Entanglement Entropy and Holography

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Outline

1. Density Matrices & Entanglement Entropy
   - Density Matrices
   - Entanglement Entropy
2. Holography
   - Holography
   - Entanglement Entropy & the AdS/CFT Correspondence
3. Entanglement Entropy Calculations
   - Calculations in AdS_{2+1}/CFT_{1+1}
   - Higher Dimension Calculations
   - Finite Temperature CFT's
4. Holographic Quantum Quenches
   - Quantum Quench
   - CFT_{1+1} Quantum Quench.
   - CFT_{2+1} Quantum Quench
5. Summary

Outline

Density Matrices

- The most general quantum state can be entangled & mixed. A pure state alone cannot describe the statistical mixing of states.
- The density matrix ρ for a state $|\psi\rangle \implies \rho = |\psi\rangle\langle\psi|$.
- Mixed state density matrix $\implies$ weighted sum of pure density matrices $\rho^{mix} = \sum_i p_i \rho^{pure} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.
- $p_i \implies$ probability to measure subsystem in state $|\psi_i\rangle$.

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Entanglement Entropy

- A measure of how strongly entangled two systems are is the entanglement entropy.
- $|\Psi\rangle$ with full density matrix ρ divided into two subsystems A & B.
- Subsystem A individually described by its reduced density matrix ρ_A.
- ρ_A for subsystem A obtained by "tracing out" degrees of freedom in B $\rho_A = -Tr_B(\rho_{Tot})$.
- Then the Entanglement Entropy of subsystem A with subsystem B is $S(\rho_A) = -Tr(\rho_A \log \rho_A)$.

Entanglement Entropy

- Wish to use entanglement entropy to probe QFT's.
- Calculating entanglement entropy's for arbitrary QFT $\implies$ difficult in general.
- QFT $\implies$ Large/Infinite number of degrees of freedom $\implies$ tracing out process hard.
- Holography provides a prescription to calculate entanglement entropy.

Outline

Holography

- Holography describes a duality between a theory on a $d+1$-dimensional manifold M to a different theory of the d-dimensional boundary ∂M.
- Boundary theory degrees of freedom are completely described by degrees of freedom in bulk.
- The most famous form of Holography is the Anti-de Sitter(AdS)/Conformal Field Theory(CFT) correspondence.

The AdS/CFT Correspondence

- AdS/CFT: Quantum gravity on $AdS_{d+2} \Leftrightarrow \text{CFT}_{d+1}$ on $\mathbb{R}^{1,d}$.
- Proposed by Maldacena (1984) arising from string theory on $AdS_5 \times \mathbf{S}^5 \Leftrightarrow \mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills.
- Yang-Mills: $\mathcal{L} = \frac{1}{g_{YM}^2} \text{Tr}\,(F_{\mu\nu}F^{\mu\nu})$.
- 't Hooft coupling $\lambda = Ng_{YM}^2 = \textit{fixed}$ as $N \to \infty$, $g_{YM} \to 0$.
- The string coupling $g_{st} \propto g_{YM}^2$
- The "holographic dictionary": Weakly coupled gravity ($g_{st} \ll 1$) $\Leftrightarrow$ Large-N $\mathcal{N} = 4$ Yang-Mills.

The AdS/CFT Correspondence

- Conformal field theory: Special type of QFT with additional symmetries called conformal symmetries.
- On $\mathbb{R}^{1,d}$ they consist of:

$$\begin{aligned} \text{Translations}: &\quad x^\mu \to x^\mu + a^\mu \\ \text{Rotations}: &\quad x^\mu \to \omega_\mu^\nu x^\nu \\ \text{Boosts}: &\quad x^\mu \to \lambda x^\mu \\ \text{Special Conformal Transformations}: &\quad x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b.x + b^2 x^2}. \end{aligned}$$

- In $d > 2$ conformal transformations form a finite dimensional group spanned by $SO(2, d+1)$ (in $d < 2$ the number of generators of the conformal algebra is infinite).

The $AdS/$CFT Correspondence

- Anti-de Sitter space: Solution to Einsteins equations, in $(d+2)$ dimensions, with action

$$S = \frac{1}{16\pi G_N^{(d+2)}} \int d^{d+2}y(R - 2\Lambda). \tag{1}$$

- Such a solution can be embedded in a $(d+3)$-dimensional coordinate system with $\Lambda = -\frac{(d)(d+1)}{2\alpha^2}$

$$\sum_{i=1}^{d+1} \chi_i^2 - {\chi_{d+2}}^2 - {\chi_{d+3}}^2 = -\alpha^2, \tag{2}$$

- $\implies SO(2, d+1)$ symmetry.

The AdS/CFT Correspondence

- AdS_{d+2} boundary is a copy of $\mathbb{R}^{1,d}$.
- AdS_{d+2} in the half space Poincaré coordinates

$$ds^2 = \frac{1}{y^2}\left(dy^2 - dt^2 + \sum_{i=1}^{d} dx_i^2\right). \tag{3}$$

- AdS boundary $y \to 0$

$$ds^2 \to -dt^2 + \sum_{i=1}^{d} dx_i^2. \tag{4}$$

- AdS_{d+2} boundary conformally equivalent to $\mathbb{R}^{1,d}$.
- $\implies$ two ways to build $SO(2, d+1)$ theory: QFT on AdS or CFT $\partial(AdS)$.

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Entanglement Entropy & the AdS/CFT Correspondence

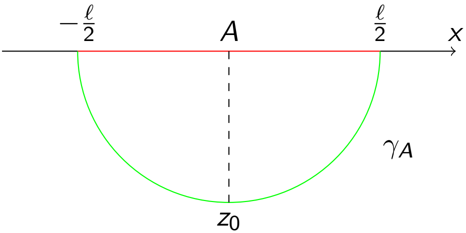
- Proposal by Ryu and Takayanagi: entanglement entropy obeys an area law.
- Entanglement entropy of CFT region A is given by the area of the minimal area surface γ_A extending from A into bulk AdS

$$S_A = \frac{Area\left(\gamma_A\right)}{4G_N^{(d+2)}}. \tag{5}$$

Outline

Infinite Line in $\mathrm{AdS}_{2+1}/\mathrm{CFT}_{1+1}$

- CFT_{1+1} on $\mathbb{R}^{1,1}$ entanglement entropy, fixed time slice on a region A of length ℓ.
- In AdS_{2+1} minimal area surface γ_A is given by geodesic in AdS_{2+1}.
- AdS_{2+1} Poincaré coordinates (3)



Infinite Line in AdS_{2+1}/CFT_{1+1}

- The geodesic length is given by minimising the action

$$Length\,(\gamma_A) = 2R \int_{\epsilon}^{\frac{\ell}{2}} \frac{1}{z}\sqrt{1+\left(\frac{dx}{dz}\right)^2}\,dz = 2R\log\left(\frac{\ell}{\epsilon}\right). \qquad (6)$$

- Applying (5) the entanglement entropy is

$$S_A = \frac{c}{3}\log\left(\frac{\ell}{\epsilon}\right). \qquad (7)$$

- UV cutoff ϵ imposed close to boundary $z \to 0$.
- c is the CFT central charge or conformal anomaly.

Outline

$\mathrm{CFT}_{d+1}/\mathrm{AdS}_{d+2}$ Infinite Strip

- Entanglement entropy of infinite strip of CFT_{d+1} region A of length $L \to \infty$ in $(d-1)$ spatial dimensions, finite width ℓ in the remaining spatial dimension.

- Poincaré coordinates (3), minimal area hypersurface γ_A given by minimising

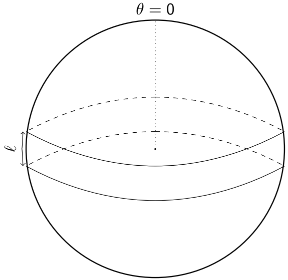
$$Area(\gamma_A) = 2\int_0^L dx_2 \ldots \int_0^L dx_d \int_0^{z_0} \sqrt{g|_{induced}}\, dz \tag{8}$$

$$= 2R^d L^{d-1} \int_0^{z_0} \frac{1}{z^d} \sqrt{1 + \left(\frac{dx_1}{dz}\right)^2}\, dz. \tag{9}$$

$$S_A = \frac{R^d}{2(d-1)G_N^{(d+2)}} \left[\left(\frac{L}{\epsilon}\right)^{d-1} - 2^{d-1}\pi^{d/2} \left(\frac{\Gamma\left(\frac{d+1}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)} \right)^d \left(\frac{L}{\ell}\right)^{d-1} \right]. \tag{10}$$

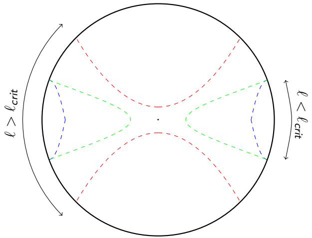
Spherical strip in $AdS_{3+1}/\mathrm{CFT}_{2+1}$

- Minimal surfaces in AdS space with compact boundaries exhibit some interesting behaviour.
- Examine a spherical strip of CFT_{2+1} on $\mathbf{S}^2$ geometry at fixed t.
- A is a spherical strip of height ℓ for $AdS_{3+1}/\ \mathrm{CFT}_{2+1}$.



Spherical strip in $AdS_{3+1}/\mathrm{CFT}_{2+1}$

- Periodicity $\ell \sim \ell + \pi \implies$ phase transition in minimal surfaces.
- Solve numerically $\implies$ 3 classes of solutions:
- Two connected solutions: one close to the boundary, another extending deep into bulk close to $r = 0$. Latter solution unfavoured as it does not minimize the action.
- One disconnected solution appearing as two unconnected caps in each hemisphere.



Spherical strip in $AdS_{3+1}/\mathrm{CFT}_{2+1}$

$$Area(\gamma_A) = 4\pi R^2 \int_{r0}^{\infty} r \sin\theta \sqrt{\frac{1}{1+r^2} + r^2 \left(\frac{d\theta}{dr}\right)^2} dr. \qquad (11)$$

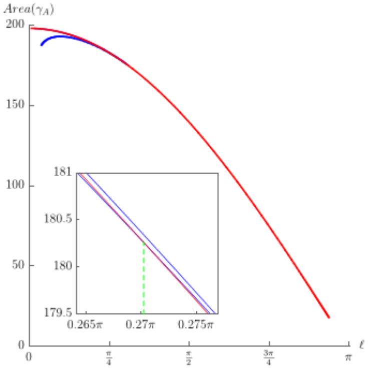


Figure: $Area(\gamma_A)$ as a function of ℓ: (Blue) Connected, (Red) Disconnected, (Inset) zoomed near transition point.

Spherical strip in $AdS_{3+1}/\mathrm{CFT}_{2+1}$

- Indicative of a first order phase transition of minimal surfaces for $\ell = \ell_{crit} \simeq 0.27\pi$.
- Interpretation on CFT side unclear.
- Similar type of picture to surfaces found by Klebanov, Kutasov, and Murugan (http://arxiv.org/abs/0709.2140).

Outline

Finite temperature CFT's

- The entanglement entropy prescription applied to CFT's at thermal equilibrium at inverse temperature β_{CFT}.
- Holographic dual to such a CFT is given by a black hole in AdS.

CFT_{1+1} at finite temperature

- Holographic dual to a CFT_{1+1} at inverse temperature β_{CFT} is the non-rotating Euclidean BTZ black hole

$$ds^2 = \frac{r^2 - r_+^2}{R^2} dt^2 + \frac{R^2}{r^2 - r_+^2} dr^2 + r^2 dx^2. \tag{12}$$

- To ensure a smooth geometry the periodicity $t \sim t + \frac{2\pi R}{r_+}$ is imposed to obtain a $\mathbf{S}^1 \times \mathbb{R}^1$ boundary geometry.
- Taking the boundary limit $r \to \infty$, $\beta_{CFT} = \frac{2\pi R}{r_+}$.

CFT^{1+1} at finite temperature

- The entangled region A is of length ℓ on $\mathbb{R}^{1,1}$ at fixed t.
- The geodesics length is

$$Length\,(\gamma_A) = 2\int_{r_0}^{\frac{\beta_{CFT} r_+}{2\pi\epsilon}} \sqrt{\frac{R^2}{r^2 - r_+^2} + r^2\left(\frac{dx}{dr}\right)^2}\, dr. \tag{13}$$

$$S_A = \frac{c}{3}\log\left[\frac{\beta_{CFT}}{\pi\epsilon}\sinh\left(\frac{\pi\ell}{\beta_{CFT}}\right)\right]. \tag{14}$$

- For empty AdS $\beta_{CFT} \to \infty$ and $S_A \to \frac{c}{3}\log\frac{\ell}{\epsilon}$ (the vacuum result).
- Large $\ell \implies S_A$ becomes extensive quantity, thermal entropy.

Outline

Quantum Quench

- Apply entanglement entropy prescription to dynamical backgrounds to study a CFT undergoing a quantum quench.
- Quantum quench $\implies$ process of taking a state $|\psi(0)\rangle$ and suddenly changing a parameter in the Hamiltonian at $t = 0 \implies$ pushes system far away from equilibrium.
- Entanglement entropy used to track entanglement propagation over time as the system thermalises to a new equilibrium state $|\psi(t)\rangle$.
- Causality $\implies$ system cannot reach true thermal equilibrium. System relaxes to an equilibrium-like state locally.

- The holographic dual to such a process described by *AdS*-Vaidya solutions.
- *AdS*-Vaidya describes infall of null dust along an ingoing lightcone coordinate v resulting in the formation of a black hole.
- AdS_{d+1}-Vaidya solution to Einstein's equations with metric, boundary $\mathbb{R}^{1,d-1}$

$$ds^2 = -\left(r^2 - \frac{m(v)}{r^{d-1}}\right) dv^2 + 2dvdr + r^2 \sum_{i=1}^{d-1} dx_i^2. \tag{15}$$

Black hole formation

- Pick a mass function to interpolate smoothly between 0 and m

$$m(v) = m\frac{1+\tanh\left(\frac{v}{a}\right)}{2}. \tag{16}$$

- Parameter a controls gradient of the black hole formation process.

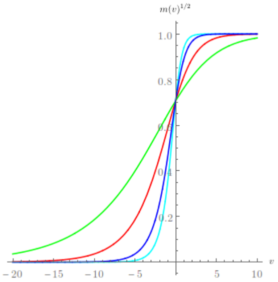


Figure: Apparent horizon location in AdS_{2+1}-Vaidya for $a = \frac{1}{3}, \frac{1}{2}, 1, 2$.

Outline

CFT_{1+1} Quantum Quench.

- Repeat analysis by Abajo-Arrastia, Aparicio, and Lopez (arxiv.org/abs/1006.4090).
- AdS_{2+1}-Vaidya models a CFT_{1+1} perturbed from it's vacuum state at boundary time $t = 0$.
- Geometry models a CFT without a mass gap $\implies$ infinite correlation length $\implies$ excitations spread over entire length of system for $t \to \infty$.
- CFT_{1+1} admits holomorphic factorisation $\implies$ left and right moving modes move away from each other and are non-interacting.
- Conformal invariance $\implies$ excitations move with velocity $v^2 = 1$ and causality $\implies$ after time t excitations a distance $\zeta = 2t$ apart.

CFT_{1+1} Quantum Quench

- Entanglement entropy is determined by the minimal area surface γ_A in bulk extending from a region of length ℓ
- Geodesic length

$$L(\ell, t) = \int_0^\ell \sqrt{r^2 + 2r'v' - (r^2 - m(v))v'^2}\,dx. \tag{17}$$

- Minimising we see the geodesic length can be written in a compact form

$$L(\ell, t) = \frac{2}{r_0} \int_\epsilon^{\frac{\ell}{2}} r(x)^2 dx. \tag{18}$$

- Solve numerically with boundary conditions $v(0) = v(\ell) = t$, $r(0) = r(\ell) = \infty$.

CFT$_{1+1}$ Quantum Quench

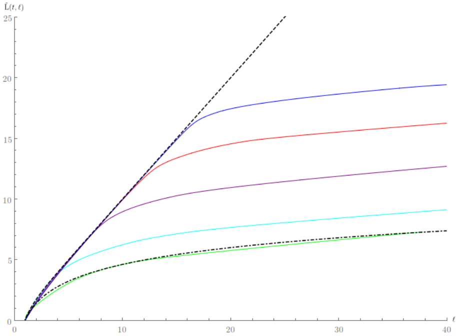


Figure: Plot of the UV independent part of the AdS_3-Vaidya geodesic length $\tilde{L}(t,\ell)$ against ℓ for $t = 0,2,4,6,8$ (bottom to top) the dot-dashed line represents the vacuum result and the dashed line the thermal result.

CFT$_{1+1}$ Quantum Quench

- Deduce thermalisation time in CFT$_{1+1}$ is

$$t_T \simeq \frac{\ell}{2}. \tag{19}$$

- Holomorphic factorisation $\implies$ excitations after time t spread over a region $\zeta = 2t = \ell \implies t_T = \frac{\ell}{2}$.
- Excitations are non-interacting.
- Result expected and decided entirely by CFT$_{1+1}$ kinematics & causality.
- Analysis closely matches analytical results found by P.Calabrese and J.Cardy arxiv.org/abs/cond-mat/0503393.

Outline

CFT_{2+1} Quantum Quench

- Leads to studying a less trivial theory.
- Repeat quantum quench analysis on infinite strip geometry of length $L \to \infty$, width ℓ for a CFT_{2+1}.
- CFT's in $> (1+1)$-dimensions $\implies$ no holomorphic factorisation $\implies$ interacting excitations.

CFT_{2+1} Quantum Quench

- In this geometry excitations spread across entire length of L for all t.
- Causality $\implies$ excitations after a time t spread over a maximum area $\zeta \leq 2tL$.

CFT_{2+1} Quantum Quench

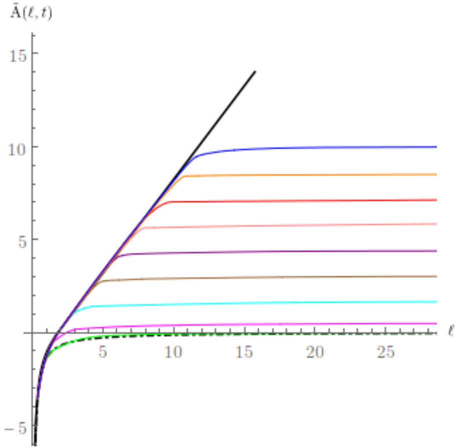


Figure: Plot of the UV independent part of the AdS_4-Vaidya minimal surface area $\tilde{A}(t, \ell)$ against ℓ for $a = \frac{1}{3}$ and $t = 0,1,..,8$ (bottom to top) the dashed line represents the vacuum result and the solid black line the thermal result.

CFT_{2+1} Quantum Quench

- Plotting ℓ for which the entanglement entropy thermalises gives the thermalisation time.

$$t_T = f(\beta, \ell) \simeq \frac{2}{3}\ell \tag{20}$$

- Infer no β dependence, cannot build a dimensionless quantity from ℓ and β with $t_T \propto \ell$.
- Additionally, L cannot be used either as t_T is invariant under scaling of L as L is purely a large multiplicative constant to the entanglement entropy.

CFT$_{2+1}$ Quantum Quench

- The value of $t_T = \frac{2}{3}\ell \implies$ excitations interact.
- Causality: $\zeta \leq 2tL$.
- Evaluated at $t = t_T$: $\zeta \leq 2t_T L = \frac{4}{3}\ell L$.
- Causality $\implies$ excitations must interact which induces longer thermalisation time.

Summary

- Applied and reproduced entanglement entropy's for some static geometries.
- Applied the prescription to a geometry with a compact boundary $\implies$ phase transition between minimal surfaces.
- Used holographic entanglement entropy to model a CFT_{1+1} undergoing a quantum quench to and reproduced previously found result $t_T = \frac{1}{2}\ell$.
- Repeated the quantum quench analysis for a more interesting geometry for a CFT_{2+1} to find $t_T \simeq \frac{2}{3}\ell \implies$ possibly follows $t_T \simeq \frac{d-1}{d}\ell$.