists in Ω'.

In the case $d < 1$, $f(x, y) = 0$ gives

$$y = \frac{(1-x)d}{x} \leqq 1 - x,$$

which implies $d \leqq x < 1$, so we have $0 < y \leqq 1 - d$. On the other hand, $g(x, y) = 0$ and $y \neq 0$ lead to

$$x = \frac{(b-c)y + (c+d)}{\theta + c}$$

and by substituting it into $f(x, y) = 0$ we obtain an equation $h(y) = 0$ at most quadratic with respect to y, where

$$h(y) := (c-b)y^2 - (c + d - cd + db)y + d(\theta - d).$$

Note that

$$h(0) = (\theta - d)d,$$

$$h(1-d) = -(1-d)b - (1-\theta)d < 0 \quad (\text{since } b > 0).$$

Suppose $\theta < d < 1$. If $c - b \geqq 0$, the graph of $v = h(u)$ is concave, so $h(0) < 0$ and $h(1-d) < 0$ yield that $h(y) < 0$ whenever $0 < y \leqq 1 - d$. Otherwise, i.e., if $c - b < 0$, since

$$c + d - cd + bd = c(1-d) + b + bd > 0,$$

we find

$$h'(y) = 2(c-b)y - (c + d - cd + bd) < 0 \quad \text{whenever } 0 < y \leqq 1 - d.$$

Hence $h(y)$ is strictly decreasing and it follows that $h(y) < h(0) < 0$. In both cases, under the condition $\theta < d < 1$, $h(y) < 0$ holds whenever $0 < y \leqq 1 - d$ and we see that $h(y) = 0$ has no solution in the range.

Suppose $0 < d < \theta$, then we have

$$h(0) = (\theta - d)d > 0,$$

$$h\left(\frac{\theta - d}{\theta + b}\right) = -\frac{\{(1-\theta)d + (1-d)b\}(\theta + c)(\theta - d)}{(\theta + b)^2} < 0,$$

$$h(1-d) < 0.$$

Hence, noting

$$(1-d) - \frac{\theta - d}{\theta + b} = \frac{(1-d)b + (1-\theta)d}{\theta + b} > 0,$$

we find that the equation $h(y)=0$ has the only solution y^* in $(0, 1-d]$, which satisfies $0 < y^* < ((\theta - d)/(\theta + b))(< 1-d)$. Set

$$x^* := \frac{(b-c)y^* + c + d}{\theta + c},$$

then the inequalities

$$x^* = \frac{by^* + d + c(1-y^*)}{\theta + c} > 0, \quad 1 - x^* - y^* = \frac{\theta - d - (\theta + b)y^*}{\theta + c} > 0$$

hold and it follows that $(x^*, y^*) \in \Omega'$, which is the only interior equilibrium, which we could call *rumor-endemic equilibrium* (REE).

Next, let us discuss the local stability of REE. Assuming $0 < d < \theta$, we examine the eigenvalues of the Jacobian matrix $M := D\mathbf{F}(\mathbf{x}^*)$ at $\mathbf{x}^*$, which is given by

$$M = \begin{pmatrix} -d - y^* & -x^* \\ (\theta + c)y^* & -d - c + (\theta + c)x^* + (-2b + 2c)y^* \end{pmatrix}.$$

Then a little calculation gives rise to

$$\det M = -y^* h'(y^*) > 0$$

and

$$(\theta - d)\,\mathrm{tr}\, M = y^* K \quad \text{where } K := (c-b)y^* - \{c(1-\theta) + (1+b)\theta\}.$$

Now, let us examine the sign of K. If $c - b \leqq 0$ then it is clear that $K < 0$. Otherwise, we see that

$$y^* = \frac{d + c + bd - cd - \sqrt{D_0}}{2(c-b)},$$

where

$$D_0 := (d + c + bd - cd)^2 - 4(c-b)(\theta - d)d$$
$$= (-d + c + bd - cd)^2 + 4d\{c(1-\theta) + b\theta\}.$$

Since

$$2K = -d + c + bd - cd - \sqrt{D_0} - 2c(1-\theta) - 2(\theta - d) - 2b\theta,$$

we have $K < 0$ clearly if $-d + c + bd - cd \leqq 0$, and this holds even if $-d + c + bd - cd > 0$, because $\sqrt{D_0} > -d + c + bd - cd$, which follows from the form of D_0.

Hence we obtain $\mathrm{tr}\, M < 0$. From it and $\det M > 0$ we can conclude that the real part of any eigenvalue of M is negative and so REE is locally asymptotically stable.

Let us apply Dulac–Bendixson Criterion (cf. [6]) to exclude the possibility of a periodic orbit or a cyclic chain of equilibria, i.e., a piecewise smooth closed curve consisting of finitely many equilibria and of orbits connecting them. It is convenient to write the system in term of y and z as follows:

$$y' = y\{\theta(1-y-z) - by - cz - d\},$$
$$z' = y\{(1-\theta)(1-y-z) + by + cz\} - dz.$$

We define a Dulac function $\rho(y,z) := (yz)^{-1}$ on the domain $\{(y,z) \in \mathbb{R}_+^2 \mid y > 0, z > 0, y+z < 1\}$. Then we have

$$\frac{\partial}{\partial y}(\rho(y,z)y\{\theta(1-y-z) - by - cz - d\}) + \frac{\partial}{\partial z}(\rho(y,z)(y\{(1-\theta)(1-y-z) + by + cz\} - dz))$$
$$= -\left\{\frac{\theta+b}{z} + \frac{(1-\theta)(1-y)+by}{z^2}\right\},$$

and this is strictly negative on the domain. Hence we can use Dulac–Bendixson Criterion.

It follows from the above results and Poincaré–Bendixson trichotomy (cf. [6]) that, if $0<d<\theta$, $\mathbf{x}^*$ is globally asymptotically stable in Ω'.

Therefore, we have the following results:

Theorem 1. *Concerning system* (2.5),

(i) *if $d > \theta$, then RFE is the only equilibrium and globally asymptotically stable in Ω, and*
(ii) *if $0 < d < \theta$, then the system has the only equilibrium $\mathbf{x}^* = (x^*, y^*)$ in Ω', which is globally asymptotically stable in Ω'.*

By applying the theory of asymptotically autonomous differential equations (cf. [43,44] and [41, Appendix F]), we find that the solutions of system (2.4) show the same type of large-time behavior as the limit system, unless $\mu = \alpha\theta$. Hence $Y(t)$ converges to 0 or a positive number as $t \to \infty$ and this system has no undamped oscillation. And we also find that $R_0 := \theta/d = \alpha\theta/\mu$ is the threshold of system (2.4). $\tau_Y := \mu^{-1}$ is the mean sojourn time spent in the spreader class with no rumor-class transition. Hence, $R_0 := (\alpha\theta)\tau_Y$ gives the average number of susceptibles that a spreader can let into spreader class during the time τ_Y, provided that the whole population is susceptible. In this context, one could also call it *basic reproduction number*.

3. Age-independent models for the transmission of a variable rumor

In this section, we consider the effect of the modification of a rumor on its transmission.

So far we did not take into consideration the possibility of the transition from the stifler class into the susceptible class, but in reality it is thought to exist. It is partly because the rumor gradually slips out of the memory of stiflers and so they become regarded as susceptibles when they hear the same rumor again.

However, what is thought to be more important for the transition is the effect of the modification of a rumor. Some rumors are modified in the communication process, which we call *variable*. Even if someone knows such a variable rumor at one time, substantially he does not know the modified one after a long period of time.

In general, the transition is considered to be dependent on how long he/she has been in the stifler class, his/her age and the speed at which the rumor is modified. For simplicity, these factors are ignored in the models below, and we assume that $\eta Z(t)\Delta t$ stiflers become susceptibles during the small interval $(t, t+\Delta t)$, where η is a positive constant number. This means that the transition rate from the stifler class into the susceptible class is independent of stifler's age and exponentially distributed with respect to the duration. For the above assumption we referred to those used in the influenza model [18,33,48].

First, we consider the transmission of a variable rumor in a closed population. Our model takes the following form:

$$\begin{aligned}\dot{X}(t) &= -\alpha X(t)\frac{Y(t)}{N(t)} + \eta Z(t),\\ \dot{Y}(t) &= \alpha\theta X(t)\frac{Y(t)}{N(t)} - \beta Y(t)\frac{Y(t)}{N(t)} - \gamma Y(t)\frac{Z(t)}{N(t)},\\ \dot{Z}(t) &= \alpha(1-\theta)X(t)\frac{Y(t)}{N(t)} + \beta Y(t)\frac{Y(t)}{N(t)} + \gamma Y(t)\frac{Z(t)}{N(t)} - \eta Z(t).\end{aligned} \tag{3.1}$$

Noting the total population $N(t)$ is a constant number N_0, after scaling time, we obtain the equations for the new scaled dependent variables $x(t) := X(t)/N(t),\ y(t) := Y(t)/N(t),\ z(t) := Z(t)/N(t)$:

$$\begin{aligned}x' &= -xy + k(1-x-y),\\ y' &= y\{\theta x - by - c(1-x-y)\},\end{aligned} \tag{3.2}$$

where $k := \eta/\alpha$ is a positive constant number and $'$ denotes the derivative with respect to dimensionless time.

The equilibria and the dynamics of system (3.2) are as follows:

Theorem 2. *System* (3.2) *has RFE* $\mathbf{x}^\circ = (1, 0)$, *which is asymptotically unstable in* Ω'. *And it has the unique REE* $\mathbf{x}^* = (x^*, y^*)$ *in* Ω', *which is asymptotically globally stable in* Ω'.

Next, we consider the transmission of a variable rumor in a population with constant immigration and emigration. Then we obtain the following system:

$$\begin{aligned}\dot{X}(t) &= B - \mu X(t) - \alpha X(t)\frac{Y(t)}{N(t)} + \eta Z(t),\\ \dot{Y}(t) &= -\mu Y(t) + \alpha\theta X(t)\frac{Y(t)}{N(t)} - \beta Y(t)\frac{Y(t)}{N(t)} - \gamma Y(t)\frac{Z(t)}{N(t)},\\ \dot{Z}(t) &= -\mu Z(t) + \alpha(1-\theta)X(t)\frac{Y(t)}{N(t)} + \beta Y(t)\frac{Y(t)}{N(t)} + \gamma Y(t)\frac{Z(t)}{N(t)} - \eta Z(t).\end{aligned} \tag{3.3}$$

As in the case of system (2.4), first let us discuss the two-dimensional limiting scaled system

$$\begin{aligned}x' &= d(1-x) - xy + k(1-x-y),\\ y' &= y\{\theta x - by - c(1-x-y) - d\}.\end{aligned} \tag{3.4}$$

The equilibria and the dynamics of system (3.4) are as follows:

Theorem 3. *Concerning system* (3.4),

(i) *if* $d > \theta$, *then RFE* $\mathbf{x}^\circ = (1, 0)$ *is the only equilibrium and globally asymptotically stable in* Ω, *and*
(ii) *if* $0 < d < \theta$, *then the system has the unique REE* $\mathbf{x}^* = (x^*, y^*)$ *in* Ω', *which is globally asymptotically stable in* Ω'. *RFE is asymptotically unstable in* Ω'.

We omit the proof of Theorems 2 and 3, since it is the same as that of Theorem 1. See [20] for details.

4. Age-structured model for rumor transmission

Henceforth, we consider the transmission of a constant rumor in a closed age-structured population under the demographic growth. Let $a \in [0, \omega]$, where the number $\omega(<\infty)$ denotes the life span of the population, and $X(t,a), Y(t,a), Z(t,a)$ be the age-density functions at time t of the susceptible class, the spreader class, and the stifler class, respectively. Let $P(t,a) := X(t,a) + Y(t,a) + Z(t,a)$ be the age-density of the total number of individuals, then the total size of the population is given by $N(t) := \int_0^\omega P(t,a)\,\mathrm{d}a$.

The basic system can be formulated as follows:

$$
\begin{aligned}
&(\partial_t + \partial_a)X(t,a) = -(\mu(a) + \lambda_1(t,a))X(t,a),\\
&(\partial_t + \partial_a)Y(t,a) = \lambda_1(t,a)\theta(a)X(t,a) - (\mu(a) + \lambda_2(t,a))Y(t,a),\\
&(\partial_t + \partial_a)Z(t,a) = \lambda_1(t,a)(1-\theta(a))X(t,a) + \lambda_2(t,a)Y(t,a) - \mu(a)Z(t,a),\\
&X(t,0) = \int_0^\omega m(a)P(t,a)\,\mathrm{d}a,\\
&Y(t,0) = 0, \quad Z(t,0) = 0,\\
&X(0,a) = X_0(a), \quad Y(0,a) = Y_0(a), \quad Z(0,a) = Z_0(a).
\end{aligned}
\tag{4.1}
$$

$\mu(a), m(a)$ stand for the age-specific natural death rate and fertility rate, respectively. $(X_0(a), Y_0(a), Z_0(a))$ is a given initial data. $\lambda_1(t,a)$ is the force of transition into the spreader class on a susceptible individual aged a at time t and defined by

$$
\lambda_1(t,a) := \frac{1}{N(t)}\int_0^\omega \alpha(a,\sigma)Y(t,\sigma)\,\mathrm{d}\sigma,
$$

where $\alpha(a,\sigma)$ is the transmission rate between a susceptible individual aged a and a spreader aged σ. $\theta(a)$ stands for the probability the susceptible individual aged a who gets to know about the rumor becomes a spreader. $\lambda_2(t,a)$ is the force of transition into the stifler class on a spreader aged a at time t and defined by

$$
\lambda_2(t,a) := \frac{1}{N(t)}\int_0^\omega \{\beta(a,\sigma)Y(t,\sigma) + \gamma(a,\sigma)Z(t,\sigma)\}\,\mathrm{d}\sigma,
$$

where $\beta(a,\sigma)$ is the transmission rate between a spreader aged a and another one aged σ, while $\gamma(a,\sigma)$ is the transmission rate between a spreader aged a and a stifler aged σ.

It follows from (4.1) that $P(t,a)$ satisfies the McKendrick equation

$$
\begin{aligned}
&(\partial_t + \partial_a)P(t,a) = -\mu(a)P(t,a),\\
&P(t,0) = \int_0^\omega m(a)P(t,a)\,\mathrm{d}a,\\
&P(0,a) = P_0(a) := X_0(a) + Y_0(a) + Z_0(a).
\end{aligned}
\tag{4.2}
$$

Note that we implicitly assume that there is no true interaction between demography and the spread of the rumor. Hence, it is convenient to introduce the fractional age distribution for each rumor-class as follows:

$$
x(t,a) := \frac{X(t,a)}{P(t,a)}, \quad y(t,a) := \frac{Y(t,a)}{P(t,a)}, \quad z(t,a) := \frac{Z(t,a)}{P(t,a)}.
$$

Then the new system for the fractional age distributions is given as follows:

$$
\begin{aligned}
&(\partial_t + \partial_a)x(t,a) = -\lambda_1(t,a)x(t,a),\\
&(\partial_t + \partial_a)y(t,a) = \lambda_1(t,a)\theta(a)x(t,a) - \lambda_2(t,a)y(t,a),\\
&(\partial_t + \partial_a)z(t,a) = \lambda_1(t,a)(1-\theta(a))x(t,a) + \lambda_2(t,a)y(t,a),\\
&x(t,0) = 1, \quad y(t,0) = 0, \quad z(t,0) = 0,\\
&\lambda_1(t,a) = \int_0^\omega \alpha(a,\sigma)\psi(t,\sigma)y(t,\sigma)\,\mathrm{d}\sigma,\\
&\lambda_2(t,a) = \int_0^\omega \psi(t,\sigma)\{\beta(a,\sigma)y(t,\sigma) + \gamma(a,\sigma)z(t,\sigma)\}\,\mathrm{d}\sigma,
\end{aligned}
\tag{4.3}
$$

where $\psi(t,a)$ is defined by

$$
\psi(t,a) := \frac{P(t,a)}{\int_0^\omega P(t,a)\,\mathrm{d}a}.
$$

According to the stable population theory (see, for example, [15,16]), as $t \to \infty$, ψ converges to the persistent normalized age distribution uniformly with respect to a:

$$
\lim_{t\to\infty} \psi(t,a) = c(a) := \frac{\mathrm{e}^{-\lambda_0 a}\mathscr{F}(a)}{\int_0^\omega \mathrm{e}^{-\lambda_0 a}\mathscr{F}(a)\,\mathrm{d}a},
$$

where λ_0 denotes the intrinsic rate of natural increase, $\mathscr{F}(a)$ is the survival rate defined by

$$
\mathscr{F}(a) := \exp\left(-\int_0^a \mu(\sigma)\,\mathrm{d}\sigma\right),
$$

and $c(a)$ is called relatively stable age distribution. Note that $\int_0^\omega c(a)\,\mathrm{d}a = 1$.

In the following we assume that the stable age distribution is already attained. Then system (4.3) is rewritten as the autonomous system below:

$$
\begin{aligned}
&(\partial_t + \partial_a)x(t,a) = -\lambda_1(t,a)x(t,a),\\
&(\partial_t + \partial_a)y(t,a) = \lambda_1(t,a)\theta(a)x(t,a) - \lambda_2(t,a)y(t,a),\\
&(\partial_t + \partial_a)z(t,a) = \lambda_1(t,a)(1-\theta(a))x(t,a) + \lambda_2(t,a)y(t,a),\\
&x(t,0) = 1, \quad y(t,0) = 0, \quad z(t,0) = 0,\\
&\lambda_1(t,a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)y(t,\sigma)\,\mathrm{d}\sigma,\\
&\lambda_2(t,a) = \int_0^\omega c(\sigma)\{\beta(a,\sigma)y(t,\sigma) + \gamma(a,\sigma)z(t,\sigma)\}\,\mathrm{d}\sigma.
\end{aligned}
\tag{4.4}
$$

We mainly consider system (4.4) under the condition

$$
x(t,a) \geqq 0, \quad y(t,a) \geqq 0, \quad z(t,a) \geqq 0, \quad x(t,a) + y(t,a) + z(t,a) = 1. \tag{4.5}
$$

Under this condition, we can formally exclude the susceptible class from the basic system. That is, instead of the basic system (4.4), we can consider the following system with linear homogeneous boundary conditions, which is more convenient to consider the well-definedness of the time evolution problem:

$$(\partial_t+\partial_a)y(t,a)=\lambda_1(t,a)\theta(a)\{1-y(t,a)-z(t,a)\}-\lambda_2(t,a)y(t,a),$$

$$(\partial_t+\partial_a)z(t,a)=\lambda_1(t,a)(1-\theta(a))\{1-y(t,a)-z(t,a)\}+\lambda_2(t,a)y(t,a),$$

$$y(t,0)=0,\quad z(t,0)=0,$$

$$\lambda_1(t,a)=\int_0^\omega \alpha(a,\sigma)c(\sigma)y(t,\sigma)\,\mathrm{d}\sigma,$$

$$\lambda_2(t,a)=\int_0^\omega c(\sigma)\{\beta(a,\sigma)y(t,\sigma)+\gamma(a,\sigma)z(t,\sigma)\}\,\mathrm{d}\sigma. \tag{4.6}$$

The state space of this system is

$$\Omega=\{(y,z)\in(L^1_+(0,\omega))^2\mid y+z\leqq 1\},$$

where

$$L^1_+(0,\omega):=\{f\in L^1(0,\omega)\mid f(a)\geqq 0\text{ a.e.}\}$$

is a positive cone of $L^1(0,\omega)$, and $E_+:=(L^1_+(0,\omega))^2$ is a positive cone of $E:=(L^1(0,\omega))^2$.

Let us define an unbounded linear operator A on E as follows:

$$(A\phi)(a):=\begin{pmatrix}-\mathrm{d}/\mathrm{d}a & 0\\ 0 & -\mathrm{d}/\mathrm{d}a\end{pmatrix}\begin{pmatrix}\phi_1(a)\\ \phi_2(a)\end{pmatrix},$$

where $\phi={}^{\mathrm{t}}(\phi_1,\phi_2)$ and the domain of A is defined by

$$\mathscr{D}(A):=\{\phi={}^{\mathrm{t}}(\phi_1,\phi_2)\in E\mid \phi_1,\phi_2\text{ are absolutely continuous on }[0,\omega],\ \phi_1(0)=\phi_2(0)=0\}.$$

Let F be a nonlinear operator on E defined by

$$F(\phi)(a):=\begin{pmatrix}\lambda_1[a\mid\phi_1]\theta(a)\{1-\phi_1(a)-\phi_2(a)\}-\lambda_2[a\mid\phi_1,\phi_2]\phi_1(a)\\ \lambda_1[a\mid\phi_1]\{1-\theta(a)\}\{1-\phi_1(a)-\phi_2(a)\}+\lambda_2[a\mid\phi_1,\phi_2]\phi_1(a)\end{pmatrix},$$

where $\lambda_1[a\mid\phi_1]$, $\lambda_2[a\mid\phi_1,\phi_2]$ are defined by

$$\lambda_1[a\mid\phi]:=\int_0^\omega \alpha(a,\sigma)c(\sigma)\phi(\sigma)\,\mathrm{d}\sigma,$$

$$\lambda_2[a\mid\phi_1,\phi_2]:=\int_0^\omega c(\sigma)\{\beta(a,\sigma)\phi_1(\sigma)+\gamma(a,\sigma)\phi_2(\sigma)\}\,\mathrm{d}\sigma.$$

Let us define an E-valued function $u={}^{\mathrm{t}}(y,z)$. Then system (4.6) can be formulated as a semilinear Cauchy problem on the Banach space E:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t)=Au(t)+F(u(t)),\quad u(0)=u_0. \tag{4.7}$$

Lemma 4. *The operator A generates a C_0-semigroup $\{\mathrm{e}^{tA}\}_{t\geqq 0}$ and the state space Ω is positively invariant with respect to the semiflow defined by $\{\mathrm{e}^{tA}\}_{t\geqq 0}$.*

Proof. Let $\{T(t)\}_{t\geqq 0}$ be a nilpotent translation C_0-semigroup on $L^1(0,\omega)$ induced by

$$(T(t)\phi)(a):=\begin{cases}0, & t-a>0,\\ \phi(a-t), & a-t>0.\end{cases} \tag{4.8}$$

Furthermore, let $\{\tilde{T}(t)\}_{t \geqq 0}$ be a C_0-semigroup on E defined by

$$\tilde{T}(t)\begin{pmatrix}\phi_1(a)\\ \phi_2(a)\end{pmatrix} := \begin{pmatrix}(T(t)\phi_1)(a)\\ (T(t)\phi_2)(a)\end{pmatrix}.$$

Then, it is clear that A is the generator of the C_0-semigroup $\{\tilde{T}(t)\}_{t \geqq 0}$. Since $\{\tilde{T}(t)\}_{t \geqq 0}$ is a translation semigroup and $(\tilde{T}(t)\phi)(a) = {}^{\mathrm{t}}(0, 0)$ whenever $t > a$, obviously Ω is positively invariant with respect to the semiflow induced by $\{\mathrm{e}^{tA}\}_{t \geqq 0}$. $\square$

Let $\|\cdot\|$ be the usual norm on $L^1(0, \omega)$ and $\|\cdot\|_E$ be a norm on E defined by

$$\|\phi\|_E := \max\{\|\phi_1\|, \|\phi_2\|\}, \quad \phi = {}^{\mathrm{t}}(\phi_1, \phi_2) \in E.$$

Let $L^\infty_+(D)$ $(D \subset \mathbb{R}^N)$ be a positive cone of $L^\infty(D)$ defined by

$$L^\infty_+(D) := \{f \in L^\infty(D) \mid f(a) \geqq 0 \text{ a.e.}\}.$$

Assumption 5. $\alpha, \beta, \gamma \in L^\infty_+((0, \omega) \times (0, \omega))$, and $\theta, c \in L^\infty_+(0, \omega)$.

Under this assumption, set

$$\alpha^\infty := \text{ess sup}\,\alpha, \quad \beta^\infty := \text{ess sup}\,\beta, \quad \gamma^\infty := \text{ess sup}\,\gamma, \quad c^\infty := \text{ess sup}\,c.$$

Lemma 6. *Under Assumption 5, the map $F|_\Omega : \Omega \to E$ is Lipschitz continuous and there exists a number $k > 0$ such that $(\mathrm{Id}|_E + kF)(\Omega) \subset \Omega$.*

Proof. First, let us prove the first part. Let $\phi = {}^{\mathrm{t}}(\phi_1, \phi_2) \in \Omega$, $\psi = {}^{\mathrm{t}}(\psi_1, \psi_2) \in \Omega$ and $F(\phi) = {}^{\mathrm{t}}(F_1(\phi), F_2(\phi))$. Applying the triangle inequality, we get the evaluation as follows:

$$\begin{aligned}\|F_1(\phi) - F_1(\psi)\| &= \int_0^\omega |\lambda_1[a|\phi_1]\theta(a)(1 - \phi_1(a) - \phi_2(a)) - \lambda_2[a|\phi_1, \phi_2]\phi_1(a) \\ &\qquad - \lambda_1[a|\psi_1]\theta(a)(1 - \psi_1(a) - \psi_2(a)) + \lambda_2[a|\psi_1, \psi_2]\psi_1(a)|\,\mathrm{d}a \\ &\leqq \int_0^\omega \{\lambda_1[a|\phi_1]\theta(a)|(1 - \phi_1(a) - \phi_2(a)) - (1 - \psi_1(a) - \psi_2(a))| \\ &\quad + \theta(a)(1 - \psi_1(a) - \psi_2(a))|\lambda_1[a|\phi_1] - \lambda_1[a|\psi_1]| \\ &\quad + \lambda_2[a|\phi_1, \phi_2]|\phi_1(a) - \psi_1(a)| + \psi_1(a)|\lambda_2[a|\phi_1, \phi_2] - \lambda_2[a|\psi_1, \psi_2]|\}\,\mathrm{d}a.\end{aligned}$$

Concerning the last line of the above evaluation, from the inequalities

$$\phi_1(a), \phi_2(a), \psi_1(a), \psi_2(a) \geqq 0, \quad \phi_1(a) + \phi_2(a) \leqq 1, \quad \psi_1(a) + \psi_2(a) \leqq 1$$

we see that

$$\lambda_1[a|\phi_1] \leqq \int_0^\omega \alpha^\infty c(\sigma)\,\mathrm{d}\sigma = \alpha^\infty, \tag{4.9}$$

$$|\lambda_1[a|\phi_1] - \lambda_1[a|\psi_1]| \leqq \int_0^\omega \alpha(a, \sigma)c(\sigma)|\phi_1(\sigma) - \psi_1(\sigma)|\,\mathrm{d}\sigma \leqq \alpha^\infty c^\infty \|\phi_1 - \psi_1\|,$$

$$\lambda_2[a|\phi_1, \phi_2] \leqq \int_0^\omega c(\sigma)(\beta^\infty + \gamma^\infty)\,\mathrm{d}\sigma \leqq \beta^\infty + \gamma^\infty,$$

$$|\lambda_2[a|\phi_1,\phi_2]-\lambda_2[a|\psi_1,\psi_2]| \leqq \int_0^\omega c(\sigma)\beta(a,\sigma)|\phi_1(\sigma)-\psi_1(\sigma)|\,\mathrm{d}\sigma + \int_0^\omega c(\sigma)\gamma(a,\sigma)|\phi_2(\sigma)-\psi_2(\sigma)|\,\mathrm{d}\sigma$$
$$\leqq c^\infty\beta^\infty\|\phi_1-\psi_1\| + c^\infty\gamma^\infty\|\phi_2-\psi_2\|.$$

Hence we have

$$\begin{aligned}\|F_1(\phi)-F_1(\psi)\| &\leqq \alpha^\infty(\|\phi_1-\psi_1\|+\|\phi_2-\psi_2\|)+\alpha^\infty c^\infty\omega\|\phi_1-\psi_1\| \\ &\quad + (\beta^\infty+\gamma^\infty)\|\phi_1-\psi_1\| + c^\infty\beta^\infty\omega\|\phi_1-\psi_1\| + c^\infty\gamma^\infty\omega\|\phi_2-\psi_2\| \\ &\leqq (2\alpha^\infty+\alpha^\infty c^\infty\omega+\beta^\infty+\gamma^\infty+c^\infty\beta^\infty\omega+c^\infty\gamma^\infty\omega)\|\phi-\psi\|_E.\end{aligned}$$

The same evaluation gives rise to

$$\|F_2(\phi)-F_2(\psi)\| \leqq (2\alpha^\infty+\alpha^\infty c^\infty\omega+\beta^\infty+\gamma^\infty+c^\infty\beta^\infty\omega+c^\infty\gamma^\infty\omega)\|\phi-\psi\|_E.$$

Therefore, we obtain

$$\|F(\phi)-F(\psi)\|_E \leqq K\|\phi-\psi\|_E \quad \text{where } K := 2\alpha^\infty+\alpha^\infty c^\infty\omega+\beta^\infty+\gamma^\infty+c^\infty\beta^\infty\omega+c^\infty\gamma^\infty\omega,$$

which implies that $F|_\Omega$ is Lipschitz continuous.

To prove the second part, let $\phi = {}^{\mathrm{t}}(\phi_1,\phi_2)\in\Omega$, then we have

$$\phi_1(a)+kF_1(\phi)(a)+\phi_2(a)+kF_2(\phi)(a) = k\lambda_1[a|\phi_1](1-\phi_1(a)-\phi_2(a))+\phi_1(a)+\phi_2(a),$$

which is less than or equal to 1 whenever $0<k<(\alpha^\infty)^{-1}$ because of (4.9). It is obvious that $\phi_2(a)+kF_2(\phi)(a)\geqq 0$. Since

$$\phi_1(a)+kF_1(\phi)(a) = (1-k\lambda_2[a|\phi_1,\phi_2])\phi_1(a)+k\lambda_1[a|\phi_1]\theta(a)(1-\phi_1(a)-\phi_2(a))$$

and $0\leqq\lambda_2[a|\phi_1,\phi_2]\leqq\beta^\infty+\gamma^\infty$, $\phi_1(a)+kF_1(\phi)(a)\geqq 0$ holds whenever $k<(\beta^\infty+\gamma^\infty)^{-1}$.

Therefore, if we choose $k>0$ so small that

$$0<k<\min\{(\alpha^\infty)^{-1},\ (\beta^\infty+\gamma^\infty)^{-1}\}, \tag{4.10}$$

then we find that $(\mathrm{Id}|_E+kF)(\Omega)\subset\Omega$. $\square$

According to [3], the Cauchy problem (4.7) can be rewritten as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \left(A-\frac{1}{k}\right)u(t)+\frac{1}{k}(\mathrm{Id}+kF)u(t), \quad u(0)=u_0\in\Omega,$$

where k is chosen such as $0<k<1$ and (4.10) are satisfied. Its mild solution is given as the solution of the integral equation (the variation-of-constants formula, see [31, Chapter 6]):

$$u(t) = \mathrm{e}^{-k^{-1}t}\,\mathrm{e}^{tA}u_0 + k^{-1}\int_0^t \mathrm{e}^{-k^{-1}(t-s)}\,\mathrm{e}^{(t-s)A}\{u(s)+kF(u(s))\}\,\mathrm{d}s.$$

Let $\{S(t)u_0\}_{t\geqq 0}$ be the semiflow induced by this mild solution. On the other hand, let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of E defined by $u^0(t):=u_0$ and

$$u^{n+1}(t) := \mathrm{e}^{-k^{-1}t}\,\mathrm{e}^{tA}u_0 + k^{-1}\int_0^t \mathrm{e}^{-k^{-1}(t-s)}\,\mathrm{e}^{(t-s)A}\{u^n(s)+kF(u^n(s))\}\,\mathrm{d}s.$$

If $u^n\in\Omega$, then $\mathrm{e}^{tA}u_0,\ \mathrm{e}^{(t-s)A}\{u^n(s)+kF(u^n(s))\}\in\Omega$, which implies $u^{n+1}\in\Omega$ since Ω is convex. It follows from the Lipschitz continuity of F that u^n uniformly converges to the mild solution $S(t)u_0$. Hence, u^n converges to $S(t)u_0$ in E and we have $S(t)u_0\in\Omega$ since Ω is closed. Thus we obtain the following theorem:

Theorem 7. *The Cauchy problem* (4.7) *has a unique mild solution $S(t)u_0$, and Ω is positively invariant with respect to the semiflow $\{S(t)u_0\}_{t\geqq 0}$. If $u_0\in\mathscr{D}(A)$, then $S(t)u_0$ gives a classical solution.*

5. Existence of the REE

In this section, we consider the condition for the existence of rumor-endemic equilibria of system (4.4). We denote the density vector at the REE by (x^*, y^*, z^*), and the forces of rumor-class transition λ_1^*, λ_2^*. They must satisfy the following system:

$$\frac{\mathrm{d}}{\mathrm{d}a}x^*(a) = -\lambda_1^*(a)x^*(a), \tag{5.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}y^*(a) = \lambda_1^*(a)\theta(a)x^*(a) - \lambda_2^*(a)y^*(a), \tag{5.1b}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}z^*(a) = \lambda_1^*(a)(1-\theta(a))x^*(a) + \lambda_2^*(a)y^*(a), \tag{5.1c}$$

$$x^*(0) = 1, \quad y^*(0) = 0, \quad z^*(0) = 0, \tag{5.1d}$$

$$\lambda_1^*(a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)y^*(\sigma)\,\mathrm{d}\sigma, \tag{5.1e}$$

$$\lambda_2^*(a) = \int_0^\omega c(\sigma)\{\beta(a,\sigma)y^*(\sigma) + \gamma(a,\sigma)z^*(\sigma)\}\,\mathrm{d}\sigma. \tag{5.1f}$$

By formal integration, we obtain the following expressions:

$$x^*(a) = \mathrm{e}^{-\int_0^a \lambda_1^*(\sigma)\,\mathrm{d}\sigma},$$

$$y^*(a) = \int_0^a \mathrm{e}^{-\int_\sigma^a \lambda_2^*(\tau)\,\mathrm{d}\tau}\lambda_1^*(\sigma)\theta(\sigma)\,\mathrm{e}^{-\int_0^\sigma \lambda_1^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma,$$

$$z^*(a) = \int_0^a \left\{\lambda_1^*(b)(1-\theta(b))\,\mathrm{e}^{-\int_0^b \lambda_1^*(\sigma)\,\mathrm{d}\sigma} + \lambda_2^*(b)\int_0^b \mathrm{e}^{-\int_\sigma^b \lambda_2^*(\tau)\,\mathrm{d}\tau}\lambda_1^*(\sigma)\theta(\sigma)\,\mathrm{e}^{-\int_0^\sigma \lambda_1^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right\}\mathrm{d}b.$$

Substituting them into (5.1e) and (5.1f) gives the following nonlinear integral equations:

$$\lambda_1^*(a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma \lambda_2^*(\tau)\,\mathrm{d}\tau}\lambda_1^*(b)\theta(b)\,\mathrm{e}^{-\int_0^b \lambda_1^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma, \tag{5.2a}$$

$$\begin{aligned}\lambda_2^*(a) = {} & \int_0^\omega \beta(a,\sigma)c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma \lambda_2^*(\tau)\,\mathrm{d}\tau}\lambda_1^*(b)\theta(b)\,\mathrm{e}^{-\int_0^b \lambda_1^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma \\ & + \int_0^\omega \gamma(a,\sigma')c(\sigma')\left(\int_0^{\sigma'}\left\{\lambda_1^*(b)(1-\theta(b))\,\mathrm{e}^{-\int_0^b \lambda_1^*(\sigma)\,\mathrm{d}\sigma}\right.\right. \\ & \left.\left. + \lambda_2^*(b)\int_0^b \mathrm{e}^{-\int_\sigma^b \lambda_2^*(\tau)\,\mathrm{d}\tau}\lambda_1^*(\sigma)\theta(\sigma)\,\mathrm{e}^{-\int_0^\sigma \lambda_1^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right\}\mathrm{d}b\right)\mathrm{d}\sigma'.\end{aligned} \tag{5.2b}$$

Let Φ be the nonlinear operator on E defined by

$$\Phi_1(u)(a) := \int_0^\omega \alpha(a,\sigma)c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau}u_1(b)\theta(b)\,\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma, \tag{5.3a}$$

$$\begin{aligned}\Phi_2(u)(a) := {} & \int_0^\omega \beta(a,\sigma)c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau}u_1(b)\theta(b)\,\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma \\ & + \int_0^\omega \gamma(a,\sigma')c(\sigma')\left(\int_0^{\sigma'}\left\{u_1(b)(1-\theta(b))\,\mathrm{e}^{-\int_0^b u_1(\sigma)\,\mathrm{d}\sigma}\right.\right. \\ & \left.\left. + u_2(b)\int_0^b \mathrm{e}^{-\int_\sigma^b u_2(\tau)\,\mathrm{d}\tau}u_1(\sigma)\theta(\sigma)\,\mathrm{e}^{-\int_0^\sigma u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right\}\mathrm{d}b\right)\mathrm{d}\sigma',\end{aligned} \tag{5.3b}$$

$$\Phi(u) = {}^{\mathrm{t}}(\Phi_1(u),\ \Phi_2(u)), \quad u = {}^{\mathrm{t}}(u_1,\ u_2) \in E. \tag{5.3c}$$

We find that Φ is a positive operator on E and $\Phi(0) = 0$. Let $T : E \to E$ be the Fréchet derivative of Φ at 0, then T is given as follows:

$$Tu = {}^{\mathrm{t}}(T_1 u,\ T_2 u), \quad u \in E,$$

$$(T_1 u)(a) := \int_0^\omega \phi_1(a, b) u_1(b)\, \mathrm{d}b, \tag{5.4a}$$

$$(T_2 u)(a) := \int_0^\omega \phi_2(a, b) u_1(b)\, \mathrm{d}b, \tag{5.4b}$$

$$\phi_1(a, b) := \theta(b) \int_b^\omega \alpha(a, \sigma) c(\sigma)\, \mathrm{d}\sigma, \tag{5.4c}$$

$$\phi_2(a, b) := \theta(b) \int_b^\omega \beta(a, \sigma) c(\sigma)\, \mathrm{d}\sigma + (1 - \theta(b)) \int_b^\omega \gamma(a, \sigma) c(\sigma)\, \mathrm{d}\sigma. \tag{5.4d}$$

Now, let us define a linear operator $\tilde{T}$ on $L^1(0, \omega)$ by

$$(\tilde{T} u)(a) := T_1\, {}^{\mathrm{t}}(u, 0)(a) = \int_0^\omega \phi_1(a, b) u(b)\, \mathrm{d}b, \quad u \in L^1(0, \omega). \tag{5.5}$$

If $v = {}^{\mathrm{t}}(v_1, v_2) \in E$ is an eigenvector of T corresponding to $\lambda \neq 0$, then v_1 is an eigenvector of $\tilde{T}$ corresponding to λ since

$$(\tilde{T} v_1)(a) = (T_1 v)(a) = \lambda v_1(a).$$

Meanwhile, if $v_1 \in L^1(0, \omega)$ is an eigenvector of $\tilde{T}$ corresponding to $\lambda \neq 0$, then $v = {}^{\mathrm{t}}(v_1, v_2) \in E$ is an eigenvector of T corresponding to λ, where v_2 is expressed in terms of v_1 as follows:

$$v_2(a) := \lambda^{-1} \int_0^\omega \phi_2(a, b) v_1(b)\, \mathrm{d}b.$$

In particular, $v = {}^{\mathrm{t}}(v_1, v_2) \in E$ is a positive eigenvector of T corresponding to $\lambda \neq 0$ if and only if $v_1 \in L^1(0, \omega)$ is a positive eigenvector of $\tilde{T}$ corresponding to $\lambda \neq 0$. In addition, T does not have any eigenvectors corresponding to 1 in E_+ if and only if $\tilde{T}$ doesn't have any eigenvectors corresponding to 1 in $L^1_+(0, \omega)$.

In the following we make some assumptions deriving some important properties of $\tilde{T}$ and Φ:

Assumption 8. (i) $\theta(a) > 0$ and $c(a) > 0$ for almost all $a \in (0, \omega)$.

(ii) There exist a number $b_0 \in (0, \omega)$ and a number $\varepsilon_0 > 0$ such that $\alpha(a, \sigma) \geqq \varepsilon_0$ for almost all $(a, \sigma) \in (0, \omega) \times (\omega - b_0, \omega)$.

(iii) $\alpha(a, \sigma)$, $\beta(a, \sigma)$, $\gamma(a, \sigma)$, $\theta(a)$ are extended as 0 when a or σ is in $\mathbb{R} \backslash (0, \omega)$, then the following holds uniformly with respect to σ:

$$\lim_{h \to 0} \int_0^\omega |\alpha(a + h, \sigma) - \alpha(a, \sigma)|\, \mathrm{d}a = 0,$$

$$\lim_{h \to 0} \int_0^\omega |\beta(a + h, \sigma) - \beta(a, \sigma)|\, \mathrm{d}a = 0,$$

$$\lim_{h \to 0} \int_0^\omega |\gamma(a + h, \sigma) - \gamma(a, \sigma)|\, \mathrm{d}a = 0,$$

$$\lim_{h \to 0} \int_0^\omega |\theta(a + h) - \theta(a)|\, \mathrm{d}a = 0.$$

Now, let us summarize some ideas from positive operator theory. For more detail, the reader may refer to [40]. As for basic results on Banach lattices and positive operators, see, for example, [28].

Let X be an ordered vector space and X_+ its positive cone. For $u, v \in X$, $u \leqslant v$ if and only if $v - u \in X_+$. X_+ is called *total* if $X_+ - X_+$ is dense in X. Let Y be an ordered vector space and Y_+ its positive cone. The dual of X is denoted by X^*. An operator $T : X \to Y$ is called *positive* if $TX_+ \subset Y_+$. A subset of X^* consisting of all positive linear functionals on X is called the *dual cone*, which we denote by X_+^*.

We write $F(u)$ as $\langle F, u\rangle$ for $u \in X$, $F \in X^*$. A positive linear functional $f \in X_+^*$ is called *strictly positive* if $\langle f, u\rangle > 0$ for all $u \in X_+\backslash\{0\}$. $u \in X_+$ is called *nonsupporting point* if $\langle f, u\rangle > 0$ for all $f \in X_+^*\backslash\{0\}$. Let T be a positive linear bounded operator on X. T is called *nonsupporting* if for all $u \in X_+\backslash\{0\}$ and for all $f \in X_+^*\backslash\{0\}$ there exists a positive integer p depending on u, f such that $\langle f, T^n u\rangle > 0$ whenever $n \geqq p$. T is called *semi-nonsupporting* if for all $u \in X_+\backslash\{0\}$ and for all $f \in X_+^*\backslash\{0\}$ there exists a positive integer p depending on u, f such that $\langle f, T^p u\rangle > 0$. It is obvious that, if T is nonsupporting, it is semi-nonsupporting.

Given a linear operator T on E, $r(T)$ stands for its spectral radius, $s(T)$ the spectral bound of T, $\rho(T)$ the resolvent set of T, $\sigma(T)$ the spectrum of T and $P_\sigma(T)$ the point spectrum of T. T^* stands for the dual operator of T.

Theorem 9 (*Klein and Rutman [22]*). *Let X be a Banach space and X_+ its positive cone, which is total. Let T be a compact positive linear operator on X satisfying $r(T) > 0$.*

Then $r(T)$ is an eigenvalue of T and the corresponding eigenvector $\psi \in X_+\backslash\{0\}$ exists.

Theorem 10 (*Sawashima [40]*). *Let X be a Banach space and X_+ its positive cone, which is total. Let T be a semi-nonsupporting positive linear operator on X. We assume that $r(T)$ is a pole of the resolvent $R(\lambda, T) = (\lambda - T)^{-1}$. Then the following holds*:

(i) *$r(T) \in P_\sigma(T)\backslash\{0\}$ and $r(T)$ is a simple pole of the resolvent.*
(ii) *The eigenspace corresponding to $r(T)$ is one-dimensional subspace of X spanned by a quasi-interior point $\psi \in X_+$. If $\phi \in X_+$, $c \in \mathbb{R}$ satisfy $T\phi = c\phi$, then $c = r(T)$ and there exists a real number $k > 0$ such that $\phi = k\psi$.*
(iii) *$r(T) \in P_\sigma(T^*)$ and the eigenspace of T^* corresponding to $r(T)$ is a one-dimensional subspace of X^* spanned by a strictly positive functional f.*

By applying the method of the Krasnoselskii's fixed point theorem [21, Theorem 4.11], we have the following (see [17, Proposition 4.6]):

Theorem 11. *Let Ψ be a positive nonlinear operator on a real Banach space X with a positive cone X_+. We assume that $\Psi(0) = 0$ and Ψ has the strong Fréchet derivative $T := \Psi'(0)$, which has a positive eigenvector $v_0 \in X_+$ corresponding to the eigenvalue $\lambda_0 > 1$ and no eigenvector corresponding to the eigenvalue 1 in X_+. In addition, we assume that Ψ is completely continuous and $\Psi(X_+)$ is bounded.*

Then, Ψ has a non-zero positive fixed point.

Let X be an ordered vector space. X is called a *vector lattice* if $x \vee y := \sup\{x, y\}$, $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in X$. The norm $\| \ \|$ on a vector lattice X is called a *lattice norm* if $|x| \leqslant |y|$ implies $\|x\| \leqq \|y\|$, where $|x| := x \vee (-x)$. A *Banach lattice* is a Banach space X endowed with an ordering $\leqslant$ such that $(X, \leqslant)$ is a vector lattice and the norm on X is a lattice norm.

The following comparison theorem is due to [26].

Theorem 12. *Let X be a Banach lattice. For positive linear bounded operators S, T on X, the following holds*:

(i) *$S \leqslant T$ implies $r(S) \leqslant r(T)$.*
(ii) *Moreover, if S, T are semi-nonsupporting and compact, then $S \leqslant T$, $S \neq T$, $r(T) \neq 0$ imply $r(S) < r(T)$.*

After the above preparations, we firstly consider the properties of $\tilde{T}$ defined by (5.5).

Lemma 13. *$\tilde{T}$ is nonsupporting.*

Proof. Since $\phi_1(a,b) \geqq 0$ holds for all $a, b \in (0,\omega)$, if $u \geqq 0$ then $\tilde{T}u \geqq 0$, that is, $\tilde{T}$ is positive. Let

$$s(\xi) := \begin{cases} \varepsilon_0 & \text{if } \xi \in (\omega - b_0, \omega), \\ 0 & \text{otherwise,} \end{cases} \tag{5.6}$$

then $\alpha(a,\sigma) \geqq s(\sigma)$ for all $a, \sigma \in \mathbb{R}$. Let f_0 be a linear functional on $L^1(0,\omega)$ defined by

$$\langle f_0, u \rangle := \int_0^\omega \theta(b) \left(\int_b^\omega s(\sigma)c(\sigma)\,\mathrm{d}\sigma \right) u(b)\,\mathrm{d}b. \tag{5.7}$$

The assumption $s, c, \theta \in L_+^\infty(0,\omega)$ implies that $f_0 \in (L^1(0,\omega))_+^*$. Moreover, since

$$\int_b^\omega s(\sigma)c(\sigma)\,\mathrm{d}\sigma > 0 \quad \text{for all } b \in (0,\omega)$$

holds, we find that f_0 is strictly positive.

Let $u \in L_+^1(0,\omega)$. $\alpha(a,\sigma) \geqq s(\sigma)$ yields that $\tilde{T}u \geqq \langle f_0, u\rangle e$, where $e(a) \equiv 1,\ a \in \mathbb{R}$. Hence

$$\tilde{T}^{n+1}u \geqq \langle f_0, u\rangle \langle f_0, e\rangle^n e \quad \text{for all } n \in \mathbb{N},$$

and for all $F \in (L^1(0,\omega))_+^* \backslash \{0\}$ and for all $n \in \mathbb{N}$ the following holds:

$$\langle F, \tilde{T}^n u\rangle \geqq \langle f_0, u\rangle \langle f_0, e\rangle^{n-1} \langle F, e\rangle > 0.$$

Therefore, $\tilde{T}$ is nonsupporting. □

Lemma 14. *$\tilde{T}$ is compact.*

Proof. First observe that

$$\begin{aligned} \int_0^\omega |\phi_1(a+h,b) - \phi_1(a,b)|\,\mathrm{d}a &= \int_0^\omega \left| \theta(b) \int_b^\omega (\alpha(a+h,\sigma) - \alpha(a,\sigma))c(\sigma)\,\mathrm{d}\sigma \right| \mathrm{d}a \\ &\leqq \int_0^\omega \left(\int_b^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)| c(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}a \\ &\leqq \int_0^\omega \left\{ \int_0^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)|\,\mathrm{d}a \right\} c(\sigma)\,\mathrm{d}\sigma. \end{aligned}$$

Assumption 8 implies that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)|\,\mathrm{d}a < \varepsilon \quad \text{for all } \sigma \in \mathbb{R}$$

holds whenever $|h| < \delta$. This gives

$$\int_0^\omega |\phi_1(a+h,b) - \phi_1(a,b)|\,\mathrm{d}a < \varepsilon \int_0^\omega c(\sigma)\,\mathrm{d}\sigma = \varepsilon.$$

Now, let $\{u_\lambda \mid \lambda \in \Lambda\}$ be a bounded subset of $L^1(0,\omega)$. Note that $\|u_\lambda\| < C_0$ for some positive constant C_0. Then we have

$$\begin{aligned} \int_0^\omega |(\tilde{T}u_\lambda)(a+h) - (\tilde{T}u_\lambda)(a)|\,\mathrm{d}a &\leqq \int_0^\omega \left(\int_0^\omega |\phi_1(a+h,b) - \phi_1(a,b)|\,\mathrm{d}a \right) |u_\lambda(b)|\,\mathrm{d}b \\ &\leqq \varepsilon \|u_\lambda\| < C_0 \varepsilon. \end{aligned}$$

From the above evaluation and the well-known compactness criteria in L^1 (see, for example, [51, p. 275]), we see that $\{\tilde{T}u_\lambda \mid \lambda \in \Lambda\}$ is precompact in $L^1(0,\omega)$, hence $\tilde{T}$ is compact. □

According to Lemmas 13, 14 and Theorems 9, 10, the spectral radius $r(\tilde{T})$ of the operator $\tilde{T}$ is the only positive eigenvalue with a positive eigenvector $u_0 \in L^1_+(0,\omega)$ which is a nonsupporting point. Moreover, $r(\tilde{T})$ is an eigenvalue of $\tilde{T}^*$ with a strictly positive eigenfunctional F_0.

Secondly, let us consider the properties of Φ defined by (5.3c).

Lemma 15. *Φ is completely continuous, and there exists a constant $M_0 > 0$ such that $\|\Phi(u)\| \leqq M_0$ whenever $u \in E_+$.*

Proof. Let $\{u_\lambda = {}^{\mathrm{t}}(u^1_\lambda, u^2_\lambda) \in E \mid \lambda \in \Lambda\}$ be a bounded subset of E. By definition $\|u_\lambda\| < C_0$ holds for some constant $C_0 > 0$. Then we have

$$\int_0^\omega |\Phi_1(u_\lambda)(a+h) - \Phi_1(u_\lambda)(a)|\,\mathrm{d}a \leqq \int_0^\omega \mathrm{d}a \int_0^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)| c(\sigma) \times \left\{ \int_0^\sigma \mathrm{e}^{-\int_b^\sigma u^2_\lambda(\tau)\,\mathrm{d}\tau} |u^1_\lambda(b)| \theta(b)\, \mathrm{e}^{-\int_0^b u^1_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b \right\} \mathrm{d}\sigma.$$

Now, if $0 \leqq b \leqq \sigma \leqq \omega$ then it follows from

$$\left| -\int_0^b u^1_\lambda(\tau)\,\mathrm{d}\tau \right| \leqq \int_0^b |u^1_\lambda(\tau)|\,\mathrm{d}\tau \leqq \|u^1_\lambda\| \leqq C_0$$

that $\mathrm{e}^{-\int_0^b u^1_\lambda(\tau)\,\mathrm{d}\tau} \leqq \mathrm{e}^{C_0}$. Similarly we have $\mathrm{e}^{-\int_b^\sigma u^2_\lambda(\tau)\,\mathrm{d}\tau} \leqq \mathrm{e}^{C_0}$. Hence it follows that

$$\begin{aligned} &\int_0^\omega |\Phi_1(u_\lambda)(a+h) - \Phi_1(u_\lambda)(a)|\,\mathrm{d}a \\ &\quad \leqq \mathrm{e}^{2C_0} \int_0^\omega \mathrm{d}a \int_0^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)| c(\sigma) \left\{ \int_0^\sigma |u^1_\lambda(b)| \theta(b)\,\mathrm{d}b \right\} \mathrm{d}\sigma \\ &\quad = \mathrm{e}^{2C_0} \int_0^\omega \mathrm{d}a \int_0^\omega \mathrm{d}b \left\{ \int_b^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)| c(\sigma)\,\mathrm{d}\sigma \right\} \theta(b) |u^1_\lambda(b)|\,\mathrm{d}b. \end{aligned}$$

Assumption 8 implies that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)|\,\mathrm{d}a < \varepsilon \quad \text{for all } \sigma \in \mathbb{R} \text{ whenever } |h| < \delta.$$

With this, a similar estimation as in the proof of Lemma 14 gives

$$\int_0^\omega |\Phi_1(u_\lambda)(a+h) - \Phi_1(u_\lambda)(a)|\,\mathrm{d}a < \mathrm{e}^{2C_0} C_0 \varepsilon.$$

Hence, $A_1 := \{\Phi_1(u_\lambda) \mid \lambda \in \Lambda\}$ is precompact in $L^1(0,\omega)$. Similarly we find that $A_2 := \{\Phi_2(u_\lambda) \mid \lambda \in \Lambda\}$ is also precompact in $L^1(0,\omega)$.

Let $A := \{\Phi(u_\lambda) \mid \lambda \in \Lambda\}$, then $A \subset A_1 \times A_2$ implies $\bar{A} \subset \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$, where $\bar{A}$ stands for the closure of A. Since $\overline{A_1}$, $\overline{A_2}$ are compact, $\overline{A_1 \times A_2}$ is compact, so $\bar{A}$, which is a closed subset of $\overline{A_1 \times A_2}$, is also compact. The above proves that Φ is completely continuous.

Next, for any $u = {}^{\mathrm{t}}(u_1, u_2) \in E_+$, it follows that

$$\begin{aligned}
\|\Phi_1(u)\| &= \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \left(\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau} u_1(b)\theta(b)\,\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b \right) \mathrm{d}\sigma \right\} \\
&\leqq \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \left(\int_0^\sigma u_1(b)\,\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b \right) \mathrm{d}\sigma \right\} \\
&= \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \Big[-\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau} \Big]_{b=0}^{b=\sigma} \mathrm{d}\sigma \right\} \\
&= \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \left(1 - \mathrm{e}^{-\int_0^\sigma u_1(\tau)\,\mathrm{d}\tau} \right) \mathrm{d}\sigma \right\} \\
&\leqq \alpha^\infty \int_0^\omega \mathrm{d}a \int_0^\omega c(\sigma)\,\mathrm{d}\sigma = \alpha^\infty \omega.
\end{aligned}$$

Similarly we find that

$$\|\Phi_2(u)\| \leqq (\beta^\infty + 2\gamma^\infty)\omega.$$

Let $M_0 := (\alpha^\infty + \beta^\infty + 2\gamma^\infty)\omega$, then we see that $\|\Phi(u)\| \leqq M_0$. □

After the above preparations, we can prove the following threshold results:

Theorem 16. (i) *If $r(\tilde{T}) \leqq 1$, then $u = 0$ is the only solution of $u = \Phi(u)$ in E_+, i.e., RFE is the only equilibrium of the system.*

(ii) *If $r(\tilde{T}) > 1$, $u = \Phi(u)$ has at least one solution in $E_+\backslash\{0\}$, i.e., the system has at least one REE.*

Proof. Suppose $r(\tilde{T}) \leqq 1$. We assume that $u = \Phi(u)$ for some $u = {}^{\mathrm{t}}(u_1, u_2) \in E_+\backslash\{0\}$. Then we have $u_1 = \Phi_1(u)$. Note that $u_1 \in L^1_+(0,\omega)\backslash\{0\}$, because $u_1 = 0$ implies $u = \Phi(u) = 0$, which contradicts the assumption $u \in E_+\backslash\{0\}$.

Since $0 < \sigma < b$ implies $\mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau}\,\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau} < 1$, it follows that

$$\tilde{T}u_1 - u_1 = \tilde{T}u_1 - \Phi_1(u) \in L^1_+(0,\omega)\backslash\{0\},$$

hence $\langle F_0, \tilde{T}u_1 - u_1 \rangle > 0$ because F_0 is strictly positive. On the other hand, observe

$$\begin{aligned}
\langle F_0, \tilde{T}u_1 - u_1 \rangle &= \langle \tilde{T}^* F_0, u_1 \rangle - \langle F_0, u_1 \rangle \\
&= (r(T) - 1)\langle F_0, u_1 \rangle,
\end{aligned}$$

Since $r(T) - 1 \leqq 0$ and $\langle F_0, u_1 \rangle > 0$, we have $\langle F_0, \tilde{T}u_1 - u_1 \rangle \leqq 0$, which is a contradiction. Therefore $u = 0$ is the only solution of $u = \Phi(u)$ in E_+.

Next suppose $r(\tilde{T}) > 1$. Then we see that T has a positive eigenvector corresponding to $r(\tilde{T})$ and no eigenvector in E_+ corresponding to the eigenvalue 1. In addition, Φ has the properties stated in Lemma 15. Therefore, Theorem 11 implies that Φ has a non-zero positive fixed point. □

From the above result, we can regard $r(\tilde{T})$ as the basic reproduction number of this system, which is denoted by R_0 in the following.

6. Stability of the RFE

In this section, we consider the stability of RFE.

The first element $y(t)$ of $u(t)$ in (4.7) satisfies the abstract equation on $L^1(0,\omega)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = By(t) + Py(t)\cdot(1-y(t)-z(t)) - \lambda_2[a \mid y, z]y(t),$$

$$y(0) = y_0 \in L^1(0,\omega),$$

where $z(t)$ is regarded as given and we define as follows:

$$\mathscr{D}(B) := \{u \in L^1(0,\omega) \mid u \quad \text{is absolutely continuous on } [0,\omega],\ u(0)=0\},$$

$$Bu(a) := -\frac{\mathrm{d}}{\mathrm{d}a}u(a),\ \ u \in \mathscr{D}(B),$$

$$Pu(a) := \theta(a)\int_0^\omega \alpha(a,\sigma)c(\sigma)u(\sigma)\,\mathrm{d}\sigma.$$

Let $\{T(t)\}_{t\geqq 0}$ be the nilpotent translation semigroup on $L^1(0,\omega)$ defined by (4.8). Note that B generates the semigroup. Let $C(t)$ be the bounded operator on $L^1(0,\omega)$ defined by

$$C(t)u := (Pu)\cdot(1-y(t)-z(t)) - \lambda_2[a \mid y, z]y(t).$$

For any $u \in L^1_+(0,\omega)$, we have $C(t)u \leqslant Pu$, because $x(t) = 1-y(t)-z(t) \leqslant 1$, $Pu(a) \geqq 0$ and $\lambda_2[a \mid y,z]y(t) \leqslant 0$. Since $T(t)$, $t \geqq 0$ is a positive operator on $L^1(0,\omega)$, we have

$$T(s)C(t)u \leqslant T(s)Pu \quad \text{for all } s,t \geqq 0 \ \text{ and } \ \text{for all } u \in L^1_+(0,\omega).$$

Hence, rewriting the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = By(t) + C(t)y(t)$$

with the variation-of-constants formula gives

$$\begin{aligned} y(t) &= T(t)y_0 + \int_0^t T(t-s)C(s)y(s)\,\mathrm{d}s \\ &\leqslant T(t)y_0 + \int_0^t T(t-s)Py(s)\,\mathrm{d}s \end{aligned}$$

for all $t \geqq 0$. Hence, if we denote the C_0-semigroup generated by $B+P$ by $\{W(t)\}_{t\geqq 0}$, we obtain

$$0 \leqslant y(t) \leqslant W(t)y_0 \quad \text{for all } t \in \mathbb{R}_+. \tag{6.1}$$

Now, let us consider the spectrum of $B+P$.

Lemma 17. *For any $\lambda \in \rho(B+P)$, the resolvent $R(\lambda, B+P)$ is compact and expressed with a compact operator $\hat{T}_\lambda$ as follows*:

$$(R(\lambda, B+P)u)(a) = \int_0^a \mathrm{e}^{-\lambda(a-\tau)}((I-\hat{T}_\lambda)^{-1}u)(\tau)\,\mathrm{d}\tau, \tag{6.2}$$

$$\sigma(B+P) = P_\sigma(B+P) = \{\lambda \in \mathbb{C} \mid 1 \in \sigma(\hat{T}_\lambda)\}, \tag{6.3}$$

where $u \in L^1(0,\omega)$, $a \in \mathbb{R}_+$.

Proof. For $u \in L^1(0,\omega)$, let $v := R(\lambda, B+P)u$ and let us express v in terms of u.

$(\lambda - (B + P))v = u$ is rewritten as follows:

$$v'(a) = -\lambda v(a) + X(a) + u(a),$$

where

$$X(a) := \theta(a) \int_0^\omega \alpha(a, \sigma) c(\sigma) v(\sigma) \, d\sigma, \quad a \in \mathbb{R}_+.$$

$v \in \mathscr{D}(B)$ implies $v(0) = 0$, so the above integral–differential equation can be transformed as follows:

$$v(a) = \int_0^a e^{-\lambda(a-\tau)} (X(\tau) + u(\tau)) \, d\tau.$$

If we multiply $\alpha(\tau', a)c(a)$ on both sides of the above equation, integrate with respect to a on $(0, \omega)$ and multiply $\theta(\tau')$, we get

$$\begin{aligned} X(\tau') &= \theta(\tau') \int_0^\omega \alpha(\tau', a) c(a) \left(\int_0^a e^{-\lambda(a-\tau)} (X(\tau) + u(\tau)) \, d\tau \right) da \\ &= \int_0^\omega \tilde{\phi}_\lambda(\tau', \tau)(X(\tau) + u(\tau)) \, d\tau, \end{aligned}$$

where

$$\tilde{\phi}_\lambda(a, b) := \theta(a) \int_b^\omega e^{-\lambda(\sigma - b)} \alpha(a, \sigma) c(\sigma) \, d\sigma.$$

For $f \in L^1(0, \omega)$, let

$$(\hat{T}_\lambda f)(a) := \int_0^\omega \tilde{\phi}_\lambda(a, b) f(b) \, db,$$

then we get the equation with respect to X as follows:

$$X = \hat{T}_\lambda X + \hat{T}_\lambda u \quad \Leftrightarrow \quad (I - \hat{T}_\lambda)X = \hat{T}_\lambda u.$$

Hence, X is uniquely determined if and only if $1 \in \rho(\hat{T}_\lambda)$, when

$$X = (I - \hat{T}_\lambda)^{-1} \hat{T}_\lambda u.$$

Noting that

$$X + u = (I - \hat{T}_\lambda)^{-1}(\hat{T}_\lambda + I - \hat{T}_\lambda)u = (I - \hat{T}_\lambda)^{-1} u,$$

we derive Eq. (6.2).

With Assumption 8, the same evaluation as in the proof of Lemma 14 yields that $\hat{T}_\lambda$ is compact. From (6.2) we see that $R(\lambda, B + P)$ is the integral operator whose integral kernel is a continuous function, hence it is a compact operator on $L^1(0, \omega)$. So, $B + P$ has a compact resolvent, from which (6.3) follows. □

Let $\Sigma := \{\lambda \in \mathbb{C} \mid 1 \in \sigma(\hat{T}_\lambda)\}$.

Assumption 18. There exist an interval $(b_1, b_2) \subset (\omega - b_0, \omega)$ and a number $\varepsilon' > 0$ for which

$$\theta(b) \geqq \varepsilon' \quad \text{for all } b \in (b_1, b_2).$$

Theorem 19. *$r(\hat{T}_{s(B+P)}) = 1$ holds, where $s(B + P)$ means the spectral bound of $B + P$.*

To show the above theorem, we prepare the following lemmas in advance:

Lemma 20. $\lim_{\lambda\to-\infty} r(\hat{T}_\lambda) = +\infty$ *and* $\lim_{\lambda\to\infty} r(\hat{T}_\lambda) = 0$.

Proof. We define $s(\xi)$ by (5.6) and f_λ by

$$\langle f_\lambda, u\rangle := \int_0^\omega \theta(b)\left(\int_b^\omega s(\sigma)c(\sigma)\,\mathrm{e}^{-\lambda(\sigma-b)}\,\mathrm{d}\sigma\right)u(b)\,\mathrm{d}b.$$

Note that, when $\lambda=0$, f_λ equals to f_0 defined by (5.7). The same argument as in the proof of Lemma 13 shows that f_λ is a strictly positive linear functional on $L^1(0,\omega)$ and

$$\hat{T}_\lambda u \geqslant \langle f_\lambda, u\rangle e,$$

from which it follows that $\hat{T}_\lambda$ is nonsupporting. In addition, in Lemma 17 we saw that $\hat{T}_\lambda$ is compact. Hence, $r(\hat{T}_\lambda)$ is an eigenvalue of $\hat{T}_\lambda^*$ with a strictly positive eigenfunctional F_λ. Then we have

$$\langle F_\lambda, \hat{T}_\lambda u\rangle \geqslant \langle f_\lambda, u\rangle\langle F_\lambda, e\rangle \Leftrightarrow r(\hat{T}_\lambda)\langle F_\lambda, u\rangle \geqslant \langle f_\lambda, u\rangle\langle F_\lambda, e\rangle.$$

Suppose $u=e$, then F_λ is strictly positive, from which $\langle F_\lambda, e\rangle > 0$ follows. Hence we get

$$r(\hat{T}_\lambda) \geqq \langle f_\lambda, e\rangle.$$

With Assumptions 8 and 18, we see that

$$\begin{aligned}\langle f_\lambda, e\rangle &= \int_0^\omega s(\sigma)c(\sigma)\left(\int_0^\sigma \theta(b)\,\mathrm{e}^{-\lambda(\sigma-b)}\,\mathrm{d}b\right)\mathrm{d}\sigma\\ &\geqq \int_{b_1}^\omega \varepsilon c(\sigma)\left(\int_{b_1}^{b_2} \varepsilon'\,\mathrm{e}^{-\lambda(\sigma-b)}\,\mathrm{d}b\right)\mathrm{d}\sigma\\ &= \int_{b_1}^\omega \varepsilon c(\sigma)\varepsilon'\,\frac{\mathrm{e}^{-\lambda(\sigma-b_1)}(1-\mathrm{e}^{-\lambda(b_1-b_2)})}{-\lambda}\,\mathrm{d}\sigma.\end{aligned}$$

As λ tends to $-\infty$, $\mathrm{e}^{-\lambda(\sigma-b_1)}(1-\mathrm{e}^{-\lambda(b_1-b_2)})/(-\lambda)$ tends to infinity, hence $\langle f_\lambda, e\rangle$ tends to infinity from Fatou's lemma, and $r(\hat{T}_\lambda)$ also tends to infinity.

Next, observe that

$$(\hat{T}_\lambda u)(a) \leqq \alpha^\infty \int_0^\omega \left(\int_b^\omega \mathrm{e}^{-\lambda(\sigma-b)}c(\sigma)\,\mathrm{d}\sigma\right)u(b)\,\mathrm{d}b.$$

The same argument as above gives

$$\begin{aligned}r(\hat{T}_\lambda) &\leqq \alpha^\infty \int_0^\omega \left(\int_b^\omega \mathrm{e}^{-\lambda(\sigma-b)}c(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}b\\ &= \alpha^\infty \int_0^\omega c(\sigma)\,\frac{1-\mathrm{e}^{-\lambda\sigma}}{\lambda}\,\mathrm{d}\sigma.\end{aligned}$$

Then, Lebesgue's dominated convergence theorem yields that $\int_0^\omega c(\sigma)(1-\mathrm{e}^{-\lambda\sigma})/\lambda\,\mathrm{d}\sigma \to 0$ as $\lambda\to\infty$, from which we have $r(\hat{T}_\lambda)\to 0$. □

Lemma 21. $\lambda \mapsto r(\hat{T}_\lambda)$ *is continuous and strictly decreasing. In addition,* $r(\hat{T}_\lambda)=1$ *has the unique root* λ_0 *in* $\mathbb{R}$, *which satisfies* $\lambda_0 \in \Sigma$.

Proof. $r(\hat{T}_\lambda)$ is a point spectrum of $\hat{T}_\lambda$ but not a accumulating point of $\sigma(\hat{T}_\lambda)$. The mapping $\mathbb{R} \ni \lambda \mapsto \hat{T}_\lambda \in \mathscr{L}(L^1(0,\omega))$, where $\mathscr{L}(L^1(0,\omega))$ is endowed with the topology induced by the operator norm, is continuous. Hence $\lambda \mapsto r(\hat{T}_\lambda)$ is continuous (see, for example, [19, IV-Section 3.5]).

In addition, $\hat{T}_\lambda$ is nonsupporting and compact for all $\lambda \in \mathbb{R}$, and if $\lambda < \lambda'$ then $\hat{T}_\lambda \geqslant \hat{T}_{\lambda'}$, $\hat{T}_\lambda \neq \hat{T}_{\lambda'}$ and $r(\hat{T}_\lambda) > 0$, so Theorem 12 implies that $r(\hat{T}_\lambda) > r(\hat{T}_{\lambda'})$.

Hence we find that $\lambda \mapsto r(\hat{T}_\lambda)$ is continuous and strictly decreasing. From it and Lemma 20, we see that $r(\hat{T}_\lambda) = 1$ has the unique root λ_0 in $\mathbb{R}$. Since $\hat{T}_{\lambda_0}$ is nonsupporting and compact, we have $1 = r(\hat{T}_{\lambda_0}) \in \sigma(\hat{T}_{\lambda_0})$, which yields that $\lambda_0 \in \Sigma$. $\square$

We referred the idea of the proof of the following lemma to [14, Theorem 6.13].

Lemma 22. Re $\lambda < \lambda_0$ *for all* $\lambda \in \Sigma \backslash \{\lambda_0\}$.

Proof. For all $\lambda \in \Sigma$, $1 = \sigma(\hat{T}_\lambda)$ and so $\hat{T}_\lambda v = v$ for some $v \in L^1(0, \omega)$.

For $u \in L^1(0, \omega)$ we define $|u| \in L^1_+(0, \omega)$ by $|u|(a) := |u(a)|$. Then we see that

$$\begin{aligned} |v|(a) &= |\hat{T}_\lambda v(a)| \\ &\leqq \int_0^\omega |v(b)|\theta(a) \left(\int_b^\omega |\,\mathrm{e}^{-\lambda(\sigma-b)}|\alpha(a,\sigma)c(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}b \\ &= \hat{T}_{\mathrm{Re}\,\lambda}|v|(a) \quad \text{for all } a \in (0,\omega), \end{aligned}$$

i.e., $|v| = |\hat{T}_\lambda v| \leqslant \hat{T}_{\mathrm{Re}\,\lambda}|v|$. Let $F_{\mathrm{Re}\,\lambda}$ be a strictly positive eigenfunctional of $\hat{T}^*_{\mathrm{Re}\,\lambda}$ corresponding to $r(\hat{T}_{\mathrm{Re}\,\lambda})$, then we have

$$r(\hat{T}_{\mathrm{Re}\,\lambda})\langle F_{\mathrm{Re}\,\lambda}, |v|\rangle \geqq \langle F_{\mathrm{Re}\,\lambda}, |v|\rangle.$$

From $\langle F_{\mathrm{Re}\,\lambda}, |v|\rangle > 0$ it follows that $r(\hat{T}_{\mathrm{Re}\,\lambda}) \geqq 1$. Since $\lambda \mapsto r(\hat{T}_\lambda)$ is strictly decreasing by Lemma 21, we obtain that Re $\lambda \leqq \lambda_0$.

Now suppose Re $\lambda = \lambda_0$. Then we have $|v| \leqslant \hat{T}_{\lambda_0}|v|$. If $\hat{T}_{\lambda_0}|v| > |v|$, letting F_{λ_0} be a strictly positive eigenfunctional of $\hat{T}^*_{\lambda_0}$ corresponding to $r(\hat{T}_{\lambda_0}) = 1$, we see that

$$\langle F_{\lambda_0}, |v|\rangle < \langle F_{\lambda_0}, \hat{T}_{\lambda_0}|v|\rangle = \langle \hat{T}^*_{\lambda_0} F_{\lambda_0}, |v|\rangle = \langle F_{\lambda_0}, |v|\rangle,$$

which is a contradiction. Hence we have $\hat{T}_{\lambda_0}|v| = |v|$. Let $v_0 \in L^1_+(0, \omega)$ be a nonsupporting eigenvector of $\hat{T}_{\lambda_0}$ corresponding to the eigenvalue 1, then $|v| = c_0 v_0$ for some $c_0 > 0$. Hence

$$v(a) = v_0(a) \cdot c_0\,\mathrm{e}^{\mathrm{i}k(a)}$$

for some function $k : (0, \omega) \to \mathbb{R}$. Observe

$$|\hat{T}_\lambda v|(a) = |v|(a) = c_0 v_0(a) = c_0 \hat{T}_{\lambda_0} v_0(a),$$

which is rewritten in terms of integration as follows:

$$\left| \int_0^\omega \mathrm{d}b \int_b^\omega g(\sigma, b)\,\mathrm{d}\sigma \right| = \int_0^\omega \mathrm{d}b \int_b^\omega |g(\sigma, b)|\,\mathrm{d}\sigma$$
$$\text{where } g(\sigma, b) := \theta(a) v_0(b) c_0\,\mathrm{e}^{\mathrm{i}k(b)}\,\mathrm{e}^{-\lambda(\sigma-b)}\alpha(a,\sigma)c(\sigma). \tag{6.4}$$

From (6.4) it follows that $g(\sigma, b) = g_0(\sigma, b)\,\mathrm{e}^{\mathrm{i}k_1}$ for some positive function $g_0(\sigma, b)$ and some constant real number k_1 independent of σ, b (see, for example, [39, Theorem 1.33]). Hence we may assume

$$k(b) - (\mathrm{Im}\,\lambda)(\sigma - b) = k_1,$$

then the following holds:

$$\begin{aligned}\hat{T}_\lambda v(a) &= \int_0^\omega \theta(a) v_0(b) c_0 \,\mathrm{e}^{\mathrm{i}k(b)} \left(\int_b^\omega \mathrm{e}^{-\lambda(\sigma-b)} \alpha(a,\sigma) c(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}b \\ &= \int_0^\omega \theta(a) v_0(b) c_0 \left(\int_b^\omega \mathrm{e}^{-\mathrm{Re}\,\lambda(\sigma-b)+\mathrm{i}k_1} \alpha(a,\sigma) c(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}b \\ &= \mathrm{e}^{\mathrm{i}k_1} c_0 \hat{T}_{\lambda_0} v_0(a) \\ &= c_0 \,\mathrm{e}^{\mathrm{i}k_1} v_0(a),\end{aligned}$$

from which it follows that

$$c_0 \,\mathrm{e}^{\mathrm{i}k(a)} v_0(a) = v(a) = \hat{T}_\lambda v(a) = c_0 \,\mathrm{e}^{\mathrm{i}k_1} v_0(a),$$

i.e., $\mathrm{e}^{(\mathrm{Im}\,\lambda)(\sigma-b)} = 1$ holds whenever $0 \leqq b \leqq \sigma \leqq \omega$. Therefore we have $\mathrm{Im}\,\lambda = 0$ and this completes the proof. □

Proof of Theorem 19. Lemma 17 implies that $s(B+P) = \sup\{\mathrm{Re}\,\lambda \mid \lambda \in \Sigma\}$, hence from Lemma 22 we have $\lambda_0 = s(B+P)$. Lemma 21 implies that $1 = r(\hat{T}_{\lambda_0})$, therefore we obtain the conclusion. □

After the above preparations we can prove the global stability for RFE in the case $R_0 < 1$.

Theorem 23. *If* $R_0 = r(\tilde{T}) < 1$, *the trivial equilibrium* $(x, y, z) = (1, 0, 0)$ *of system* (4.4) *is globally asymptotically stable in the state space* $\{(x, y, z) \in (L^1_+(0,\omega)^3) \mid x + y + z = 1\}$.

Proof. The translation C_0-semigroup $\{T(t)\}_{t \geqq 0}$ is nilpotent, so it is eventually norm continuous. In addition, with Assumption 8, the same evaluation as in the proof of Lemma 14 yields that P is compact. Hence $\{W(t)\}_{t \geqq 0}$, generated by $B + P$, is also eventually norm continuous [28, A-II, Theorem 1.30]. Therefore we can apply the spectral mapping theorem [28, A-III, Theorem 6.6] and we obtain $\omega_0(B+P) = s(B+P)$, where $\omega_0(B+P)$ denotes the growth bound of the semigroup $\{W(t)\}_{t \geqq 0}$.

On the other hand, observe that

$$r(\hat{T}_0) = r(\tilde{T}) < 1 = r(\hat{T}_{s(B+P)}).$$

Since $\lambda \mapsto r(\hat{T}_\lambda)$ is strictly decreasing, we have $s(B+P) < 0$.

Hence, $\omega_0(B+P) < 0$ and $\|W(t)\| \to 0$ as $t \to \infty$. From (6.1) it follows that $y(t) \to 0$ as $t \to \infty$. It is easily seen from (4.4) that $\lambda_1(t, \cdot) \to 0$ and $x(t) \to 1$. This completes the proof. □

Finally, in the case $R_0 > 1$, the following holds:

Theorem 24. *If* $R_0 > 1$, *then the trivial equilibrium* $(x, y, z) = (1, 0, 0)$ *of system* (4.4) *is unstable*.

Proof. Let us consider the abstract equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = By(t) + Py(t) \cdot (1 - y(t) - z(t)) - \lambda_2[a \mid y, z] y(t),$$

where $y(0) = y_0 \in L^1(0, \omega)$. The linearization of its right-hand side at 0 gives $(B+P)y(t)$. Since

$$r(\hat{T}_0) = R_0 > 1 = r(\hat{T}_{s(B+P)}),$$

we have $s(B+P) > 0$, from which it follows that $B + P$ has an eigenvalue whose real part is positive. This completes the proof. □

7. Stability of the REE

Let us investigate the local stability of REE (x^*, y^*, z^*) under the condition $R_0 > 1$ and the proportionate mixing assumption (PMA):

Assumption 25. α, β, γ are expressed as follows:

$$\alpha(a,\sigma) = \alpha_1(a)\alpha_2(\sigma), \quad \beta(a,\sigma) = \beta_1(a)\beta_2(\sigma), \quad \gamma(a,\sigma) = \gamma_1(a)\gamma_2(\sigma).$$

$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in L^\infty_+(0,\omega)$, and β_1 and γ_1 are linearly independent in $L^\infty(0,\omega)$.

The assumption $\alpha(a,\sigma) = \alpha_1(a)\alpha_2(\sigma)$ means that there is no correlation between the age of a susceptible individual and the age of a spreader in their contact. The interpretation of the other assumptions is the same. Although PMA is simplistic because people usually mix differently, our analysis would be far more difficult without PMA.

Let $\alpha_1^\infty :=$ ess sup α_1, etc. Notice that we can calculate R_0 under PMA. In fact, we see that

$$(\tilde{T}u)(a) = \alpha_1(a)\int_0^\omega u(b)\theta(b)\left(\int_b^\omega \alpha_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}b$$

and substituting α_1 for u gives

$$(\tilde{T}\alpha_1)(a) = \alpha_1(a)\int_0^\omega \alpha_1(b)\theta(b)\left(\int_b^\omega \alpha_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}b,$$

which yields through the change of integral order that

$$R_0 = \int_0^\omega \alpha_2(\sigma)c(\sigma)\theta(b)\left(\int_0^\sigma \alpha_1(b)\theta(b)\,\mathrm{d}b\right)\mathrm{d}\sigma. \tag{7.1}$$

In this section, we fix the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ and θ so that $R_0 = 1$ holds, and we rewrite this α_1 as $\alpha_1^\star$. By definition we have

$$\int_0^\omega \alpha_2(\sigma)c(\sigma)\left(\int_0^\sigma \alpha_1^\star(b)\theta(b)\,\mathrm{d}b\right)\mathrm{d}\sigma = 1. \tag{7.2}$$

Let $\alpha_1(a) := \varepsilon\alpha_1^\star(a)$, where ε is the bifurcation parameter here and $R_0 = \varepsilon$. PMA implies that λ_1^* and λ_2^* have finite-dimensional ranges, i.e.,

$$\lambda_1^*(a) = c_1\alpha_1(a) = c_1\varepsilon\alpha_1^\star(a), \quad \lambda_2^*(a) = c_2\beta_1(a) + c_3\gamma_1(a),$$

where

$$c_1 := \int_0^\omega \alpha_2(\sigma)c(\sigma)y^*(\sigma)\,\mathrm{d}\sigma, \quad c_2 := \int_0^\omega \beta_2(\sigma)c(\sigma)y^*(\sigma)\,\mathrm{d}\sigma, \quad c_3 := \int_0^\omega \gamma_2(\sigma)c(\sigma)z^*(\sigma)\,\mathrm{d}\sigma.$$

Note that the condition $c_1 = 0$ implies $c_2 = c_3 = 0$ and corresponds to RFE and that the condition $c_1 > 0$ corresponds to REE. Then, (5.2a) and (5.2b) is expressed as a nonlinear system for c_1, c_2, c_3 corresponding to REE:

$$\Theta_j(c_1, c_2, c_3; \varepsilon) = 0, \quad j = 1, 2, 3,$$

where

$$\Theta_1(c_1,c_2,c_3;\varepsilon) := \varepsilon\int_0^\omega \alpha_2(\sigma)c(\sigma)\left(\int_0^\sigma \mathrm{e}^{-c_2\int_b^\sigma \beta_1(\tau)\,\mathrm{d}\tau - c_3\int_b^\sigma \gamma_1(\tau)\,\mathrm{d}\tau}\alpha_1^\star(b)\theta(b)\,\mathrm{e}^{-c_1\varepsilon\int_0^b \alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right)\mathrm{d}\sigma - 1, \tag{7.3a}$$

$$\Theta_2(c_1,c_2,c_3;\varepsilon) := c_1\varepsilon\int_0^\omega \beta_2(\sigma)c(\sigma)\left(\int_0^\sigma \mathrm{e}^{-c_2\int_b^\sigma \beta_1(\tau)\,\mathrm{d}\tau - c_3\int_b^\sigma \gamma_1(\tau)\,\mathrm{d}\tau}\alpha_1^\star(b)\theta(b)\,\mathrm{e}^{-c_1\varepsilon\int_0^b \alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right)\mathrm{d}\sigma - c_2, \tag{7.3b}$$

$$\Theta_3(c_1, c_2, c_3; \varepsilon) := c_1\varepsilon \int_0^\omega \gamma_2(\sigma')c(\sigma') \left\{ \int_0^\sigma \alpha_1^\star(b)(1-\theta(b))\,\mathrm{e}^{-c_1\varepsilon\int_0^b \alpha_1^\star(\tau)\,\mathrm{d}\tau} \right.$$
$$+ (c_2\beta_1(b) + c_3\gamma_1(b)) \left(\int_0^b \mathrm{e}^{-c_2\int_b^\sigma \beta_1(\tau)\,\mathrm{d}\tau - c_3\int_b^\sigma \gamma_1(\tau)\,\mathrm{d}\tau} \alpha_1^\star(\sigma)\theta(\sigma) \right.$$
$$\left.\left. \times \mathrm{e}^{-c_1\varepsilon\int_0^b \alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \right) \mathrm{d}b \right\} \mathrm{d}\sigma' - c_3. \tag{7.3c}$$

By using (7.2), we see that $\Theta_j(0,0,0;1)=0$ for $j=1,2,3$. Let us use the implicit function theorem to find a solution $(c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon))$ bifurcated at the point $\varepsilon = 1$ from the trivial solution $(c_1, c_2, c_3) = (0, 0, 0)$. Let $M(c_1, c_2, c_3; \varepsilon)$ be the Jacobian matrix of the mapping

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \mapsto \begin{pmatrix} \Theta_1(c_1, c_2, c_3; \varepsilon) \\ \Theta_2(c_1, c_2, c_3; \varepsilon) \\ \Theta_3(c_1, c_2, c_3; \varepsilon)) \end{pmatrix},$$

i.e.,

$$M(c_1, c_2, c_3; \varepsilon) := \left(\frac{\partial \Theta_i}{\partial c_j}(c_1, c_2, c_3; \varepsilon) \right)_{i,j=1,2,3}.$$

At the bifurcation point, this matrix takes the form

$$M(0,0,0;1) = \begin{pmatrix} \frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) & \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) & \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1) \\ \frac{\partial\Theta_2}{\partial c_1}(0,0,0;1) & -1 & 0 \\ \frac{\partial\Theta_3}{\partial c_1}(0,0,0;1) & 0 & -1 \end{pmatrix},$$

where

$$\frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) = -\int_0^\omega \alpha_2(\sigma)c(\sigma) \int_0^\sigma \alpha_1^\star(b)\theta(b) \int_0^b \alpha_1^\star(\tau)\,\mathrm{d}\tau\,\mathrm{d}b\,\mathrm{d}\sigma < 0,$$
$$\frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) = -\int_0^\omega \alpha_2(\sigma)c(\sigma) \int_0^\sigma \alpha_1^\star(b)\theta(b) \int_0^b \beta_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}b\,\mathrm{d}\sigma \leqq 0,$$

$$\frac{\partial\Theta_1}{\partial c_3}(0,0,0;1) = -\int_0^\omega \alpha_2(\sigma)c(\sigma) \int_0^\sigma \alpha_1^\star(b)\theta(b) \int_0^b \gamma_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}b\,\mathrm{d}\sigma \leqq 0,$$
$$\frac{\partial\Theta_2}{\partial c_1}(0,0,0;1) = \int_0^\omega \beta_2(\sigma)c(\sigma) \int_0^\sigma \alpha_1^\star(b)\theta(b)\,\mathrm{d}b\,\mathrm{d}\sigma \geqq 0,$$
$$\frac{\partial\Theta_3}{\partial c_1}(0,0,0;1) = \int_0^\omega \gamma_2(\sigma)c(\sigma) \int_0^\sigma \alpha_1^\star(b)(1-\theta(b))\,\mathrm{d}b\,\mathrm{d}\sigma \geqq 0.$$

It follows that

$$D_0 := \det M(0,0,0;1) = \frac{\partial \Theta_1}{\partial c_1}(0,0,0;1) + \frac{\partial \Theta_1}{\partial c_2}(0,0,0;1)\frac{\partial \Theta_2}{\partial c_1}(0,0,0;1) + \frac{\partial \Theta_1}{\partial c_3}(0,0,0;1)\frac{\partial \Theta_3}{\partial c_1}(0,0,0;1)$$

is strictly negative. Hence, we can apply the implicit function theorem to show the existence of a branching solution $(c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon))$ with $c_1(1) = c_2(1) = c_3(1) = 0$ when $\varepsilon > 1$ is small enough. In addition, we see that

$$\begin{pmatrix} c_1'(1) \\ c_2'(1) \\ c_3'(1) \end{pmatrix} = -M(0,0,0;1)^{-1} \begin{pmatrix} \dfrac{\partial \Theta_1}{\partial \varepsilon}(0,0,0;1) \\ \dfrac{\partial \Theta_2}{\partial \varepsilon}(0,0,0;1) \\ \dfrac{\partial \Theta_3}{\partial \varepsilon}(0,0,0;1) \end{pmatrix} = -\frac{1}{D_0} \begin{pmatrix} 1 \\ \dfrac{\partial \Theta_2}{\partial c_1}(0,0,0;1) \\ \dfrac{\partial \Theta_3}{\partial c_1}(0,0,0;1) \end{pmatrix}.$$

REE (x^*, y^*, z^*) depends on λ_1^* and λ_2^*, i.e., $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon)$ when $\varepsilon > 1$ is small enough, which admits such expression as

$$x^*(a) = x^*(a;\varepsilon), \quad y^*(a) = y^*(a;\varepsilon), \quad z^*(a) = z^*(a;\varepsilon).$$

Then, we have

$$\frac{\partial x^*}{\partial \varepsilon}(a;1) = -c_1'(1)\int_0^a \alpha_1^\star(\sigma)\,\mathrm{d}\sigma,$$

$$\frac{\partial y^*}{\partial \varepsilon}(a;1) = c_1'(1)\int_0^a \alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{d}\sigma,$$

$$\frac{\partial z^*}{\partial \varepsilon}(a;1) = c_1'(1)\int_0^a \alpha_1^\star(\sigma)(1-\theta(\sigma))\,\mathrm{d}\sigma.$$

Then, let us return into the discussion of the local stability of REE. Let

$$x(t,a) = x^*(a) + \bar{x}(t,a), \quad y(t,a) = y^* + \bar{y}(t,a), \quad z(t,a) = z^* + \bar{z}(t,a)$$

be a solution of system (4.4). $(\bar{x}(t,a), \bar{y}(t,a), \bar{z}(t,a))$ denote the small perturbations from REE. Note that

$$\bar{x}(t,0) = \bar{y}(t,0) = \bar{z}(t,0) = 0, \tag{7.4}$$

$$\bar{x}(t,a) + \bar{y}(t,a) + \bar{z}(t,a) = 0. \tag{7.5}$$

The small perturbations satisfy the following equations:

$$\begin{aligned}
(\partial_t + \partial_a)\bar{x}(t,a) &= -\bar{x}(t,a)(\lambda_1^*(a) + \bar{\lambda}_1(t,a)) - x^*(a)\bar{\lambda}_1(t,a), \\
(\partial_t + \partial_a)\bar{y}(t,a) &= \bar{x}(t,a)\theta(a)(\lambda_1^*(a) + \bar{\lambda}_1(t,a)) + x^*(a)\theta(a)\bar{\lambda}_1(t,a) \\
&\quad - \bar{y}(t,a)(\lambda_2^*(a) + \bar{\lambda}_2(t,a)) - y^*(a)\bar{\lambda}_2(t,a), \\
(\partial_t + \partial_a)\bar{z}(t,a) &= \bar{x}(t,a)(1-\theta(a))(\lambda_1^*(a) + \bar{\lambda}_1(t,a)) + x^*(a)(1-\theta(a))\bar{\lambda}_1(t,a) \\
&\quad + \bar{y}(t,a)(\lambda_2^*(a) + \bar{\lambda}_2(t,a)) + y^*(a)\bar{\lambda}_2(t,a),
\end{aligned} \tag{7.6}$$

where

$$\bar{\lambda}_1(a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)\bar{y}(\sigma)\,\mathrm{d}\sigma,$$

$$\bar{\lambda}_2(a) = \int_0^\omega c(\sigma)\{\beta(a,\sigma)\bar{y}(\sigma) + \gamma(a,\sigma)\bar{z}(\sigma)\}\,\mathrm{d}\sigma.$$

We can formulate (7.6) as an abstract semilinear problem on the Banach space $(L^1(0,\omega))^3$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + G(u(t)), \quad u(t) = {}^{\mathrm{t}}(\bar{x}(t,\cdot), \bar{y}(t,\cdot), \bar{z}(t,\cdot)). \tag{7.7}$$

The generator A is defined by

$$(A\phi)(a) := \begin{pmatrix} -\mathrm{d}/\mathrm{d}a & 0 & 0 \\ 0 & -\mathrm{d}/\mathrm{d}a & 0 \\ 0 & 0 & -\mathrm{d}/\mathrm{d}a \end{pmatrix} \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \phi_3(a) \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$

where the domain of A is defined by

$$\begin{aligned} \mathscr{D}(A) := \{&\phi = {}^{\mathrm{t}}(\phi_1, \phi_2, \phi_3) \in (L^1(0,\omega))^3 \,|\, \phi_1, \phi_2, \phi_3 \text{ are absolutely continuous on } [0,\omega], \\ &\phi_1(0) = \phi_2(0) = \phi_3(0) = 0,\ \phi_1(a) + \phi_2(a) + \phi_3(a) = 0\}. \end{aligned}$$

The nonlinear term G is defined by

$$\begin{aligned} G(u) &:= {}^{\mathrm{t}}(G_1(u), G_2(u), G_3(u)), \\ G_1(u) &:= -u_1(\lambda_1^* + P_\alpha u_2) - x^* P_\alpha u_2, \\ G_2(u) &:= u_1\theta(\lambda_1^* + P_\alpha u_2) + x^*\theta P_\alpha u_2 - u_2(\lambda_2^* + P_\beta u_2 + P_\gamma u_3) - y^*(P_\beta u_2 + P_\gamma u_3), \\ G_3(u) &:= u_1(1-\theta)(\lambda_1^* + P_\alpha u_2) + x^*(1-\theta)P_\alpha u_2 + u_2(\lambda_2^* + P_\beta u_2 + P_\gamma u_3) + y^*(P_\beta u_2 + P_\gamma u_3), \end{aligned}$$

where, for $f \in L^1(0,\omega)$,

$$\begin{aligned} (P_\alpha f)(a) &:= \int_0^\omega \alpha(a,\sigma)c(\sigma)f(\sigma)\,\mathrm{d}\sigma, \\ (P_\beta f)(a) &:= \int_0^\omega \beta(a,\sigma)c(\sigma)f(\sigma)\,\mathrm{d}\sigma, \\ (P_\gamma f)(a) &:= \int_0^\omega \gamma(a,\sigma)c(\sigma)f(\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{7.8}$$

The linearized equation of (7.7) around $u = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = (A + C)u(t),$$

where the bounded linear operator C is the Fréchet derivative of $G(u)$ at $u = 0$ given by

$$Cu := \begin{pmatrix} -u_1\lambda_1^* - x^* P_\alpha u_2 \\ u_1\theta\lambda_1^* + x^*\theta P_\alpha u_2 - u_2\lambda_2^* - y^*(P_\beta u_2 + P_\gamma u_3) \\ u_1(1-\theta)\lambda_1^* + x^*(1-\theta)P_\alpha u_2 + u_2\lambda_2^* + y^*(P_\beta u_2 + P_\gamma u_3) \end{pmatrix}.$$

Now let us consider the resolvent equation for $A+C$:

$$(\zeta-(A+C))v=u, \quad v\in\mathscr{D}(A),\ \ u\in(L^1(0,\omega))^3,\ \ \zeta\in\mathbb{C}. \tag{7.9}$$

Then we have

$$v_1'(a)=-\zeta v_1(a)-\lambda_1^*(a)v_1(a)-x^*(a)(P_\alpha v_2)(a)+u_1(a), \tag{7.10a}$$

$$\begin{aligned}v_2'(a)=&-\zeta v_2(a)+\lambda_1^*(a)\theta(a)v_1(a)+\theta(a)x^*(a)(P_\alpha v_2)(a)-\lambda_2^*(a)v_2(a)\\&-y^*(a)(P_\beta v_2)(a)-y^*(a)(P_\gamma v_3)(a)+u_2(a),\end{aligned} \tag{7.10b}$$

$$\begin{aligned}v_3'(a)=&-\zeta v_3(a)+\lambda_1^*(a)(1-\theta(a))v_1(a)+(1-\theta(a))x^*(a)(P_\alpha v_2)(a)\\&+\lambda_2^*(a)v_2(a)+y^*(a)(P_\beta v_2)(a)+y^*(a)(P_\gamma v_3)(a)+u_3(a).\end{aligned} \tag{7.10c}$$

From (7.10a) and $v_1(0)=0$, we obtain

$$v_1(a)=\int_0^a\{-x^*(\tau)(P_\alpha v_2)(\tau)+u_1(\tau)\}\,\mathrm{e}^{-\int_\tau^a(\zeta+\lambda_1^*(r))\,\mathrm{d}r}\,\mathrm{d}\tau. \tag{7.11}$$

From (7.10b) and $v_2(0)=0$, we have

$$\begin{aligned}v_2(a)=\int_0^a&\{\lambda_1^*(\sigma)\theta(\sigma)v_1(\sigma)+\theta(\sigma)x^*(\sigma)(P_\alpha v_2)(\sigma)-y^*(\sigma)(P_\beta v_2)(\sigma)\\&-y^*(\sigma)(P_\gamma v_3)(\sigma)+u_2(\sigma)\}\,\mathrm{e}^{-\zeta(a-\sigma)}\,\mathrm{e}^{-\int_\sigma^a\lambda_2^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma.\end{aligned} \tag{7.12}$$

Let

$$\xi_1:=\int_0^\omega\alpha_2(\sigma)c(\sigma)v_2(\sigma)\,\mathrm{d}\sigma,\quad \xi_2:=\int_0^\omega\beta_2(\sigma)c(\sigma)v_2(\sigma)\,\mathrm{d}\sigma,\quad \xi_3:=\int_0^\omega\gamma_2(\sigma)c(\sigma)v_3(\sigma)\,\mathrm{d}\sigma, \tag{7.13}$$

Assumption 25 implies

$$(P_\alpha v_2)(a)=\xi_1\alpha_1(a),\quad (P_\beta v_2)(a)=\xi_2\beta_1(a),\quad (P_\gamma v_3)(a)=\xi_3\gamma_1(a).$$

Inserting (7.11) and (7.12) into (7.13) yields a three-dimensional system as

$$(I-\Phi(\zeta,\varepsilon))\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}, \tag{7.14}$$

where I is the 3×3 unit matrix and $\Phi(\zeta,\varepsilon)=(\phi_{ij}(\zeta,\varepsilon))_{1\leqq i,j\leqq 3}$ a 3×3 matrix. $\phi_{ij}(\zeta,\varepsilon)$ and η_i $(1\leqq i,\ j\leqq 3)$ are defined as follows:

$$\begin{aligned}\phi_{11}(\zeta,\varepsilon)=&\int_0^\omega\alpha_2(r)c(r)\left(\int_0^r\theta(\sigma)x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\&-\int_0^\omega\alpha_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\right.\\&\left.\times\left(\int_0^\sigma\varepsilon\alpha_1^\star(\tau')\,\mathrm{e}^{-\zeta(\sigma-\tau')}\,\mathrm{e}^{-c_1(\varepsilon)\int_0^\sigma\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r,\end{aligned}$$

$$\phi_{12}(\zeta,\varepsilon)=-\int_0^\omega\alpha_2(r)c(r)\left(\int_0^r y^*(\sigma;\varepsilon)\beta_1(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r,$$

$$\phi_{13}(\zeta,\varepsilon)=-\int_0^\omega\alpha_2(r)c(r)\left(\int_0^r y^*(\sigma;\varepsilon)\gamma_1(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r,$$

$$\begin{aligned}\phi_{21}(\zeta,\varepsilon) &= \int_0^\omega \beta_2(r)c(r)\left(\int_0^r \theta(\sigma)x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&\quad-\int_0^\omega \beta_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\right.\\
&\quad\left.\times\left(\int_0^\sigma \varepsilon\alpha_1^\star(\tau')\,\mathrm{e}^{-\zeta(\sigma-\tau')}\,\mathrm{e}^{-c_1(\varepsilon)\int_0^\sigma \varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r,\end{aligned}$$

$$\phi_{22}(\zeta,\varepsilon) = -\int_0^\omega \beta_2(r)c(r)\left(\int_0^r y^*(\sigma;\varepsilon)\beta_1(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r,$$

$$\phi_{23}(\zeta,\varepsilon) = -\int_0^\omega \beta_2(r)c(r)\left(\int_0^r y^*(\sigma;\varepsilon)\gamma_1(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r,$$

$$\begin{aligned}\phi_{31}(\zeta,\varepsilon) &= \int_0^\omega \gamma_2(r)c(r)\left(\int_0^r x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_1(\varepsilon)\int_\sigma^r \varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&\quad-\int_0^\omega \gamma_2(r)c(r)\left(\int_0^r \theta(\sigma)x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&\quad+\int_0^\omega \gamma_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\right.\\
&\quad\left.\times\left(\int_0^\sigma \varepsilon\alpha_1^\star(\tau')\,\mathrm{e}^{-\zeta(\sigma-\tau')}\,\mathrm{e}^{-c_1(\varepsilon)\int_0^\sigma \varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r,\end{aligned}$$

$$\phi_{32}(\zeta,\varepsilon) = \int_0^\omega \gamma_2(r)c(r)\left(\int_0^r y^*(\sigma;\varepsilon)\beta_1(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r,$$

$$\phi_{33}(\zeta,\varepsilon) = \int_0^\omega \gamma_2(r)c(r)\left(\int_0^r y^*(\sigma;\varepsilon)\gamma_1(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r,$$

$$\begin{aligned}\eta_1 &= \int_0^\omega \alpha_2(r)c(r)\left(\int_0^r u_2(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&\quad+\int_0^\omega \alpha_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\right.\\
&\quad\left.\times\left(\int_0^\sigma u_1(\tau')\,\mathrm{e}^{-\zeta(\sigma-\tau')}\,\mathrm{e}^{-c_1(\varepsilon)\int_{\tau'}^\sigma \varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r,\end{aligned}$$

$$\begin{aligned}\eta_2 &= \int_0^\omega \beta_2(r)c(r)\left(\int_0^r u_2(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&\quad+\int_0^\omega \beta_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\right.\\
&\quad\left.\times\left(\int_0^\sigma u_1(\tau')\,\mathrm{e}^{-\zeta(\sigma-\tau')}\,\mathrm{e}^{-c_1(\varepsilon)\int_{\tau'}^\sigma \varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r,\end{aligned}$$

$$
\begin{aligned}
\eta_3 = &-\int_0^{\omega} \gamma_2(r)c(r)\left(\int_0^r u_1(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_1(\varepsilon)\int_\sigma^r \varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r \\
&-\int_0^{\omega} \gamma_2(r)c(r)\left(\int_0^r u_2(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r \\
&-\int_0^{\omega} \gamma_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r \beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r \gamma_1(\tau)\,\mathrm{d}\tau}\right. \\
&\times\left.\left(\int_0^\sigma u_1(\tau')\,\mathrm{e}^{-\zeta(\sigma-\tau')}\,\mathrm{e}^{-c_1(\varepsilon)\int_{\tau'}^\sigma \varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r.
\end{aligned}
$$

$\phi_{ij}(0,1)$ are calculated as follows:

$$
\begin{aligned}
&\phi_{11}(0,1) = \int_0^{\omega} \alpha_2(r)c(r)\left(\int_0^r \theta(\sigma)\alpha_1^\star(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r = 1, \\
&\phi_{21}(0,1) = \int_0^{\omega} \beta_2(r)c(r)\left(\int_0^r \theta(\sigma)\alpha_1^\star(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r = \frac{\partial\Theta_2}{\partial c_1}(0,0,0;1), \\
&\phi_{31}(0,1) = \int_0^{\omega} \gamma_2(r)c(r)\left(\int_0^r (1-\theta(\sigma))\alpha_1^\star(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r = \frac{\partial\Theta_3}{\partial c_1}(0,0,0;1), \\
&\phi_{ij}(0,1) = 0 \quad (i=1,2,3,\ \ j=2,3).
\end{aligned}
$$

In addition, we observe that

$$
\begin{aligned}
&\frac{\partial\phi_{11}}{\partial\zeta}(0,1) = \int_0^{\omega} \alpha_2(r)c(r)\left(\int_0^r \theta(\sigma)\alpha_1(\sigma)(-r+\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r < 0, \\
&\frac{\partial\phi_{ij}}{\partial\zeta}(0,1) = 0 \quad (i=1,2,3,\ \ j=2,3), \\
&\frac{\partial\phi_{11}}{\partial\varepsilon}(0,1) = \frac{\partial\Theta_1}{\partial c_1}(0,0,0;1)\{1+c_1'(1)\} + 1 + \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) + \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1), \\
&\frac{\partial\phi_{12}}{\partial\varepsilon}(0,1) = c_1'(1)\,\frac{\partial\Theta_1}{\partial c_2}(0,0,0;1), \\
&\frac{\partial\phi_{13}}{\partial\varepsilon}(0,1) = c_1'(1)\frac{\partial\Theta_1}{\partial c_3}(0,0,0;1).
\end{aligned}
$$

Now we denote the determinant of $I-\Phi(\zeta,\varepsilon)$ by $f(\zeta,\varepsilon)$. We see that the roots of $f(\zeta,1)$ give the eigenvalues of linearized system at RFE. If $\varepsilon=1$, then $f(0,1)=0$ and we can see that

$$
\frac{\partial f}{\partial\zeta}(0,1) = -\frac{\partial\phi_{11}}{\partial\zeta}(0,1) > 0.
$$

Then the implicit function theorem implies that the equation $f(\zeta,\varepsilon)=0$ can be solved locally as $\zeta=\zeta(\varepsilon)$ with $\zeta(\varepsilon)=0$. At the same time, it is rather easy to see that

$$
\begin{aligned}
\frac{\partial f}{\partial\varepsilon}(0,1) &= -\frac{\partial\phi_{11}}{\partial\varepsilon}(0,1) - \frac{\partial\phi_{12}}{\partial\varepsilon}(0,1)\phi_{21}(0,1) - \frac{\partial\phi_{13}}{\partial\varepsilon}(0,1)\phi_{31}(0,1) \\
&= -D_0c_1'(1) - 1 - \frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1) \\
&= -\frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1) > 0,
\end{aligned}
$$

which means that the dominant eigenvalue goes to the left half complex plane as ε increases small enough from 1. The well-known technique based on the Rouché's theorem (see [15, Proposition 4.1]) yields that $\zeta(\varepsilon)$ is the dominant root of f as long as $|\varepsilon - 1|$ is small enough.

Now, let us examine the relationship between the dominant eigenvalue and the local stability of REE. Let $\Sigma^* = \sigma(A+C)$ be the spectrum of $A + C$, then the following holds:

Lemma 26. (i) Σ^* *can be rewritten as follows*:

$$\Sigma^* = P_\sigma(A + C) = \{\zeta \in \mathbb{C} \mid f(\zeta, \varepsilon) = 0\}.$$

(ii) *The linearized semigroup* $\{e^{t(A+C)}\}_{t \geqq 0}$ *is eventually compact and*

$$\omega_0(A + C) = s(A + C) := \sup\{\operatorname{Re} \zeta \mid \zeta \in P_\sigma(A + C)\}.$$

Proof. The resolvent equation (7.9) is solvable if and only if $I - \Phi(\zeta, \varepsilon)$ is invertible, i.e., $f(\zeta, \varepsilon) = 0$. Hence we have

$$\rho(A + C) = \{\zeta \in \mathbb{C} \mid f(\zeta, \varepsilon) \neq 0\}, \quad \Sigma^* = \{\zeta \in \mathbb{C} \mid f(\zeta, \varepsilon) = 0\}.$$

It is easy to show that the resolvent $R(\zeta, A + C)$ is compact for any $\zeta \in \rho(A + C)$, from which it follows that $\Sigma^* = P_\sigma(A + C)$ and we have shown the first assertion.

Let us prove the second assertion. The linearized generator $A + C$ is decomposed as $A + C_1 + C_2$, where

$$C_1 u := {}^t(-\lambda_1^* u_1, \theta\lambda_1^* u_1 - \lambda_2^* u_2, (1 - \theta)\lambda_1^* u_1 + \lambda_2^* u_2)$$

and

$$C_2 u := \begin{pmatrix} -x^*(P_\alpha u_2) \\ \theta x^*(P_\alpha u_2) - y^*(P_\beta u_2 + P_\gamma u_3) \\ (1 - \theta)x^*(P_\alpha u_2) + y^*(P_\beta u_2 + P_\gamma u_3) \end{pmatrix}.$$

We easily find that the operator $A + C_1$ is the generator of a multistate stable population with a finite age interval (see [16]). Hence it follows that the population semigroup generated by $A + C_1$ is eventually compact and its essential growth $\omega_1(A + C_1)$ is $-\infty$. Since C_2 is compact due to Assumption 8, we have $\omega_1(A + C_1 + C_2) = \omega_1(A + C_1) = -\infty$ (see [50, Proposition 4.14]). Then the second assertion is obtained from [50, Proposition 4.13]. $\square$

From the above results and the principle of linearized stability (see, for example, [50, Proposion 4.17]), we conclude:

Theorem 27. *Under Assumption* 25, *the REE bifurcates forward from RFE and is locally asymptotically stable if* $R_0 > 1$ *and* $|R_0 - 1|$ *is small enough.*

8. Uniform strong persistence

In this section we show that $R_0 > 1$ implies uniform strong rumor persistence under PMA.

PMA with Assumption 8(ii) implies that $\alpha_1(a) > 0$ for almost every $a \in (0, \omega)$ and $\alpha_2(\sigma) > 0$ for almost every $\sigma \in (\omega - b_0, \omega)$. Moreover, it implies that

$$\lambda_1(t, a) = \alpha_1(a)\psi_1(t) \quad \text{where } \psi_1(t) := \int_0^\omega \alpha_2(\sigma)c(\sigma)y(t, \sigma)\,d\sigma \tag{8.1}$$

and

$$\lambda_2(t, a) = \beta_1(a)\psi_2(t) + \gamma_1(a)\psi_3(t),$$

where

$$\psi_2(t) := \int_0^\omega \beta_2(\sigma)c(\sigma)y(t, \sigma)\,d\sigma, \quad \psi_3(t) := \int_0^\omega \gamma_2(\sigma)c(\sigma)z(t, \sigma)\,d\sigma. \tag{8.2}$$

By integrating along characteristics we obtain from the partial derivative equation for $y(t,a)$ in (4.4) that

$$y(t,a)=\int_0^a \alpha_1(\sigma)\psi_1(t-a+\sigma)\theta(\sigma)x(t-a+\sigma,\sigma)\,\mathrm{e}^{-\int_\sigma^a \lambda_2(t-a+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \tag{8.3}$$

if $t>a$, and

$$y(t,a)=y_0(a-t)\,\mathrm{e}^{-\int_0^t \lambda_2(\tau,a-t+\tau)\,\mathrm{d}\tau}+x_0(a-t)\int_0^t \{\alpha_1(a-t+s)\psi_1(s)$$
$$\times\,\theta(a-t+s)\,\mathrm{e}^{-\int_0^s \lambda_1(\tau,a-t+\tau)\,\mathrm{d}\tau-\int_s^t \lambda_2(\tau,a-t+\tau)\,\mathrm{d}\tau}\}\,\mathrm{d}s$$

if $t<a$. If $t>\omega$, by substituting this into (8.1) and letting $r=s-\sigma$, we have

$$\begin{aligned}\psi_1(t)&=\int_0^\omega \alpha_2(s)c(s)\left\{\int_0^s \alpha_1(s-r)\psi_1(t-r)\theta(s-r)x(t-r,s-r)\,\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}r\right\}\mathrm{d}s\\ &=\int_0^\omega \psi_1(t-r)\left\{\int_r^\omega \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t-r,s-r)\,\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}\mathrm{d}r.\end{aligned}$$

if $t<\omega$. Let $c(a)=0$ for $a>\omega$, then we have

$$\psi_1(t)=\int_0^t \psi_1(t-r)\left\{\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t-r,s-r)\,\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}\mathrm{d}r \tag{8.4}$$

if $t>\omega$. Similarly, if $t<\omega$, then we have

$$\begin{aligned}\psi_1(t)=&\int_0^t \psi_1(t-r)\left\{\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t-r,s-r)\,\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}\mathrm{d}r\\ &+\int_t^\omega \alpha_2(a)c(a)y_0(a-t)\,\mathrm{e}^{-\int_0^t \lambda_2(\tau,a-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}a\\ &+\int_0^t \psi_1(t-r)\left\{\int_t^\omega \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x_0(s-t)\right.\\ &\left.\times\mathrm{e}^{-\int_0^{t-r} \lambda_1(\tau,s-t+\tau)\,\mathrm{d}\tau-\int_{t-r}^t \lambda_2(\tau,s-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}\mathrm{d}r.\end{aligned}$$

Let $\psi_{1,b}(t):=\psi_1(b+t)$ for $b\geqq 0$, then from the way $c(a)$ is extended for $a>\omega$ we see that

$$\psi_{1,b}(t)=\int_0^t \psi_{1,b}(t-r)\left\{\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t+b-r,s-r)\,\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau+b,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}\mathrm{d}r \tag{8.5}$$

for sufficiently large t.

Lemma 28. (1) *If* $t>a$, *then* $x(t,a)>0$. *And if* $t<a$ *and* $x_0(a-t)>0$, *then* $x(t,a)>0$.
(2) *If* $y(t,a)>0$, *then* $y(t+\sigma,a+\sigma)>0$ *for all* $\sigma\in(0,\omega-a)$.

Proof. From the partial derivative equation for $x(t,a)$ in (4.4) we have

$$x(t,a)=x(t-a,0)\,\mathrm{e}^{-\int_0^a \lambda_1(t-a+\tau,\tau)\,\mathrm{d}\tau}$$

for $t>a$. This and $x(t-a,0)=1$ imply that $x(t,a)>0$ for $t>a$. The case $t<a$ and $x_0(a-t)>0$ is as well.

From the partial derivative equation for $y(t,a)$ in (4.4) and $\lambda_1(t,a)\theta(a)x(t,a) \geqq 0$, we have

$$(\partial_t + \partial_a)y(t,a) \geqq -\lambda_2(t,a)y(t,a).$$

Integrating along characteristics gives

$$y(t+\sigma, a+\sigma) \geqq y(t,a)\,\mathrm{e}^{-\int_0^\sigma \lambda_2(t+\tau,a+\tau)\,\mathrm{d}\tau} > 0,$$

thus we have the conclusion. □

The following lemma states that spreaders never clear out unless there are no spreaders from start.

Lemma 29. *If $\int_0^\omega y(0,a)\,\mathrm{d}a > 0$, then $\int_0^\omega y(t,a)\,\mathrm{d}a > 0$ for all $t > 0$.*

Proof. Suppose $\int_0^\omega y(t,a)\,\mathrm{d}a = 0$ for some $t = t_0 > 0$. Then we have $y(t_0,a) = 0$ for almost every $a \in (0,\omega)$.

If t_0 is sufficiently large, then we find from Lemma 28 that $y(t_0 - t_1, a) = 0$ for all $t_1 \in (0, b_0)$ and almost every $a \in (0, \omega - t_1)$, where b_0 is defined in Assumption 8(ii). In addition, the integrand of (8.3) is strictly positive except ψ_1 for almost every $\sigma \in (0,a)$, hence we obtain $\psi_1(t) = 0$ for almost every $t \in (t_0 - a, t_0)$. Since this holds for almost every $a \in (0,\omega)$, we find $\psi_1(t) = 0$ for almost every $t \in (t_0 - \omega, t_0)$. The fact that $\alpha_2(\sigma)c(\sigma)$ in (8.1) is strictly positive for almost every $\sigma \in (\omega - b_0, \omega)$ implies that $y(t,a) = 0$ for almost every $t \in (t_0 - \omega, t_0)$ and almost every $a \in (\omega - b_0, \omega)$. Then we find again from Lemma 28 that $y(t_0 - t_1, a) = 0$ for all $t_1 \in (0, b_0)$ and almost every $a \in (\omega - t_1, \omega)$. Therefore it follows that $y(t,a) = 0$ for all $t \in (t_0 - b_0, t_0]$ and almost every $a \in (0,\omega)$.

By iterating this discussion finite times, we obtain $y(0,a) = 0$ for almost every $a \in (0,\omega)$, where the above discussion should be modified a little when $t < \omega$. This contradicts the assumption $\int_0^\omega y(0,a)\,\mathrm{d}a > 0$, thus we obtain the conclusion. □

The rumor is called *uniformly weakly persistent* if there exists some $\varepsilon > 0$, which does not depend on the initial data $(x(0,\cdot), y(0,\cdot), z(0,\cdot))$ and satisfies

$$\rho^\infty := \limsup_{t\to\infty} \int_0^\omega y(t,a)\,\mathrm{d}a > \varepsilon,$$

provided that $\int_0^\omega y(0,a)\,\mathrm{d}a > 0$. The rumor is called *uniformly strongly persistent* if there exists some $\varepsilon > 0$, which does not depend on the initial data $(x(0,\cdot), y(0,\cdot), z(0,\cdot))$ and satisfies

$$\rho_\infty := \liminf_{t\to\infty} \int_0^\omega y(t,a)\,\mathrm{d}a > \varepsilon,$$

provided that $\int_0^\omega y(0,a)\,\mathrm{d}a > 0$. It is clear that uniform strong persistence implies uniform weak persistence.

First, let us show the weak result. We referred the idea of its proof to [46].

Theorem 30. *If $R_0 > 1$, then the rumor is uniformly weakly persistent.*

Proof. Assume that for any $\varepsilon > 0$ there exist some $T_0 > 0$ and some appropriate initial condition such that $\int_0^\omega y(t,a)\,\mathrm{d}a \leqq \varepsilon$ for all $t \geqq T_0$. We can choose T_0 to be so large that (8.5) holds for all $t \geqq T_0$. By the definition of $\psi_1(t)$ and $\psi_2(t)$, it is easily seen that

$$\psi_1(t) \leqq \alpha_2^\infty c^\infty \varepsilon, \quad \psi_2(t) \leqq \beta_2^\infty c^\infty \varepsilon$$

for $t \geqq T_0$. Next, let us obtain the upper bound for $\psi_3(t)$. The conditions that $z(t,a) \leqq 1$ and $\int_0^\omega c(\sigma)\,\mathrm{d}\sigma = 1$ imply that

$$\lambda_1(t,a) \leqq \alpha^\infty c^\infty \varepsilon, \quad \lambda_2(t,a) \leqq \beta^\infty c^\infty \varepsilon + \gamma^\infty$$

for $t \geqq T_0$. It follows that

$$(\partial_t + \partial_a)z(t,a) \leqq \alpha^\infty c^\infty \varepsilon + (\beta^\infty c^\infty \varepsilon + \gamma^\infty)y(t,a)$$

for $t \geqq T_0$. Hence, if $t-a \geqq T_0$, integrating the above inequality along characteristics gives

$$z(t,a) \leqq \alpha^{\infty} c^{\infty} a\varepsilon + (\beta^{\infty} c^{\infty} \varepsilon + \gamma^{\infty}) \int_0^a y(t-a+\tau,\tau)\,\mathrm{d}\tau.$$

Therefore, if $t \geqq T_0 + \omega$, we have

$$\begin{aligned}
\psi_3(t) &\leqq \alpha^{\infty} c^{\infty} \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^{\infty} c^{\infty} \varepsilon + \gamma^{\infty}) \int_0^{\omega} \left(\int_0^{\sigma} y(t-\sigma+\tau,\tau)\,\mathrm{d}\tau \right) \mathrm{d}\sigma \\
&= \alpha^{\infty} c^{\infty} \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^{\infty} c^{\infty} \varepsilon + \gamma^{\infty}) \int_0^{\omega} \left(\int_0^{\omega-u} y(t-u,\tau)\,\mathrm{d}\tau \right) \mathrm{d}u \\
&\leqq \alpha^{\infty} c^{\infty} \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^{\infty} c^{\infty} \varepsilon + \gamma^{\infty}) \int_0^{\omega} \varepsilon\,\mathrm{d}u \\
&= \alpha^{\infty} c^{\infty} \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^{\infty} c^{\infty} \varepsilon + \gamma^{\infty})\omega\varepsilon.
\end{aligned}$$

In what follows, we can assume for simplification that, for any $\varepsilon > 0$, there exist some $T_0 > 0$ and some appropriate initial condition such that (8.5) and

$$\int_0^{\omega} y(t,a)\,\mathrm{d}a \leqq \varepsilon, \quad \psi_i(t) \leqq \varepsilon, \quad i = 1,2,3$$

hold for $t \geqq T_0$. Integrating the partial derivative equation for $x(t,a)$ in (4.4) along characteristics gives

$$\begin{aligned}
x(t+b-r,s-r) &= \mathrm{e}^{-\int_0^{s-r} \psi_1(t+b-s+\tau)\alpha_1(\tau)\,\mathrm{d}\tau} \\
&\geqq \mathrm{e}^{-\varepsilon\int_0^{s-r} \alpha_1(\tau)\,\mathrm{d}\tau}
\end{aligned} \tag{8.6}$$

for $t > T_0$, $0 \leqq r \leqq s \leqq t$ and $b \geqq 0$. Hence it follows from (8.5) that

$$\psi_{1,b}(t) \geqq \int_0^t \psi_{1,b}(t-r) \left\{ \int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{e}^{-\varepsilon\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau}\,\mathrm{e}^{-\varepsilon\int_{s-r}^{s}(\beta_1(\tau)+\gamma_1(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s \right\} \mathrm{d}r.$$

This implies that

$$\begin{aligned}
\psi_{1,b+T}(t) &= \psi_{1,b}(t+T) \\
&\geqq \int_0^{t+T} \psi_{1,b+T}(t-r) \left\{ \int_r^{t+T} \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{e}^{-\varepsilon\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau} \right. \\
&\qquad \left. \times \mathrm{e}^{-\varepsilon\int_{s-r}^{s}(\beta_1(\tau)+\gamma_1(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s \right\} \mathrm{d}r \\
&\geqq \int_0^{t} \psi_{1,b+T}(t-r) \left\{ \int_r^{r+T} \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{e}^{-\varepsilon\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau} \right. \\
&\qquad \left. \times \mathrm{e}^{-\varepsilon\int_{s-r}^{s}(\beta_1(\tau)+\gamma_1(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s \right\} \mathrm{d}r
\end{aligned}$$

for all $t \geqq 0$ if we take sufficiently large $T > 0$. Taking Laplace transforms lead to

$$\widehat{\psi_{1,b+T}}(\lambda) \geqq \widehat{\psi_{1,b+T}}(\lambda) F(\varepsilon,\lambda,T), \tag{8.7}$$

where

$$F(\varepsilon,\lambda,T) := \int_0^{\infty} \mathrm{e}^{-\lambda r} \left\{ \int_r^{r+T} \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{e}^{-\varepsilon\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau-\varepsilon\int_{s-r}^{s}(\beta_1(\tau)+\gamma_1(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s \right\} \mathrm{d}r. \tag{8.8}$$

Since ψ_1 is bounded, the Laplace transform of $\psi_{1,b+T}(\lambda)$ is defined for all $\lambda \geqq 0$. Moreover,

$$\lim_{T\to\infty} F(0,0,T) = \int_0^\infty \left\{ \int_r^\infty \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{d}s \right\} \mathrm{d}r$$
$$= \int_0^\omega \left\{ \int_r^\omega \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{d}s \right\} \mathrm{d}r,$$

which is equal to $R_0 > 1$ through the change of variables: $\sigma = s,\ \ b = s - r$. Since $F(\varepsilon, \lambda, T)$ is continuous, we see that $F(\varepsilon, \lambda, T) > 1$ if $\varepsilon, \lambda > 0$ are chosen small enough and T large enough. Then (8.7) implies that $\widehat{\psi_{1,b+T}}(\lambda) = 0$, which means $\psi_{1,b+T} = 0$ a.e. on $[0, \infty)$, i.e., $\psi_1(t) = 0$ for almost every $t > b + T$.

Take some $t = T_1 > b + T$ for which $\psi_1(T_1) = 0$ holds, then we have $y(T_1, a) = 0$ for almost every $a \in (\omega - b_0, \omega)$. On the other hand, if we take any p, q such that $0 < p < q < \omega - b_0$ and $q - p < b_0$, then $(p + \tau, q + \tau) \subset (\omega - b_0, \omega)$ for any $\tau \in (\omega - b_0 - p, \omega - q)$. There exists some $\tau \in (\omega - b_0 - p, \omega - q)$ such that $\psi_1(T_1 + \tau) = 0$, so that $y(T_1 + \tau, a) = 0$ for almost every $a \in (p + \tau, q + \tau)$, which implies by means of Lemma 28 that $y(T_1, a) = 0$ for almost every $a \in (p, q)$. Hence $y(T_1, a) = 0$ for almost every $a \in (0, \omega)$, from which it follows that $\int_0^\omega y(T_1, a)\,\mathrm{d}a = 0$.

However, we have $\int_0^\omega y(T_1, a)\,\mathrm{d}a > 0$, because of the assumption $\int_0^\omega y(0, a)\,\mathrm{d}a > 0$ and Lemma 29. This contradicts the above result. $\square$

Next, let us show that uniform weak persistence implies uniform strong persistence for system (4.4).

Let $\Phi : \mathbb{R}_+ \times \tilde{\Omega} \to \tilde{\Omega}$ be the semiflow induced by system (4.4) via

$$\Phi(t, {}^{\mathrm{t}}(x_0, y_0, z_0)) = {}^{\mathrm{t}}(x(t, \cdot), y(t, \cdot), z(t, \cdot)),$$

where

$$\tilde{\Omega} := \{ {}^{\mathrm{t}}(x, y, z) \in (L^1_+(0, \omega))^3 \mid x + y + z = 1 \}$$

is the state space of the system.

For sufficiently large $T_0 (> \omega)$, set

$$B := \{ \Phi(t, {}^{\mathrm{t}}(x_0, y_0, z_0)) \mid t > T_0, \quad {}^{\mathrm{t}}(x_0, y_0, z_0) \in \tilde{\Omega} \}.$$

Then we have the following:

Lemma 31. *B is precompact in $\tilde{\Omega}$.*

Proof. It is obvious that B is bounded. Observe that Assumption 8 with PMA implies that for any ε there exists $\delta > 0$ such that

$$\int_0^\omega |\alpha_1(a + h) - \alpha_1(a)|\,\mathrm{d}a < \varepsilon, \qquad \int_0^\omega |\theta(a + h) - \theta(a)|\,\mathrm{d}a < \varepsilon$$

whenever $|h| < \delta$. We can assume that $\delta < \varepsilon$.

Let $\Phi(t, {}^{\mathrm{t}}(x_0, y_0, z_0)) = {}^{\mathrm{t}}(x(t, \cdot), y(t, \cdot), z(t, \cdot)) \in B$. Let $P_j\, {}^{\mathrm{t}}(u_1, u_2, u_3) := u_j\ (j = 1, 2, 3)$ be the projection operator on $(L^1(0, \omega))^3$.

First, we find that, if $a, a+h \in (0, \omega)$, then

$$
\begin{aligned}
|x(t,a+h)-x(t,a)| &= |\,\mathrm{e}^{-\int_0^{a+h}\lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau}-\mathrm{e}^{-\int_0^{a}\lambda_1(t-a+\tau,\tau)\,\mathrm{d}\tau}| \\
&\leqq \left|\int_0^{a+h}\lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau-\int_0^{a}\lambda_1(t-a+\tau,\tau)\,\mathrm{d}\tau\right| \\
&= \left|\int_0^{h}\lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau+\int_0^{a}(\alpha_1(\tau+h)-\alpha_1(\tau))\psi_1(t-a+\tau)\,\mathrm{d}\tau\right| \\
&\leqq \left|\int_0^{h}\lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau\right|+\int_0^{a}|\alpha_1(\tau+h)-\alpha_1(\tau)|\psi_1(t-a+\tau)\,\mathrm{d}\tau \\
&\leqq \alpha^{\infty}|h|+\alpha_2^{\infty}\varepsilon \\
&\leqq (\alpha^{\infty}+\alpha_2^{\infty})\varepsilon.
\end{aligned}
$$

This implies that, if $h > 0$, then

$$
\begin{aligned}
\int_0^{\omega}|x(t,a+h)-x(t,a)|\,\mathrm{d}a &= \int_0^{\omega-h}|x(t,a+h)-x(t,a)|\,\mathrm{d}a+\int_{\omega-h}^{\omega}|0-x(t,a)|\,\mathrm{d}a \\
&\leqq (\alpha^{\infty}+\alpha_2^{\infty})\omega\varepsilon+h \\
&\leqq \{(\alpha^{\infty}+\alpha_2^{\infty})\omega+1\}\varepsilon.
\end{aligned}
\tag{8.9}
$$

The case $h < 0$ is as well. Hence, applying the well-known compactness criteria in L^1 yields that P_1B is precompact in $P_1\tilde{\Omega}$.

Next, let us see that the same argument leads to the precompactness of P_2B. If $a, a+h \in (0, \omega)$, then

$$
\begin{aligned}
y(t,a+h)-y(t,a) &= \int_0^{h}\lambda_1(t-a-h+\sigma,\sigma)\theta(\sigma)x(t-a-h+\sigma,\sigma)\,\mathrm{e}^{-\int_\sigma^{a+h}\lambda_2(t-a-h+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \\
&\quad+\int_0^{a}\psi_1(t-a+\sigma)\{\alpha_1(\sigma+h)\theta(\sigma+h)x(t-a+\sigma,\sigma+h) \\
&\quad\times\mathrm{e}^{-\int_\sigma^{a}\lambda_2(t-a+\tau,\tau+h)\,\mathrm{d}\tau}-\alpha_1(\sigma)\theta(\sigma)x(t-a+\sigma,\sigma)\,\mathrm{e}^{-\int_\sigma^{a}\lambda_2(t-a+\tau,\tau)\,\mathrm{d}\tau}\}\,\mathrm{d}\sigma.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|y(t,a+h)-y(t,a)| &\leqq \alpha^{\infty}|h|+\alpha_2^{\infty}\int_0^{a}|\alpha_1(\sigma+h)-\alpha_1(\sigma)|\,\mathrm{d}\sigma+\alpha^{\infty}\int_0^{a}|\theta(\sigma+h)-\theta(\sigma)|\,\mathrm{d}\sigma \\
&\quad+\alpha^{\infty}\int_0^{a}|x(t-a+\sigma,\sigma+h)-x(t-a+\sigma,\sigma)|\,\mathrm{d}\sigma \\
&\quad+\alpha^{\infty}\int_0^{a}|\,\mathrm{e}^{-\int_\sigma^{a}\lambda_2(t-a+\tau,\tau+h)\,\mathrm{d}\tau}-\mathrm{e}^{-\int_\sigma^{a}\lambda_2(t-a+\tau,\tau)\,\mathrm{d}\tau}|\,\mathrm{d}\sigma \\
&\leqq \alpha^{\infty}\varepsilon+\alpha_2^{\infty}\varepsilon+\alpha^{\infty}\varepsilon+\{(\alpha^{\infty}+\alpha_2^{\infty})a+1\}\varepsilon \\
&\quad+\alpha^{\infty}\int_0^{a}\left\{\int_\sigma^{a}|\lambda_2(t-a+\tau,\tau+h)-\lambda_2(t-a+\tau,\tau)|\,\mathrm{d}\tau\right\}\,\mathrm{d}\sigma,
\end{aligned}
$$

where the evaluation similar to (8.9) is made. Let us evaluate the last term of the above inequality. We have

$$
\begin{aligned}
&\lambda_2(t-a+\tau,\tau+h)-\lambda_2(t-a+\tau,\tau) \\
&\quad=\psi_2(t-a+\tau)(\beta_1(\tau+h)-\beta_1(\tau))+\psi_3(t-a+\tau)(\gamma_1(\tau+h)-\gamma_1(\tau)),
\end{aligned}
$$

and the evaluation

$$\int_{\sigma}^{a} |\psi_2(t-a+\tau)(\beta_1(\tau+h)-\beta_1(\tau))|\,\mathrm{d}\tau \leqq \beta_2^{\infty}\varepsilon \tag{8.10}$$

holds whenever $|h|<\delta$ if $\delta>0$ is chosen small enough. This is true if $\beta_2^{\infty}=0$, because then $\beta_2(\sigma)=0$ for almost every $\sigma\in(0,\omega)$ and it follows that $\psi_2(t)=0$ for all t. Otherwise, $\beta_2(\sigma)>0$ for some $\sigma\in(0,\omega)$, so that Assumption 8 implies that

$$\int_{0}^{\omega} |\beta_1(a+h)-\beta_1(a)|\,\mathrm{d}a<\varepsilon$$

holds whenever $|h|<\delta$ if $\delta>0$ is chosen small enough. This gives the evaluation (8.10). The same argument leads to the evaluation

$$\int_{\sigma}^{a} |\psi_3(t-a+\tau)(\gamma_1(\tau+h)-\gamma_1(\tau))|\,\mathrm{d}\tau \leqq \gamma_2^{\infty}\varepsilon.$$

whenever $|h|<\delta$ if $\delta>0$ is chosen small enough. Hence we have

$$\begin{aligned}\int_{0}^{a}\left\{\int_{\sigma}^{a}|\lambda_2(t-a+\tau,\tau+h)-\lambda_2(t-a+\tau,\tau)|\,\mathrm{d}\tau\right\}\mathrm{d}\sigma &\leqq \int_0^a(\beta_2^{\infty}+\gamma_2^{\infty})\varepsilon\,\mathrm{d}\sigma\\ &\leqq(\beta_2^{\infty}+\gamma_2^{\infty})\varepsilon a,\end{aligned}$$

which yields that $|y(t,a+h)-y(t,a)|\leqq C_0\varepsilon$ for some constant $C_0>0$.

The same evaluation as (8.9) implies that

$$\int_{0}^{\omega}|y(t,a+h)-y(t,a)|\,\mathrm{d}a\leqq(C_0\omega+1)\varepsilon.$$

Therefore we obtain the precompactness of P_2B.

Then it is obvious that P_3B is precompact and we have the conclusion. □

Theorem 32. *If $R_0>1$, then the rumor is uniformly strongly persistent.*

Proof. Since we have already shown that the rumor is uniformly weakly persistent, the assertion is a direct consequence of [47, Theorem A.32] (also see [45]). All we have to do is to make sure the "compactness condition".

It is clear that the autonomous semiflow Φ is continuous. We define $\rho:\tilde{\Omega}\to\mathbb{R}_+$ by

$$\rho({}^{\mathrm{t}}(x,y,z)):=\int_0^{\omega}y(a)\,\mathrm{d}a,\quad {}^{\mathrm{t}}(x,y,z)\in\tilde{\Omega}.$$

Then we find that ρ is uniformly continuous on $\tilde{\Omega}$. Lemma 29 implies that

$$\rho(\Phi(t,{}^{\mathrm{t}}(x,y,z)))>0\quad\text{for all } t>0\ \text{ whenever } \rho(\Phi({}^{\mathrm{t}}(x,y,z)))>0.$$

By the definition of B, we have $\Phi(t,{}^{\mathrm{t}}(x,y,z))\to\bar{B}$ as $t\to\infty$. In addition, Lemma 31 implies that $\bar{B}$ is compact. Therefore we obtain the conclusion. □

9. Discussion

In this paper we have examined rumor transmission models motivated by S-I-R type epidemic models.

We have derived the global behavior of the age-independent rumor transmission models which are extensions of the deterministic Daley–Kendall model. The result is that there is no undamped oscillation and the solution converges to some equilibrium as $t\to\infty$.

In addition, we have shown that, in the age-structured transmission model of a constant rumor, there exists a threshold value $R_0:=r(\tilde{T})$ given as the spectral radius of the positive linear operator $\tilde{T}$ and that RFE is the only equilibrium and globally asymptotically stable if $R_0 < 1$ and at least one REE exists if and only if $R_0 > 1$. Moreover, assuming PMA, we have shown that $R_0 > 1$ implies that the rumor is uniformly strongly persistent and REE is locally asymptotically stable if $|R_0 - 1|$ is small enough.

Assuming PMA is a huge simplification. Without PMA, even uniform strong persistence or the local stability of REE is a difficult problem.

As for the age-structured model considered in this paper, how many REEs exist in the case $R_0 > 1$ is left as open problems, which should be investigated in the near future. It would be also an interesting open problem whether the stable REE could lose its stability and lead a bifurcation of periodic or chaotic solutions. Moreover, the age-structured model for a variable rumor remains to be analyzed.

Our rumor transmission models could be extended to several directions: one way is to introduce more fine structures such as how the transition rate from the stifler into the susceptible class depends on duration in the case of a variable rumor. Another important way of extension is to introduce the effect on mass communication, which could be considered as both rumor-source and rumor-“vaccination”. Moreover, it would be interesting to consider the case that several conflicting rumors are transmitted [10,5,13,30], which would be useful in the control of a rumor which is troublesome for an individual or an organization.

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