Instanton Counting for $\mathcal{N} = 1$ Class $\mathcal{S}_k$

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Outline

- Review: Instanton counting in 4d $\mathcal{N} = 2$ gauge theories
- $\mathcal{N} = 1$ theories of Class $\mathcal{S}_k$
- Z_{inst} from 2d Superconformal Index (SCI) computation

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Instanton counting in 4d $\mathcal{N} = 2$ theories

Nekrasov's partition function

Path integral of (topologically twisted) $\mathcal{N} = 2$ theories in Ω-background [Nekrasov '02]

$$\mathbb{T}^2_{\epsilon_{1,2}} : (z_1, z_2) \mapsto (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2) , \quad \mathbb{R}^4 \text{ limit } \epsilon_{1,2} \to 0$$

- Equivariant Localisation : path integral localises onto saddle points $\leftrightarrow$ ASD instanton configurations $F = - \star F$

$$Z_{\text{Nek}} = \int_{\langle\phi\rangle = a} \mathcal{D}[\phi, \dots] \, e^{-S} = Z_{\text{pert}} \, Z_{\text{inst}}$$

Instanton Counting in 4d $\mathcal{N} = 2$ theories

Instantons classified by topological number $K = \frac{1}{4\pi^2} \int \mathrm{tr}\, F \wedge F \in \mathbb{Z}$

$$Z_{\text{inst}} = \sum_{K=0}^{\infty} \mathbf{q}^K Z_{\text{inst}}^{(K)} \,, \quad Z_{\text{inst}}^{(K)} = \int_{\mathcal{M}_{\text{inst}}^{(K)}} d\mu \, e^{-S_{\text{inst}}^{(K)}}$$

- $Z_{\text{inst}}^{(K)}$ partition function of a $0d$ σ-model into K-instanton moduli space $\mathcal{M}_{\text{inst}}^{(K)} \circlearrowleft \mathbb{T}^2_{\epsilon_{1,2}}$

ADHM-construction:

If gauge theory realised as WV theory on Dp-brane $\implies$ σ-model is WV theory on D$(p-4)$-branes

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Class $\mathcal{S}_k$

Class $\mathcal{S}$

Compactify 6d $\mathcal{N} = (2,0)$ SCFT on Riemann surface $\mathcal{C} \implies$ 4d $\mathcal{N} = 2$ theory [Gaiotto, Moore, Neitzke '09], [Gaiotto '09]

Class $\mathcal{S}_k$ - Generalisation of Class $\mathcal{S}$

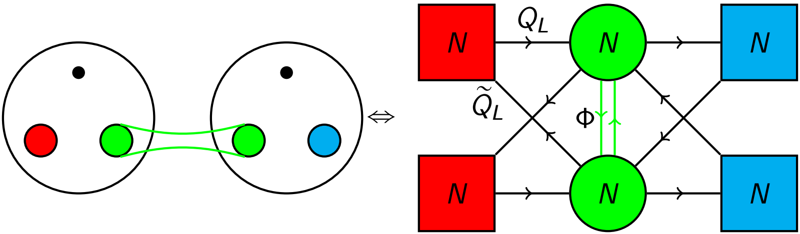
Instead: Compactify a 6d $\mathcal{N} = (1,0)$ SCFT on $\mathcal{C} \implies$ 4d $\mathcal{N} = 1$ theory [Gaiotto, Razamat '15].

- 6d $\mathcal{N} = (1,0)$ theory we use is a $\mathbb{Z}_k$ orbifold of the 6d $\mathfrak{g} = A_{N-1}$ $\mathcal{N} = (2,0)$ SCFT
- WV theory on N M5-branes sitting at tip of transverse $\mathbb{C}^2/\mathbb{Z}_k$ singularity

Example

$\mathcal{C}$ = four punctured sphere

- For $k = 2$ - $\mathcal{N} = 1$ superconformal quiver theory



- $W \sim \left(Q_L \Phi \widetilde{Q}_L - \widetilde{Q}_R \Phi Q_R \right) - \frac{i\tau}{8\pi^2} \operatorname{tr} W^\alpha W_\alpha$

Type IIA description

The quiver theory can be embedded in Type IIA [Hanany, Witten '96], [Witten '97]

	$\mathbb{C}^2$				$\mathbb{S}^1$	$\mathbb{C}^2/\mathbb{Z}_k$				$\mathbb{R}$
N D4	–	–	–	–	–	·	·	·	·	·
NS5	–	–	–	–	·	·	–	–	·	·
K D0	·	·	·	·	–	·	·	·	·	·

- Instantons can be realised as D$(p-4)$-branes

$$\boxed{K \text{ instantons in a D4-brane} \equiv K \text{ D0-branes}}$$

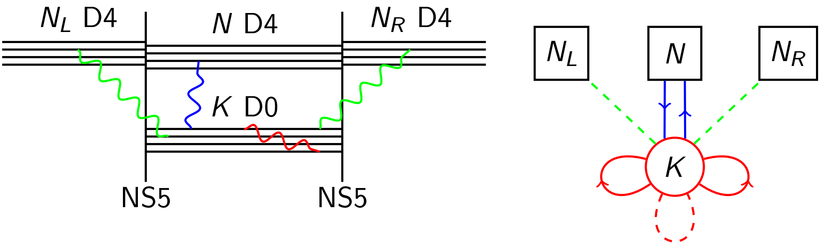
- D0 Higgs branch equivalent to moduli space of instantons [Douglas '95, '96] [Witten '95]

$$\mathcal{M}_{\text{inst}}^{(K)} \cong \mathcal{M}_{\text{Higgs}}^{K\text{ D0}}\,, \quad \text{ADHM equations} \equiv \{F = D = 0\}$$

D0-brane Worldvolume theory

- Before the orbifold: 2 supercharges $\mathcal{Q}_+$, $\tilde{\mathcal{Q}}_+$ (reduction of 2d $\mathcal{N} = (0,2)$)
- Lift the 0d theory to a 2d $U(K)$ gauge theory on $\mathbb{S}^1 \times \mathbb{S}^1$ (2×T-duality: D0 → D2). Radii β_1, β_2 $\Longrightarrow$ Superconformal index

$$Z_{\text{inst}}^{(K)} = \lim_{\beta_1, \beta_2 \to 0} I_{\text{SCI}}$$



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Z_{inst} from a 2d Superconformal Index computation

Computing the Superconformal Index

- Witten Index graded by 'fugacities' for $\mathcal{Q}_+$-commuting global symmetries

$$I_{\text{SCI}} = \text{Tr}(-1)^F q^{H_-} v^{2j_1} t^{2j_D} \mathbf{x}^{\mathbf{f}} \,, \quad q = e^{-\beta_1/\beta_2}$$

[Witten '82], [Romelsburger '05], [Kinney, Maldacena, Minwalla, Raju '05]

- $\mathcal{Q}_+$-commuting global symmetry = maximal torus of:

$$f = U(1)_- \times SU(2)_1 \times SU(2)_D \times SU(N_L) \times SU(N) \times SU(N_R)$$

- Receives contribution only from states with

$$\delta = \left\{ \mathcal{Q}_+, \mathcal{Q}_+^\dagger \right\} = 0$$

Z_{inst} from a 2d Superconformal Index computation

- Fact: SCI is independent of continuous couplings $\Leftrightarrow$ we can compute it in the free theory using letter counting

$$\boxed{I_{\text{SCI}} = \int \left[d\mu_{U(K)}(\mathbf{z})\right] \mathrm{PE}\left[\sum_{M\in\{\text{multiplets}\}} i_M(q,\mathbf{z},\dots)\right]}$$

- After identifying the fugacities with the gauge theory parameters e.g. $v = e^{\beta_1\epsilon_+/2}$, $2\epsilon_\pm = \epsilon_1 \pm \epsilon_2$

- We recover Nekrasov's expression

$$\lim_{\beta_1,\beta_2\to 0} I_{\text{SCI}} \equiv Z^{(K)}_{\text{inst}}$$

Orbifolding to Class $\mathcal{S}_k$

After the $\mathbb{Z}_k$ orbifold

- Supersymmetry is broken by $1/2$, however $\mathcal{Q}_+$ is preserved
- We can compute the Witten Index ('Superconformal Index') of the orbifolded 2d theory
- After orbifold: $\mathcal{Q}_+$-commuting global symmetry $=$ maximal torus of:

$$C_f(\mathbb{Z}_k) = U(1)_- \times SU(2)_1 \times U(1)_D \times SU(N_L)^k \times SU(N)^k \times SU(N_R)^k$$

- We can compute for the orbifold theory

$$I_{\text{SCI}}^{\text{orb}} = \text{Tr}(-1)^F q^{H_-} v^{2j_1} t^{2j_D} \mathbf{x}^{\mathbf{f}}$$

Orbifolding to Class $\mathcal{S}_k$

- "Orbifold" single letter index [Douglas, Moore '96]

$$i_M^{\text{orb}} := \frac{1}{|\mathbb{Z}_k|} \sum_{\gamma \in \mathbb{Z}_k} i_M(q, \mathbf{z}, \dots; \gamma)$$

$$\boxed{I_{\text{SCI}}^{\text{orb}} = \int [d\mu_G(\mathbf{z})] \, \mathrm{PE}\left[\sum_{M \in \{\text{multiplets}\}} i_M^{\text{orb}}(q, \mathbf{z}, \dots) \right]}$$

- Taking the limit

$$\lim_{\beta_1, \beta_2 \to 0} I_{\text{SCI}}^{\text{orb}} = Z_{\text{inst}}^{(K)}$$

- Expected to equal integration over the moduli space of $K = \{K_1, K_2, \dots, K_k\}$ instantons for this $\mathcal{N} = 1$ theory

Summary

- Repackaged the $\mathcal{N} = 2$ instanton partition function into a 2d SCI computation - free field combinatorics computation!
- We computed partition functions for matrix model describing instantons for a class of special $\mathcal{N} = 1$ theories.

Explicit expressions for Z_{inst}

$$\lim_{\beta_1,\beta_2\to 0} I_{\text{SCI}} = \int [du] \prod_{I,J=1}^{K} \frac{u'_{IJ}\left(u_{IJ}-2\epsilon_+\right)}{\left(u_{IJ}+\epsilon_1\right)\left(u_{IJ}+\epsilon_2\right)} \times \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{\left(u_I-m_{L,A}\right)\left(u_I-m_{R,A}\right)}{\left(u_I-a_A-\epsilon_+\right)\left(u_I-a_A+\epsilon_+\right)}$$

$$\lim_{\beta_1,\beta_2\to 0} I_{\text{SCI}}^{\text{orb}} \propto \prod_{i=1}^{k} \int \prod_{I=1}^{K_i} du_{i,I} \prod_{I=1}^{K_i} \frac{u'_{ii,IJ} \prod_{j\neq i} \prod_{J=1}^{K_j} \left(u_{ij,IJ}-2\epsilon_+\right)}{\prod_{j=1}^{k} \prod_{J=1}^{K_j} \left(u_{ij,IJ}+\epsilon_1\right)\left(u_{ij,IJ}+\epsilon_2\right)} \times \prod_{j=1}^{k}\prod_{I=1}^{K_i}\prod_{A=1}^{N} \frac{\left(u_{i,I}-\widetilde{m}_{L,j,A}\right)\left(u_{i,I}-\widetilde{m}_{R,j,A}\right)}{\left(u_{i,I}-\widetilde{a}_{j,A}-\epsilon_+\right)\left(u_{i,I}-\widetilde{a}_{j,A}+\epsilon_+\right)}$$