Holography Notes

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1. Intro

Throughout we will always use the upper half Poincaré patch metric of AdS space which in $d+1$ dimensions can be described by the ball B_{d+1} with conformal boundary S_d at $x_0 = \infty$.

$$ds^2 = \frac{1}{x_0^2}\left(dx_0^2 + \sum_{i=1}^{d} dx_i^2\right). \tag{1.1}$$

2. AdS Massless scalar

Consider a massless scalar with action

$$I(\phi) = \frac{1}{2}\int_{B_{d+1}} d^{d+1}y\sqrt{g}|d\phi|^2 \tag{2.1}$$

Which obeys the Laplace equation

$$D_i D^i \phi = 0 \tag{2.2}$$

Take the boundary value ϕ_0 to be the value of ϕ at the boundary $x_0 = \infty$ and is a source for a field $\mathcal{O}$ which is the boundary field theory. We wish to solve for ϕ in terms of only boundary terms and then evaluate the action (2.1) in an aim to compute 2-point function for the field $\mathcal{O}$ on the boundary.
Take point P to be a point on the boundary at $x_0 = \infty$. The boundary conditions and metric has translational invariance in the x_i directions. We look for a Green's function solution G. Green's functions have the property

$$D_i D^i G(x, x') = \delta(x - x') \tag{2.3}$$

and for a function $v(x)$ with a classical source $u(x)$ at the boundary

$$v(x) = \int u(x')G(x, x')dx'. \tag{2.4}$$

The Laplace-Beltrami equation for a metric $g_{\mu\nu}$ is given by

$$D_i D^i \phi = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\right)\phi = 0 \tag{2.5}$$

where $g = det(g_{\mu\nu})$ and $g^{\mu\nu}$ is the inverse metric tensor.
Therefore the Laplace equation with metric (1.1) for the Greens function $G(x_0)$ reads

$$x_0^{(d+1)} \frac{\partial}{\partial x_0} \left(x_0^{-(d+1)} x_0^2 \frac{\partial}{\partial x_0} G(x_0) \right) = 0 \tag{2.6}$$

$$\frac{\partial}{\partial x_0} \left(x_0^{-d+1} \frac{\partial}{\partial x_0} G(x_0) \right) = 0 \tag{2.7}$$

This is solved by $G(x_0) = A + Bx_0^d$ with A and B arbitrary constants, we require $G(x_0)$ to vanish at $x_0 = 0$, therefore take $A = 0$. As $x_0 \to \infty$, $G(x_0) \to \infty$ but we may make a $SO(1, d+1)$ conformal transformation which allows us to map any point at infinity to a finite point. The point P can be mapped to the origin with the transformation

$$x_i \to \frac{x_i}{x_0^2 + \sum_{j=1}^d x_j^2}. \tag{2.8}$$

under this transformation $G(x_0)$ transforms as

$$G(x) \to B \frac{x_0^d}{(x_0^2 + \sum_{j=1}^d x_j^2)^d}. \tag{2.9}$$

Then $\phi(x_0, x_i)$ is given by

$$\phi(x_0, x_i) = \int \phi_0(x_i') G(x_i, x_i') d\underline{x}' \tag{2.10}$$

$$= B \int \phi_0(x_i') \frac{x_0^d}{(x_0^2 + |\underline{x} - \underline{x}'|^2)^d} d\underline{x}' \tag{2.11}$$

where $|\underline{x}|^2 = \sum_{j=1}^d x_j^2$. By applying integration by parts the action (2.1) may be written as

$$I(\phi) = \frac{1}{2} \int_{B_{d+1}} d^{d+1}y \sqrt{g} d(\phi d\phi) - \int_{B_{d+1}} d^{d+1}y \sqrt{g} \phi dd\phi \tag{2.12}$$

$$= \frac{1}{2} \int_{S_d} d^d y x_0^{-(d+1)} (\phi d\phi) \hat{n}_i \tag{2.13}$$

where the second term of (2.12) vanishes due to the equation of motion and the exterior derivative squared $d^2 = 0$ (2.2).

$$I(\phi) = \frac{1}{2} \lim_{\epsilon \to 0} \int_{T_\epsilon} d\underline{x} x_0^{-(d+1)} (\phi g^{ij} \nabla_j \phi) \hat{n}_i = \frac{1}{2} \lim_{\epsilon \to 0} \int_{T_\epsilon} d\underline{x} x_0^{-d} \phi (\underline{\hat{n}}.\underline{\nabla}) \phi, \tag{2.14}$$

where $\hat{n}$ is a normal vector pointing perpendicular to the surface T_ϵ satisfying $g_{\mu\nu} n^\mu n^\nu = 1$. Therefore $\underline{\hat{n}}.\underline{\nabla}\phi = x_0 (\frac{\partial \phi}{\partial x_0})$. Following (2.11) for $x_0 \to 0$ we acquire

$$\frac{\partial \phi}{\partial x_0} \approx dB x_0^{d-1} \int d\underline{x}' \frac{\phi_0(x_i')}{|\underline{x} - \underline{x}'|^{2d}} + O(x_0^{d+1}). \tag{2.15}$$

Subbing in to (2.14) and recalling that for $x_0 = \epsilon \to 0$, $\phi \to \phi_0$

$$I(\phi) \approx \frac{Bd}{2} \int d\underline{x} d\underline{x}' x_0^{-d} \phi_0(\underline{x}) \frac{x_0^d}{|\underline{x} - \underline{x}'|^{2d}} \phi_0(\underline{x}') \tag{2.16}$$

$$= \frac{Bd}{2} \int d\underline{x} d\underline{x}' \frac{\phi_0(\underline{x})\phi_0(\underline{x}')}{|\underline{x} - \underline{x}'|^{2d}} \tag{2.17}$$

This is of the form of a two-point function $\langle \mathcal{O}(\underline{x})\mathcal{O}(\underline{x}')\rangle$ of a conformal field with conformal dimension d. We have calculated the two point function for a conformal field $\mathcal{O}$ on the boundary by evaluating the AdS action in the bulk

3. AdS Gauge Field

We can perform an analogous exercise for the case of a free gauge theory with gauge group $U(1)$ on the metric (1.1), Look for one form solution A, i.e. the field is a vector field that carries one Lorentz index, $A = f(x_0)dx^i$ for a fixed $1 \le i \le d$. We must solve for Maxwell's equations

$$d(*F) = d(*dA) = 0. \tag{3.1}$$

With action

$$I(A) = \frac{1}{2} \int_{B_{d+1}} F \wedge *F. \tag{3.2}$$

The two form dA is given by

$$dA = \frac{\partial f(x_0)}{\partial dx_\mu} dx^\mu \wedge dx^i = \frac{\partial f(x_0)}{\partial dx_0} dx^0 \wedge dx^i. \tag{3.3}$$

Which gives the Hodge dual, which takes a k-form to a $(d-k)$-form

$$*dA = \frac{1}{2!} \epsilon_{\mu\nu 0 i} \sqrt{g} f'(x_0) dx^0 \wedge dx^i \tag{3.4}$$

$$= \frac{(-1)^i}{2!} \sqrt{g} x_0^4 f'(x_0) dx^1 \wedge dx^2 ... \widehat{dx^i} ... \wedge dx^d$$

$$= \frac{(-1)^i}{2!} x_0^{-d+3} f'(x_0) dx^1 \wedge dx^2 ... \widehat{dx^i} ... \wedge dx^d \tag{3.5}$$

where $\widehat{dx^i}$ means that dx^i is omitted from the wedge product. The factor $(-1)^i$ comes from "anti-commuting" dx^i through the wedge product $(i-1)$ times using $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. Maxwell's equations (3.1) gives

$$d(*dA) = \frac{(-1)^i}{2!} \frac{\partial}{\partial x_\mu} \left(x_0^{-d+3} f'(x_0) \right) dx^\mu \wedge dx^2 ... \widehat{dx^i} ... \wedge dx^d$$

$$= \frac{(-1)^i}{2!} \frac{\partial}{\partial x_0} \left(x_0^{-d+3} f'(x_0) \right) dx^0 \wedge dx^2 ... \widehat{dx^i} ... \wedge dx^d. \tag{3.6}$$

To satisfy (3.1) we must pick $f(x_0)$ such that $\frac{\partial}{\partial x_0}\left(x_0^{-d+3}f'(x_0)\right) = 0$, this gives $f''(x_0) = x_0^1(d-3)f'(x_0)$ this gives $f(x_0) = Bx_0^{d-2}$ with B an undetermined constant which we may pick to be $B = \frac{d-1}{d-2}$, giving

$$A = \frac{d-1}{d-2}x_0^{d-2}dx^i. \tag{3.7}$$

As for the massless scalar, A diverges at the boundary $x_0 \to \infty$ which we may resolve by performing the transformation (2.8) giving

$$\begin{aligned} A &= \frac{d-1}{d-2}\left(\frac{x_0}{x_0^2+|\underline{x}^2|}\right)^{d-2} d\left(\frac{x^i}{x_0^2+|\underline{x}|^2}\right) && (3.8)\\ &= \frac{d-1}{d-2}\frac{x_0^{d-2}}{(x_0^2+|\underline{x}^2|)^{d-1}}\left(dx_i - \frac{2x_i^2dx_i}{x_0^2+|\underline{x}^2|} - \frac{2x_0x_idx_0}{x_0^2+|\underline{x}^2|}\right). && (3.9)\end{aligned}$$

Due to the gauge freedom in A we may make a gauge transformation, as Maxwell's equations (3.1) are clearly invariant up to the adding of a term of the form of an exterior derivative of a scalar 0-form f to A (as $d(df) = 0$ by definition). Therefore we may add to (3.9) a term of the form $d(f)$

$$\begin{aligned} d(f) &= -\frac{1}{d-2}d\left(\frac{x_0^{d-2}x_i}{(x_0^2+|\underline{x}|^2)^{d-1}}\right) && (3.10)\\ &= -\frac{x_0^{d-2}}{(x_0^2+|\underline{x}|^2)^{d-1}}\left(\frac{x_idx_0}{x_0} + \frac{dx_i}{(d-2)} - \frac{(d-1)}{(d-2)}\frac{2x_i^2dx_i}{(x_0^2+|\underline{x}^2|)} - \frac{(d-1)}{(d-2)}\frac{2x_0x_idx_0}{(x_0^2+|\underline{x}^2|)}\right). && (3.11)\end{aligned}$$

Adding (3.9) and (3.11) we obtain

$$A = \frac{x_0^{d-2}}{(x_0^2+|\underline{x}|^2)^{d-1}}\left(dx_i - \frac{x_idx_0}{x_0}\right). \tag{3.12}$$

If we want a solution of Maxwell's equations at $x_0 = 0$ coinciding with the source $A_0 = \sum_{i=1}^d a_idx^i$ on the boundary, then using the Green's function (3.12) we have

$$\begin{aligned} A(x_0,\underline{x}) &= \int d\underline{x}'A(\underline{x},\underline{x}')A_0(\underline{x}') && (3.13)\\ &= x_0^{d-2}\int d\underline{x}'\frac{a_i(\underline{x}')dx^i}{(x_0^2+|\underline{x}-\underline{x}'|^2)^{d-1}} - x_0^{d-3}dx_0\int d\underline{x}'\frac{a_i(\underline{x}')(x^i-x'^i)}{(x_0^2+|\underline{x}-\underline{x}'|^2)^{d-1}}. && (3.14)\end{aligned}$$

Then the field strength tensor F is given by

$$\begin{aligned} F = dA =& (d-1)x_0^{d-3}dx_0\int d\underline{x}'\frac{a_i(\underline{x}')dx^i}{(x_0^2+|\underline{x}-\underline{x}'|^2)^{d-1}}\\ &- 2(d-1)x_0^{d-1}dx_0\int d\underline{x}'\frac{a_i(\underline{x}')dx^i}{(x_0^2+|\underline{x}-\underline{x}'|^2)^{d}}\\ &- 2(d-1)x_0^{d-3}dx_0\int d\underline{x}'\frac{a_i(\underline{x}')(x^i-x'^i)(x_k-x'_k)dx^k}{(x_0^2+|\underline{x}-\underline{x}'|^2)^{d}}\\ &+ \ldots \end{aligned} \tag{3.15}$$

where ... are terms with no dx_0 in.
By applying integration by parts from the product rule for a k-form ω, $d(\omega \wedge \alpha) = (d\omega) \wedge \alpha + (-1)^k \omega \wedge (d\alpha)$.

$$\begin{aligned} I(A) &= \frac{1}{2} \int_{B_{d+1}} dA \wedge *F && (3.16) \\ &= \frac{1}{2} \int_{S_d} A \wedge *F + \frac{1}{2} \int_{B_{d+1}} A \wedge d(*F) && (3.17) \\ &= \frac{1}{2} \lim_{\epsilon \to 0} \int_{T_\epsilon} A \wedge *F, && (3.18) \end{aligned}$$

again, the second term in (3.17) vanishes due to Maxwell's equations (3.1) with T_ϵ the surface where $x_0 = \epsilon \to 0$. $*F$ is given by

$$\begin{aligned} *F = (d-1)(-1)^i \int d\underline{x}' \frac{a_i(\underline{x}') d\omega^i}{(x_0^2 + |\underline{x} - \underline{x}'|^2)^{d-1}} \\ - 2(d-1)(-1)^i x_0^4 \int d\underline{x}' \frac{a_i(\underline{x}') d\omega^i}{(x_0^2 + |\underline{x} - \underline{x}'|^2)^{d}} \\ - 2(d-1)(-1)^i \int d\underline{x}' \frac{a_k(\underline{x}')(x_i - x_i')(x^k - x'^k) d\omega^i}{(x_0^2 + |\underline{x} - \underline{x}'|^2)^{d}} \\ + \ldots, \end{aligned} \tag{3.19}$$

where $d\omega^i = dx^1 \wedge dx^2 ... \widehat{dx^i} ... \wedge dx^d$ where $\widehat{dx^i}$ means that dx^i is omitted from the wedge product so $dx^j \wedge d\omega^i = (-1)^j \delta^{ij}$. As in the massless scalar case, as $x_0 = \epsilon \to 0$, $A \to A_0$, then in this limit (3.18) gives

$$\begin{aligned} I(A) &= (d-1)(-1)^i \int d\underline{x} d\underline{x}' \frac{a_i(\underline{x}') a_j(\underline{x}) dx^j \wedge d\omega^i}{|\underline{x} - \underline{x}'|^{2d-2}} \\ &\quad - 2(d-1)(-1)^i \int d\underline{x}' \frac{a_k(\underline{x}') a_j(\underline{x})(x_i - x_i')(x^k - x'^k) dx^j \wedge d\omega^i}{|\underline{x} - \underline{x}'|^{2d}} \\ &= (d-1) \int d\underline{x} d\underline{x}' a_i(\underline{x}) a_j(\underline{x}') \left(\frac{\delta^{ij}}{|\underline{x} - \underline{x}'|^{2d-2}} - \frac{2(x^i - x'^i)(x^j - x'^j)}{|\underline{x} - \underline{x}'|^{2d}} \right) \end{aligned} \tag{3.20}$$

This is of the form of a two-point function for a conserved current. We began with a vector field A_μ with a local $U(1)$ gauge symmetry in the bulk AdS and we obtain the form of a conserved current J_μ two point function with a global $U(1)$ symmetry on the boundary $\partial(AdS)$ (i.e. (3.20) is invariant under global rotation of its i,j indices)

4. AdS Massive scalar

We may consider the case of a massive scalar field with mass m in AdS$_{d+1}$ space with action

$$I(\phi) = \frac{1}{2} \int d^{d+1} y \sqrt{g} (|d\phi|^2 + m^2 \phi^2). \tag{4.1}$$

ϕ obeys the field equation

$$(D_iD^i + m^2)\phi = 0 \tag{4.2}$$

writing the metric (1.1) in hyperbolic coordinates $d\tilde{s}^2 = dy^2 + (\sinh^2 y)d\Omega^2$ the field equation (4.2) takes the form

$$\left(\frac{-1}{\sinh^d y}\frac{d}{dy}(\sinh^d y)\frac{d}{dy} + \frac{L^2}{\sinh^2 y} + m^2\right)\phi = 0 \tag{4.3}$$

Where the L^2 term is the angular part of the Laplacian which we can disregard for large y, in this limit $\sinh y \to \frac{1}{2}e^y$, therefore we look for a solution of the form $\phi = e^{\lambda y}$. Then we obtain

$$\lambda(\lambda + d) = m^2 \tag{4.4}$$

We limit m^2 to the region of which the roots are real, and with the larger and smaller roots $\lambda_+ \geq -\frac{d}{2}$ and $\lambda_- \leq -\frac{d}{2}$ picking a linear combination of the two roots then the solution at the boundary behaves as $e^{\lambda_+ y}$. As in the massless case we would like to find a solution which approaches a constant at the boundary, however $e^{\lambda_+ y}$ clearly diverges at the boundary to remedy this we can pick a function f on B_{d+1} which approaches zero on the boundary, e.g. $f = e^{-y}$ and then we look for solutions of the form

$$\phi \sim f^{-\lambda_+}\phi_0 \tag{4.5}$$

with ϕ_0 the boundary value By studying the scaling we see that if $f \to e^{\omega}f$ then $d\tilde{s}^2 \to e^{2\omega}d\tilde{s}^2$ therefore $\phi_0 \to e^{\omega\lambda_+}\phi_0$ from this we deduce that ϕ_0 has mass dimension $-\lambda_+$.

As in the massless case we wish to compute the two point function of the field $\mathcal{O}$ on the boundary, $\mathcal{O}$ will couple to the boundary value of the massive AdS scalar field ϕ_0 as $\int \phi_0\mathcal{O}$ which must have conformal dimension d, therefore $\mathcal{O}$ must have conformal dimension $(d + \lambda_+)$.

As in the massless case we look for Green's function $G(x_0)$ which obeys the wave equation (4.2) and is a delta function at the point P on the boundary, the solutions must also behave as (4.5) on the boundary located at $x_0 = \infty$. Using the metric as in (1.1). With this metric, and noting the translational invariance, the massive field equation is

$$\left(-x_0^{d+1}\frac{d}{dx_0}x_0^{-d+1}\frac{d}{dx_0} + m^2\right)G(x_0) = 0. \tag{4.6}$$

Picking $G(x_0) = x_0^A$ and using (4.4) $m^2 = \lambda_+(d + \lambda_+)$ then we must choose $A = d + \lambda_+$ to give $G(x_0) = x_0^{d+\lambda_+}$. Which vanishes at $x_0 = 0$ as required, the solution diverges at $x_0 = \infty$ which may be remedied by performing the transformation (2.8) to map the point P to the origin, then

$$G(x) = \frac{x_0^{d+\lambda_+}}{(x_0^2 + |\underline{x}|^2)^{d+\lambda_+}} \tag{4.7}$$

Using the Green's function G $\phi(x)$ is given by

$$\phi(x) = c\int d\underline{x}'\frac{x_0^{d+\lambda_+}}{(x_0^2 + |\underline{x}|^2)^{d+\lambda_+}}\phi_0(\underline{x}') \tag{4.8}$$

For $x_0 \to 0$ then $\phi \to x_0^{-\lambda_+}\phi_0$. By applying integration by parts to the action (4.1) we obtain

$$I(\phi) = \frac{1}{2}\int_{S_d} d^d y x_0^{-(d+1)}(\phi d\phi)\hat{n}_i - \frac{1}{2}\int_{B_{d+1}} d^{d+1} y x_0^{-(d+1)}\left(\phi dd\phi + m^2\phi^2\right). \tag{4.9}$$

In the limit $x_0 = \epsilon \to 0$ then (4.9)

$$I(\phi) = \frac{1}{2}\lim_{\epsilon\to 0}\int_{T_\epsilon} d\underline{x} x_0^{-d}\phi(\underline{n}.\underline{\nabla})\phi \tag{4.10}$$

with $(n.\nabla)\phi = x_0(\frac{d\phi}{dx_0})$ and in the limit $x_0 = \epsilon \to 0$ we acquire

$$\frac{\partial\phi}{\partial x_0} \approx c(d+\lambda_+)x_0^{d+\lambda_+-1}\int d\underline{x}'\frac{\phi_0(x_i')}{|\underline{x}-\underline{x}'|^{2(d+\lambda_+)}} + O(x_0^{d+\lambda_++1}). \tag{4.11}$$

Noting that $\phi \to x_0^{-\lambda_+}\phi_0(x)$ as $x_0 \to 0$, thus evaluating the action in this limit (4.10) we get

$$I(\phi) = \frac{c(d+\lambda_+)}{2}\int d\underline{x}d\underline{x}'\frac{x_0^{-d}x_0^{-\lambda_+}\phi_0(\underline{x})x_0^{d+\lambda_+-1}\phi_0(\underline{x}')}{|\underline{x}-\underline{x}'|^{2(d+\lambda_+)}} \tag{4.12}$$

$$I(\phi) = \frac{c(d+\lambda_+)}{2}\int d\underline{x}d\underline{x}'\frac{\phi_0(\underline{x})\phi_0(\underline{x}')}{|\underline{x}-\underline{x}'|^{2(d+\lambda_+)}} \tag{4.13}$$

This is the form of a two-point function of a conformal field $\mathcal{O}$ of conformal dimension $\Delta = d+\lambda_+$. Thus the dimension of a conformal field on the boundary that is the holographic dual to a field of mass m in AdS_{d+1} $\Delta = d+\lambda_+$ is is related to the mass as the largest root of $m^2 = \Delta(\Delta - d)$