In[63]:=
```
Quit;
```

In[64]:=
```
ClearAll;
```

In[65]:=
```
Action = (((r * Sin[θ[r]])^2) * ((1 + (r^2))^(-1) + ((r * (θ'[r]))^2)))^(1 / 2)
```

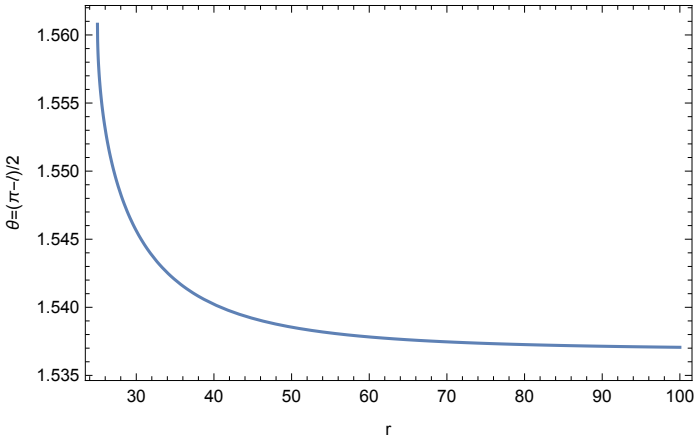
Out[65]=
$$\sqrt{r^2 \, \text{Sin}[\theta[r]]^2 \left(\frac{1}{1+r^2} + r^2 \, \theta'[r]^2\right)}$$

In[66]:=
```
RHSr = FullSimplify[D[Action, θ[r]]]; (*Compute Euler Lagrange Equations*)
```

In[67]:=
```
LHSr = FullSimplify[Dt[D[Action, θ'[r]], r]];
```

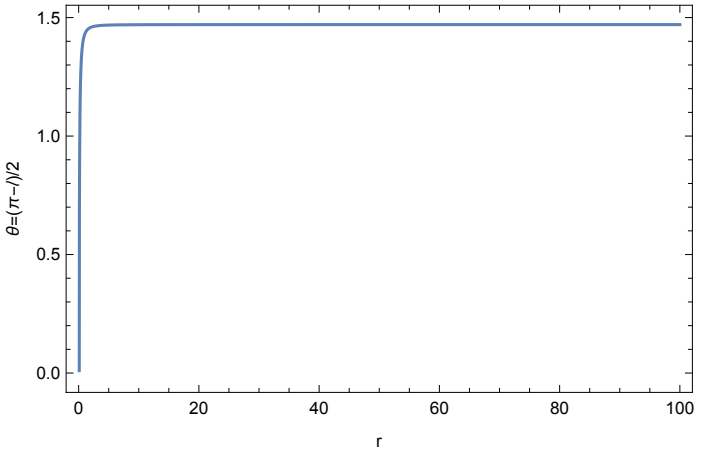
In[68]:=
```
r0j = 25;
Jointed = NDSolve[{RHSr == LHSr, θ[r0j] == Pi / 2 - 0.01, θ'[r0j] == -80}, θ, {r, r0j, 100}];
(*Solve for the solution which does not cross θ=0*)
Plot[Evaluate[θ[r] /. Jointed], {r, r0j, 100}, PlotRange -> All, Frame → True, FrameLabel → {"r", "θ=(π-ℓ)/2"}]
```

Out[70]=



In[71]:=
```
r0d = 0.1;
DisJointed = NDSolve[{RHSr == LHSr, θ[r0d] == 0.01, θ'[r0d] == 80}, θ, {r, r0d, 100}];
(*Solve for the solution which crosses from one side at θ=0 r =r0 *)
Plot[Evaluate[θ[r] /. DisJointed], {r, r0d, 100}, PlotRange → All, Frame → True, FrameLabel → {"r", "θ=(π-ℓ)/2"}]
```

Out[73]=

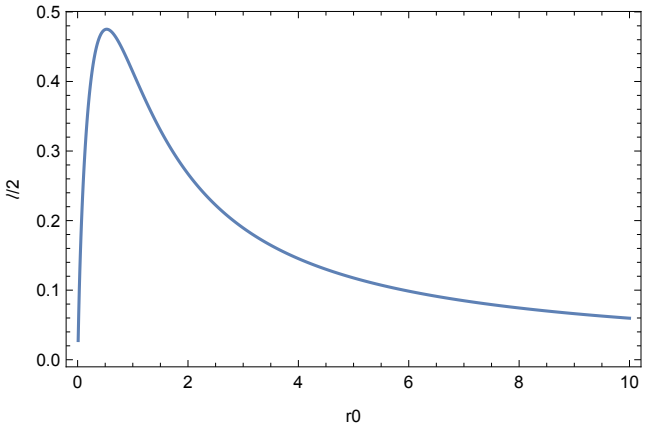


In[74]:=
```
beg = 0.01;
end = 10.0;
step = 0.01;
```

In[77]:=
```
JointedTab = Flatten[Table[{r0, (Pi / 2 - θ[100])} /.
      NDSolve[{RHSr == LHSr, θ[r0] == Pi / 2 - 0.0001, θ'[r0] == -80}, θ, {r, r0, 1000}], {r0, beg, end, step}], 1];
```

In[78]:=
```
ListLinePlot[JointedTab, PlotRange → All, Frame → True, FrameLabel → {"r0", "ℓ/2"}]
(*Plot the variation of ℓ/2 with r for the jointed solution. we see that ℓ/2 is bounded between 0 and ~0.5rad*)
```

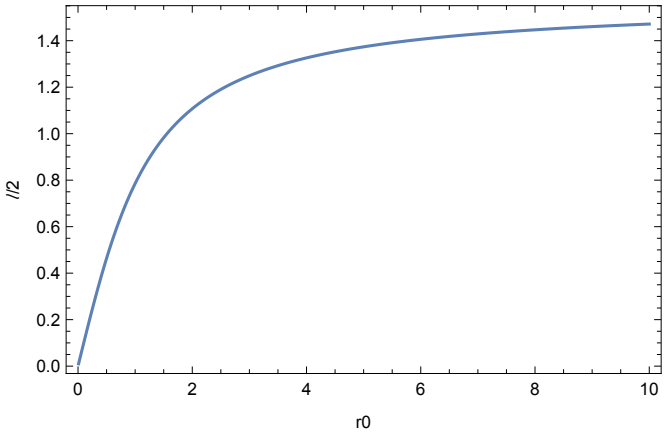
Out[78]=



In[79]:=
```
DisJointedTab = Flatten[Table[
    {r0, Pi / 2 - θ[100]} /. NDSolve[{RHSr == LHSr, θ[r0] == 0.001, θ'[r0] == 80}, θ, {r, r0, 1000}], {r0, beg, end, step}], 1];
```

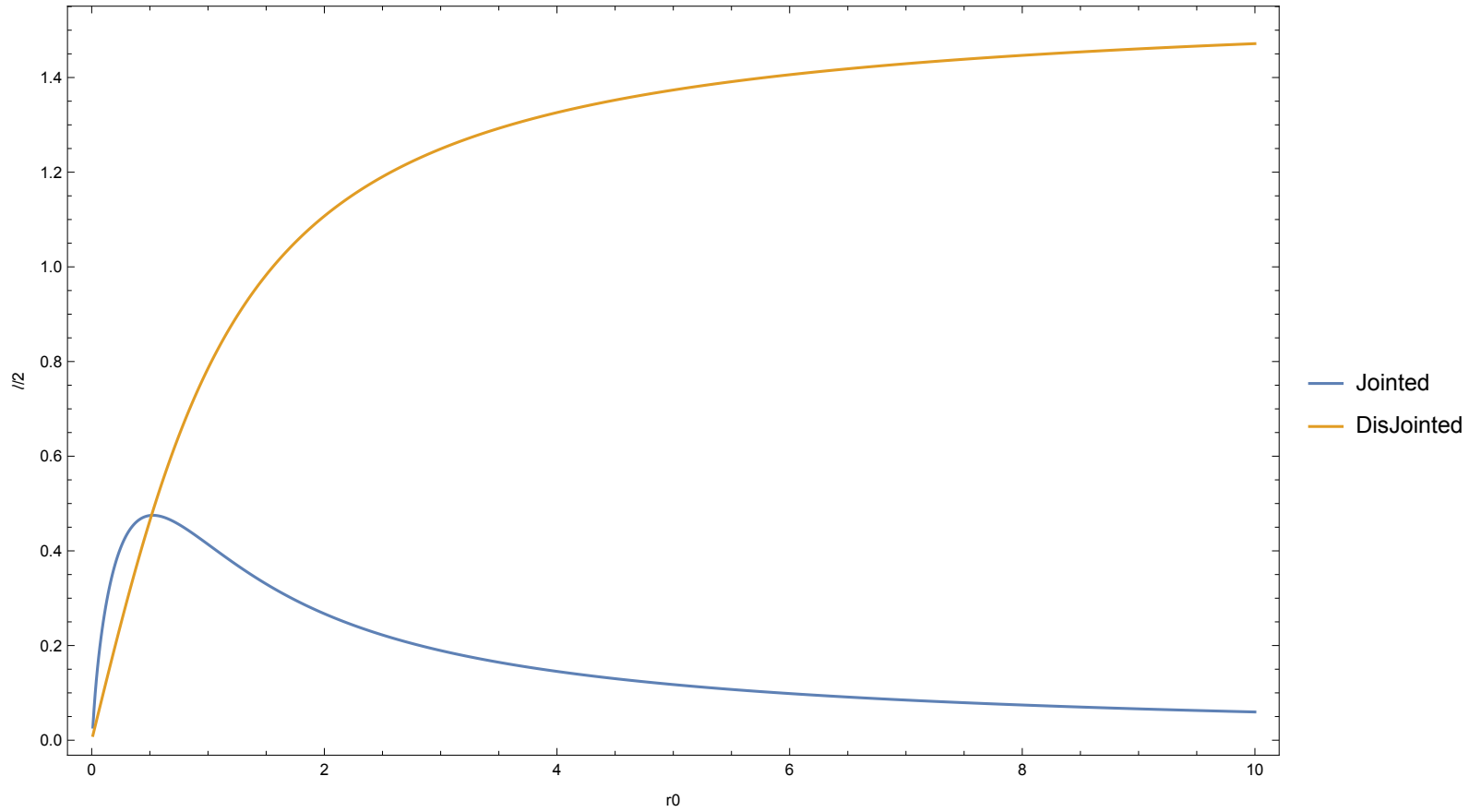
In[80]:=
```
ListLinePlot[DisJointedTab, PlotRange → All, Frame → True, FrameLabel → {"r0", "ℓ/2"}]
(*Plot the variation of ℓ/2 with r for the disjointed solution.*)
```

Out[80]=



In[81]:=
```
ListLinePlot[{JointedTab, DisJointedTab}, PlotRange → All,
 Frame → True, FrameLabel → {"r0", "l/2"}, PlotLegends → {"Jointed", "DisJointed"}]
```

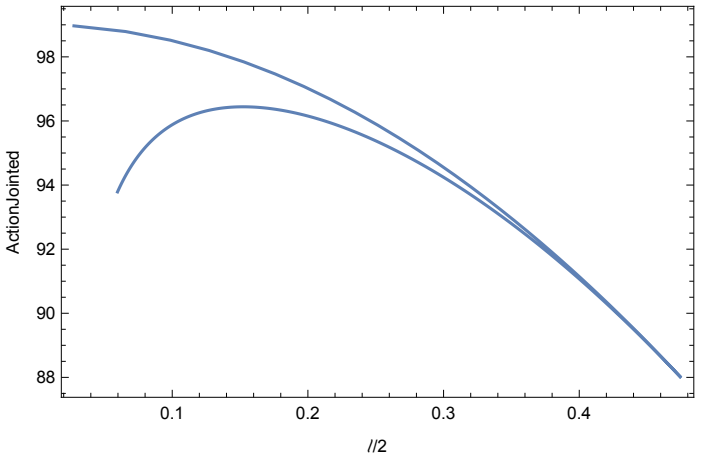
Out[81]=



In[82]:=
```
tabl = actjointed = Partition[
    Flatten[Table[{(Pi / 2 - θ[100]) /. NDSolve[{RHSr == LHSr, θ[r0] == Pi / 2 - 0.0001, θ'[r0] == -80}, θ, {r, r0, 100}],
       NIntegrate[Action /. NDSolve[{RHSr == LHSr, θ[r0] == Pi / 2 - 0.0001, θ'[r0] == -80}, θ, {r, r0, 100}],
        {r, r0, 100}]}, {r0, beg, end, step}], 2], 2];
```

In[83]:=
```
ListLinePlot[tab/actjointed, PlotRange → All, Frame → True, FrameLabel → {"ℓ/2", "ActionJointed"}]
```

Out[83]=

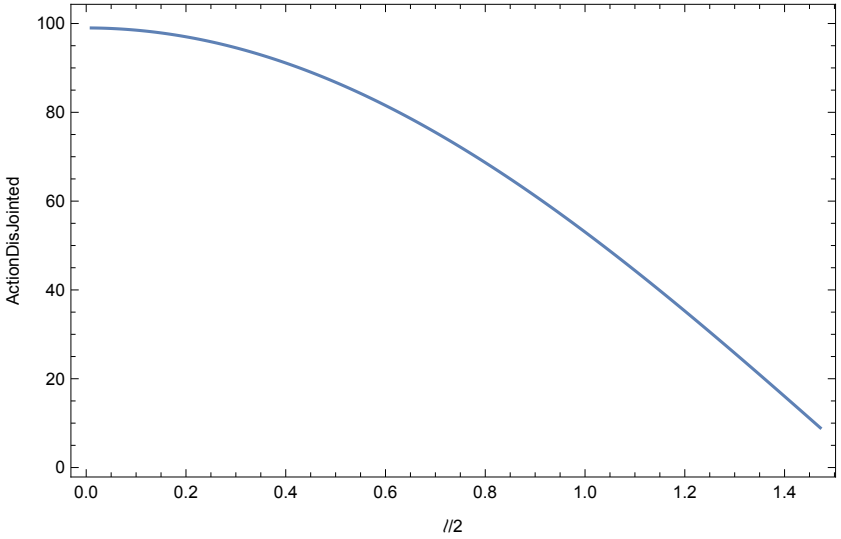


In[84]:=
```
(*We get two values for the action for the jointed case. corresponding to 2
 possible values for r0 for each ℓ/2 value, The action for small r0 ie the surface which
 extends very close to the centre is never prefered as it comes with higher energy cost*)
```

In[85]:=
```
tab/actdisjointed =
  Partition[Flatten[Table[{(Pi / 2 - θ[100]) /. NDSolve[{RHSr == LHSr, θ[r0] == 0.001, θ'[r0] == 80}, θ, {r, r0, 100}],
      NIntegrate[Action /. NDSolve[{RHSr == LHSr, θ[r0] == 0.001, θ'[r0] == 80}, θ, {r, r0, 100}], {r, r0, 100}]}, {r0,
      beg, end, step}], 2], 2];
```

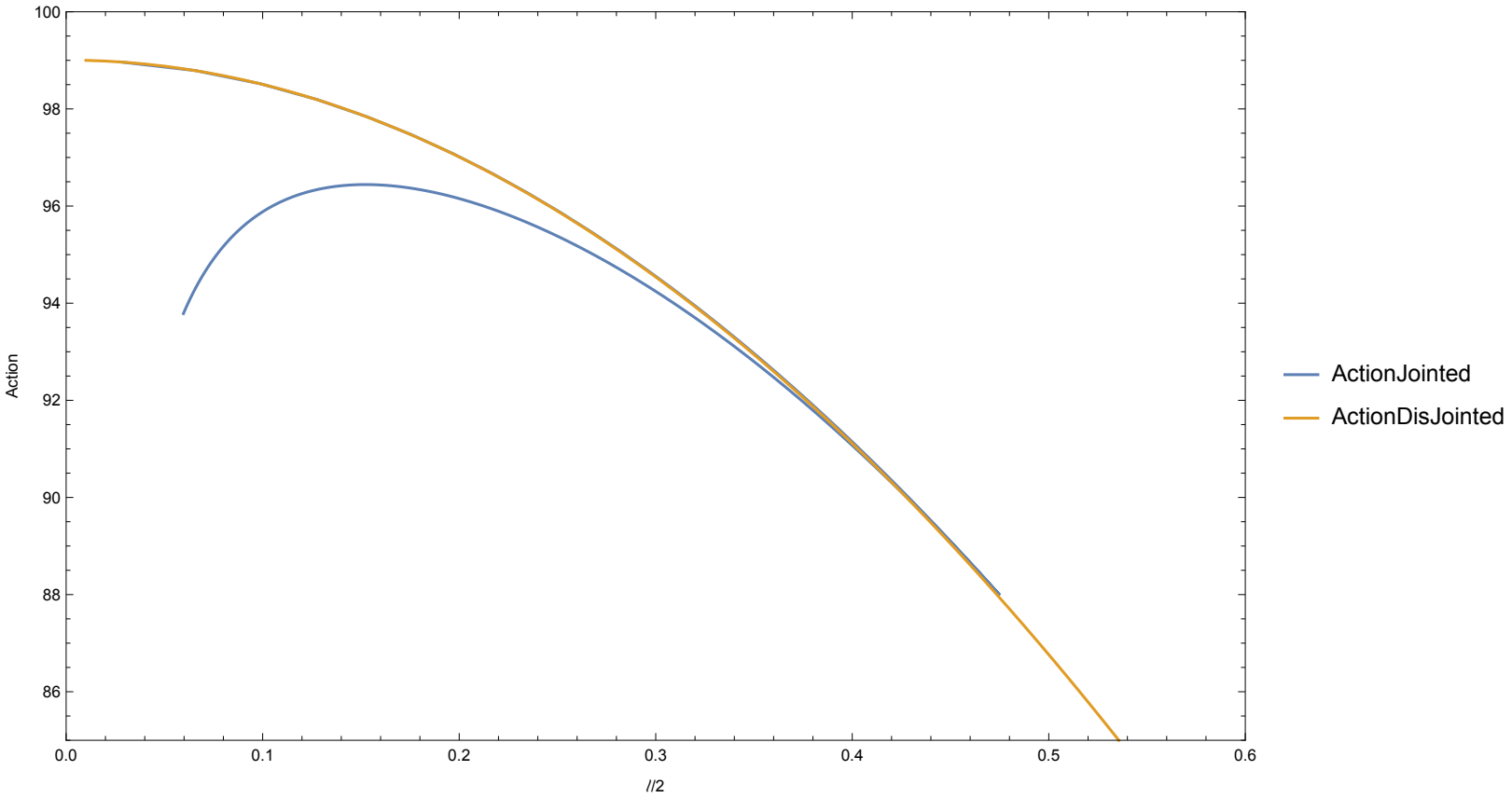
In[86]:=
```
ListLinePlot[tab[actdisjointed, PlotRange → All, Frame → True, FrameLabel → {"/2", "ActionDisJointed"}]
```

Out[86]=



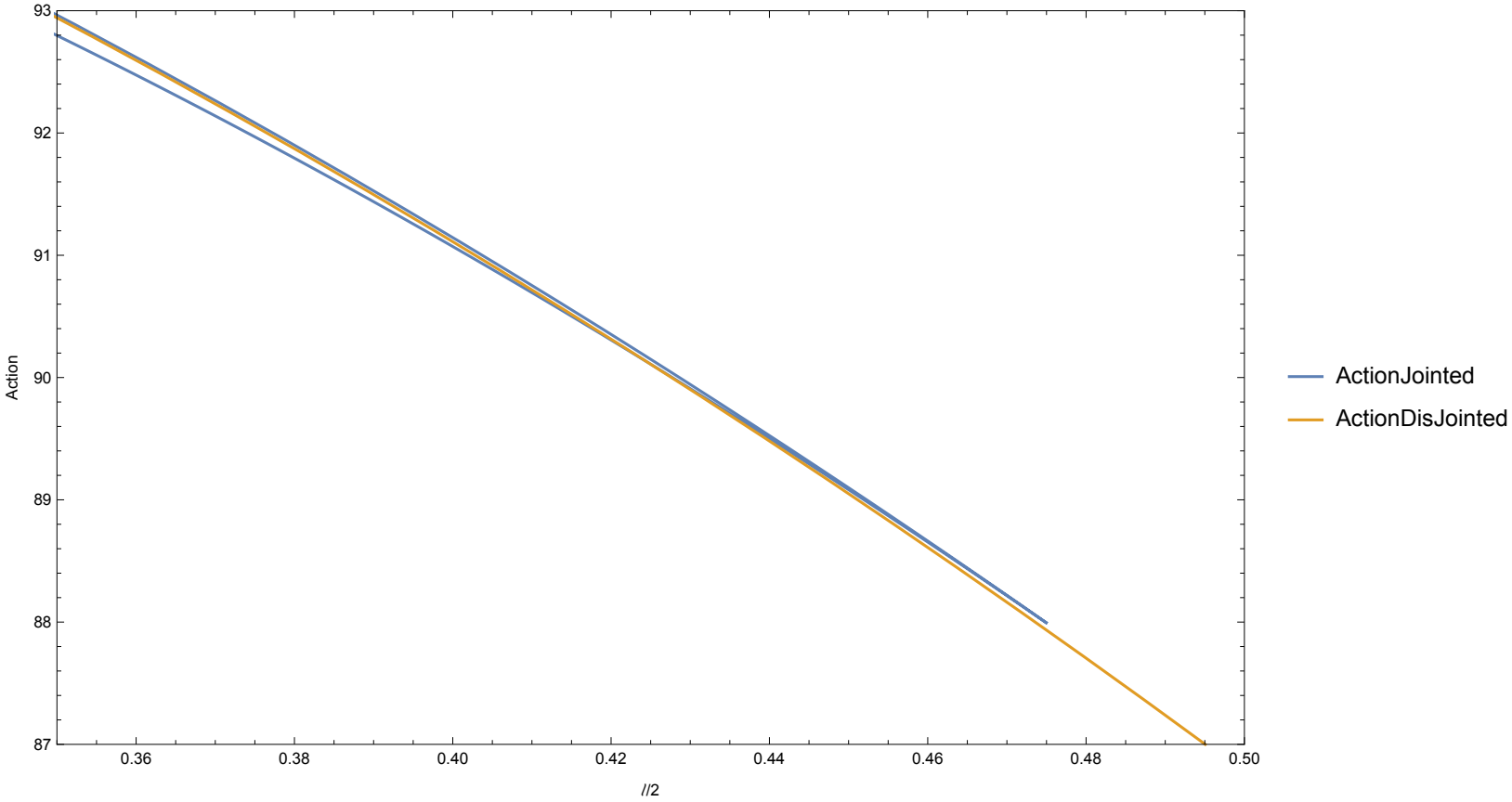
In[87]:=
```
ListLinePlot[{tab[actjointed, tab[actdisjointed}, PlotLegends → {"ActionJointed", "ActionDisJointed"},
 Frame → True, FrameLabel → {"ℓ/2", "Action"}, PlotRange → {{0, 0.6}, {85, 100}}](**)
```

Out[87]=



In[88]:=
```
ListLinePlot[{tab⁄actjointed, tab⁄actdisjointed}, PlotLegends → {"ActionJointed", "ActionDisJointed"},
 Frame → True, FrameLabel → {"⁄/2", "Action"}, PlotRange → {{0.35, 0.5}, {87, 93}}]
 (*Demonstrates that there exists a phase transition as the action crosses between the two cases.*)
```

Out[88]=



In[91]:=
```
ListLinePlot[{tab⁄actjointed, tab⁄actdisjointed}, PlotLegends → {"ActionJointed", "ActionDisJointed"},
 Frame → True, FrameLabel → {"⁄/2", "Action"}, PlotRange → {{0.40, 0.44}, {88, 91}}]
 (*Demonstrates that there exists a phase transition as the action crosses between the two cases.*)
```

Out[91]=