August 9, 2019

Solutions to Peskin & Schroeder

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2.1 Classical Electromagnetism

2.1.a

From the action

$$S = \int d^4x \mathcal{L} = \int d^4x - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.1}$$

from which the Euler-Lagrange equations yield the equations of motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 = -\partial_\mu F^{\mu\nu}, \tag{2.2}$$

which give the two inhomogeneous Maxwell equations:

$$\partial_\mu F^{\mu 0} = \partial_0 F^{00} + \partial_i F^{i0} = -\underline{\nabla} \cdot \underline{E} = 0 \tag{2.3}$$

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = -\frac{\partial}{\partial t} \underline{E} - \underline{\nabla} \times \underline{B} = 0. \tag{2.4}$$

The homogeneous Maxwell equations may be obtained either through use of the Bianchi identity on (2.2) or using the explicit forms in terms of A for the E and B fields,

$$\underline{E} = -\underline{\nabla} \cdot \phi - \frac{\partial \underline{A}}{\partial t}, \qquad \underline{B} = \underline{\nabla} \times \underline{A}. \tag{2.5}$$

Then,

$$\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot \underline{\nabla} \times \underline{A} = 0 \tag{2.6}$$

$$\underline{\nabla} \times \underline{E} = -\underline{\nabla} \times \underline{\nabla}\phi - \frac{\partial}{\partial t} \underline{\nabla} \times \underline{A} = -\frac{\partial}{\partial t} \underline{B}. \tag{2.7}$$

2.1.b

Borrowing equation 2.17

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - \eta^{\mu\nu} \mathcal{L} \tag{2.8}$$

$$= - F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{2.9}$$

Which is indeed no symmetric. Thus we add on a term of the form $\partial_\lambda K^{[\lambda\mu]\nu}$, where the square brackets denote anti-symmetric indices, which will still preserve the conservation equations $\partial_\mu T^{\mu\nu}$ because $\partial_\mu \partial_\lambda K^{\lambda\mu\nu} = -\partial_\lambda \partial_\mu K^{\mu\lambda\nu} = -\partial_\mu \partial_\lambda K^{\lambda\mu\nu} = 0$. We choose $K^{\lambda\mu\nu} = F^{\mu\lambda} A^\mu$. Therefore,

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda A^\nu F^{\mu\lambda} \tag{2.10}$$

$$= T^{\mu\nu} + A^\nu \partial_\lambda F^{\mu\lambda} + F^{\mu\lambda} \partial_\lambda A^\nu \tag{2.11}$$

$$= F^{\mu\lambda} F^{\lambda\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}, \tag{2.12}$$

where in the third line we make use of (2.2).

The energy density is given by the zero-zero component of the energy-momentum tensor

$$\mathcal{E} = \hat{T}^{00} \tag{2.13}$$
$$= F^{0\lambda}F^{\lambda 0} + \frac{1}{4}F_{\lambda\rho}F^{\lambda\rho} \tag{2.14}$$
$$= -\eta_{\lambda\rho}F^{0\lambda}F^{0\rho} + \frac{1}{4}(2\eta_{\lambda\rho}F^{0\lambda}F^{0\rho} + F^{ij}F_{ij}) \tag{2.15}$$
$$= \frac{1}{2}E^2 + \frac{1}{4}\epsilon^{ijk}\epsilon_{ijl}B^k B_l \tag{2.16}$$
$$= \frac{1}{2}E^2 + \frac{2}{4}\delta_k^l B^k B_l \tag{2.17}$$
$$= \frac{1}{2}(E^2 + B^2). \tag{2.18}$$

The momentum density is given by the zero-i component

$$S^i = \hat{T}^{0i} = \eta_{\lambda\rho}F^{0\lambda}F^{\rho i} + 0 = F^{0j}F^{ij} = E^j\epsilon^{ijk}B^k = (\underline{E} \times \underline{B})^i. \tag{2.19}$$

2.2 Complex scalar field

The action is

$$S = \int d^4x\mathcal{L} = \int d^4x \left(|\partial_\mu\phi|^2 - m^2|\phi|^2\right), \quad \phi \in \mathbb{C}. \tag{2.20}$$

2.2.a

We have $\pi(\underline{x}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi(\underline{x})}}$, then,

$$\pi^*(\underline{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*(\underline{x}))} = \partial^0\phi(\underline{x}), \quad \pi(\underline{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(\underline{x}))} = \partial^0\phi^*(\underline{x}). \tag{2.21}$$

Then

$$H \equiv \int d^3x(\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}) \tag{2.22}$$
$$= \int d^3x(2\dot{\phi}^*\dot{\phi} - \mathcal{L}) \tag{2.23}$$
$$= \int d^3x(\pi^*\pi + \underline{\nabla}\phi^* \cdot \underline{\nabla}\phi + m^2\phi^*\phi). \tag{2.24}$$

The canonical commutation relations are

$$\left[\phi(\underline{x}), \pi(\underline{y})\right] = \mathrm{i}\delta^{(3)}(\underline{x} - \underline{y}), \quad \left[\phi^*(\underline{x}), \pi^*(\underline{y})\right] = -\mathrm{i}\delta^{(3)}(\underline{x} - \underline{y}) \tag{2.25}$$

while all others vanish.

The Heisenburg equation of motion for the ϕ field is given by

$$\mathrm{i}\frac{\partial}{\partial t}\phi(\underline{x},t) =[\phi(\underline{x},t), H(\underline{x}',t)] \tag{2.26}$$

$$= \left[\phi(\underline{x},t), \int d^3x'(\pi^*(\underline{x}',t)\pi(\underline{x}',t) + \underline{\nabla}\phi^*(\underline{x}',t)\cdot\underline{\nabla}\phi(\underline{x}',t) + m^2\phi^*(\underline{x}',t)\phi(\underline{x}',t))\right] \tag{2.27}$$

$$= \int d^3x'\pi^*(\underline{x}',t)[\phi(\underline{x},t),\pi(\underline{x}',t)] \tag{2.28}$$

$$= \int d^3x'\pi^*(\underline{x}',t)\mathrm{i}\delta^{(3)}(\underline{x}-\underline{x}') \tag{2.29}$$

$$=\mathrm{i}\pi^*(\underline{x},t) \tag{2.30}$$

where to get to the second line have used the relation $[A,BC]=[A,B]C+B[A,C]$. Then,

$$\mathrm{i}\frac{\partial}{\partial t}\pi^*(\underline{x},t) = \left[\pi^(\underline{x},t), \int d^3x'(\pi^*(\underline{x}',t)\pi(\underline{x}',t) - \phi^*(\underline{x}',t)\nabla^2\phi(\underline{x}',t) + m^2\phi^*(\underline{x}',t)\phi(\underline{x}',t))\right] \tag{2.31}$$

$$= \int d^3x'\left(-\mathrm{i}\delta^{(3)}(\underline{x}-\underline{x}')(-\nabla^2+m^2)\phi(\underline{x}',t)\right) \tag{2.32}$$

$$=-\mathrm{i}(-\nabla^2+m^2)\phi(\underline{x},t), \tag{2.33}$$

thus we arrive at the Klien Gordon equation

$$\frac{\partial^2}{\partial t^2}\phi(\underline{x},t) = (\nabla^2 - m^2)\phi(\underline{x},t). \tag{2.34}$$

2.2.b

We introduce the raising lowering operators in terms of the Fourier components of fields

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\left(a_{\underline{p}} + b^\dagger_{-\underline{p}}\right)e^{\mathrm{i}\underline{p}\cdot\underline{x}} \tag{2.35}$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\left(b_{\underline{p}} + a^\dagger_{-\underline{p}}\right)e^{\mathrm{i}\underline{p}\cdot\underline{x}} \tag{2.36}$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3}(-\mathrm{i})\sqrt{\frac{E_{\underline{p}}}{2}}\left(b_{\underline{p}} - a^\dagger_{-\underline{p}}\right)e^{\mathrm{i}\underline{p}\cdot\underline{x}} \tag{2.37}$$

$$\pi^*(x) = \int \frac{d^3p}{(2\pi)^3}(-\mathrm{i})\sqrt{\frac{E_{\underline{p}}}{2}}\left(a_{\underline{p}} - b^\dagger_{-\underline{p}}\right)e^{\mathrm{i}\underline{p}\cdot\underline{x}}. \tag{2.38}$$

with

$$[a_{\underline{p}}, a^\dagger_{\underline{q}}] = [b_{\underline{p}}, b^\dagger_{\underline{q}}] = (2\pi)^3\delta^{(3)}(\underline{p}-\underline{q}). \tag{2.39}$$

Subbing eqs. (2.35) to (2.38) into (2.24) and making use of (2.39) we obtain

$$H = \int d^3x \int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{2}e^{\mathrm{i}(\underline{p}+\underline{q})\cdot\underline{x}}\left[-\sqrt{E_{\underline{p}}E_{\underline{q}}}\left(a_{\underline{p}} - b^\dagger_{-\underline{p}}\right)\left(b_{\underline{q}} - a^\dagger_{-\underline{q}}\right) + \frac{-\underline{p}\cdot\underline{q}+m^2}{\sqrt{E_{\underline{p}}E_{\underline{q}}}}\left(b_{\underline{p}} + a^\dagger_{-\underline{p}}\right)\left(a_{\underline{q}} + b^\dagger_{-\underline{q}}\right)\right] \tag{2.40}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}E_{\underline{p}}\left[-\left(a_{\underline{p}} - b^\dagger_{-\underline{p}}\right)\left(b_{\underline{q}} - a^\dagger_{-\underline{q}}\right) + \left(b_{\underline{p}} + a^\dagger_{-\underline{p}}\right)\left(a_{\underline{q}} + b^\dagger_{-\underline{q}}\right)\right] \tag{2.41}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}E_{\underline{p}}\left[a^\dagger_{\underline{p}}a_{\underline{p}} + b^\dagger_{\underline{p}}b_{\underline{p}} + a_{\underline{p}}a^\dagger_{\underline{p}} + b_{\underline{p}}b^\dagger_{\underline{p}}\right] \tag{2.42}$$

$$= \int \frac{d^3p}{(2\pi)^3}E_{\underline{p}}\left[a^\dagger_{\underline{p}}a_{\underline{p}} + b^\dagger_{\underline{p}}b_{\underline{p}} + \frac{1}{2}\left[a_{\underline{p}}, a^\dagger_{\underline{p}}\right] + \frac{1}{2}\left[b_{\underline{p}}, b^\dagger_{\underline{p}}\right]\right]. \tag{2.43}$$

Then, dropping the infinite additive constant $\frac{1}{2}\left[a_{\underline{p}}, a^\dagger_{\underline{p}}\right] + \frac{1}{2}\left[b_{\underline{p}}, b^\dagger_{\underline{p}}\right] = \left[a_{\underline{p}}, a^\dagger_{\underline{p}}\right] \propto \delta^{(3)}(0)$ gives the final result

$$H = \int \frac{d^3p}{(2\pi)^3}E_{\underline{p}}\left[a^\dagger_{\underline{p}}a_{\underline{p}} + b^\dagger_{\underline{p}}b_{\underline{p}}\right]. \tag{2.44}$$

2.2.c

The action (2.20) is invariant under the global transformation $\phi(x) \to e^{\mathrm{i}\alpha}\phi(x)$, then by Nother's theorem we have a conserved current j^μ given by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^* = \mathrm{i}\alpha(\partial^\mu\phi^*\phi - \phi^*\partial^\mu\phi). \tag{2.45}$$

Choosing the arbitrary constant infront of the charge operator

$$Q = \int d^3x\frac{-1}{2}j^0 \tag{2.46}$$

$$= \int d^3x\frac{-\mathrm{i}}{2}(\partial^0\phi^*\phi - \phi^*\partial^0\phi) \tag{2.47}$$

$$= \int d^3x\frac{\mathrm{i}}{2}(\phi^*\pi^* - \pi\phi) \tag{2.48}$$

$$= \int d^3x \int \frac{d^3qd^3p}{(2\pi)^6}\frac{1}{4}\sqrt{\frac{E_{\underline{p}}}{E_{\underline{q}}}}e^{i(\underline{p}+\underline{q})\cdot\underline{x}}\left[\left(b_{\underline{q}} + a^\dagger_{-\underline{q}}\right)\left(a_{\underline{p}} - b^\dagger_{-\underline{p}}\right) - \left(b_{\underline{p}} - a^\dagger_{-\underline{p}}\right)\left(a_{\underline{q}} + b^\dagger_{-\underline{q}}\right)\right] \tag{2.49}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[a^\dagger_{\underline{p}}a_{\underline{p}} - b^\dagger_{\underline{p}}b_{\underline{p}}\right], \tag{2.50}$$

2.2.d

With two complex fields the Lagrangian becomes

$$\mathcal{L} = \partial_\mu\phi^*_a\partial^\mu\phi_a - m^2\phi^*_a\phi_a \tag{2.51}$$

Now $\mathcal{L}$ is invariant under both $U(1)$ and $SU(2)$ "rotations" $\phi_a \to R_{ab}\phi_b$ the $U(1)$ rotations are given by $R_{ab} = e^{i\alpha\mathbb{I}_{ab}}$ and the $SU(2)$ rotations given by $R_{ab} = e^{\mathrm{i}\alpha_i T^i_{ab}}$ with $i = 1,..,3$ and the T^i's are $SU(2)$ generators $T^i = \frac{1}{2}\sigma^i i$.

$\mathcal{L}$ is clearly invariant under $U(1)$ as in part (c) and under $SU(2)$

$$
\begin{aligned}
\mathcal{L} \to & \partial_\mu e^{-\mathrm{i}\alpha_i T^{i*}_{ac}} \phi_c^* e^{\mathrm{i}\alpha_i T^i_{ab}} \partial^\mu \phi_a - m^2 e^{-\mathrm{i}\alpha_i T^{i*}_{ac}} \phi_a^* e^{\mathrm{i}\alpha_i T^i_{ab}} \phi_b & (2.52) \\
\simeq & \partial_\mu \phi_c^* \left(1 - \frac{\mathrm{i}}{2}\alpha_i \sigma^i_{ca}\right)\left(1 + \frac{\mathrm{i}}{2}\alpha_i \sigma^i_{ab}\right) \partial^\mu \phi_a - m^2 \phi_a^* \left(1 - \frac{\mathrm{i}}{2}\alpha_i \sigma^i_{ca}\right)\left(1 + \frac{\mathrm{i}}{2}\alpha_i \sigma^i_{ab}\right)\phi_b & (2.53) \\
= & \mathcal{L} + O(\alpha^2). & (2.54)
\end{aligned}
$$

Then, using (2.45) the $U(1)$ conserved current is

$$
j^\mu_{U(1)} = \mathrm{i}\alpha(\partial^\mu \phi_a^* \mathbb{I}_{ab} \phi_b - \phi_a^* \mathbb{I}_{ab} \partial^\mu \phi_b). \tag{2.55}
$$

and the $SU(2)$

$$
j^{\mu i}{}_{SU(2)} = \frac{\mathrm{i}}{2}\alpha_i(\partial^\mu \phi_a^* \sigma^i_{ab} \phi_b - \phi_a^* \sigma^i_{ab} \partial^\mu \phi_b). \tag{2.56}
$$

Which gives the respective charges

$$
\begin{aligned}
Q_{U(1)} &= \int d^3x j^0_{U(1)} = \int d^3x \frac{\mathrm{i}}{2}(\phi_a^* \pi_a^* - \pi_a \phi_a) & (2.57) \\
Q^i_{SU(2)} &= \int d^3x j^{0i}{}_{U(1)} = \int d^3x \frac{\mathrm{i}}{2}(\phi_a^* \sigma^i_{ab} \pi_b^* - \pi_a \sigma^i_{ab} \phi_b). & (2.58)
\end{aligned}
$$

$\mathcal{L}$ is actually invariant under a larger symmetry group, $SO(4)$ which has $\frac{4}{2}(4-1) = 6$ generators corresponding to six conserved currents.

For the case of n complex fields $a = 0....n-1$, we can make the $SO(2n)$ symmetry manifest by making the substitution $\phi_a = \frac{1}{\sqrt{2}}(\eta_{2a} + \mathrm{i}\eta_{2a+1})$ where $\eta_i \in \mathbb{R}$, then

$$
\mathcal{L} = \frac{1}{2}\sum_{i=0}^{2n-1} \partial_\mu \eta_i \partial^\mu \eta_i + \frac{1}{2}\sum_{i=0}^{2n-1} (\eta_i)^2 \tag{2.59}
$$

is invariant under $SO(2n)$ rotations. $SO(2n)$ has $\frac{2n(2n-1)}{2}$ generators with each generator corresponding to a conserved current.

2.3

To compute the integral we transform to polar coordinates and we take $(x-y)$ to be a spacelike interval so that $(x-y)^2 = -r^2$

$$
\begin{aligned}
\int \frac{d^3p}{(2\pi)^3} \frac{e^{\mathrm{i}(x-y)p}}{2E_p} &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{dp d\theta d\phi}{(2\pi)^3} \frac{p^2 \sin\theta}{2E_p} e^{ipr\cos\theta} & (2.60) \\
&= \frac{1}{2\mathrm{i}r(2\pi)^2} \int_0^\infty dp \frac{p}{E_p}\left(e^{ipr} - e^{-ipr}\right) & (2.61) \\
&= \frac{-\mathrm{i}}{2r(2\pi)^2} \int_{-\infty}^\infty dp \frac{p}{\sqrt{p^2+m^2}} e^{ipr}. & (2.62)
\end{aligned}
$$

The integral has branch cuts

3 Chapter 3

3.1 Lorentz group

3.1.a Write out the commutation relations for the operators L^i, K^i and the verify the combinations $\mathbf{J}_+, \mathbf{J}_-$ satisfy the angular momentum algebra.

$$[L^i, L^j] = \mathrm{i}\epsilon^{ijk}L^k \tag{3.1}$$

$$\begin{aligned}
[L^i, K^j] =& \frac{1}{2}\epsilon^{imn}[J^{mn}, J^{0j}] && (3.2)\\
=& \frac{\mathrm{i}}{2}\epsilon^{imn}\left(g^{n0}J^{mj} - g^{m0}J^{nj} - g^{nj}J^{m0} + g^{mj}J^{n0}\right) && (3.3)\\
=& \frac{\mathrm{i}}{2}\epsilon^{imn}(-2J^{m0}) && (3.4)\\
=& -\mathrm{i}\epsilon^{ijm}K^m. && (3.5)
\end{aligned}$$

$$\begin{aligned}
[K^i, K^j] =& [J^{0i}, J^{0j}] && (3.6)\\
=& \mathrm{i}(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) && (3.7)\\
=& -\mathrm{i}J^{ij} && (3.8)\\
=& -\frac{\mathrm{i}}{2}\epsilon^{ijk}\epsilon^{kjm}J^{lm} = -\epsilon^{ijk}L^k, && (3.9)
\end{aligned}$$

in the last line we use the fact that $J^{ij} = -J^{ji}$ and that $\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$. And then

$$\begin{aligned}
[J_+^i, J_-^j] =& \frac{1}{4}[L^i + \mathrm{i}K^i, L^j - \mathrm{i}K^j] && (3.10)\\
=& \frac{1}{4}\left(\mathrm{i}\epsilon^{ijm}L^m - \mathrm{i}\epsilon^{ijm}L^m + \epsilon^{ijm}K^m - \epsilon^{ijm}K^m\right) && (3.11)\\
=& 0. && (3.12)
\end{aligned}$$

$$\begin{aligned}
[J_+^i, J_+^j] =& \frac{1}{4}[L^i + \mathrm{i}K^i, L^j + \mathrm{i}K^j] && (3.13)\\
=& \frac{2}{4}\mathrm{i}\epsilon^{ijk}(L^k + \mathrm{i}K^k) = \mathrm{i}\epsilon^{ijk}J_+^k && (3.14)
\end{aligned}$$

and,

$$[J_-^i, J_-^j] = \mathrm{i}\epsilon^{ijk}J_-^k \tag{3.15}$$

thus $\boldsymbol{J}_+, \boldsymbol{J}_-$ satisfy the commutation relations of angular momentum.

3.1.b

We have $\boldsymbol{J}_+ \in (1/2, 0)$ and $\boldsymbol{J}_- \in (0, 1/2)$ representations.

For $\boldsymbol{J} = \boldsymbol{\sigma}/2$ we see from the commutation relations (3.14), (3.15) that in the $(1/2, 0)$ representation we have $L^k_{(1/2,0)} = \sigma^k/2$, $K^k_{(1/2,0)} = -\mathrm{i}\sigma^k/2$ and $L^k_{(0,1/2)} = \sigma^k/2$, $K^k_{(0,1/2)} = \mathrm{i}\sigma^k/2$.

So for ϕ_+ transforming in the $(1/2, 0)$ representation

$$\phi_+ \to (1 - \mathrm{i}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}/2 + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}/2)\phi_+, \tag{3.16}$$

and for ϕ_- transforming in the $(1/2, 0)$ representation

$$\phi_- \to (1 - \mathrm{i}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}/2 + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}/2)\phi_-. \tag{3.17}$$

Therefore, ϕ_+ transforms like a Left-handed spinor while ϕ_- transforms like a Right-handed spinor.

3.1.c

Using the relation $\boldsymbol{\sigma}^T = -\sigma_2 \boldsymbol{\sigma} \sigma_2$ allows us to rewrite the ψ_L transformation law in the form

$$\psi_L^T \sigma_2 \to \psi_L^T \sigma_2 (1 + \frac{\mathrm{i}}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma}). \tag{3.18}$$

We may represent a 2x2 matrix transforming in the $(1/2, 1/2)$ representation which transforms as $\psi_L^T \sigma_2$ on the RHS and as ψ_R on the LHS, the matrix may be parameterised by

$$\begin{pmatrix} V^0 + V^3 & V^1 - \mathrm{i}V^2 \\ V^1 + \mathrm{i}V^2 & V^0 - V^3 \end{pmatrix} = V^\mu \bar{\sigma}_\mu = V^0 + V^i \sigma^i. \tag{3.19}$$

Therefore under the $(1/2, 1/2)$ rep the object $V^\mu \bar{\sigma}_\mu$ transforms, using equation (3.37), as

$$V^\mu \bar{\sigma}_\mu \to (1 - \frac{\mathrm{i}}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma}) V^\mu \bar{\sigma}_\mu (1 + \frac{\mathrm{i}}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma}) \tag{3.20}$$

$$= V^0(1 + 2 \cdot \frac{1}{2}\beta_j \sigma_j) + V^i(\sigma_i + \frac{\mathrm{i}}{2}\sigma_i \theta_j \sigma_j + \frac{1}{2}\beta_j \sigma_i \sigma_j - \frac{\mathrm{i}}{2}\theta_j \sigma_j \sigma_i + \frac{1}{2}\beta_j \sigma_j \sigma_i) + O(\theta_j^2, \beta_j^2) \tag{3.21}$$

$$= V^0(1 + \beta_j \sigma_j) + V^i(\sigma_i + \frac{\mathrm{i}}{2}\theta_j [\sigma_i, \sigma_j] + \frac{1}{2}\beta_j \{\sigma_i, \sigma_j\}) + O(\theta_j^2, \beta_j^2) \tag{3.22}$$

$$= V^0(1 + \beta_j \sigma_j) + V^i(\sigma_i - \mathrm{i}\epsilon_{ijk}\theta_j \sigma_k + \beta_i) + O(\theta_j^2, \beta_j^2). \tag{3.23}$$

Now, if V^μ is indeed a 4-vector it will transform under LT's as $V^\mu \to \Lambda^\mu_\nu V^\nu$ with $\Lambda^\mu \nu$ given, in infinitesimal form, by equation 3.19

$$\Lambda^\mu_\nu = \delta^\mu_\nu - g^{\mu\rho}\frac{\mathrm{i}}{2}\omega_{\alpha\beta}(\mathcal{J}^{\alpha\beta})_{\rho\nu} \tag{3.24}$$

with $(\mathcal{J}^{\alpha\beta})_{\mu\nu} = \mathrm{i}(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu)$ and $\omega_{ij} = \epsilon_{ijk}\theta_k$, $\omega_{0i} = -\omega_{i0} = \beta_i$ is antisymmetric.

$$V^0 \to \Lambda^0_\nu V^\nu = V^0 + \frac{1}{2}\omega_{\alpha\beta}(\delta^\alpha_0 \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_0)V^\nu \tag{3.25}$$

$$= V^0 + \frac{1}{2}(\omega_{0i}V^i - \omega_{i0}V^i) \tag{3.26}$$

$$= V^0 + \beta_i V^i \tag{3.27}$$

$$V^i\sigma_i \to \Lambda^i_\nu V^\nu =(\sigma^i_\nu - \frac{\mathrm{i}}{2}\omega_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^i_\nu)V^\nu\sigma_i \tag{3.28}$$

$$=V^i\sigma_i + \frac{1}{2}\omega_{\alpha\beta}(\delta^\alpha_i\delta^\beta_j - \delta^\alpha_j\delta^\beta_i)V^j\sigma_j + \frac{1}{2}\omega_{\alpha\beta}(\delta^\alpha_i\delta^\beta_0 - \delta^\alpha_0\delta^\beta_i)V^0\sigma_i \tag{3.29}$$

$$=V^i\sigma_i + \epsilon_{ijk}V^j\sigma_i + \beta_i V^0\sigma_i \tag{3.30}$$

So in total

$$V^\mu\bar{\sigma}_\mu \to V^i(\sigma_i - \epsilon_{ijk}\sigma_j\theta_k + \beta_i) + V^0(1+\beta_i\sigma_i) \tag{3.31}$$

which matches the transformation (3.23), thus $V^\mu 0$ is indeed a 4-vector

3.2 Gordon Identity

3.3 Spinor Products

k^μ_0, k^μ_1 are fixed 4-vectors with $k_0^2 = 0$, $k_1^2 = -1$ and $k_0 \cdot k_1 = 0$. u_{L0} is a LH spinor with momentum k_0 and $u_{R0} = \not{k}_1 u_{L0}$. For lightlike p, $p^2 = 0$

$$u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0}, \qquad u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0} \tag{3.32}$$

3.3.a Show that $\not{k}_0 u_{R0}$ and show $\not{p}u_L(p) = \not{p}u_R(p) = 0$

$$\not{k}_0 u_{R0} = \not{k}_0\not{k}_1 u_{L0} = 2(k_0\cdot k_1) - \not{k}_1\not{k}_0 u_{L0} = 0 \tag{3.33}$$

by the Dirac equation $\not{k}_0 u_{L0} = 0$

$$\not{p}u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}p^2 u_{R0} = 0 \tag{3.34}$$

$$\not{p}u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}p^2 u_{L0} = 0 \tag{3.35}$$

using $\not{p}\not{p} = p^2 = 0$.

3.3.b For $k_0 = (E,0,0,-E)$, $k_1 = (0,1,0,0)$ construct u_{L0}, u_{R0}, $u_L(p)$ and $u_R(p)$ explicitly

Letting $u_{L0} = (A,B,C,D)$, in the Weyl representation

$$u_{R0} = \not{k}_1 u_{L0} = \gamma^1 u_{L0} = (D,C,-B,-A) \tag{3.36}$$

subjected to the constraints

$$\not{k}_0 u_{R0} = E(0,-2A,2D,0) = 0, \qquad \not{k}_0 u_{L0} = E(0,2D,2A,0) = 0 \tag{3.37}$$

We may pick $C = 0$, and then normalising according to $u^\dagger_{L0}u_{L0} = 2E$, gives $B = \sqrt{2E}$, thus the final forms are

$$u_{L0} = (0,\sqrt{2E},0,0), \qquad (0,0,-\sqrt{2E},0). \tag{3.38}$$

Then, using $p^\mu = (p^0, p^1, p^2, p^3)$,

$$u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}} \begin{pmatrix} 0 & 0 & p^0+p^3 & p^1+\mathrm{i}p^2 \\ 0 & 0 & p^1+\mathrm{i}p^2 & p^0-p^3 \\ p^0-p^3 & p^1+\mathrm{i}p^2 & 0 & 0 \\ p^1-\mathrm{i}p^2 & p^0+p^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2E} \\ 0 \end{pmatrix} = \frac{-1}{\sqrt{2p\cdot k_0}} \begin{pmatrix} p^0+p^3 \\ p^1+\mathrm{i}p^2 \\ 0 \\ 0 \end{pmatrix} \tag{3.39}$$

$$u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}} \begin{pmatrix} 0 & 0 & p^0+p^3 & p^1+\mathrm{i}p^2 \\ 0 & 0 & p^1+\mathrm{i}p^2 & p^0-p^3 \\ p^0-p^3 & p^1+\mathrm{i}p^2 & 0 & 0 \\ p^1-\mathrm{i}p^2 & p^0+p^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix} = \frac{-1}{\sqrt{2p\cdot k_0}} \begin{pmatrix} 0 \\ 0 \\ -p^1+\mathrm{i}p^2 \\ p^0+p^3 \end{pmatrix} \tag{3.40}$$

3.3.c Define $s(p1,p2) = \bar{u}_R(p_1)u_L(p_2)$, $t(p1,p2) = \bar{u}_L(p_1)u_R(p_2)$.

$$(s(p,q))^* = (\bar{u}_R(p)u_L(q))^* = u_L^\dagger(q)(\gamma^0)^\dagger u_R(p) = \bar{u}_L(q)u_p(p) = t(q,p) \tag{3.41}$$

and from the explicit forms we can easily see that $s(p,q) = -s(q,p)$.

$$\begin{aligned}
|s(p,q)|^2 =& \frac{1}{(p^0+p^3)(q^0+q^3)}\big((q^0+q^3)(p^1-\mathrm{i}p^2) - (q^1-\mathrm{i}q^2)(p^0+p^3)\big) \\
&\times \big((p^1+\mathrm{i}p^2)(q^0+q^3) - (p^0+p^3)(q^1+\mathrm{i}q^2)\big) && (3.42)\\
=& \frac{1}{(p^0+p^3)(q^0+q^3)}\Big[(q^0+q^3)^2({p^1}^2+{p^2}^2) + (p^0+p^3)^2({q^1}^2+{q^2}^2) \\
&-2(q^0+q^3)(p^0+p^3)(p^1q^2+p^2q^2)\Big] && (3.43)\\
=&(p^0-p^3)(q^0+q^3) + (p^0+p^3)(q^0-q^3) - 2(p^1q^1+p^2q^2) && (3.44)\\
=&2(p^0q^0 - p^1q^1 - p^2q^2 - p^3q^3) = 2p\cdot q. && (3.45)
\end{aligned}$$

in the third line we have used the fact that p, q are lightlike ${p^1}^2 + {p^2}^2 = {p^0}^2 - {p^3}^2$.

3.4 Majorana fermions

χ_a transforms as the upper two components of a Dirac spinor (ψ_L)

3.4.a Show the Majorana equation $\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi - \mathrm{i}m\sigma_2\chi^*$ is relativistically invariant and implies the KG equation.

Under Lorentz transformation the Majorana equation transforms as

3.4.b Show that the action S is real and that the variation of S w.r.t χ yields the Majorana equation

The action is

$$S = \int d^4x \left[\chi^\dagger \mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi + \frac{\mathrm{i}m}{2}(\chi^T\sigma_2\chi - \chi^\dagger\sigma_2\chi^*)\right] \tag{3.46}$$

taking the conjugate gives]

4 Chapter 4

5 Chapter 5

6 Chapter 6

6.1 Rosenbluth formula

6.2

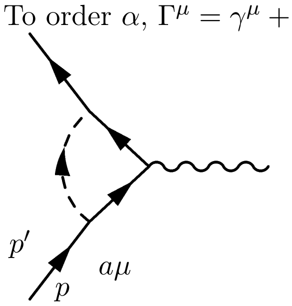
6.3 Exotic contributions to $g-2$

6.3.a

The Higgs field couples to the electron via

$$H_{\text{int}} = \int d^3x \frac{\lambda}{\sqrt{2}} h\overline{\psi}\psi. \tag{6.1}$$

To order α, $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$.



$$\begin{aligned}
\overline{u}(p')\delta\Gamma^\mu u(p) =& \overline{u}(p') \int \frac{d^4k}{(2\pi)^4} \left(\frac{-\mathrm{i}\lambda}{\sqrt{2}}\right)^2 \frac{\mathrm{i}}{(k-p)^2 - m_h^2} \frac{\mathrm{i}(\not{k}' + m)}{k'^2 - m^2} \gamma^\mu \frac{\mathrm{i}(\not{k} + m)}{k^2 - m^2} u(p) && (6.2)\\
=& \frac{-\mathrm{i}\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\overline{u}(p')(\not{k} + \not{q} + m)\gamma^\mu(\not{k} + m)u(p)}{((k-p)^2 - m_h^2)((k+q)^2 - m^2)(k^2 - m^2)} && (6.3)\\
=& \frac{\mathrm{i}\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\overline{u}(p')(\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu + m\not{q}\gamma^\mu + 2mk^\mu)u(p)}{((k-p)^2 - m_h^2)((k+q)^2 - m^2)(k^2 - m^2)} && (6.4)
\end{aligned}$$

The denominator may be re-written as

$$\frac{1}{((k-p)^2 - m_h^2)((k+q)^2 - m^2)(k^2 - m^2)} = \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3} \tag{6.5}$$

with,

$$\begin{aligned}
D =& x(k^2 - m^2) + y((k+q)^2 - m^2) + z((k-p)^2 - m_h^2) && (6.6)\\
=& k^2 - (x+y)m^2 + q^2y + 2yk\cdot q + zp^2 - 2zk\cdot p - zm_h^2. && (6.7)
\end{aligned}$$

Letting $l \equiv k + yq - zp$ so that,

$$D = l^2 - (z-1)^2m^2 - zm_h^2 + yq^2(1-y) + 2zyq\cdot p = l^2 - \Delta \tag{6.8}$$

with $\Delta = (z-1)^2m^2 + zm_h^2 - yq^2(1+y) - 2zyq\cdot p$

and the Numerator may be written in the form $A\gamma^\mu + B(p'+p)^\mu + Cq^\mu$ using Dirac algebra and the Dirac equation $\not{p}u(p) = mu(p)$ with $C=0$ by the Ward identity. The *Mathematica* package *FeynCalc* [1] is useful for calculations such as these.

$$\overline{u}(p')(\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu + m\not{q}\gamma^\mu + 2mk^\mu)u(p) \tag{6.9}$$

$$= \overline{u}(p')\left[\not{l}\gamma^\mu\not{l} + (z\not{p} + (1-y)\not{q})\gamma^\mu(z\not{p} - y\not{q}) + m^2\gamma^\mu + 2m(zp^\mu - yq^\mu) + m\not{q}\gamma^\mu\right]u(p) \tag{6.10}$$

$$= \overline{u}(p')\left[\not{l}\gamma^\mu\not{l} + \gamma^\mu(-zm^2 - zyq^2 - (z+1)m\not{q} + y(1-y)q^2) + 2m(zp^\mu + 2(1-y)q^\mu + z^2p^\mu)\right]u(p) \tag{6.11}$$

$$= \overline{u}(p')\left[\not{l}\gamma^\mu\not{l} + \gamma^\mu((y-y^2-zy)q^2 + (2z+3-z^2)m^2) + \frac{2m^2}{2m}(z^2-1)(p'^\mu + p^\mu)\right]u(p) \tag{6.12}$$

Where we have used the symmetric integration formulas (A.41), (A.44) and (A.45)

By the Gordon identity:

$$\overline{u}(p')\left[\gamma^\mu - \frac{p'^\mu + p^\mu}{2m}\right]u(p) = \overline{u}(p')\left[\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]u(p). \tag{6.13}$$

Giving the final form for the numerator

$$\overline{u}(p')\left[\gamma^\mu(\frac{1}{2}l^2 + (y-y^2-zy)q^2 + (2z+1+z^2)m^2) + 2m^2(z^2-1)\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{6.14}$$

The form factors are

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) \tag{6.15}$$

Therefore we have the correction to $F_2(q^2)$

$$\delta F_2(q^2) = \frac{\mathrm{i}\lambda^2}{2}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m^2(z^2-1)}{[l^2-\Delta]^3} \tag{6.16}$$

which we must evaluate at $q^2=0$

$$\delta F_2(q^2=0) = 2\mathrm{i}\lambda^2\int_0^1 dxdydz\delta(x+y+z-1)\frac{-\mathrm{i}m^2(z^2-1)}{(3-1)(3-2)(4\pi)^2\Delta} \tag{6.17}$$

$$= \frac{2m^2\lambda^2}{(4\pi)^2}\int_0^1 dxdydz\delta(x+y+z-1)\frac{(z^2-1)}{(z-1)^2m^2 + zm_h^2} \tag{6.18}$$

$$= \frac{2m^2\lambda^2}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\frac{(z-1)(z+1)}{(z-1)^2m^2 + zm_h^2} \tag{6.19}$$

$$= \frac{2m^2\lambda^2}{(4\pi)^2}\int_0^1 dz\frac{(z-1)^2(z+1)}{(z-1)^2m^2 + zm_h^2} \tag{6.20}$$

$$= \frac{-\lambda^2}{(4\pi)^2}\int_0^1 dz\frac{(z-1)^2(z+1)}{(z-1)^2 + z(m_h^2/m^2)} \tag{6.21}$$

Using the approximation $m_h^2 \gg m^2$ So

$$\delta F_2(q^2=0) \simeq \frac{\lambda^2}{4\pi^2}\int_0^1 dz\left(\frac{1}{1+z(m_h/m)^2} + \frac{1-z-z^2}{(m_h/m)^2)}\right) = \frac{\lambda^2 m^2}{4\pi^2 m_h^2}\left(\log\frac{m_h^2}{m^2} + \frac{1}{6}\right). \tag{6.22}$$

6.3.b

If $a = \frac{g-2}{2}$. $a_{expt} \simeq a_{higgs} + q_{QED}$. So, plugging in $\lambda = 3 \times 10^{-6}$ and $m_h > 60\text{GeV}$

$$|a_{expt} - q_{QED}| = |a_{higgs}| = |\frac{\lambda^2 m^2}{4\pi^2 m_h^2}\left(\log\frac{m_h^2}{m^2} + \frac{1}{6}\right)| \simeq 10^{-14} < 10^{-10} \tag{6.23}$$

6.3.c Now we compute the contribution of a pseudoscalar particle called the axion to $g-2$

$$H_{int} = \int d^3x \frac{\mathrm{i}\lambda}{\sqrt{2}} a\overline{\psi}\gamma^5\psi. \tag{6.24}$$

$$\begin{aligned}
&\overline{u}(p')\delta\Gamma^\mu u(p) && (6.25)\\
&= \overline{u}(p')\int \frac{d^4k}{(2\pi)^4}\left(\frac{-\mathrm{i}\lambda}{\sqrt{2}}\right)^2 \frac{\mathrm{i}}{(k-p)^2 - m_a^2}\gamma^5 \frac{\mathrm{i}(\not k' + m)}{k'^2 - m^2}\gamma^\mu \frac{\mathrm{i}(\not k + m)}{k^2 - m^2}\gamma^5 u(p) && (6.26)\\
&= \frac{\mathrm{i}\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\left[\gamma^5(\not k' + m)\gamma^\mu(\not k + m)\gamma^5\right]u(p)}{((k-p)^2 - m_a^2)((k+q)^2 - m^2)(k^2 - m^2)}. && (6.27)
\end{aligned}$$

The denominator is precisely that of (6.5) but with $m_h \to m_a$. The Numerator may be rearranged in a similar fashion to the Higgs result

$$\overline{u}(p')\left[\gamma^5(\not k' + m)\gamma^\mu(\not k + m)\gamma^5\right]u(p) = \overline{u}(p')\left[\gamma^\mu(-\frac{1}{2}l^2 - m^2(z-1)^2 + y(z-y+y^2)q^2) + 2m^2(z-1)^2\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right. \tag{6.28}$$

The $\frac{g-2}{2}$ contribution is then

$$\begin{aligned}
\delta F_2(q^2 = 0) =& \frac{\mathrm{i}\lambda^2}{2}\int_0^1 dxdydz\delta(x+y+z-1)\int\frac{d^dl}{(2\pi)^4}\frac{2}{[l^2-\Delta]^3}2m^2(z-1)^2 && (6.29)\\
=& \frac{m^2\lambda^2}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy \frac{(z-1)^2}{m^2(z-1)^2 + zm_a^2} && (6.30)\\
=& \frac{\lambda^2}{(4\pi)^2}\int_0^1 dz \frac{(z-1)^3}{(z-1)^2 + zm_a^2/m^2} && (6.31)\\
\simeq& \frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\left[\frac{1}{1 + z(m_a^2/m^2 - 2)} + \frac{z^2 + 1 - 2z}{m_a^2/m^2}\right] && (6.32)\\
=& \frac{\lambda^2}{(4\pi)^2}\left[\frac{1}{m_a^2/m^2 - 2}\log\left(\frac{m_a^2}{m^2} - 1\right) - \frac{2}{3}\frac{m^2}{m_a^2}\right]. && (6.33)
\end{aligned}$$

7 Chapter 7

7.1

7.2 Alternative regulators in QED

7.2.a Compute δZ_1 and δZ_2 with a momentum cutoff Λ and show $\delta Z_1 \neq \delta Z_2$

$\delta Z_1 = -\delta F_1(q^2 = 0)$, $\delta Z_2 = \frac{d\Sigma_2}{d\not p}|_{\not p = m}$

The naievly regulated integrals that we will need are:

$$\int_0^\Lambda \frac{d^4l}{(2\pi)^4}\frac{1}{[l^2-\Delta]^2} = \mathrm{i}(-1)^2\int d\Omega_4 \int_0^\Lambda \frac{dl_E}{(2\pi)^4}\frac{l_E^3}{[l_E^2+\Delta]^2} \tag{7.1}$$

$$= \frac{\mathrm{i}2\pi^2}{(2\pi)^4}\int_\Delta^{\Lambda+\Delta}\frac{dk}{2l_E}\frac{l_E^3}{k^2} \tag{7.2}$$

$$= \frac{\mathrm{i}}{(4\pi)^2}\int_\Delta^{\Lambda^2+\Delta} dk\frac{k-\Delta}{k^2} \tag{7.3}$$

$$= \frac{\mathrm{i}}{(4\pi)^2}\left[\log(\frac{\Lambda^2}{\Delta}+1)-\frac{\Lambda^2}{\Lambda^2+\Delta}\right]. \tag{7.4}$$

$$\int_0^\Lambda \frac{d^4l}{(2\pi)^4}\frac{1}{[l^2-\Delta]^3} = \frac{-\mathrm{i}}{(4\pi)^2}\int_\Delta^{\Lambda^2+\Delta} dk\frac{k-\Delta}{k^3} = \frac{-\mathrm{i}}{(4\pi)^2}\frac{\Lambda^4}{2(\Lambda^2+\Delta)^2\Delta} \tag{7.5}$$

$$\int_0^\Lambda \frac{d^4l}{(2\pi)^4}\frac{l^2}{[l^2-\Delta]^3} = \frac{\mathrm{i}\pi^2}{(2\pi)^4}\int_\Delta^{\Lambda^2+\Delta} dk\frac{(k-\Delta)^2}{k^3} = \frac{\mathrm{i}}{(4\pi)^2}\left[\log(\frac{\Lambda^2}{\Delta}+1)+\frac{4\Delta\Lambda^2+3\Delta^2}{2(\Lambda^2+\Delta)^2}-\frac{3}{2}\right] \tag{7.6}$$

where in all instances we have used the substitution $k = l_E^2+\Delta$.

Now, from equation 6.47

$$-\delta Z_1 = F_1(q^2=0) = 2\mathrm{i}e^2\int_0^\Lambda \frac{d^4l}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{-l^2+2(1-4z+z^2)m^2}{[l^2-(1-z)^2m^2]^3} \tag{7.7}$$

$$= \frac{e^2}{8\pi^2}\int_0^1 dz(1-z)\left[\log(\frac{\Lambda^2}{(1-z)^2m^2}+1)+\frac{2(1-z)^4m^4\Lambda^2+3(1-z)^6m^6+\Lambda^4(1-4z+}{(1-z)^2m^2(\Lambda^2+(1-z)^2m^2)^2}\right. \tag{7.8}$$

and from equation 7.17

$$-\mathrm{i}\Sigma_2(\not p) = -2e^2\int_0^1 dz\int_0^\Lambda \frac{d^4l}{(2\pi)^4}\int -z\not p+2m[l^2-(1-z)^2m^2+z(1-z)p^2]^2 \tag{7.9}$$

$$= \frac{\mathrm{i}e^2}{8\pi^2}\int_0^1 dz(z\not p-2m)\left[\log(\frac{\Lambda^2}{(1-z)^2m^2-z(1-z)p^2}+1)-\frac{\Lambda^2}{\Lambda^2+(1-z)^2m^2-z(1-z)p^2}\right]. \tag{7.10}$$

7.2.b Recompute δZ_1 and δZ_2 using dimensional regularisation.

$$\delta F_1(q^2=0) = 4\mathrm{i}e^2\int \frac{d^dl}{(2\pi)^d}\int_0^1 dxdydz\delta(x+y+z-1)\frac{\frac{-(d-2)}{d}l^2+(1-4z+z^2)m^2}{[l^2-((1-z)^2m^2+z\mu^2)]^3} \tag{7.11}$$

$$\underset{d\to 4-\epsilon}{=} \frac{2e^2}{(4\pi)^{2-\epsilon/2}}\int_0^1 dxdydz\delta(x+y+z-1)\left((2-\epsilon)\frac{\Gamma(\epsilon/2)}{((1-z)^2m^2+z\mu^2)^{\epsilon/2}}-m^2(z^2-4z+1)\frac{}{((1-}\right. \tag{7.12}$$

$$\simeq \frac{-2e^2}{(4\pi)^2}\int_0^1 dz(1-z)\left((\frac{2}{\epsilon}-\gamma-2-\log((1-z)^2m^2+z\mu^2)-\frac{m^2(z^2-4z+1)}{(1-z)^2m^2+z\mu^2}\right)+\mathcal{O}(\epsilon) \tag{7.13}$$

$$-\mathrm{i}\Sigma_2(\not{p}) = -e^2\int_0^1 dz \int \frac{d^dl}{(2\pi)^2}\frac{(2-d)\not{p}+dm}{[l^2+z(1-z)p^2-(1-z)m^2-z\mu^2]^2} \tag{7.14}$$

$$= \frac{-\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1 dz((2-d)\not{p}+dm)\frac{\Gamma(2-d/2)}{(-z(1-z)p^2+(1-z)m^2+z\mu^2)^{2-d/2}} \tag{7.15}$$

$$\underset{d\to 4-\epsilon}{=} \frac{-\mathrm{i}e^2}{(4\pi)^2}\int_0^1 dz(-(2-\epsilon)\not{p}+(4-\epsilon)m)(\frac{2}{\epsilon}-\gamma)\left(1+\frac{\epsilon}{2}\log 4\pi-\frac{\epsilon}{2}\log(-z(1-z)p^2+(1-z)m^2+z\mu^2)\right. \tag{7.16}$$

Therefore

$$\delta Z_2 = \frac{d\Sigma}{d\not{p}}|_{\not{p}=m} = \frac{-2e^2}{(4\pi)^2}\int_0^1 dzz\left[\frac{\epsilon}{2}-\gamma-1+\log 4\pi-\log((1-z)^2m^2+z\mu^2)-\frac{2(z-2)-(z-1)m^2}{(1-z)^2+z\mu^2}\right] \tag{7.17}$$

Then

$$\delta Z_1-\delta Z_2 = \frac{e^2}{8\pi^2}\int_0^1 dz$$
$$\left[(1-2z)\left(\frac{2}{\epsilon}-\gamma+\log 4\pi+\log((1-z)^2m^2+z\mu^2)\right)+\frac{m^2(1-z)(1-z^2)}{(1-z)^2m^2+z\mu^2}-2-z\right] \tag{7.18}$$

Using $\int_0^1(2z-1)dz = 0$ and the integration by parts formula below equation 7.32 this reduces to

$$\delta Z_1-\delta Z_2 = \frac{e^2}{8\pi^2}\int_0^1 dz\left[\frac{(1-z)(m^2(1-z)^2+z\mu^2+m^2(z^2-1)+m^2(1-z^2))}{(1-z)^2m^2+z\mu^2}-2(1-z)+z\right] \tag{7.19}$$

$$= \frac{e^2}{8\pi^2}\int_0^1 dz(2z-1) = 0 \tag{7.20}$$

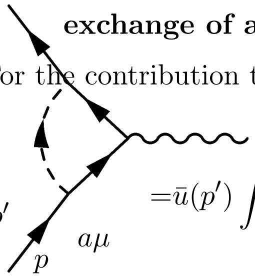
and therefore, to order α, $Z_1 = Z_2$ is preserved when using dimensional regularization.

7.3 Fermions coupled to QED and Yukawa

$$H_{int} = \int d^3x\frac{\lambda}{\sqrt{2}}\phi\overline{\psi}\psi+\int d^3xeA_\mu\overline{\psi}\gamma^\mu\psi \tag{7.21}$$

7.3.a Verify that the contribution to Z_1 form the vertex diagram from a virtual ϕ equals the contribution to Z_2 from the fermion propagator with the exchange of a virtual ϕ

. For the contribution to Z_1 we must calculate the diagram



$$=\bar{u}(p')\int\frac{d^dk}{(2\pi)^d}\left(\frac{i\lambda}{\sqrt{2}}\right)^2\frac{\mathrm{i}}{(p-k)^2-m_\phi^2}\frac{\mathrm{i}(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\gamma^\mu\frac{\mathrm{i}(\not{k}+m)}{k^2-m^2}u(p) \tag{7.22}$$

$$\equiv\mathrm{i}\lambda^2\int_0^1 dxdydz\delta(x+y+z-1)\int\frac{d^dk}{(2\pi)^d}\frac{\overline{u}(p')[(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)]u(p)}{D^3} \tag{7.23}$$

with

$$D=x(k^2-m^2)+y((k+1)^2-m^2)+z((p-k)^2-m_\phi^2)=l^2-\Delta \tag{7.24}$$

$l\equiv k+yq-zp$, $\Delta=(z-1)^2m^2+y(1-y)q^2+zm_\phi^2-2zyq\cdot p$.

Now we turn our attention to the numerator of (7.23)

$$\overline{u}(p')[(\not{k}+\not{q}+m)\gamma^\mu(\not{p}+m)]u(p)=\overline{u}(p')[(\not{l}+(1-y)\not{q}+z\not{p}+m)\gamma^\mu(\not{l}-y\not{q}+z\not{p}+m)]u(p) \tag{7.25}$$

8 Chapter 8

9 Chapter 9

9.1 Scalar QED

The Lagrangian is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+(D_\mu\phi)^*(D^\mu\phi)-m^2\phi^*\phi \tag{9.1}$$

with $D_\mu=\partial_\mu+\mathrm{i}eA_\mu$. The propagator is given by

$$\langle 0|\,T\phi^*(x_1)\phi(x_2)\,|0\rangle=\frac{\int\mathcal{D}\phi^*\mathcal{D}\phi e^{\mathrm{i}S_s}\phi^*(x_1)\phi(x_2)}{\int\mathcal{D}\phi^*\mathcal{D}\phi e^{\mathrm{i}S_s}} \tag{9.2}$$

$$S_s=\int d^4x\mathcal{L}_s=\int d^4x[\partial_\mu\phi^*\partial^\mu\phi-m^2\phi^*\phi] \tag{9.3}$$

We can rewrite

$$\phi=\frac{1}{\sqrt{2}}(\phi_1+\mathrm{i}\phi_2),\qquad\phi^*=\frac{1}{\sqrt{2}}(\phi_1-\mathrm{i}\phi_2) \tag{9.4}$$

where $\phi_1,\phi_2\in\mathbb{R}$. So

$$\int d^4x\mathcal{L}_s=\int d^4x\frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2-m^2(\phi_1^2+\phi_2^2)]. \tag{9.5}$$

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16 Chapter 16

16.1 Arnowitt-Fickler gauge

16.2 Scalar field with non-Abelian charge.

16.2.a

The Lagrangian is given by:

$$\mathcal{L} = -\frac{1}{4}\operatorname{tr} F^a_{\mu\nu}F^{b\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi \tag{16.1}$$

where

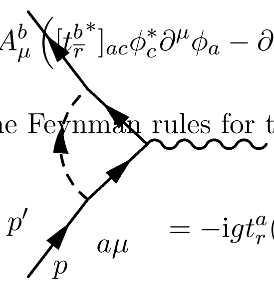
$$\begin{aligned} F^a_{\mu\nu} =& \partial_\mu A^a_\nu - \partial_\mu A^a_\nu + g f^{abc} A^b_\mu A^c_\nu \\ D_\mu =& \partial_\mu - \mathrm{i} g A^a_\mu t^a_r. \end{aligned} \tag{16.2}$$

ϕ transforms under an irreducible representation r of the gauge group generated by the t^a_r's and ϕ^* transforms under the conjugate representation $\overline{r}$ generated by $t^a_{\overline{r}} = (t^a_r)^*$. The gauge fields transform under the adjoint representation which may be generated by the structure constants themselves.

Aswell as the standard Feynman rules given by equations (16.5), (A.1), (A.1), (A.11-A.14) we have scalar-gauge boson interactions contained within the second term in (16.1)

$$(D_\mu\phi)^*(D^\mu\phi) = \partial_\mu\phi^*_a\partial^\mu\phi_a + \mathrm{i}gA^b_\mu\left([t^{b*}_{\overline{r}}]_{ac}\phi^*_c\partial^\mu\phi_a - \partial^\mu\phi^*_a[t^b_r]_{ac}\phi_c\right) + g^2A^b_\mu A^{d\mu}[t^{b*}_{\overline{r}}]_{ac}\phi^*_c[t^d_r]_{ae}\phi_e. \tag{16.3}$$

From which we may read of the Feynman rules for the Scalar-Boson vertices



$$= -\mathrm{i}gt^a_r(p+p')^\mu \tag{16.4}$$

$$= \mathrm{i}g^2g^{\mu\nu}(t^at^b + t^bt^a) \tag{16.5}$$

16.2.b

To calculate the β function, we must calculate the counterterms δ_1, δ_2 and δ_3. The counterterm vertices are given by

$$= \mathrm{i}(p^2\delta_2 - \delta_m) \tag{16.6}$$

$$a \qquad b \qquad = -\mathrm{i}\delta^{ab}(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3 \tag{16.7}$$

$$p \quad p' \quad a\mu \qquad = -\mathrm{i}gt_r^a(p+p')^\mu\delta_1 \tag{16.8}$$

Renormalized according to

$$p^2 \qquad \big|_{p^2=-M^2} = 0 \tag{16.9}$$

$$\big|_{q^2=0, p^2=-M^2} = -\mathrm{i}2gt_r^a p^\mu \tag{16.10}$$

$$\big|_{p^2=-M^2} = 0 \tag{16.11}$$

Starting with δ_3 we must calculate

$$a \qquad b \; = \underset{(1)}{} + \underset{(2)}{} + \underset{(3)}{} + \underset{(4)}{} + \underset{(5)}{} + \underset{(6)}{} + \underset{(7)}{} \tag{16.12}$$

diagrams (2)-(4) are given by equation (16.71)

$$(2)-(4) = \mathrm{i}(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta^{ab}\frac{5g^2}{3(4\pi)^2}C_2(G)\left(\frac{2}{\epsilon} - \log M^2\right) \tag{16.13}$$

In the massless limit

$$
\begin{aligned}
(5) =&(-\mathrm{i}g)^2 \operatorname{tr} \int \frac{d^dk}{(2\pi)^4} \frac{\mathrm{i}}{(k+q)^2} \frac{\mathrm{i}}{k^2} t_r^a t_t^b (2k+q)^\mu (2k+q)^\nu && (16.14)\\
=&g^2 C(r) \delta^{ab} \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{(2l^\mu - q^\mu(2x-1))(2l^\nu - q^\nu(2x-1))}{[l^2 - x(x-1)p^2]^2} && (16.15)\\
=&\frac{\mathrm{i}g^2 C(r)\delta^{ab}}{(4\pi)^{d/2}} \int_0^1 dx \left(\frac{2g^{\mu\nu} x(1-x) q^2}{1-d/2} + (2x-1)^2 q^\mu q^\nu \right) \frac{\Gamma(2-d/2)}{(x(x-1)q^2)^{2-d/2}} && (16.16)\\
\underset{d\to 4-\epsilon}{\simeq}& \frac{-\mathrm{i}\delta^{ab} g^2 C(r)}{(4\pi)^2} \frac{1}{3} (g^{\mu\nu} q^2 - q^\mu q^\nu) \left(\frac{2}{\epsilon} - \log M^2 \right). && (16.17)
\end{aligned}
$$

$$
\begin{aligned}
(6) =&(\mathrm{i}g^2) \int \frac{d^dk}{(2\pi)^d} \left(\frac{\mathrm{i}}{k^2} \right)^2 (t_r^a t_r^b + t_r^b t_r^a) && (16.18)\\
\underset{d\to 4}{=}& 0 && (16.19)
\end{aligned}
$$

giving

$$
\delta_3 = \frac{g^2}{(4\pi)^2} \left(\frac{5}{3} C_2(G) - \frac{1}{3} C(r) \right) \left(\frac{2}{\epsilon} - \log M^2 \right). \tag{16.20}
$$

For δ_2 we must calculate

$$
a \quad b = \underset{(8)}{\quad} + \underset{(9)}{\quad} + \underset{(10)}{\quad} + \underset{(11)}{\quad} \tag{16.21}
$$

$$
\begin{aligned}
(9) =&\frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \delta^{ab} \frac{-\mathrm{i}g^{\mu\nu}}{k^2} \mathrm{i}g^2 (t_r^a t_r^b + t_r^b t_r^a) g^{\mu\nu} && (16.22)\\
=&d g^2 C_2(r) \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} = 0 && (16.23)
\end{aligned}
$$

$$
\begin{aligned}
(10) =& \int \frac{d^dk}{(2\pi)^d} (-\mathrm{i}g)^2 t_r^a (p+k)^\mu \frac{-\mathrm{i}g_{\mu\nu}}{(k-p)^2} \frac{\mathrm{i}\delta^{ab}}{k^2} t_r^b (p+k)^\nu && (16.24)\\
=& -g^2 \frac{d^dk}{(2\pi)^d} C_2(r) \frac{(p+k)^2}{(k-p)^2 k^2} && (16.25)\\
=& -g^2 C_2(r) \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{l^2 + (1+x)^2 p^2}{[l^2 - x(x-1)p^2]^2} && (16.26)\\
=& \frac{-\mathrm{i}g^2 C_2(r)}{(4\pi)^2} \int_0^1 dx \left[(1+x)^2 p^2 - \frac{d}{2} \frac{x(x-1)p^2}{1-d/2} \right] \frac{\Gamma(2-d/2)}{(x(x-1)p^2)^{2-d/2}} && (16.27)\\
\underset{d\to 4-\epsilon}{\simeq}& \frac{-\mathrm{i}g^2 C_2(r)}{(4\pi)^2} p^2 \left(\frac{2}{\epsilon} - \log p^2 \right) && (16.28)
\end{aligned}
$$

Therefore,

$$
\delta_2 = \frac{-g^2 C_2(r)}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log M^2 \right) \tag{16.29}
$$

For δ_1 we must calculate

p

$$q = 0 \quad \underline{c\mu} \qquad + \qquad + \qquad + \tag{16.30}$$

p

(12) (13) (14) (15)

$$(13) = \int \frac{d^dk}{(2\pi)^d}(-\mathrm{i}g)^2 t_r^a (2p-k)^\rho \frac{-\mathrm{i}g_{\rho\nu}\delta^{ab}}{k^2} \frac{\mathrm{i}}{(p-k)^2} \frac{\mathrm{i}}{(p-k)^2}(2p-2k)^\mu)(-\mathrm{i}g)t_r^c t_r^b (2p-k)^\nu \tag{16.31}$$

$$\begin{aligned}
=& -\mathrm{i}g^3[C_2(r) - \frac{1}{2}C_2(G)]t^c \int \frac{d^dk}{(2\pi)^d} \frac{2(2p-k)^2(p-k)^\mu}{k^2(p-k)^2(p-k)^2} && (16.32)\\
=& -4\mathrm{i}g^3[C_2(r) - \frac{1}{2}C_2(G)]t^c p^\mu \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{l^2(x+\frac{2}{d}(1+x)) + x(1+x)^2p^2}{[l^2 - x(x-1)p^2]^3} && (16.33)\\
=& \frac{-2\mathrm{i}g^3}{(2\pi)^{d/2}}[C_2(r) - \frac{1}{2}C_2(G)]t^c p^\mu \int_0^1 dx(1-x)\left[\frac{d}{2}(x+\frac{2}{d}(1+x)) + (2-\frac{d}{2})\frac{(1+x)^2}{1-x}\right] \frac{\Gamma(2-d/2)}{(x(x-1)p^2)^{2-d/2}} && (16.34)\\
\underset{d\to 4-\epsilon}{\sim}& \frac{-2\mathrm{i}g^3}{(2\pi)^2}[C_2(r) - \frac{1}{2}C_2(G)]t^c p^\mu \left(\frac{2}{\epsilon} - \log -p^2\right). && (16.35)
\end{aligned}$$

$$(14) = \begin{aligned} &\int \frac{d^dk}{(2\pi)^d}(-\mathrm{i}g)^2 t_r^b (p+k)^\rho \frac{-\mathrm{i}g_{\rho\rho'}}{(p-k)^2} \frac{-\mathrm{i}g_{\nu\nu'}}{(p-k)^2} \frac{\mathrm{i}}{k^2} t_r^a (p+k)^\nu \\ &\times g f^{abc}\left[g^{\nu'\rho'}(-2p+2k)^\mu + g^{\rho'\mu}(p-k)^{\nu'} + g^{\mu\nu'}(p-k)^{\rho'}\right]\end{aligned} \tag{16.36}$$

$$\begin{aligned}
=& g^3\frac{4}{2}C_2(G)t_r^c \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu(p^2 + k\cdot p) - p^\mu(k^2 + k\cdot p)}{(p-k)^2(p-k)^2k^2} && (16.37)\\
=& 4g^3 C_2(G)t_r^c p^\mu \int_0^1 dx(1-x) \int \frac{d^dl}{(2\pi)^d} \frac{l^2(\frac{1}{d}-1)}{[l^2 - x(x-1)p^2]^3} && (16.38)\\
=& \frac{2\mathrm{i}g^3}{(4\pi)^{d/2}} p^\mu C_2(G)t_r^c \int_0^1 dx(1-x)\frac{d}{2}(\frac{1}{d}-1)\frac{\Gamma(2-d/2)}{(x(x-1)p^2)^{2-d/2}} && (16.39)\\
\underset{d\to -\epsilon}{\sim}& \frac{-2\mathrm{i}g^3}{(4\pi)^2}\frac{3}{2}p^\mu C_2(G)t_r^c \left(\frac{2}{\epsilon} - \log -p^2\right). && (16.40)
\end{aligned}$$

Therefore

$$\delta_1 = \frac{-g^2}{(4\pi)^2}[C_2(r) + C_2(G)]\left(\frac{2}{\epsilon} - \log M^2\right) \tag{16.41}$$

Collecting (16.41), (16.29) and (16.20) gives the β function

$$\begin{aligned}
\beta(g) =& gM\frac{\partial}{\partial M}(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3) && (16.42)\\
=& \frac{-g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}C(r)\right). && (16.43)
\end{aligned}$$

16.3 Counterterm relations

16.3.a

The ghost counterterms are given by

$$a \quad p \quad b = \mathrm{i}\delta^{ab}p^2\delta_2^c \tag{16.44}$$

$$c \quad b\mu = -gf^{abc}p^\mu\delta_1^c \tag{16.45}$$

p

a

and we renormalize according to

$$a \quad p \quad b|_{p^2=-M^2} = 0 \tag{16.46}$$

a

$$p \quad b\mu|_{q^2=0,p^2=-M^2} = -gf^{abc}p^\mu \tag{16.47}$$

c

$$\tag{16.48}$$

We will also require the following structure constant identity which may be obtained via the jacobi identity:

$$f^{cde}f^{bmd}f^{aem} = (f^{bde}f^{mcd} + f^{mde}f^{cbd})f^{aem} \tag{16.49}$$

$$= f^{bde}f^{mcd}f^{aem} + C_2(G)f^{abc} \tag{16.50}$$

$$= -f^{cde}f^{bmd}f^{aem} + C_2(G)f^{abc} \tag{16.51}$$

$$\implies f^{cde}f^{bmd}f^{aem} = \frac{1}{2}C_2(G)f^{abc} \tag{16.52}$$

To calculate δ_2^c we must calculate

$$a \quad b = \quad + \quad + \tag{16.53}$$

(16) (17) (18)

$$
\begin{aligned}
(17) =&(-g)^2 \int \frac{d^dk}{(2\pi)^d} f^{eca} k^\mu \frac{-\mathrm{i}g_{\mu\nu}\delta^{cd}}{(p-k)^2} \frac{\mathrm{i}\delta^{ef}}{k^2} f^{bdf} p^\nu && (16.54)\\
=&-g^2 C_2(G)\delta^{ab} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{xp^2}{[l^2 - x(x-1)p^2]^2} && (16.55)\\
=&\frac{-g^2 C_2(G)}{(4\pi)^{d/2}} \delta^{ab} \int_0^1 dx\mathrm{i}xp^2 \frac{\Gamma(2-d/2)}{(x(x-1)p^2)^{2-d/2}} && (16.56)\\
\underset{d\to 4-\epsilon}{\sim}&\frac{-\mathrm{i}g^2 C_2(G)}{2(4\pi)^2} \delta^{ab} p^2 \left(\frac{2}{\epsilon} - \log M^2\right). && (16.57)
\end{aligned}
$$

Therefore

$$
\delta_2^c = \frac{\mathrm{i}g^2 C_2(G)}{2(4\pi)^2} \left(\frac{2}{\epsilon} - \log M^2\right). \tag{16.58}
$$

a

p $\quad b\mu \;=\; \quad + \quad + \quad +$ (16.59)

c

(19) (20) (21) (22)

$$
\begin{aligned}
(20) =&(-g)^3 \int \frac{d^dk}{(2\pi)^d} f^{dec} k^\rho f^{mbd} k^\mu f^{afm} p^\sigma \frac{-\mathrm{i}g_{\rho\sigma}\delta^{ef}}{(p-k)^2} \frac{\mathrm{i}}{k^2} \frac{\mathrm{i}}{k^2} && (16.60)\\
=&-\mathrm{i}g^3 \frac{1}{2} C_2(G) f^{abc} \int \frac{d^dk}{(2\pi)^d} \frac{(k\cdot p)k^\mu}{k^2 k^2 (p-k)^2} && (16.61)\\
=&-2\mathrm{i}g^3 \frac{1}{2} C_2(G) f^{abc} \int_0^1 dx(1-x) \int \frac{d^dl}{((2\pi)^d} \frac{(\frac{1}{d}l^2 + xp^2)p^\mu}{[l^2 - x(x-1)p^2]^3} && (16.62)\\
=&\frac{g^3}{(4\pi)^{d/2}} \frac{1}{2} C_2(G) f^{abc} p^\mu \int_0^1 dx(1-x) \left[\frac{1}{2} + \frac{xp^2(2-d/2)}{x(x-1)p^2}\right] \frac{\Gamma(2-d/2)}{(x(x-1)p^2)^{2-d/2}} && (16.63)\\
\underset{d\to 4-\epsilon}{\sim}&\frac{-g^3}{8(4\pi)^2} C_2(G) f^{abc} p^\mu \left(\frac{2}{\epsilon} - \log -p^2\right) && (16.64)
\end{aligned}
$$

$$
\begin{aligned}
(21) =& \begin{array}{l} g^3 \displaystyle\int \frac{d^dk}{(2\pi)^d} f^{edc} k^\rho f^{afe} p^{\rho'} f^{dbf} \frac{-\mathrm{i}g_{\rho\rho'}}{(p-k)^2} \frac{-\mathrm{i}g_{\nu\nu'}}{(p-k)^2} \frac{\mathrm{i}}{k^2} \\ \left[g^{\rho'\mu}(p-k)^\nu + g^{\mu\nu}(p-k)^{\rho'} + 2g^{\rho'\nu}(p-k)^\nu\right] \end{array} && (16.65)\\
=&\mathrm{i}g^3 \frac{1}{2} C_2(G) f^{abc} \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu((p-k)\cdot p) + p^\mu((p-k)\cdot p) + 2(p-k)^\mu(k\cdot p)}{(p-k)^2(p-k)^2k^2} && (16.66)\\
=&\mathrm{i}g^3 \frac{1}{2} C_2(G) f^{abc} 2 \int_0^1 dx(1-x) \int \frac{d^dl}{(2\pi)^d} \frac{p^\mu(xp^2 + 3l^2/d)}{[l^2 + x(x-1)p^2]^3} && (16.67)\\
=&\frac{\mathrm{i}g^3}{2(4\pi)^2} C_2(G) f^{abc} \int_0^1 dx(1-x) \left[\frac{3\mathrm{i}}{2} - \frac{\mathrm{i}(2-d/2)xp^2}{x(x-1)p^2}\right] \frac{\Gamma(2-d/2)}{(x(x-1)p^2)^{2-d/2}} && (16.68)\\
\underset{d\to 4-\epsilon}{\sim}&\frac{-3g^3}{8(4\pi)^2} C_2(G) f^{abc} \left(\frac{2}{\epsilon} - \log -p^2\right) && (16.69)
\end{aligned}
$$

Which gives

$$\delta_1^c = \frac{g^2}{2(4\pi)^2} C_2(G) \left(\frac{2}{\epsilon} - \log M^2\right) \tag{16.70}$$

and therefore that

$$\begin{aligned} \delta_1^c - \delta_2^c = & \frac{g^2}{(4\pi)^2} C_2(G) \left(\frac{2}{\epsilon} - \log M^2\right) && (16.71) \\ = & \delta_1 - \delta_2. && (16.72) \end{aligned}$$

16.3.b Compute the 3 gauge-boson vertex counter term δ^{3g}.

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18 Chapter 18

19 Chapter 19

20 Chapter 20

21 Chapter 21

22 Chapter 22

23 Chapter 23

24 Chapter 24

References

[1] *FeynCalc*, . http://feyncalc.org/.