Exact results for Class $\mathcal{S}_k$

Thomas Bourton

DESY

SCGSC 2018

Based on [hep-th/1712.01288]

Outline

Motivation

Exact results for 4d $\mathcal{N} = 2$ & Class $\mathcal{S}$

$\mathcal{N} = 1$ theories & Class $\mathcal{S}_k$

Class $\mathcal{S}_k$ construction

Some known results

The instanton partition function

Relation to 2d CFT

Outline

Motivation

Exact results for 4d $\mathcal{N} = 2$ & Class $\mathcal{S}$

$\mathcal{N} = 1$ theories & Class $\mathcal{S}_k$

Class $\mathcal{S}_k$ construction

Some known results

The instanton partition function

Relation to 2d CFT

4d $\mathcal{N} = 2$ theories

Seiberg-Witten theory

- Low energy effective action encoded in algebraic curve. [Seiberg, Witten '94]

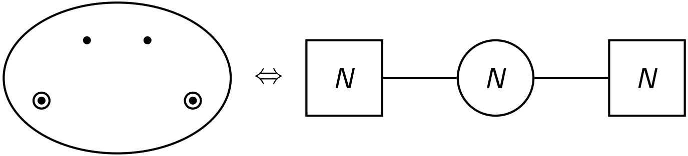
Gaiotto classification: Class $\mathcal{S}$

- Compactify 6d $\mathcal{N} = (2, 0)$ SCFT (stack of N M5 branes) on a torus [Vafa '94]
- Gives rise to 4d $\mathcal{N} = 4$ Super Yang-Mills theory. Gauge coupling τ = torus complex structure
- $\mathbb{SL}(2, \mathbb{Z})$ S-duality Group = Modular transformations of torus!

4d $\mathcal{N} = 2$ theories

- More generally: Compactify 6d $(2,0)$ on Riemann surface $\mathcal{C}$ [Gaiotto '09] [Gaiotto, Moore, Neitzke '09]
- Rephrase field theory operations as operations on $\mathcal{C}$. e.g. Generalised S-duality frames $\leftrightarrow$ Pair of Pants decomposition

$\mathcal{C}$ = four punctured sphere $\Leftrightarrow$ $\mathcal{N} = 2$ SQCD



4d $\mathcal{N} = 2$ theories

Localisation techniques

- Exact evaluation of full path integral on various manifolds e.g. $\mathbb{S}^4$, $\mathbb{S}^1 \times \mathbb{S}^3$. [Pestun '07] [Romelsberger '05] [Kinney, Maldacena, Minwalla, Raju '05] [...]

- Idea of Localisation: $S \to S + t \int d^4x \, \mathcal{Q}V$

$$Z_{\mathbb{S}^4}(t) = \int d\mu \, e^{-S - t \int d^4x \, \mathcal{Q}V} \,, \quad \partial_t Z_{\mathbb{S}^4}(t) = 0$$

  $t \to \infty$

- $\mathbb{S}^4$ partition function localises on instanton configurations $F = \pm \star F$

$$Z_{\mathbb{S}^4} = \int d\mathbf{a} \, Z_{\text{1-loop}}(\mathbf{a}) |Z_{\text{inst}}(\mathbf{a})|^2$$

4d $\mathcal{N} = 2$ theories

- Nekrasov's instanton counting [Nekrasov '02]: path integral $Z^{(K)}_{\text{inst}}$ over all instanton configurations with topological number $K = \frac{1}{4\pi^2} \int \text{tr}\, F \wedge F$

$$Z_{\text{inst}}(\mathbf{a}) = \sum_{K=0}^{\infty} \mathbf{q}^K Z^{(K)}_{\text{inst}}\,, \quad Z^{(K)}_{\text{inst}} = \int_{\mathcal{M}^{(K)}_{\text{inst}}} d\mu\, e^{-S^{(K)}_{\text{inst}}}$$

- $Z_{\text{1-loop}}$ quadratic fluctuations around each saddle point

AGT(W)-correspondence

- Protected quantities often computed by a 2d QFT on $\mathcal{C}$

$\mathbb{S}^4$ partition function

- Computed by correlator of 2d Liouville/Toda CFT. [Alday, Gaiotto, Tachikawa '09] [Wyllard '09]
- Instanton partition function Z_{inst} = Conformal blocks of Liouville/Toda. $Z_{\text{1-loop}}$ = three point function coefficients

$\mathbb{S}^1 \times \mathbb{S}^3$ partition function (superconformal index)

- Equal to a correlator of a 2d TQFT. [Gadde, Pomoni, Rastelli, Razamat '09]

Outline

Motivation

Exact results for 4d $\mathcal{N} = 2$ & Class $\mathcal{S}$

$\mathcal{N} = 1$ theories & Class $\mathcal{S}_k$

Class $\mathcal{S}_k$ construction

Some known results

The instanton partition function

Relation to 2d CFT

Class $\mathcal{S}_k$

Natural generalisation of Class $\mathcal{S}$

Instead: Compactify a 6d $\mathcal{N} = (1,0)$ on $\mathcal{C}$.

- The 6d $\mathcal{N} = (1,0)$ theory we use is a $\mathbb{Z}_k$ orbifold of the 6d $\mathcal{N} = (2,0)$ theory [Gaiotto, Razamat '15]
- The $\mathbb{Z}_k$ breaks supersymmetry by 1/2, leaving $\mathcal{N} = 1$ in 4d

	$\mathbb{C}^2$				$\mathcal{C}$		$\mathbb{C}^2_\perp$				$\mathbb{S}^1$
N M5	–	–	–	–	–	–	·	·	·	·	·
$\mathbb{Z}_k$	·	·	·	·	·	·	×	×	×	×	·

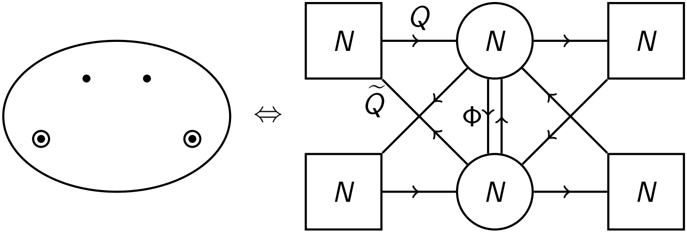
$$\mathbb{Z}_k : (v_\perp, w_\perp) \mapsto (e^{2\pi i/k} v_\perp, e^{-2\pi i/k} w_\perp)$$

- 4d theories again classified by punctured Riemann surfaces $\mathcal{C}$

Example

$\mathcal{C}$ = four punctured sphere

- For $k = 2$ - $\mathcal{N} = 1$ superconformal quiver



- $W \simeq \left(Q\Phi\widetilde{Q} - \widetilde{Q}\Phi Q\right) - \frac{i\tau}{8\pi^2}\,\mathrm{tr}\, W^{\alpha} W_{\alpha}$

Outline

Some known results for Class $\mathcal{S}_k$

$\mathbb{S}^3 \times \mathbb{S}^1$ partition function (superconformal index)

- Can be computed via localisation
- First example of 2d/4d relation for $\mathcal{N} = 1$
- It may be computed as a correlator in a TQFT

$\mathbb{S}^4$ partition function

- No localisation computation so far
- For $\mathcal{N} = 1$: regularisation ambiguities - scheme dependent. [Gerchkovitz, Gomis, Komargodski '14]
- Derivatives of the free energy $F_{\mathbb{S}^4}$ are universal (for $\mathcal{N} = 1^*$ theory) [Bobev, Elvang, Kol, Olson, Pufu '16]

Some known results for Class $\mathcal{S}_k$

Z_{inst}?

- Z_{inst} played an important role in $\mathcal{N} = 2$ $\mathbb{S}^4$ partition function.
- Instantons in susy gauge theories can be embedded within string theory - we can use this to compute Z_{inst}

Outline

Motivation

- Exact results for 4d $\mathcal{N}=2$ & Class $\mathcal{S}$

$\mathcal{N}=1$ theories & Class $\mathcal{S}_k$

- Class $\mathcal{S}_k$ construction
- Some known results
- The instanton partition function
- Relation to 2d CFT

$\mathcal{N} = 2$ SCQCD and Dp-D$(p-4)$ system

String theory construction

- $\mathcal{N} = 2$ SQCD can be embedded in Type IIA [Witten '97]

	$\mathbb{C}^2$				$\mathbb{R}^2$		$\mathbb{S}^1$	$\mathbb{R}^3_\perp$		
N D4	–	–	–	–	·	·	–	·	·	·
NS5	–	–	–	–	–	–	·	·	·	·
K D0	·	·	·	·	·	·	–	·	·	·

- Supersymmetric instantons can be realised as D$(p-4)$-branes

$$K \text{ instantons in a D}p\text{-brane} \equiv K \text{ D}(p-4)\text{-branes}$$

- D$(p-4)$ Higgs branch equivalent to moduli space of instantons

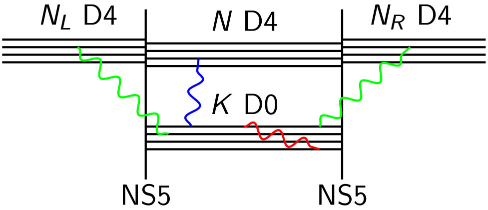
$$\mathcal{M}^{\mathrm{D}p}_{\mathrm{inst}} \cong \mathcal{M}^{\mathrm{D}(p-4)}_{\mathrm{Higgs}}$$

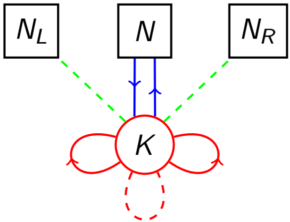
[Douglas '95] [Witten '95] [Douglas '96]

Z_{inst}

The theory on D0-branes

- Supersymmetric sigma model in zero dimensions
- (At least) 2 supercharges $\mathcal{Q}_+$, $\widetilde{\mathcal{Q}}_+$ (reduction of 2d $\mathcal{N} = (0, 2)$)
- Chiral multiplets $\{\phi,\, \psi_+\}$ (arrowed lines). Fermi multiplets $\{\psi_-\}$ (dashed lines). Vector multiplet $\sim$ Fermi multiplets





Z_{inst} from a 2d Superconformal Index computation

Rephrasing the computation

- Partition function for sigma model living on D0-branes equals $Z^{(K)}_{\text{inst}}$
- Fact: Higgs branch is invariant under circle compactification (When we have susy)
- Lift the 0d theory to a 2d theory on $\mathbb{S}^1 \times \mathbb{S}^1$ ($2\times$T-duality: D0 $\to$ D2). Radii β_1, β_2
- $\mathbb{S}^1 \times \mathbb{S}^1$ partition function is superconformal index of the 2d theory!

$$Z^{(K)}_{\text{inst}} = \lim_{\beta_1,\beta_2\to 0} I_{\text{SCI}}$$

Z_{inst} from a 2d Superconformal Index computation

Computing the superconformal index

- Counting gauge invariant operators on $\mathbb{R}^2$ graded by 'fugacities' for $\mathcal{Q}_+$-commuting global symmetries

$$I_{\text{SCI}} = \text{Tr}(-1)^F q^{H_-} v^{2j_1} t^{2j_D} \mathbf{x}^{\mathbf{f}}\,, \quad q = e^{-\beta_1/\beta_2}$$

- $\mathcal{Q}_+$-commuting global symmetry $=$ Cartan subalgebra of:

$$\mathfrak{u}(1)_- \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_D \oplus \mathfrak{su}(N_L) \oplus \mathfrak{su}(N) \oplus \mathfrak{su}(N_R)$$

- Receives contribution only from states with

$$\delta = \left\{\mathcal{Q}_+, \mathcal{Q}_+^\dagger\right\} = H_+ - \frac{1}{2}\mathcal{R}_+^{U(1)} = 0$$

- Fact: SCI is independent of continuous couplings $\Leftrightarrow$ we can compute it in the free theory

Z_{inst} from a 2d Superconformal Index computation

Letter counting

- Example: 'Letters' of Fermi multiplet in adjoint of $U(K)$

Letter	H_-	H_+	$\mathcal{R}_+^{U(1)}$	$(-1)^F$	δ	Index
ψ_-	$1/2$	0	0	-1	0	$-q^{1/2}\chi_{\text{adj.}}^{U(K)}(\mathbf{z})$
$\psi_-^\dagger$	$1/2$	0	0	-1	0	$-q^{1/2}\chi_{\text{adj.}}^{U(K)}(\mathbf{z})$
∂_-^n	n	0	0	$+1$	0	q^n
∂_+^n	0	n	0	$+1$	n	

- Single letter index $i_{\text{Fermi}}(q,\mathbf{z}) = \frac{-2q^{1/2}}{1-q}\chi_{\text{adj.}}^{U(K)}(\mathbf{z})$

$$\boxed{I_{\text{SCI}} = \int \left[d\mu_{U(K)}(\mathbf{z})\right] \mathrm{PE}\left[\sum_{\{\text{multiplets}\}} i_{multiplet}(q,\mathbf{z},\dots)\right]}$$

- Plethystic exponential: $\mathrm{PE}\left[f(x)\right] := \mathrm{Exp}\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x^n)\right]$

Z_{inst} from a 2d Superconformal Index computation

$$I_{\mathrm{SCI}} = \frac{\left(q^k;q^k\right)^{2K}}{K!} I^{(0)} \oint \prod_{I=1}^{K} \frac{dz_I}{z_I} \prod_{A=1}^{N} \prod_{I=1}^{K} \frac{\theta\left(q^{\frac{1}{2}} \frac{z_I}{x_{L,A}};q\right) \theta\left(q^{-\frac{1}{2}} \frac{z_{R,I}}{x_I};q\right)}{\theta\left(t^{-1} q^{\frac{1}{2}} \frac{z_I}{x_A};q\right) \theta\left(t q^{-\frac{1}{2}} \frac{z_I}{x_A};q\right)}$$

$$\times \prod_{I \neq J} \theta\left(\frac{z_I}{z_J};q\right) \prod_{I,J=1}^{K} \frac{\theta\left(t^{-2} q \frac{z_I}{z_J};q\right)}{\theta\left(v t^{-1} q^{\frac{1}{2}} \frac{z_I}{z_J};q\right) \theta\left(v^{-1} t^{-1} q^{\frac{1}{2}} \frac{z_I}{z_J};q\right)}.$$

$(a;q) = \prod_{i=0}^{\infty}(1-aq^i)$ and $\theta(a;q) = (a;q)(a^{-1}q;q)$

- As an aside: I_{SCI} is related to the partition function for the 6d $(2,0)$ theory on $\mathbb{R}^4 \times \mathbb{S}^1 \times \mathbb{S}^1$ [Haghighat, Iqbal, Kozcaz, Lockhart, Vafa '13]

Z_{inst} from a 2d Superconformal Index computation

After taking the limit

$$\lim_{\beta_1,\beta_2\to 0} I_{\text{SCI}} = \frac{1}{K!}\int \prod_{I=1}^{K} du_I \prod_{I,J=1}^{K} \frac{u'_{IJ}\,(u_{IJ}-2\epsilon_+)}{(u_{IJ}+\epsilon_1)\,(u_{IJ}+\epsilon_2)}$$
$$\times \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{(u_I - m_{L,A})\,(u_I - m_{R,A})}{(u_I - a_A - \epsilon_+)\,(u_I - a_A + \epsilon_+)}$$

- $\epsilon_{1,2}$ are Ω-background parameters: related to the fugacities e.g. $v = e^{\beta_1(\epsilon_1+\epsilon_2)}$
- Reproduces Nekrasov's expression for $Z_{\text{inst}}^{(K)}$

Orbifolding to Class $\mathcal{S}_k$

After the $\mathbb{Z}_k$ orbifold

- Supersymmetry is broken by 1/2, however $\mathcal{Q}_+$ is preserved
- We can again compute the 'superconformal index' of the 2d theory
- Counting gauge invariant operators on $\mathbb{R}^2/\mathbb{Z}_k$
- After orbifold: $\mathcal{Q}_+$ -commuting global symmetry = Cartan subalgebra of:

$$\mathfrak{u}(1)_- \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_D \oplus \mathfrak{su}(N_L)^k \oplus \mathfrak{su}(N)^k \oplus \mathfrak{su}(N_R)^k$$

- We can play the same games and compute

$$I_{\mathrm{SCI}}^{\mathrm{orb}} = \mathrm{Tr}(-1)^F q^{H_-} v^{2j_1} t^{2j_D} \mathbf{x}^{\mathbf{f}}$$

[TB, Pomoni '17]

Orbifolding to Class $\mathcal{S}_k$

- The orbifold breaks $U(K) \to \prod_{i=1}^{k} U(K_i)$=G

$$\boxed{I_{\text{SCI}}^{\text{orb}} = \int [d\mu_G(\mathbf{z})] \,\mathrm{PE}\left[\sum_{\{\text{multiplets}\}} i_{multiplet}^{\text{orb}}(q, \mathbf{z}, \dots)\right]}$$

- Take the limit

$$\lim_{\beta_1,\beta_2\to 0} I_{\text{SCI}}^{\text{orb}} \propto \prod_{i=1}^{k} \int \prod_{I=1}^{K_i} du_{i,I} \prod_{I=1}^{K_i} \frac{u'_{ii,IJ} \prod_{j\neq i} \prod_{J=1}^{K_j} (u_{ij,IJ} - 2\epsilon_+)}{\prod_{j=1}^{k} \prod_{J=1}^{K_j} (u_{ij,IJ} + \epsilon_1)(u_{ij,IJ} + \epsilon_2)}$$
$$\times \prod_{j=1}^{k} \prod_{I=1}^{K_i} \prod_{A=1}^{N} \frac{(u_{i,I} - \widetilde{m}_{L,j,A})(u_{i,I} - \widetilde{m}_{R,j,A})}{(u_{i,I} - \widetilde{a}_{j,A} - \epsilon_+)(u_{i,I} - \widetilde{a}_{j,A} + \epsilon_+)}$$

- We expect this to be equal to the integration over the moduli space of $\{K_1, K_2, \dots, K_k\}$ instantons for this $\mathcal{N} = 1$ theory

Outline

Motivation

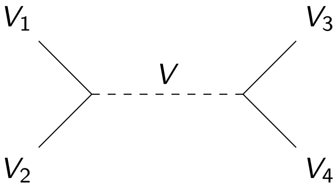
- Exact results for 4d $\mathcal{N}=2$ & Class $\mathcal{S}$

$\mathcal{N}=1$ theories & Class $\mathcal{S}_k$

- Class $\mathcal{S}_k$ construction
- Some known results
- The instanton partition function
- Relation to 2d CFT

Relation to 2d CFT

- There are Seiberg-Witten curves for these Class $\mathcal{S}_k$ theories [Coman, Pomoni, Taki, Yagi '15]
- SW curve coefficients are related to W-algebra blocks (in semi-classical $\epsilon_i \sim Q \to 0$ limit) [Mitev, Pomoni '17]

$\mathcal{B}_{1234} =$ 

- Define

$$Z^{\text{orb}}_{\text{inst}} = \sum_{\{K_i\}=1}^{\infty} \mathbf{q}^{\sum_{j=1}^{k} K_i} Z^{(\{K_i\})}_{\text{inst}} \,, \quad Z^{(\{K_i\})}_{\text{inst}} = \lim_{\beta_1,\beta_2 \to 0} I^{\text{orb}}_{\text{SCI}}$$

- $Z^{\text{orb}}_{\text{inst}} = \mathcal{B}_{1234}$ equal to to W-blocks for certain W-algebra representations

Summary

- Repackaged the $\mathcal{N} = 2$ instanton partition into a 2d SCI computation
- We computed instanton partition functions for some special $\mathcal{N} = 1$ theories.
- We independently matched a prediction based on assumption that an 'AGT'-type relation exists for these theories and matched to W-blocks.
- Outlook/Future directions
  - $\mathbb{S}^4$ partition function for Class $\mathcal{S}_k$
  - Generalisation to other orbifolds of $\Gamma = DE$ type $\rightarrow$ Class $\mathcal{S}_\Gamma$ [Heckman, Jefferson, Rudelius, Vafa '16]