September 16, 2021

# Solutions to Lie Algebras in Particle Physics 2nd Edition

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Abstract: I have written these solutions to the book by H.Georgi [1] during some free time. I make no assurances or guarantees as to the validity, completeness or correctness of any of these solutions, these were put together purely as a personal exercise.

## Contents

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# 1 Chapter 1 - Finite Groups

## 1.A Find Multiplication table for a group $G = \{3 \text{ elements}\}$ and prove that it is unique.

Consider the group of three elements $G = \{e, a, b\}$. By the four group axioms the only table we may have is

| | $e$ | $a$ | $b$ |
|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ |
| $a$ | $a$ | $b$ | $e$ |
| $b$ | $b$ | $e$ | $a$ |

because of closure we must have $aa = a^2 = b$, $ab = e$ and $bb = a$. Now suppose the table is not unique, to satisfy closure the only other property we can think of would be $aa = e$ and then $aab = b$ and $a = a^{-1} \implies aaa^{-1} = ba$, but then $a = ba \implies b = e$ and thus the group has only two elements and thus the multiplication table for a group with three elements is unique.

## 1.B Find all unique multiplication tables for groups for four elements.

For $G = \{e, a, b, c\}$ there are two unique multiplication tables. First write out

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a^2$ | $ab$ | $ac$ |
| $b$ | $b$ | $ba$ | $b^2$ | $bc$ |
| $c$ | $c$ | $ca$ | $cb$ | $c^2$ |

The first possible table which obeys the four group axoims begins with $aa = b$ and $bb = aaaa = e$, $c^2 = (ab)^2 = a^2a^4 = a^2$, $ac = e$ the table is

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $b$ | $c$ | $e$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $e$ | $a$ | $b$ |

This table describes the group $\mathbb{Z}_4$, the additive group of integers modulo 4.

The other possibility is

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

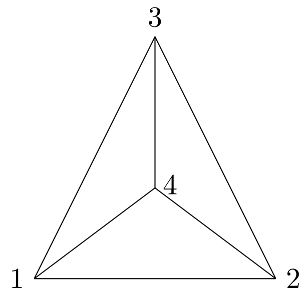
This table descibes the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**1.C Show that the defining representation of $S_n$ is reducible**

**1.D Suppose $D_1$, $D_2$ equivalent irreps of a group $G$ s. t. $D_2(g) = SD_1(g)S^{-1} \forall g \in G$, what can we say about an operator $A$ that satisfies $AD_1(g) = D_2(g)A$?**

$AD_1(g) = D_2(g)A = SD_1(g)S^{-1}A$ then $S^{-1}AD_1g = D_1(g)S^{-1}A$. Then, by Schur's lemma $S^{-1}A \propto \mathbb{I}$ so that $A \propto S$ but because we are always free to rescale the similarity transformation by an arbitrary constant $A = S$.

**1.E Find the symmetries of the regular tetrahedron invariant. Find the conjugacy classes and the characters of the irreps of the group**




The regular tetrahedron symmetries are given by the $4! = 24$ ways we can rearrange the labels $1, 2, 3, 4$ Firstly, the rotations are

$$(1,2,3,4) \to (1,2,3,4) = e \tag{1.1}$$
$$(1,2,3,4) \to (2,3,1,4) \tag{1.2}$$
$$(1,2,3,4) \to (3,1,2,4) \tag{1.3}$$
$$(1,2,3,4) \to (1,4,2,3) \tag{1.4}$$
$$(1,2,3,4) \to (1,3,4,2) \tag{1.5}$$
$$(1,2,3,4) \to (4,2,1,3) \tag{1.6}$$
$$(1,2,3,4) \to (3,2,4,1) \tag{1.7}$$
$$(1,2,3,4) \to (4,1,3,2) \tag{1.8}$$
$$(1,2,3,4) \to (2,4,3,1) \tag{1.9}$$
$$(1,2,3,4) \to (2,1,4,3) \tag{1.10}$$
$$(1,2,3,4) \to (3,2,1,4) \tag{1.11}$$
$$(1,2,3,4) \to (4,3,2,1) \tag{1.12}$$

The odd permutations are

$$(1,2,3,4) \to (2,1,3,4) \tag{1.13}$$
$$(1,2,3,4) \to (3,2,1,4) \tag{1.14}$$
$$(1,2,3,4) \to (4,2,3,1) \tag{1.15}$$
$$(1,2,3,4) \to (1,3,2,4) \tag{1.16}$$
$$(1,2,3,4) \to (1,4,3,2) \tag{1.17}$$
$$(1,2,3,4) \to (1,2,4,3) \tag{1.18}$$
$$(1,2,3,4) \to (4,3,1,2) \tag{1.19}$$
$$(1,2,3,4) \to (2,3,4,1) \tag{1.20}$$
$$(1,2,3,4) \to (3,4,1,2) \tag{1.21}$$
$$(1,2,3,4) \to (3,1,4,2) \tag{1.22}$$
$$(1,2,3,4) \to (3,4,2,1) \tag{1.23}$$
$$(1,2,3,4) \to (3,2,4,1) \tag{1.24}$$

### 1.F Analyse the normal modes of the system of four blocks sliding on a frictionless plane connected by springs.

## 2 Chapter 2 - Lie Groups

### 2.A Calculate the matrix $\mathrm{e}^{\mathrm{i}\alpha A}$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.1}$$

For some generator $X_a$, $\mathrm{e}^{\mathrm{i}\alpha_a X_a} = \sum_{k=0}^{k=\infty} \frac{(\mathrm{i}\alpha_a X_a)^k}{n!}$.

$$A^2 = AA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

$$A^3 = AAA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A \tag{2.3}$$

So,

$$A^n = \begin{cases} A & \text{if } n \text{ odd} \\ AA & \text{if } n \text{ even} \end{cases} \tag{2.4}$$

$$
\begin{aligned}
e^{i\alpha A} &= \mathbb{I} + i\alpha A + \frac{(i\alpha A)^2}{2!} + ... && (2.5)\\
&= \begin{pmatrix} \sum_{n=0}^{n=\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} & 0 & i\sum_{n=0}^{n=\infty} (-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} + ... \\ 0 & 1 & 0 \\ i\sum_{n=0}^{n=\infty} (-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} & 0 & \sum_{n=0}^{n=\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} \end{pmatrix} && (2.6)\\
&= \begin{pmatrix} \cos(\alpha) & 0 & i\sin(\alpha) \\ 0 & 1 & 0 \\ i\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} && (2.7)
\end{aligned}
$$

## 2.B $[A, B] = B$ calculate $e^{i\alpha A}\, B\, e^{-i\alpha A}$

Let

$$
B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}. \tag{2.8}
$$

Then

$$
[A, B] = \begin{pmatrix} g-c & h & i-a \\ -f & 0 & -d \\ a-i & b & c-g \end{pmatrix} = B \tag{2.9}
$$

Comparing components we get the simultaneous equations

$$
\begin{aligned}
a &= g - c && (2.10)\\
a &= i + g && (2.11)\\
a &= i - c && (2.12)\\
i &= c - g && (2.13)\\
b &= h && (2.14)\\
d &= -f && (2.15)\\
e &= 0. && (2.16)
\end{aligned}
$$

but this implies $g = -c$, $i = 2c$ so $a = c$, but then $a = g - c = -2a \implies a = 0$. So then $B$ must be given by

$$
B = \begin{pmatrix} 0 & b & 0 \\ d & 0 & -d \\ 0 & b & 0 \end{pmatrix}. \tag{2.17}
$$

So,

$$e^{i\alpha A} B e^{-i\alpha A} = \begin{pmatrix} \cos(\alpha) & 0 & i\sin(\alpha) \\ 0 & 1 & 0 \\ i\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 0 & b & 0 \\ d & 0 & -d \\ 0 & b & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha) & 0 & -i\sin(\alpha) \\ 0 & 1 & 0 \\ -i\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}. \tag{2.18}$$

$$= \begin{pmatrix} 0 & b(\cos(\alpha) + i\sin(\alpha)) & 0 \\ d(\cos(\alpha) + i\sin(\alpha)) & 0 & -d(\cos(\alpha) + i\sin(\alpha)) \\ 0 & b(\cos(\alpha) + i\sin(\alpha)) & 0 \end{pmatrix} \tag{2.19}$$

$$= \begin{pmatrix} 0 & b\,e^{i\alpha} & 0 \\ d\,e^{i\alpha} & 0 & -d\,e^{i\alpha} \\ 0 & b\,e^{i\alpha} & 0 \end{pmatrix} \tag{2.20}$$

## 2.C Carry out the expansion of $\delta_c$ in $e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_c X_c}$ to third order in $\alpha$ and $\beta$

*Note: the Mathematica package NCAlgebra [2] is very useful for calculations like these + a variety of other Non-commutative algebra problems.* Taking the logarithm of both side of $e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_c X_c}$ gives

$$i\delta_c X_c = \log\left(1 + e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1\right). \tag{2.21}$$

Then let $K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$ and then we taylor expand $\log(K+1)$.

Expanding $K$ to third order

$$\begin{aligned} K =& e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1 && (2.22)\\ =&(1 + i\alpha_a X_a - \frac{1}{2}(\alpha_a X_a)^2 - \frac{i}{6}(\alpha_a X_a)^3) + (1 + i\beta_b X_b - \frac{1}{2}(\beta_b X_b)^2 - \frac{i}{6}(\beta_b X_b)^3) - 1 && (2.23)\\ =& -\alpha_a\beta_b X_a X_b - \frac{i}{2}\alpha_a\beta_b^2 X_a X_b^2 + \frac{i}{2}\alpha_a^2\beta_b X_a^2 X_b + i\alpha_a X_a - \frac{1}{2}\alpha_a^2 X_a^2 \\ &- \frac{i}{6}\alpha_a^3 X_a^3 + i\beta_b X_b - \frac{1}{2}\beta_b^2 X_b^2 - \frac{i}{6}\beta_b^3 X_b^3 + \ldots && (2.24) \end{aligned}$$

Then, to third order

$$\begin{aligned} i\delta_c X_c =& \log(K+1) = K - \frac{1}{2}KK + \frac{1}{3}KKK + \ldots && (2.25)\\ =& i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}\alpha_a\beta_b[X_a, X_b] + \frac{i}{6}\alpha_a^2\beta_b\left(X_a X_b X_a - \frac{1}{2}X_b X_a^2 - \frac{1}{2}X_a^2 X_b\right) \\ &+ \frac{i}{6}\alpha_a\beta_b^2\left(X_b X_a X_b - \frac{1}{2}X_b^2 X_a - \frac{1}{2}X_a X_b^2\right) + \ldots && (2.26)\\ =& i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}\alpha_a\beta_b[X_a, X_b] - \frac{i}{12}\alpha_a^2\beta_b[X_a, [X_a, X_b]] - \frac{i}{12}\alpha_a\beta_b^2[X_b, [X_b, X_a]] + \ldots && (2.27) \end{aligned}$$

# 3 Chapter 3 - $\mathfrak{su}(2)$

## 3.A Use the highest weight decomposition to show $\{j\} \otimes \{s\} \sum_{\oplus l=|s-j|}^{s+j} \{l\}$

$\{j\}\{s\}$ are two spin irreps over a vector spaces $V_j$ and $V_s$ respectively, then the tensor product $\{j\} \otimes \{s\}$ acts on the vector space $V = V_j \otimes V_s$, but the representation $\{j\} \otimes \{s\} = \{j\} \otimes \mathbb{I} + \mathbb{I} \otimes \{s\}$ is not necessarily, and in general is not, an irrep on $V$. We label spin $j$ states on $V_j$ as $|j, m_j\rangle$ and spin $s$ states on $V_s$ as $|s, m_s\rangle$ with $j, s$ the highest weight states, s.t.

$$J_3^{\{j\}\otimes\{s\}} \left(|j, m_j\rangle \otimes |s, m_s\rangle\right) = (m_j + m_s) \left(|j, m_j\rangle \otimes |s, m_s\rangle\right). \tag{3.1}$$

and

$$J_3^{\{j\}} |j, m_j\rangle = m_j |j, m_j\rangle \tag{3.2}$$

$$J_3^{\{s\}} |s, m_s\rangle = m_s |s, m_s\rangle. \tag{3.3}$$

The tensor product space has a unique highest weight state

$$|j, j\rangle \otimes |s, s\rangle \equiv |j + s, j + s\rangle. \tag{3.4}$$

The main point is that in subspace of $V$, we denote this subspace by $V_{j+s} \subset V$ where the tensor product representation is an irrep. Then we may write the representation on $V$ as a direct sum of irreps $D_1 \oplus D_2 \oplus ....$

$$V = V_1 \oplus V_{j+s} \tag{3.5}$$

Now, $V_1 \subset V$ can be further decomposed. There are two states of highest weight $j + s - 1$, either $|j, j-1\rangle \otimes |s, s\rangle$ or $|j, j\rangle \otimes |s, s-1\rangle$. One of these hgihest weight states lives in the space $V_1$ while the other lives in another subspace where this tensor product is an irrep similarly this subspace is denoted $V_{j+s-1}$. So,

$$V = V_2 \oplus V_{j+s-1} \oplus V_{j+s} \tag{3.6}$$

We may keep decomposing in this fashion, for the highest weight state $|j + s, j + s - k\rangle$ with $k \leq 2s$, from which point the tensor product space reduces to a spin state. We have $k + 1$ orthogonal subspaces and 1 of these subspaces the tensor product will be an irrep thus we may construct

$$V = V_{|j-s|} \oplus V_{|j-s|+1} \oplus ... \oplus V_{j+s} \tag{3.7}$$

In which each subspace transforms irreducibly under the spin $|j - s|, |j - s| + 1, ..., j + s$ representation respectively. Thus

$$\{j\} \otimes \{s\} = \{|j - s|\} \oplus \{|j - s| + 1\} \oplus ... \oplus \{j + s\} \tag{3.8}$$

$$= \sum_{\oplus l=|s-j|}^{s+j} \{l\} \tag{3.9}$$

### 3.B Calculate $\exp(\mathrm{i}\underline{r}\cdot\underline{\sigma})$

$$\mathrm{e}^{\mathrm{i}\underline{r}.\underline{\sigma}} = \mathrm{e}^{\mathrm{i}|\underline{r}|\hat{\underline{r}}.\underline{\sigma}} \tag{3.10}$$

$$= \sum_{n=0}^{n=\infty} \frac{1}{n!}(\mathrm{i}|\underline{r}|)^n(\hat{\underline{r}}.\underline{\sigma})^n \tag{3.11}$$

Pauli matrices satisfy

$$\sigma_i^n = \begin{cases} \sigma_i & \text{if } n \text{ odd} \\ \mathbb{I} & \text{if } n \text{ even} \end{cases}, \tag{3.12}$$

thus the sum may be expressed as a sum of even & odd terms.

$$\sum_{n=0}^{n=\infty} \frac{(\mathrm{i}|\underline{r}|)^n(\hat{\underline{r}}.\underline{\sigma})^n}{n!} = \sum_{n=0}^{n=\infty} \frac{(\mathrm{i}|\underline{r}|)^{2n}(\hat{\underline{r}}.\underline{\sigma})^{2n}}{(2n)!} + \sum_{n=0}^{n=\infty} \frac{(\mathrm{i}|\underline{r}|)^{2n+1}(\hat{\underline{r}}.\underline{\sigma})^{2n+1}}{(2n+1)!} \tag{3.13}$$

$$= \sum_{n=0}^{n=\infty} \frac{(-1)^n|\underline{r}|^{2n}}{(2n)!}\mathbb{I} + \mathrm{i}\sum_{n=0}^{n=\infty} \frac{(-1)^n|\underline{r}|^{2n+1}}{(2n+1)!}(\hat{\underline{r}}.\underline{\sigma}) \tag{3.14}$$

$$= \cos(|\underline{r}|)\mathbb{I} + \mathrm{i}\sin(|\underline{r}|)\hat{\underline{r}}.\underline{\sigma} \tag{3.15}$$

### 3.C Show that the spin 1 rep is equivalent to the adjoint rep with $f_{abc} = \epsilon_{abc}$

The spin 1 rep of $\mathfrak{su}(2)$ is given by the 3 matrices

$$J_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad J_3^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.16}$$

The adjoint representation is generated by the structure constants themselves $[T_a^{adj}]_{bc} = -\mathrm{i}f_{abc} = -\mathrm{i}\epsilon_{abc}$. So,

$$T_1^{adj} = -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2^{adj} = -\mathrm{i}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3^{adj} = -\mathrm{i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

If the two representations are equivalent then $\exists S$ s. t. $S : J_a^1 \to T_a^{adj}$. Where $S$ acts "adjointly" on $T_a^{adj}$, i.e. find a matrix $S$ s. t. $J_a^1 = S^{-1}T_a^{adj}S$. Let

$$S = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}. \tag{3.18}$$

Then

$$SJ_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} b & a+c & b \\ e & d+f & e \\ h & g+i & h \end{pmatrix}. \tag{3.19}$$

$$T_1^{adj}S = -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ g & h & i \\ -d & -e & -f \end{pmatrix}. \tag{3.20}$$

Solving the algebra we get

$$S = \begin{pmatrix} a & 0 & -a \\ d & e & d \\ \frac{\mathrm{i}}{\sqrt{2}}e & \mathrm{i}\sqrt{2}d & \frac{\mathrm{i}}{\sqrt{2}e} \end{pmatrix}. \tag{3.21}$$

Then

$$SJ_1^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i}2a & 0 \\ \mathrm{i}e & 0 & -\mathrm{i}e \\ -\sqrt{2}d & 0 & \sqrt{2}d \end{pmatrix}. \tag{3.22}$$

$$T_2^{adj} S = -\mathrm{i} \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{2}}e & -\mathrm{i}\sqrt{2}d & -\frac{\mathrm{i}}{\sqrt{2}}e \\ 0 & 0 & 0 \\ a & 0 & -a \end{pmatrix}. \tag{3.23}$$

Which by comparing and setting $a = 1$ w.l.o.g. gives us a final form for $S$

$$S = \begin{pmatrix} 1 & 0 & -1 \\ \mathrm{i} & 0 & \mathrm{i} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}. \tag{3.24}$$

Which one may verify that an $S$ of this form satisfies $J_a^1 = S^{-1} T_a^{adj} S \forall a$.

### 3.D Write out the matrix elements of $\sigma_2 \otimes \eta_1$.

The formula we need is given by equation (1.105) in Georgi.

$$[D_{D_1 \otimes D_2(g)}]_{jxky} = \langle j, x| D_{D_1 \otimes D_2(g)} |k, y\rangle \equiv \langle j| D_1(g) |k\rangle \langle x| D_2(g) |y\rangle \tag{3.25}$$

We are given a basis

$$|1\rangle = |i=1\rangle |x=1\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = |i=1\rangle |x=2\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{3.26}$$

$$|3\rangle = |i=2\rangle |x=1\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |4\rangle = |i=2\rangle |x=2\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.27}$$

The non-vanishing matrix elements are

$$[\sigma_2 \otimes \eta_1]_{14} = \langle 1| \sigma_2 |2\rangle \langle 1| \eta_1 |2\rangle = -\mathrm{i} \tag{3.28}$$

$$[\sigma_2 \otimes \eta_1]_{23} = \langle 1| \sigma_2 |2\rangle \langle 2| \eta_1 |1\rangle = -\mathrm{i} \tag{3.29}$$

$$[\sigma_2 \otimes \eta_1]_{32} = \langle 2| \sigma_2 |1\rangle \langle 1| \eta_1 |2\rangle = \mathrm{i} \tag{3.30}$$

$$[\sigma_2 \otimes \eta_1]_{41} = \langle 2| \sigma_2 |1\rangle \langle 2| \eta_1 |1\rangle = \mathrm{i}. \tag{3.31}$$

Which gives the matrix

$$\sigma_2 \otimes \eta_1 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{3.32}$$

Which, in this basis, is completely equivalent to just taking the tensor product of the two matrices in the usual way.

$$\sigma_2 \otimes \eta_1 = \begin{pmatrix} 0 & -\mathrm{i}\eta_1 \\ \mathrm{i}\eta_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}. \tag{3.33}$$

### 3.E Tensor product notation

#### 3.E.a $[\sigma_a, \sigma_b\eta_c]$

$$[\sigma_a, \sigma_b\eta_c] = \sigma_a\sigma_b \otimes \eta_c - \sigma_b\sigma_a \otimes \eta_c \tag{3.34}$$

$$= [\sigma_a, \sigma_b] \otimes \eta_c \tag{3.35}$$

$$= 2\mathrm{i}\epsilon_{abd}\sigma_d \otimes \eta_c. \tag{3.36}$$

#### 3.E.b $\mathrm{Tr}\left(\sigma_a\{\eta_b, \sigma_c\eta_d\}\right)$

$$\mathrm{Tr}\left[\sigma_a\{\eta_b, \sigma_c\eta_d\}\right] = \mathrm{Tr}\left[\sigma_a\sigma_c \otimes \{\eta_b, \eta_d\}\right] \tag{3.37}$$

$$= \mathrm{Tr}\left[(\delta_{ac}\mathbb{I} + \mathrm{i}\epsilon_{ace}\sigma_e) \otimes 2\delta_{bd}\mathbb{I}\right] \tag{3.38}$$

$$= 8\delta_{ac}\delta_{bd}. \tag{3.39}$$

Where we have used $\mathrm{Tr}\left[\mathbb{I} \otimes \mathbb{I}\right] = 4$ and $\mathrm{Tr}\left[\sigma_a\right] = 0$.

#### 3.E.c $[\sigma_1\eta_1, \sigma_2\eta_2]$

$$[\sigma_1\eta_1, \sigma_2\eta_2] = \sigma_1\sigma_2 \otimes \eta_1\eta_2 - \sigma_2\sigma_1 \otimes \eta_2\eta_1 \tag{3.40}$$

$$= \mathrm{i}\sigma_3 \otimes \mathrm{i}\eta_3 - \mathrm{i}\sigma_3 \otimes \mathrm{i}\eta_3 \tag{3.41}$$

$$= 0. \tag{3.42}$$

## 4 Chapter 4 - Tensor Operators

### 4.A Calculate $\langle 3/2, -3/2, \alpha| O_{-1/2} |1, -1, \beta\rangle$ given $\langle 3/2, -1/2, \alpha| O_{1/2} |1, -1, \beta\rangle = A$

The operator $O_x$ transforms in the spin 1/2 representation

$$[J_a, O_x] = \frac{1}{2} O_y [\sigma_a]_{yx} \tag{4.1}$$

With $x, y$ running from 1/2 to $-1/2$. So $[J_3, O_{1/2}] = \frac{1}{2}O_{1/2}$ and $[J_3, O_{-1/2}] = -\frac{1}{2}O_{-1/2}$. We also have $[J^{\pm}, O_{1/2}] = \frac{1}{2\sqrt{2}}(O_{-1/2} \mp O_{-1/2})$ and $[J^{\pm}, O_{-1/2}] = \frac{1}{2\sqrt{2}}(O_{1/2} \pm O_{1/2})$. Thus we know that $O_{1/2}$ has highest weight with $+\frac{1}{2}$ $J_3$ value, thus the highest weight state of the $1 \otimes \frac{1}{2}$ tensor product space is

$$|3/2, 3/2\rangle \equiv O_{1/2} |1, 1, \beta\rangle . \tag{4.2}$$

We have

$$A = \langle 3/2, -1/2, \alpha| O_{1/2} |1, -1, \beta\rangle \tag{4.3}$$

$$= \sqrt{\frac{2}{3}} \langle 3/2, -3/2, \alpha| J^{-} O_{1/2} |1, -1, \beta\rangle \tag{4.4}$$

$$= \sqrt{\frac{2}{3}} \langle 3/2, -3/2, \alpha| [J^{-}, O_{1/2}] |1, -1, \beta\rangle \tag{4.5}$$

$$= \sqrt{\frac{2}{3}} \langle 3/2, -3/2, \alpha| \frac{1}{\sqrt{2}} O_{-1/2} |1, -1, \beta\rangle . \tag{4.6}$$

Therefore,

$$\langle 3/2, -3/2, \alpha| O_{-1/2} |1, -1, \beta\rangle = \sqrt{3}A. \tag{4.7}$$

### 4.B Construct the components $O_m$.

We have $[L^+, (r_{+1})^2] = 0$ and is therefore the $O_{+2}$ component of a spin 2 operator. The other components are given by applying the lowering operators. We define $O_{j,j} = (r_{+1})^j$ so $(r_{+1})^2 \equiv O_{2,2}$ and $[L^-, O_{j,m}] = \sqrt{(j+m)(j-m+1)/2}\,O_{j,m-1}$. Therefore,

$$O_{2,2} = (r_{+1})^2 \tag{4.8}$$

$$O_{2,1} = \frac{1}{\sqrt{2}}[L^-, O_{2,2}] = \frac{1}{\sqrt{2}}([L^-, r_{+1}]r_{+1} + r_{+1}[L^-, r_{+1}]) = \frac{1}{\sqrt{2}}(r_0 r_{+1} + r_{+1} r_0) \tag{4.9}$$

$$O_{2,0} = \frac{1}{\sqrt{3}}[L^-, O_{2,1}] = \frac{1}{\sqrt{6}}[L^-, (r_0 r_{+1} + r_{+1} r_0)] = \frac{1}{\sqrt{6}}(r_{-1} r_{+1} + 2(r_0)^2 + r_{+1} r_{-1}) \tag{4.10}$$

$$O_{2,-1} = \frac{1}{\sqrt{3}}[L^-, O_{2,0}] = \frac{1}{\sqrt{18}}[L^-, (r_{-1} r_{+1} + 2(r_0)^2 + r_{+1} r_{-1})] = \frac{1}{\sqrt{2}}(r_{-1} r_0 + r_0 r_{-1}) \tag{4.11}$$

$$O_{2,-2} = \frac{1}{\sqrt{2}}[L^-, O_{2,-1}] = \frac{1}{2}[L^-, (r_{-1} r_0 + r_0 r_{-1})] = (r_{-1})^2. \tag{4.12}$$

Letting $r_1 = \sin\theta\cos\phi$, $r_2 = \sin\theta\sin\phi$, $r_3 = \cos\theta$ so that $r_0 = r_3 = \cos\theta$ and $r_{\pm 1} = \mp(r_1 \pm \mathrm{i} r_2)\sqrt{2} = \mp\frac{\sin\theta}{\sqrt{2}}(\cos\phi \pm \mathrm{i}\sin\phi) = \mp\frac{\sin\theta}{\sqrt{2}}\,\mathrm{e}^{\pm\mathrm{i}\phi}$.

Then, looking up the spherical harmonics in a book [3] we see that

$$r_{1,m} = \sqrt{\frac{4\pi}{3}} Y_{1,m}(\theta, \phi) \tag{4.13}$$

and also

$$O_{2,m} = \sqrt{\frac{8\pi}{15}} Y_{2,m}(\theta, \phi). \tag{4.14}$$

### 4.C Calculate $e^{i\alpha_a X_a^1}$

$$X_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad X_1^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}, \quad X_1^3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \tag{4.15}$$

We write $\alpha_a X_a^1 = \alpha\hat{\alpha}_a X_a^1$ with $\alpha = \sqrt{\alpha_a\alpha_a}$ and $\hat{\alpha}_a\hat{\alpha}_a = 1$. We are also told that $(\hat{\alpha}_a X_a^1)^2$ is a projection operator, i.e. $(\hat{\alpha}_a X_a^1)^{2n} = (\hat{\alpha}_a X_a^1)^2 \,\forall\, n \in \mathbb{Z}$, and therefore $(\hat{\alpha}_a X_a^1)^{2n+1} = \hat{\alpha}_a X_a^1$. Thus, we may write the expansion

$$\begin{aligned}
e^{i\alpha_a X_a^1} &= \sum_{n=0}^{n=\infty} \frac{(i\alpha_a X_a^1)^n}{n!} && (4.16)\\
&= \sum_{n=0}^{n=\infty} \frac{(i\alpha_a X_a^1)^{2n}}{(2n)!} + \sum_{n=0}^{n=\infty} \frac{(i\alpha_a X_a^1)^{2n+1}}{(2n+1)!} && (4.17)\\
&= \sum_{n=0}^{n=\infty} \frac{(-1)^n(\alpha)^{2n}}{(2n)!}(\hat{\alpha}_a X_a^1)^2 + i\sum_{n=0}^{n=\infty} \frac{(-1)^n(\alpha)^{2n+1}}{(2n+1)!}\hat{\alpha}_a X_a^1 && (4.18)\\
&= (\hat{\alpha}_a X_a^1)^2 \cos\alpha + i(\hat{\alpha}_a X_a^1)\sin\alpha. && (4.19)
\end{aligned}$$

## 5 Chapter 5 - Isospin

### 5.A $X \to \pi^i\pi^i$ production.

We know that in terms of isospin $|\pi\pi\rangle = 1\otimes 1 = 0\oplus 1\oplus 2$ and the pions are spin-0 singlets made up of a quark-antiquark pair. The isospin 0 and 2 final states are anti-symmetric in the exchange of flavour this means that the spins are anti-aligned resulting in a zero angular momentum final state. The isospin 1 final state is symmetric under the exchange in flavour and so the spins are aligned so the final state has a non-zero angular momentum. Thus the possible final isospin states are $0, 2$ states.

### 5.B Show that the operators $T_a = \sum_{x,\alpha,m,m'} a^\dagger_{x,m,\alpha}[J_a^{j_x}]_{mm'}a_{x,m',\alpha}$ have the commutation relations of isospin generators

$$\begin{aligned}
[T_a, T_b] &= \sum_{x,\alpha,m,m',y,\beta,n,n'} [a^\dagger_{x,m,\alpha}[J_a^{j_x}]_{mm'}a_{x,m',\alpha}, a^\dagger_{y,n,\beta}[J_b^{j_y}]_{nn'}a_{y,n',\beta}] && (5.1)\\
&= \sum_{x,\alpha,m,m',y,\beta,n,n'} [J_a^{j_x}]_{mm'}[J_b^{j_y}]_{nn'}(a^\dagger_{x,m,\alpha}[a_{x,m',\alpha}, a^\dagger_{y,n,\beta}a_{y,n',\beta}]\\
&\quad - [a^\dagger_{y,n,\beta}a_{y,n',\beta}, a^\dagger_{x,m,\alpha}]a_{x,m',\alpha}). && (5.2)
\end{aligned}$$

Applying the commutation relations for bosons(-) and fermions(+) $[a_{x,m,\alpha}, a^\dagger_{x',m',\alpha'}]_\pm = \delta_{mm'}\delta_{\alpha\alpha'}\delta_{xx'}$ with all others vanishing. Then for bosons we get

$$
\begin{aligned}
[T_a, T_b] = &\sum_{x,\alpha,m,m',y,\beta,n,n'} [J_a^{j_x}]_{mm'}[J_b^{j_y}]_{nn'}(a^\dagger_{x,m,\alpha}\Big([a_{x,m',\alpha}, a^\dagger_{y,n,\beta}]a_{y,n',\beta} + a^\dagger_{y,n,\beta}[a_{x,m',\alpha}, a_{y,n,\beta}]) \\
&- (a^\dagger_{yn\beta}[a_{yn'\beta}, a^\dagger_{xm\alpha}] + [a^\dagger_{yn\beta}, a^\dagger_{xm\alpha}]a_{yn'\beta})a_{xm'\alpha}\Big) && (5.3) \\
= &\sum_{x,\alpha,m,m',y,\beta,n,n'} [J_a^{j_x}]_{mm'}[J_b^{j_y}]_{nn'}(a^\dagger_{x,m,\alpha}a_{yn'\beta}\delta_{xy}\delta_{m'n}\delta_{\alpha\beta} - a^\dagger_{y,n,\beta}a_{xm'\alpha}\delta_{xy}\delta_{mn'}\delta_{\alpha\beta}) && (5.4)
\end{aligned}
$$

rearranging and relabelling (5.4) we have

$$
\begin{aligned}
[T_a, T_b] &= \sum_{\alpha,m,y,n,n'} (a^\dagger_{y,m,\alpha}a_{yn'\alpha}[J_a^{j_y}]_{mn}[J_b^{j_y}]_{nn'} - a^\dagger_{y,m,\alpha}a_{yn'\alpha}[J_b^{j_y}]_{mn}[J_a^{j_y}]_{nn'}) && (5.5) \\
&= \sum_{\alpha,m,y,n,n'} a^\dagger_{y,m,\alpha}[J_a^{j_y}, J_b^{j_y}]_{mn'}a_{yn'\alpha} && (5.6) \\
&= \sum_{\alpha,m,y,n,n'} a^\dagger_{y,m,\alpha}[J_c^{j_y}]_{mn'}a_{yn'\alpha} && (5.7) \\
&= T_c. && (5.8)
\end{aligned}
$$

Which is indeed the commutation relation of isospin generators, and by making use of the identity $[AB, C] = A\{B, C\} - \{A, C\}B$ the same relation can also be shown to be true for fermions.

### 5.C Decay probabilities

We have $\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$ which live in the spin 3/2 isospin representation with third component of isospin $I_3 = 3/2, 1/2, -1/2, -3/2$ respectively. We denote nucleons by $|P\rangle = |1/2, 1/2\rangle$ and $|N\rangle = |1/2, -1/2\rangle$ and the pions $|\pi^+\rangle = |1, 1\rangle$, $|\pi^0\rangle = |1, 0\rangle$, $|\pi^-\rangle = |1, -1\rangle$. We decompose the $1/2 \otimes 1 = 3/2 \oplus 1/2$ states and can immediately see the probability of $\pi^+P \to \Delta^{++}$

$$
|P\pi^+\rangle = |1/2, 1/2\rangle |1, 1\rangle = |3/2, 3/2\rangle. \tag{5.9}
$$

Therefore

$$
\begin{aligned}
\mathrm{Prob}(P\pi^+ \to \Delta^{++}) &= |\langle \Delta^{++}|P\pi^+\rangle|^2 && (5.10) \\
&= |\langle 3/2, 3/2|3/2, 3/2\rangle|^2 && (5.11) \\
&= 1. && (5.12)
\end{aligned}
$$

To obtain the probability of $\pi^-P \to \Delta^0$ we apply lowering operators to the highest weight spin 3/2 state (5.9)

$$
\begin{aligned}
|3/2, 1/2\rangle &= \sqrt{\frac{2}{3}}J^- |3/2, 3/2\rangle && (5.13) \\
&= \sqrt{\frac{2}{3}}J^-(|1, 1\rangle |1/2, 1/2\rangle) && (5.14) \\
&= \sqrt{\frac{2}{3}}(|1, 0\rangle |1/2, 1/2\rangle + \frac{1}{\sqrt{2}} |1, 1\rangle |1/2, -1/2\rangle). && (5.15)
\end{aligned}
$$

Then

$$\left|\Delta^0\right\rangle = |3/2, -1/2\rangle \tag{5.16}$$
$$= \frac{1}{\sqrt{2}} J^- |3/2, 1/2\rangle \tag{5.17}$$
$$= \frac{1}{\sqrt{3}} |1, -1\rangle |1/2, 1/2\rangle + \sqrt{\frac{2}{3}} |1, 0\rangle |1/2, -1/2\rangle \tag{5.18}$$

and the orthogonal $s = -1/2$ state living in the spin 1/2 rep is given by $|v\rangle$ s. t. $\langle 3/2, -1/2|v\rangle = 0$. Thus,

$$|1/2, -1/2\rangle = \frac{1}{\sqrt{3}} |1, 0\rangle |1/2, -1/2\rangle - \sqrt{\frac{2}{3}} |1, -1\rangle |1/2, 1/2\rangle . \tag{5.19}$$

Which gives the full

$$\left|P\pi^-\right\rangle = |1/2, 1/2\rangle |1, -1\rangle \tag{5.20}$$
$$= \frac{1}{\sqrt{3}} |3/2, -1/2\rangle - \sqrt{\frac{2}{3}} |1/2, -1/2\rangle . \tag{5.21}$$

Now we can obtain the probability

$$\text{Prob}(P\pi^- \to \Delta^0) = |\left\langle \Delta^0|P\pi^-\right\rangle|^2 \tag{5.22}$$
$$= \left| \frac{1}{\sqrt{3}} \langle 3/2, -1/2|3/2, -1/2\rangle - \sqrt{\frac{2}{3}} \langle 3/2, -1/2|1/2, -1/2\rangle \right|^2 \tag{5.23}$$
$$= \frac{1}{3} \tag{5.24}$$

# 6 Chapter 6 - Roots & Weights

## 6.A Show that $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$

To show this we make use of the Jacobi identity

$$[H_i, [E_\alpha, E_\beta]] = -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] \tag{6.1}$$
$$= [E_\alpha, \beta_i E_\beta] + [\alpha_i E_\alpha, E_\beta] \tag{6.2}$$
$$= (\alpha_i + \beta_i)[E_\alpha, E_\beta]. \tag{6.3}$$

Also

$$[H_i, E_{\alpha+\beta}] = (\alpha_i + \beta_i) E_{\alpha+\beta}. \tag{6.4}$$

This implies $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$. Furthermore if $(\alpha_i + \beta_i)$ is not a root then $[E_\alpha, E_\beta] = 0$ and consequently $E_{\alpha+\beta} = 0$.

### 6.B Calculate $[E_\alpha, E_{-\alpha-\beta}]$

We have $[E_\alpha, E_\beta] = NE_{\alpha+\beta}$. Taking the h.c. we get $[E_{-\alpha}, E_{-\beta}] = NE_{-\alpha-\beta}$. So,

$$\begin{aligned}
[E_\alpha, E_{-\alpha-\beta}] &= \frac{1}{N}[E_\alpha, [E_{-\alpha}, E_{-\beta}]] && (6.5)\\
&= -\frac{1}{N}\left([E_{-\alpha}, [E_\alpha, E_{-\beta}]] + [E_\beta, [E_\alpha, E_{-\alpha}]]\right) && (6.6)\\
&= \frac{1}{N}[[E_\alpha, E_{-\alpha}], E_\beta] && (6.7)\\
&= \frac{1}{N}[\alpha.H, E_\beta] && (6.8)\\
&= -\frac{1}{N}\alpha.\beta E_{-\beta}, && (6.9)
\end{aligned}$$

where we have used the fact that $[E_\alpha, E_{-\beta}] = 0$ because $\alpha - \beta$ is not a root.

### 6.C Find the weights in the defining representation

The simple lie algebra generated by $\sigma_a \otimes \mathbb{I}$, $\sigma_a \otimes \tau_1$, $\sigma_a \otimes \tau_3$, $\mathbb{I} \otimes \tau_2$. We take the Cartan subalgebra as $H_1 = \sigma_3 \otimes \mathbb{I}$, $H_2 = \sigma_3 \otimes \tau_3$.

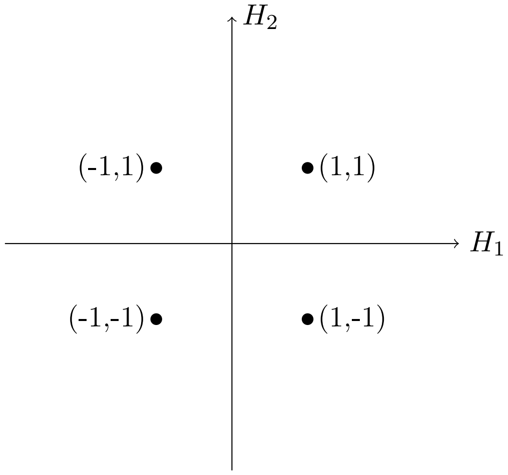
$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.10}$$

#### 6.C.a Weights in the defining representation

The eigenvalues of $H_1$ & $H_2$ are $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$. Calculating the eigenvectors $H_k V_i = \lambda_{ik} V_i$. Then the eigenvectors with their associated weights are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to (1,1) \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to (-1,1) \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to (1,-1) \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \to (-1,-1) \tag{6.11}$$

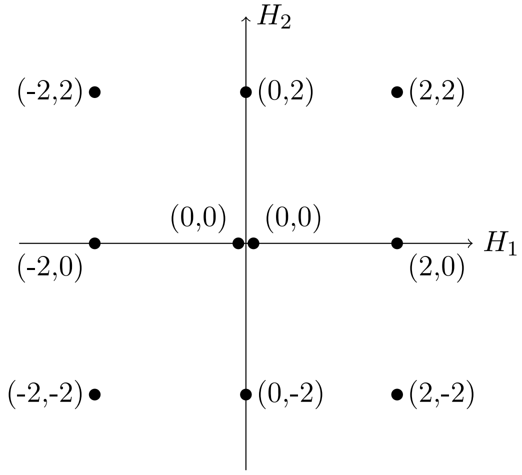
Plotting the weight diagram



### 6.C.b Weights of the adjoint representation

The weights of the adjoint representation are given by the the roots of the algebra which we can deduce by looking at the difference in weights of the defining representation from part (a). The raising and lowering operators take us between the weights in part (a), they are: $E_{\pm 2,0}$, $E_{\pm 2,\mp 2}$, $E_{\pm 2,\pm 2}$, $E_{0,\pm 2}$.

Plotting the roots on the weight diagram




## 7 Chapter 7 - $\mathfrak{su}(3)$

### 7.A Calculate $f_{147}$ & $f_{458}$ in $\mathfrak{su}(3)$

$T_a = \frac{\lambda_a}{2}$ and $[T_a, T_b] = \mathrm{i}F_{abc}T_c$.

$$[T_1, T_4] = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \mathrm{i}f_{14c}T_c \tag{7.1}$$

Inspecting the other generators we see that $f_{14c} = 0$ for $c \neq 6, 7$, so

$$[T_1, T_4] = \mathrm{i}f_{146}T_6 + \mathrm{i}f_{147}T_7 \tag{7.2}$$

$$= \frac{1}{2}\mathrm{i}f_{146}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{2}\mathrm{i}f_{147}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}. \tag{7.3}$$

This implies that $f_{146} = 0$ and $f_{147} = \frac{1}{2}$ because the structure constants are real. Because in some basis $\{e_i\}$ with the basis commutator $[e_i, e_j] = c_{ij}^k e_k$, and since a representation $D$ maps a Lie algebra to a set of linear operators $D : \mathcal{L} \to \{\text{Linear ops. on V}\}$ with Lie bracket on the generators $X_a$

$$
\begin{aligned}
[X_a, X_b] &= [D(e_a), D(x_b)] && (7.4)\\
&= D([e_a, e_b]) && (7.5)\\
&= D(c_{ab}^k e_k) && (7.6)\\
&= c_{ab}^k X_k && (7.7)\\
&\implies f_{abc}\text{'s real.} && (7.8)
\end{aligned}
$$

$$
[T_4, T_5] = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathrm{i} f_{45c} T_c. \tag{7.9}
$$

So $f_{45c} = 0$ for $c \neq 3, 8$. Therefore

$$
\begin{aligned}
[T_4, T_5] &= \mathrm{i} f_{453} T_3 + \mathrm{i} f_{458} T_8 && (7.10)\\
&= \frac{1}{2}\mathrm{i} f_{453}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2\sqrt{3}}\mathrm{i} f_{458}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. && (7.11)
\end{aligned}
$$

Therefore $f_{453} = \frac{1}{\sqrt{3}} f_{458} = \frac{1}{2}$. Thus $f_{458} = \frac{\sqrt{3}}{2}$

### 7.B Show $T_1, T_2, T_3$ generate an $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(3)$. How does the representation generated by the Gell-Mann matrices transform under the algebra?

To show this we must show that $[T_a, T_b] = \mathrm{i}\epsilon_{abc} T_c$, $a, b, c = 1..3$ with $T_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}$

$$
\begin{aligned}
[T_a, T_b] &= \frac{1}{2}\begin{pmatrix} [\sigma_a, \sigma_b] & 0 \\ 0 & 0 \end{pmatrix} && (7.12)\\
&= \mathrm{i}\epsilon_{abc} T_c && (7.13)
\end{aligned}
$$

Thus $T_1, T_2, T_3$ does indeed generate an $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(3)$. Next we calculate the weights of the subalgebra. Take the Cartan subalgebra to be $T_3$ which has eigenvalues

$\Lambda = 0, \pm\frac{1}{2}$, then the associated weights are

$$\Lambda_{\frac{1}{2}} = \frac{1}{2} \quad V_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.14}$$

$$\Lambda_0 = 0 \quad V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.15}$$

$$\Lambda_{-\frac{1}{2}} = -\frac{1}{2} \quad V_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{7.16}$$

$$\tag{7.17}$$

To determine whether the $\mathfrak{su}(3)$ representation is an irrep under the $\mathfrak{su}(2)$ subalgebra we look for invariant subspaces by applying the raising and lowering operators $T^{\pm} \equiv (T_1 \pm \mathrm{i}T_2)$.

$$T^+ V_{\frac{1}{2}} = 0 \quad \implies V_{\frac{1}{2}} \quad \text{highest weight state} \tag{7.18}$$

$$T^- V_{\frac{1}{2}} = V_{-\frac{1}{2}} \tag{7.19}$$

$$T^+ V_{-\frac{1}{2}} = V_{\frac{1}{2}} \tag{7.20}$$

$$T^- V_{-\frac{1}{2}} = 0 \tag{7.21}$$

$$T^{\pm} V_0 = 0. \tag{7.22}$$

Thus the $\mathfrak{su}(2)$ subalgebra is not irreducible on the $\mathfrak{su}(3)$ representation generated by the Gell-Mann matrices and the subalgebra decomposes into $3 = 2 \oplus 1$.

### 7.C Show $\lambda_2, \lambda_5, \lambda_7$ generate an $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(3)$. How does the representation generated by the Gell-Mann matrices transform under the algebra?

We can again easily show that $[\lambda_a, \lambda_b] = \mathrm{i}\epsilon_{abc}\lambda_c,\ a, b, c = 2, 5, 7$ and thus generate an $\mathfrak{su}(2)$ subalgebra. We pick $\lambda_2$ as the Cartan subalgebra and $\lambda^{\pm} \equiv \frac{1}{\sqrt{2}}(\lambda_5 \pm \mathrm{i}\lambda_7)$ so that $[\lambda_2, \lambda^{\pm}] = \pm\lambda^{\pm}$. The eigenvalues are given by $\Lambda = 0, \pm 1$. The associated weights are

$$\Lambda_1 = 1 \quad V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix} \tag{7.23}$$

$$\Lambda_0 = 0 \quad V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.24}$$

$$\Lambda_{-1} = -1 \quad V_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} \\ -1 \\ 0 \end{pmatrix}. \tag{7.25}$$

Applying the raising and lowering operators

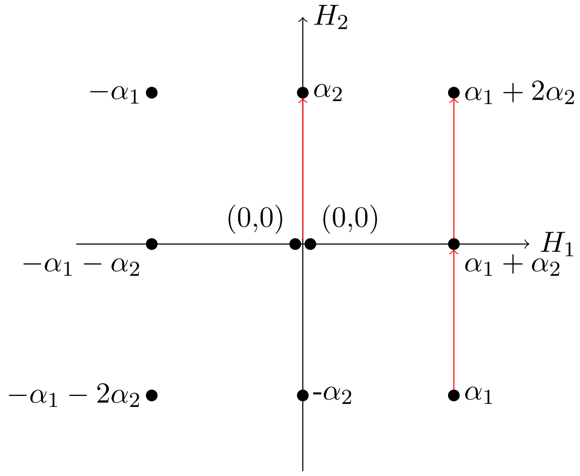
$$\lambda^+ V_1 = 0 \quad \implies V_1 \quad \text{highest weight state} \tag{7.26}$$
$$\lambda^- V_1 = V_0 \tag{7.27}$$
$$\lambda^+ V_0 = V_1 \tag{7.28}$$
$$\lambda^- V_0 = V_{-1} \tag{7.29}$$
$$\lambda^+ V_{-1} = V_0 \tag{7.30}$$
$$\lambda^- V_{-1} = 0. \tag{7.31}$$

Thus there are no invariant subspaces $\implies$ therefore the $\mathfrak{su}(2)$ subalgebra generated by $\lambda_2, \lambda_5, \lambda_7$ acts irreducibly on the Gell-Mann representation.

# 8 Chapter 8 - Simple Roots

## 8.A Find the simple roots, fundamental weights and the Dynkin diagram for the algebra in part (6.C)

From part (6.C.b) we can obtain the simple roots by picking the two lowest roots. These are $\alpha_1 = (2, -2)$ and $\alpha_2 = (0, 2)$, then plotting the roots in terms of the simple roots




The fundamental roots are given by

$$\frac{2\alpha^j.\mu^i}{{\alpha^j}^2} = \delta^{ij}. \tag{8.1}$$

Letting $\mu^i = (a^i, b^i)$

$$\mu^i.\alpha^1 = 2a^i - 2b^i \tag{8.2}$$
$$\mu^i.\alpha^2 = 2b^i \tag{8.3}$$
$${\alpha^1}^2 = 8 \tag{8.4}$$
$${\alpha^2}^2 = 4. \tag{8.5}$$

So

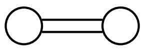
$$\frac{2\alpha^1.\mu^1}{8} = \frac{1}{2}(a^1 - b^1) = 1 \tag{8.6}$$
$$\frac{2\alpha^1.\mu^2}{8} = \frac{1}{2}(a^2 - b^2) = 0 \tag{8.7}$$
$$\frac{2\alpha^2.\mu^1}{4} = b^1 = 0 \tag{8.8}$$
$$\frac{2\alpha^2.\mu^2}{4} = b^2 = 1. \tag{8.9}$$

Therefore, $\mu^1 = (2, 0)$, $\mu^2 = (1, 1)$.

For the Dynkin diagram

$$\cos(\theta_{\alpha^1,\alpha^2}) = -\frac{\sqrt{p_1 p_2}}{2} \tag{8.10}$$
$$= \sqrt{\frac{(\alpha^1.\alpha^2)^2}{{\alpha^2}^2 {\alpha^1}^2}} \tag{8.11}$$
$$= -\frac{1}{\sqrt{2}}. \tag{8.12}$$

Then $\theta_{\alpha^1,\alpha^2} = \frac{3\pi}{4}$. So the Dynkin diagram is



Which is $C_2$ or $B_2$, $\mathfrak{so}(5) \cong \mathfrak{sp}(4)$.

## 8.B Show the algebra generated by $\sigma_a \otimes \mathbb{I}$ and $\sigma_a \otimes \eta_1$ is semisimple. Draw it's Dynkin diagram.

We know that the simple roots of a simple Lie algebra are simply connected with atleast one other simple root in the root system and they obey $\alpha^i.\alpha^j < 0$. One way of showing that the algebra is not simple, but semi-simple is to find the simple roots and then show that $\alpha^i.\alpha^j \not< 0$. First we write out the commutation relations

$$[\sigma_a \otimes \mathbb{I}, \sigma_b \otimes \mathbb{I}] = 2\mathrm{i}\epsilon_{abc}\sigma_c \otimes \mathbb{I} \tag{8.13}$$
$$[\sigma_a \otimes \eta_1, \sigma_b \otimes \eta_1] = 2\mathrm{i}\epsilon_{abc}\sigma_c \otimes \mathbb{I} \tag{8.14}$$
$$[\sigma_a \otimes \mathbb{I}, \sigma_b \otimes \eta_1] = 2\mathrm{i}\epsilon_{abc}\sigma_c \otimes \eta_1. \tag{8.15}$$

Therefore we pick the Cartan subalgebra as

$$\sigma_3 \otimes \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma_3 \otimes \eta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{8.16}$$

Which have eigenvalues $\lambda = \pm 1$ and weights eigenvectors are

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mu_1 = (1, 1), \tag{8.17}$$

$$x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mu_2 = (-1, -1), \tag{8.18}$$

$$x_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \mu_3 = (1, -1), \tag{8.19}$$

$$x_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad \mu_4 = (-1, 1). \tag{8.20}$$

From which we may construct the simple roots $\alpha = (1, 1)$, $\beta = (1, -1)$ and then $\alpha.\beta = 0 \not< 0$ contrary to that of a simply connected Lie algebra. Thus the algebra is semi simple.

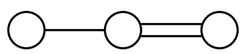
$$\cos(\theta_{\alpha,\beta}) = \sqrt{\frac{(\alpha.\beta)^2}{(\alpha)^2(\beta)^2}} = 0. \tag{8.21}$$

So $\theta_{\alpha,\beta} = \frac{\pi}{2}$. So the Dynkin diagram is

This algebra describes $A_1 \times A_1$ or $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

## 8.C Find the Cartan matrix and find the Dynkin coefficients

We are given the Dynkin diagram



with ${\alpha^1}^2 = {\alpha^2}^2$, ${\alpha^3}^2 = 1$.

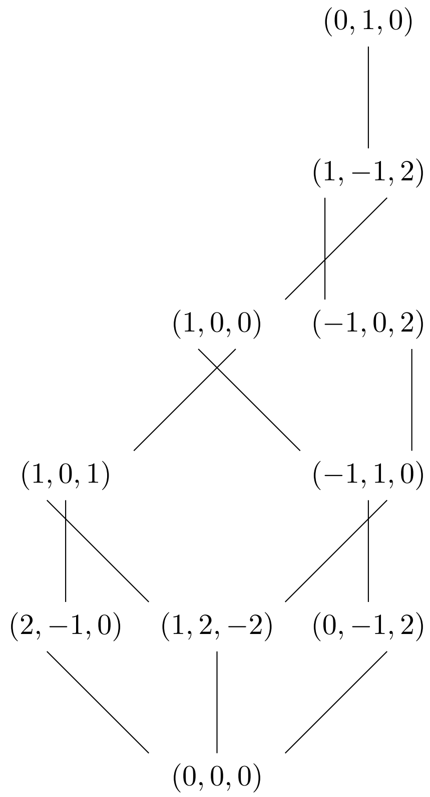
$$\cos(\theta_{\alpha_1\alpha_2}) = -\frac{1}{2} = -\sqrt{\frac{(\alpha_1.\alpha_2)^2}{(\alpha_1)^2(\alpha_2)^2}} \tag{8.22}$$

$$\cos(\theta_{\alpha_2\alpha_3}) = -\frac{1}{\sqrt{2}} = -\sqrt{\frac{(\alpha_2.\alpha_3)^2}{(\alpha_2)^2(\alpha_3)^2}}. \tag{8.23}$$

So $\alpha_1.\alpha_2 = \alpha_2.\alpha_3 = 1$. The Cartan matrix is

$$A_{ji} = \frac{2\alpha^j.\alpha^i}{{\alpha^i}^2} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \tag{8.24}$$

Which gives the Dynkin tree diagram




# 9 Chapter 9 - More $\mathfrak{su}(3)$

## 9.A With $|\mu\rangle = |\mu_1 + \mu_2\rangle$ the highest weight of the adjoint representation of $\mathfrak{su}(3)$ show that the states $|A\rangle = E_{-\alpha^1}E_{-\alpha^2}|\mu\rangle$ & $|B\rangle = E_{-\alpha^2}E_{-\alpha^1}|\mu\rangle$ are linearly independent

Consider two states $|x\rangle, |y\rangle$, $\exists a, b \neq 0\,\text{s.t.}\, a|x\rangle + b|y\rangle = 0 \iff |x\rangle, |y\rangle$ linearly dependent and we know $\frac{b^2}{a^2} = -1$ So $|x\rangle = -\frac{b}{a}|y\rangle$ and $|y\rangle = -\frac{a}{b}|x\rangle$. But then

$$\langle x|y\rangle . \langle y|x\rangle = -\frac{b}{a}\langle y|y\rangle . \left(-\frac{a}{b}\right)\langle x|x\rangle = \langle y|y\rangle . \langle x|x\rangle . \tag{9.1}$$

Therefore we wish to calculate $\langle A|A\rangle$, $\langle B|B\rangle$, $\langle B|A\rangle$ & $\langle A|B\rangle$. So, making use of $E_\alpha^\dagger = E_{-\alpha}$

$$\langle A|A\rangle = \langle \mu|E_{\alpha^2}E_{\alpha^1}E_{-\alpha^1}E_{-\alpha^2}\mu\rangle \tag{9.2}$$

$$\langle B|B\rangle = \langle \mu|E_{\alpha^1}E_{\alpha^2}E_{-\alpha^2}E_{-\alpha^1}|\mu\rangle \tag{9.3}$$

$$\langle A|B\rangle = \langle\mu|E_{\alpha^2}E_{\alpha^1}E_{-\alpha^2}E_{-\alpha^1}|\mu\rangle \tag{9.4}$$
$$= \langle\mu|([E_{\alpha^2},E_{\alpha^1}] + E_{\alpha^1},E_{\alpha^2})E_{-\alpha^2}E_{-\alpha^1}|\mu\rangle \tag{9.5}$$
$$= \langle B|B\rangle + \langle\mu|\left([[E_{\alpha^2},E_{\alpha^1}],E_{-\alpha^2}] + E_{-\alpha^2}[E_{\alpha^2},E_{\alpha^1}]\right)E_{-\alpha^1}|\mu\rangle \tag{9.6}$$
$$\begin{aligned} &= \langle B|B\rangle + \langle\mu|\big([[[E_{\alpha^2},E_{\alpha^1}],E_{-\alpha^2}],E_{-\alpha^1}] + E_{-\alpha^1}[[E_{\alpha^2},E_{\alpha^1}],E_{-\alpha^2}] \\ &\quad + E_{-\alpha^2}[[E_{\alpha^2},E_{\alpha^1}],E_{-\alpha^1}] + E_{-\alpha^2}E_{-\alpha^1}[E_{\alpha^2},E_{\alpha^1}]\big)|\mu\rangle \end{aligned} \tag{9.7}$$

So (9.7) is equal to $\langle B|B\rangle +$ non-vanishing terms, they are all non-vanishing because $\alpha^1$, $\alpha^2$ & $\alpha^1+\alpha^2$ are all roots of the adjoint representation of $\mathfrak{su}(3)$. It can also be shown that

$$\langle B|A\rangle = \langle A|A\rangle + \text{non-vanishing terms}. \tag{9.8}$$

and therefore that

$$\langle A|A\rangle\langle B|B\rangle \neq \langle B|A\rangle\langle A|B\rangle \tag{9.9}$$

thus the states are linearly independent.

## 9.B Show the the matrices $\frac{1}{2}\lambda_a\otimes\sigma_2$, $a = 1,3,4,6,8$ and $\frac{1}{2}\lambda_a\otimes\mathbb{I}$, $a = 2,5,7$ generate a reducible representation of $\mathfrak{su}(3)$ and reduce it.

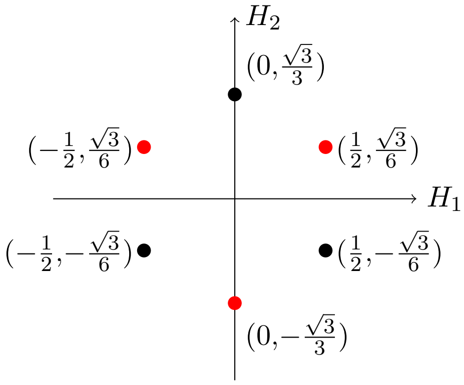
The commutation relations

$$\frac{1}{4}[\lambda_a\otimes\sigma_2,\lambda_b\otimes\sigma_2] = \frac{1}{2}\mathrm{i}f_{abc}\lambda_c\otimes\mathbb{I}, \tag{9.10}$$
$$\frac{1}{4}[\lambda_a\otimes\mathbb{I},\lambda_b\otimes\sigma_2] = \frac{1}{2}\mathrm{i}f_{abc}\lambda_c\otimes\sigma_2, \tag{9.11}$$
$$\frac{1}{4}[\lambda_a\otimes\mathbb{I},\lambda_b\otimes\mathbb{I}] = \frac{1}{2}\mathrm{i}f_{abc}\lambda_c\otimes\mathbb{I}. \tag{9.12}$$

We pick $H_1 = \frac{1}{2}\lambda_3\otimes\sigma_2$ and $H_2 = \frac{1}{2}\lambda_8\otimes\sigma_2$ as the Cartan subalgebra. Using the general result that for the tensor product of two matrices $A\otimes B$ their eigenvalues multiply $(\Lambda_{A\otimes B})_{i,j} = (\Lambda_A)_i(\Lambda_B)_j$ Then the eigenvalues of $H_1$ and $H_2$ are $(\pm\frac{1}{2},0)$ and $(\pm\frac{\sqrt{3}}{3},\pm\frac{\sqrt{3}}{6})$ respectively. The eigenvectors are given by the eigenvectors of $\sigma_2$, $V_\pm = \begin{pmatrix}1\\ \pm\mathrm{i}\end{pmatrix}$ in the various spaces. Therefore the eigenvectors and their associated weights are

$$\begin{pmatrix}V_\pm\\0\\0\end{pmatrix}\to(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{6}) \quad \begin{pmatrix}0\\V_\pm\\0\end{pmatrix}\to(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{6}) \quad \begin{pmatrix}0\\0\\V_\pm\end{pmatrix}\to(0,\pm\frac{\sqrt{3}}{3}). \tag{9.13}$$

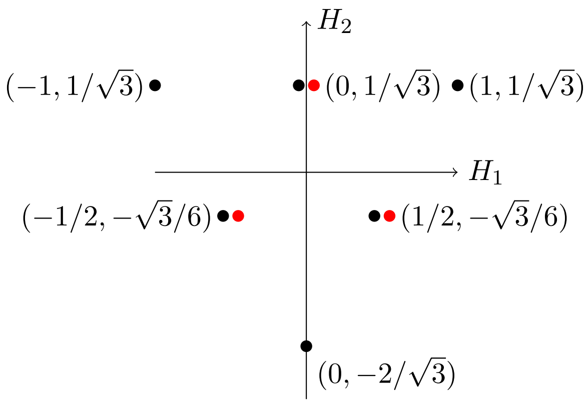
Plotting the weight diagram




So the representation reduces to $3 \oplus \bar{3}$.

### 9.C Decompose the $3 \otimes 3$ tensor product of $\mathfrak{su}(3)$

Suppose $D_1$, $D_2$ irreps of $\mathfrak{su}(3)$ on $V_1, V_2$ with $V = V_1 \otimes V_2$. $D = D_1 \otimes \mathbb{I} + \mathbb{I} \otimes D_2$ the tensor product rep on $V$, which is not necessarily an irrep on $V$. In a basis of $V_1$, $V_2$ consisting of eigenstates of the Cartan generators $D_1(H_1), D_1(H_2), D_2(H_1), D_2(H_2)$, then for some state $|\phi_i\rangle \in V_i$ a linear combination of basis states with weight $(p_i, q_i)$. Then $D_i(H_1)\,|\phi_i\rangle = p_i\,|\phi_i\rangle$, $D_i(H_2)\,|\phi_i\rangle = q_i\,|\phi_i\rangle$. Then $D(H_1)\,|\phi\rangle = D_1(H_1)\,|\phi_1\rangle \otimes |\phi_2\rangle + |\phi_1\rangle \otimes D_2(H_1)\,|\phi_2\rangle = (p_1 + p_2)\,|\phi\rangle$. Therefore the weight of the state $|\phi\rangle$ is $((p_1 + p_2), (q_1 + q_2))$ thus the highest weight state is $(\frac{1}{2} + \frac{1}{2}, \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{6}) = (1, \frac{1}{\sqrt{3}})$. Applying the raising and lowering operators $E_{\pm 1,0}, E_{\pm 1/2, \pm\sqrt{3}/2}, E_{\mp 1/2, \pm\sqrt{3}/2}$. This gives the other weights $(1/2, -\sqrt{3}/6)$, $(0, 1/\sqrt{3})$, $(1/2, -\sqrt{3}/6)$, $(0, -2/\sqrt{3})$, $(-1/2, -\sqrt{3}/6)$, $(-1, 1/\sqrt{3})$, $(0, 1/\sqrt{3})$, $(-1/2, -\sqrt{3}/6)$. Plotting on the weight diagram




Thus we see that the weights $(\pm 1/2, -\sqrt{3}/6)$ and $(0, -1/\sqrt{3})$ have multiplicity 2 and therefore the $3 \otimes 3$ tensor product representation is not irreducible on the 9-dimensional vector space. The state with multiplicity 2 correspond to the $\bar{3}$ representation of $\mathfrak{su}(3)$ while the remianing states correspond to the 6 and therefore we see the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$

# 10 Chapter 10 - Tensor Methods

## 10.A Decompose the product of tensor components $u^i v^{jk}$, $v^{jk} = v^{kj}$ is a 6 of $\mathfrak{su}(3)$.

$u^i v^{jk}$ is a $3 \otimes 6$ or $(1,0) \otimes (2,0)$. We must construct the product s. t. the terms are traceless and symmetric. The idea is to first write out the relation in terms which have definite symmetry properties

$$u^i v^{jk} = \frac{1}{3}(u^i v^{jk} + u^j v^{ik} + u^k v^{ij}) + \frac{1}{3}(2u^i v^{jk} - u^j v^{ik} - u^k v^{ij}) \tag{10.1}$$

The first term is completely symmetric in $i, j, k$, which the second term in antisymmetric in $i, j$ and $i, k$ which it is symmetric in $j, k$.

$$u^i v^{jk} = \frac{1}{3}(u^i v^{jk} + u^j v^{ik} + u^k v^{ij}) + \frac{1}{3}(\epsilon^{ikl}\epsilon_{lnm} u^n v^{jm} + \epsilon^{jil}\epsilon_{lnm} u^n v^{mk}) \tag{10.2}$$

Therefore we acquire the known decomposition $3 \otimes 6 = 10 \oplus 8$ or $(1,0) \otimes (2,0) = (3,0) \oplus (1,1)$.

## 10.B Find the matrix elements $\langle u|T_a|v\rangle$ where $|u\rangle, |v\rangle \in (1,1)$ (adjoint) representation and $T_a = \frac{1}{2}\lambda_a$

We have $|v\rangle = \left|{}^i_j\right\rangle v^j_i$. So

$$T_a |v\rangle = |T_a v\rangle = T_a v^j_i \left|{}^i_j\right\rangle = [T_a]^j_k v^k_i \left|{}^i_j\right\rangle - [T_a]^k_i v^j_k \left|{}^i_j\right\rangle \tag{10.3}$$

$$= \frac{1}{2}([\lambda_a]^j_k v^k_i - [\lambda_a]^k_i v^j_k) \left|{}^i_j\right\rangle . \tag{10.4}$$

So that the matrix elements are given by

$$\langle u|T_a|v\rangle = \frac{1}{2}\bar{u}^m_n([\lambda_a]^j_k v^k_i - [\lambda_a]^k_i v^j_k) \left\langle {}^m_n \middle| {}^i_j \right\rangle \tag{10.5}$$

$$= \frac{1}{2}\bar{u}^m_n([\lambda_a]^j_k v^k_i - [\lambda_a]^k_i v^j_k)\delta^i_m \delta^n_j \tag{10.6}$$

$$= \frac{1}{2}\bar{u}^i_j([\lambda_a]^j_k v^k_i - [\lambda_a]^k_i v^j_k). \tag{10.7}$$

## 10.C $\forall$ weights $\in 6$ of $\mathfrak{su}(3)$ find the corresponding tensor components $v^{ij}$

The 6 of $\mathfrak{su}(3)$ has highest weight $2\mu^1 = (2,0) = \left|1, \frac{1}{\sqrt{3}}\right\rangle \equiv |{}_{1,1}\rangle$ The remaining five weights are

$$(0,1) = 2\mu^1 - \alpha^1 = \left|\frac{1}{2}, -\frac{\sqrt{3}}{6}\right\rangle \equiv |{}_{1,2}\rangle \tag{10.8}$$

$$(1,-1) = 2\mu^1 - 2\alpha^1 = \left|0, -\frac{2}{\sqrt{3}}\right\rangle \equiv |{}_{1,3}\rangle \tag{10.9}$$

$$(-2,2) = 2\mu^1 - \alpha^1 - \alpha^2 = \left|0, -\frac{1}{\sqrt{3}}\right\rangle \equiv |{}_{2,2}\rangle \tag{10.10}$$

$$(-1,0) = 2\mu^1 - 2\alpha^1 - \alpha^2 = \left|-\frac{1}{2}, -\frac{\sqrt{3}}{6}\right\rangle \equiv |{}_{2,3}\rangle \tag{10.11}$$

$$(0,-2) = 2\mu^1 = \left|-1, \frac{1}{\sqrt{3}}\right\rangle \equiv |{}_{3,3}\rangle \tag{10.12}$$

The tensor components corresponding to the highest weight of the 6 is given by $v_h^{ij} = v_{11}^{ij} = N\delta_1^i\delta_1^j$, with $N = 1$. The remaining components are

$$v_{12}^{ij} = \frac{1}{\sqrt{2}}(\delta_1^i\delta_2^j + \delta_2^i\delta_1^j) \tag{10.13}$$
$$v_{13}^{ij} = \frac{1}{\sqrt{2}}(\delta_1^i\delta_3^j + \delta_3^i\delta_1^j) \tag{10.14}$$
$$v_{23}^{ij} = \frac{1}{\sqrt{2}}(\delta_2^i\delta_3^j + \delta_3^i\delta_2^j) \tag{10.15}$$
$$v_{33}^{ij} = \delta_3^i\delta_3^j \tag{10.16}$$
$$v_{22}^{ij} = \delta_2^i\delta_2^j. \tag{10.17}$$

## 10.D $\mathfrak{su}(2)$ tensor methods

### 10.D.a Repeat problem (5.C) suing the $\mathfrak{su}(2)$ tensor methods.

The equation we need is equation (10.82) in Georgi.

$$p + \pi^+ = |1/2, 1/2\rangle\,|1, 1\rangle = |3/2, 3/2\rangle = \left|\Delta^{++}\right\rangle = \binom{3}{3}^{-1/2} \left|v_{3/2,3/2}\right\rangle = \left|v_{3/2,3/2}\right\rangle. \tag{10.18}$$

Therefore $\text{Prob}(p + \pi^+ \to \Delta^{++}) = 1$.

$$\text{Prob}(p + \pi^- \to \Delta^0) = |\,\langle 3/2, -1/2|1/2, 1/2\rangle\,|1, 1\rangle\,|^2 \tag{10.19}$$
$$= |\,\langle 3/2, -1/2|\,\langle 3/2, -1/2|3/2, -1/2\rangle\,|1/2, 1/2\rangle\,|1, 1\rangle\,| \tag{10.20}$$
$$= \left|\binom{1}{1}^{1/2}\binom{2}{0}^{1/2}\binom{3}{1}^{-1/2} |3/2, -1/2\rangle\right|^2 \tag{10.21}$$
$$= \frac{1}{3}. \tag{10.22}$$

Which agrees with the result from Problem(5.C).

### 10.D.b Find the Clebsch-Gordon coefficient $\langle 3/2, 1/2|1, 3/2, 0, 1/2\rangle$ by finding the state $|3/2, 1/2\rangle$ in $3/2 \otimes 1$ and compare it to eqn(10.94).

To begin with we use $3/2 \otimes 1 = 5/2 \oplus 3/2 \oplus 1/2$. Apply the lowering operators to the highest weight state $|3/2, 3/2\rangle\,|1, 1\rangle = |5/2, 5/2\rangle$.

$$|5/2, 3/2\rangle = \sqrt{\frac{2}{5}} J^- \,|5/2, 5/2\rangle \tag{10.23}$$
$$= \sqrt{\frac{2}{5}} J^- \,|3/2, 3/2\rangle\,|1, 1\rangle \tag{10.24}$$
$$= \sqrt{\frac{2}{5}} \left( \sqrt{\frac{3}{2}}\,|3/2, 1/2\rangle\,|1, 1\rangle + |3/2, 3/2\rangle\,|1, 0\rangle \right) \tag{10.25}$$

which is a spin 3/2 state $\in$ spin the 5/2 rep and therefore the orthogonal state $|v\rangle$ is the highest weight state $\in$ spin 3/2 rep.

$$\begin{aligned} \langle v|5/2,3/2\rangle =& 0 && (10.26)\\ =& \langle v|\left(\sqrt{\frac{3}{5}}\,|3/2,1/2\rangle\,|1,1\rangle + \sqrt{\frac{2}{5}}\,|3/2,3/2\rangle\,|1,0\rangle\right) && (10.27)\end{aligned}$$

So,

$$\begin{aligned} |v\rangle =& \sqrt{\frac{2}{5}}\,|3/2,1/2\rangle\,|1,1\rangle - \sqrt{\frac{3}{5}}\,|3/2,3/2\rangle\,|1,0\rangle && (10.28)\\ \equiv& |3/2,3/2\rangle && (10.29)\end{aligned}$$

Applying the lowering operators to this state

$$\begin{aligned} |3/2,1/2\rangle =& \sqrt{\frac{2}{3}}J^-\,|3/2,3/2\rangle && (10.30)\\ =& \frac{\sqrt{2}}{\sqrt{15}}\Big(\,|3/2,-1/2\rangle\,|1,1\rangle + \sqrt{2}\,|3/2,1/2\rangle\,|1,0\rangle \\ & - \frac{3}{\sqrt{2}}\,|3/2,1/2\rangle\,|1,0\rangle - \sqrt{3}\,|3/2,3/2\rangle\,|1,-1\rangle\,\Big) && (10.31)\end{aligned}$$

So that the Clebsch-Gordon coefficient

$$\langle 3/2,1/2|1,3/2,0,1/2\rangle = \frac{1}{\sqrt{15}}. \qquad (10.32)$$

By equation (10.94)

$$\begin{aligned} &\langle s_1+s_2-1, m_1+m_2|s_1,s_2,m_1,m_2\rangle \\ &= \binom{2s_1}{s_1+m_1}^{-1/2}\binom{2s_2}{s_2+m_2}^{-1/2}\left(\frac{2s_1s_2}{s_1+s_2}\right)^{1/2}\binom{2s_1+2s_2-2}{s_1+s_2-1+m_1+m_2}^{-1/2} \\ &\cdot\left[\binom{2s_1-1}{s_1+m_1-1}\binom{2s_2-1}{s_2+m_2} - \binom{2s_1-1}{s_1+m_1}\binom{2s_2-1}{s_2+m_2-1}\right]. \end{aligned} \qquad (10.33)$$

so that

$$\begin{aligned} \langle 3/2,1/2|1,3/2,0,1/2\rangle =& \binom{3}{2}^{-1}\binom{2}{1}^{-1/2}\left(\frac{6}{5}\right)^{1/2}\left[\binom{2}{1}\binom{1}{1} - \binom{2}{2}\binom{1}{0}\right] && (10.34)\\ =& \frac{1}{\sqrt{15}}. && (10.35)\end{aligned}$$

## 10.E $\pi p$ scattering

We are given

$$\begin{aligned} \pi^+p \to \pi^+p, &\qquad A_{+p} = \langle \pi^+p|H_I|\pi^+p\rangle && (10.36)\\ \pi^-p \to \pi^-p, &\qquad A_{-p} = \langle \pi^-p|H_I|\pi^-p\rangle && (10.37)\\ \pi^-p \to \pi^0n, &\qquad A_{0n} = \langle \pi^0n|H_I|\pi^-p\rangle && (10.38)\end{aligned}$$

where $H_I$ is the interaction Hamiltonian which is approximately $\mathfrak{su}(3)$ invariant. The pion and nucleon wavefunctions can be described by $\mathfrak{su}(2)$ tensors $\pi^{ij} = \pi^{ji}$ and $N^j$ respectively. Thus the most general amplitude is

$$\langle \pi N | H_I | \pi N \rangle = A_1 \bar{\pi}^{jk} \bar{N}_l \pi^{jk} N^l + A_2 \bar{\pi}_{jk} \bar{N}_l \pi^{jl} N^k \tag{10.39}$$

### 10.E.a Write out the three scattering amplitudes in terms of $A_1$ and $A_2$

Begin by writing out the states in tensor form

$$\left|\pi^+\right\rangle = |1,1\rangle = \binom{2}{2}^{-1/2} \pi^{jk}_{11} \left|_{jk}\right\rangle = \delta^j_1 \delta^k_1 \left|_{jk}\right\rangle \tag{10.40}$$

$$\left|\pi^-\right\rangle = |1,-1\rangle = \binom{2}{0}^{-1/2} \pi^{ij}_{1,-1} \left|_{ij}\right\rangle = \delta^i_2 \delta^j_2 \left|_{ij}\right\rangle \tag{10.41}$$

$$\left|\pi^0\right\rangle = |1,0\rangle = \binom{2}{1}^{-1/2} \pi^{ij}_{1,0} \left|_{ij}\right\rangle = \frac{1}{\sqrt{2}} (\delta^i_1 \delta^j_2 + \delta^i_2 \delta^j_1) \left|_{ij}\right\rangle \tag{10.42}$$

$$|p\rangle = |1/2, 1/2\rangle = \binom{1}{1}^{-1/2} N^i \left|_i\right\rangle = \delta^i_1 \left|_i\right\rangle \tag{10.43}$$

$$|n\rangle = |1/2, -1/2\rangle = \binom{1}{0}^{-1/2} N^i \left|_i\right\rangle = \delta^i_2 \left|_i\right\rangle . \tag{10.44}$$

For $|\pi^+ p\rangle = |1,1\rangle \otimes |1/2, 1/2\rangle$

$$\begin{aligned} \left\langle \pi^+ p | H_I | \pi^+ p \right\rangle =& A_1 \delta^1_j \delta^1_k \delta^1_l \delta^j_1 \delta^k_1 \delta^l_1 + A_2 \delta^1_j \delta^1_k \delta^1_l \delta^j_1 \delta^k_1 \delta^l_1 && (10.45)\\ =& A_1 + A_2 = A_{+p}. && (10.46) \end{aligned}$$

For $|\pi^- p\rangle = |1,-1\rangle \otimes |1/2, 1/2\rangle$

$$\begin{aligned} \left\langle \pi^- p | H_I | \pi^- p \right\rangle =& A_1 \delta^2_j \delta^2_k \delta^1_l \delta^j_2 \delta^k_2 \delta^l_1 + A_2 \delta^2_j \delta^2_k \delta^1_l \delta^j_2 \delta^l_2 \delta^k_1 && (10.47)\\ =& A_1 = A_{-p}. && (10.48) \end{aligned}$$

For $\left|\pi^0 n\right\rangle = |1,0\rangle \otimes |1/2, -1/2\rangle$

$$\begin{aligned} \left\langle \pi^0 n | H_I | \pi^- p \right\rangle =& A_1 \frac{1}{\sqrt{2}} (\delta^1_j \delta^2_k + \delta^2_j \delta^1_k) \delta^2_l \delta^j_2 \delta^k_2 \delta^l_2 + A_2 \frac{1}{\sqrt{2}} (\delta^1_j \delta^2_k + \delta^2_j \delta^1_k) \delta^2_l \delta^j_2 \delta^k_1 \delta^l_2 && (10.49)\\ =& \frac{1}{\sqrt{2}} A_2 = A_{0n}. && (10.50) \end{aligned}$$

### 10.E.b Write out the scattering amplitudes in terms of $I = 3/2$ and $I = 1/2$ amplitudes by decomposing $\pi N$ states into irrep $I = 3/2$ and $I = 1/2$ reps and using Schur's lemma. Write them in terms of $A_1$ and $A_2$.

Because $H_I$ is taken to be $\mathfrak{su}(3)$ invariant so $[T_a, H_I] = 0$. Then, by Schur's lemma, $H_I \propto \mathbb{I}$. Thus,

$$\left\langle s', m' | H_I | s, m \right\rangle = \delta_{ss'} \delta_{mm'} \sum_s a_s \phi(s)^\dagger \phi(s), \tag{10.51}$$

where $\phi(s)$ are states belonging to the spin $s$ rep, in this case the spin 1/2 and 3/2 reps, and $a_s$ is the amplitude in the spin $s$ rep.

We must first write out the pion-nucleon states in terms of spin 3/2 and spin 1/2 states.

$$\left|\pi^+ p\right\rangle = |1,1\rangle \otimes |1/2,1/2\rangle = |3/2,3/2\rangle \tag{10.52}$$
$$\left|\pi^- n\right\rangle = |1,-1\rangle \otimes |1/2,-1/2\rangle = |3/2,-3/2\rangle \tag{10.53}$$

are the highest and lowest weights.

For the other states, apply the lowering operators

$$|3/2,1/2\rangle = \sqrt{\frac{2}{3}} J^- |3/2,3/2\rangle \tag{10.54}$$
$$= \sqrt{\frac{2}{3}} J^- |1,1\rangle |1/2,1/2\rangle \tag{10.55}$$
$$= \sqrt{\frac{2}{3}} |1,0\rangle |1/2,1/2\rangle + \sqrt{\frac{1}{3}} |1,1\rangle |1/2,-1/2\rangle \tag{10.56}$$
$$= \sqrt{\frac{2}{3}} \left|\pi^0 p\right\rangle + \sqrt{\frac{1}{3}} \left|\pi^+ n\right\rangle . \tag{10.57}$$

and

$$|3/2,-1/2\rangle = \sqrt{\frac{1}{2}} J^- |3/2,1/2\rangle \tag{10.58}$$
$$= \sqrt{\frac{1}{3}} |1,-1\rangle |1/2,1/2\rangle + \sqrt{\frac{2}{3}} |1,0\rangle |1/2,-1/2\rangle \tag{10.59}$$
$$= \sqrt{\frac{1}{3}} \left|\pi^- p\right\rangle + \sqrt{\frac{2}{3}} \left|\pi^0 n\right\rangle . \tag{10.60}$$

The orthogonal spin 1/2 states are then given by

$$|1/2,1/2\rangle = \sqrt{\frac{1}{3}} \left|\pi^0 p\right\rangle - \sqrt{\frac{2}{3}} \left|\pi^+ n\right\rangle \tag{10.61}$$
$$|1/2,-1/2\rangle = \sqrt{\frac{2}{3}} \left|\pi^- p\right\rangle - \sqrt{\frac{1}{3}} \left|\pi^0 n\right\rangle . \tag{10.62}$$

Now we can rearrange to find the pion-nucleon states in terms of spin 3/2 and spin 1/2 reps.

Firstly, the $m = 1/2$ states

$$\left|\pi^0 p\right\rangle = \sqrt{3}\, |1/2,1/2\rangle + \sqrt{2} \left|\pi^+ n\right\rangle \tag{10.63}$$
$$\left|\pi^0 p\right\rangle = \sqrt{\frac{3}{2}} |3/2,1/2\rangle - \frac{1}{\sqrt{2}} \left|\pi^+ n\right\rangle \tag{10.64}$$
$$\tag{10.65}$$

So,

$$\left|\pi^0 p\right\rangle = \frac{1}{\sqrt{3}} |1/2,1/2\rangle + \sqrt{\frac{2}{3}} |3/2,1/2\rangle \tag{10.66}$$
$$\left|\pi^+ n\right\rangle = -\sqrt{\frac{2}{3}} |1/2,1/2\rangle + \frac{1}{\sqrt{3}} |3/2,1/2\rangle . \tag{10.67}$$

Similarly,

$$\left|\pi^0 n\right\rangle = -\frac{1}{\sqrt{3}}\left|1/2,-1/2\right\rangle + \sqrt{\frac{2}{3}}\left|3/2,-1/2\right\rangle \tag{10.68}$$

$$\left|\pi^- p\right\rangle = \sqrt{\frac{2}{3}}\left|1/2,-1/2\right\rangle + \frac{1}{\sqrt{3}}\left|3/2,-1/2\right\rangle . \tag{10.69}$$

Now we are ready to calculate the scattering amplitudes by applying these states and (10.51). Firstly,

$$\left\langle \pi^+ p|H_I|\pi^+ p\right\rangle = \langle 3/2,3/2|H_I|3/2,3/2\rangle = a_{3/2}\langle 3/2,3/2|3/2,3/2\rangle = a_{3/2} \tag{10.70}$$

and

$$\left\langle \pi^- p|H_I|\pi^- p\right\rangle = \left(\sqrt{\frac{2}{3}}\langle 1/2,-1/2| + \frac{1}{\sqrt{3}}\langle 3/2,-1/2|\right)|H_I|\left(\sqrt{\frac{2}{3}}|1/2,-1/2\rangle + \frac{1}{\sqrt{3}}|3/2,-1/2\rangle\right) \tag{10.71}$$

$$= \frac{2}{3}a_{1/2} + \frac{1}{3}a_{3/2}. \tag{10.72}$$

Finally,

$$\left\langle \pi^0 n|H_I|\pi^- p\right\rangle = \left(-\frac{1}{\sqrt{3}}\langle 1/2,-1/2| + \sqrt{\frac{2}{3}}\langle 3/2,-1/2|\right)|H_I|\left(\sqrt{\frac{2}{3}}|1/2,-1/2\rangle + \frac{1}{\sqrt{3}}|3/2,-1/2\rangle\right) \tag{10.73}$$

$$= -\frac{\sqrt{2}}{3}a_{1/2} + \frac{\sqrt{2}}{3}a_{3/2}. \tag{10.74}$$

Then

$$A_{+P} = A_1 + A_2 = a_{3/2} \tag{10.75}$$

$$A_{-P} = A_1 = \frac{2}{3}a_{1/2} + \frac{1}{3}a_{3/2} \tag{10.76}$$

$$A_{0n} = \frac{1}{\sqrt{2}}A_2 = -\frac{\sqrt{2}}{3}a_{1/2} + \frac{\sqrt{2}}{3}a_{3/2}. \tag{10.77}$$

and solving the simultaneous equations yields

$$a_{3/2} = A_1 + A_2 \tag{10.78}$$

$$a_{1/2} = A_1 - \frac{1}{2}A_2. \tag{10.79}$$

# 11 Chapter 11 - Hypercharge & Strangeness

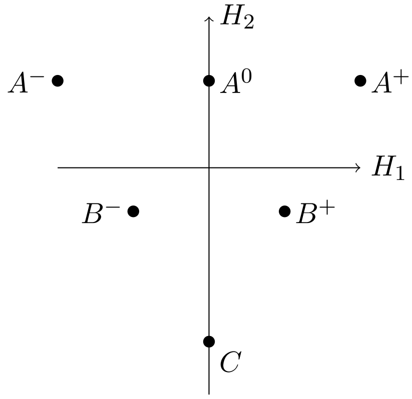
## 11.A What does the Gell-Mann-Okubo argument say about the masses of particles transforming like a 6 of $\mathfrak{su}(3)$

Now the particles transform as a 6 and the states can be described by the tensor $S^{ij}\left|_{ij}\right\rangle$. Whereas particles transforming like an 8 has two independent contributions, because $8 \otimes 8 \ni$

$... \oplus 8 \oplus 8 \oplus ...$ But now we have only one independent tensor contribution to the matrix element $\langle S|H_{MS}|S\rangle$ because $8 \otimes 6 = 15 \oplus 24 \oplus 3 \oplus 6$ contains only one 6 for the $\overline{6}$ to couple to. This contribution is

$$\overline{S}_{ij}[T_8]^j_k S^{ki} = \mathrm{Tr}\,(S^\dagger T_8 S). \tag{11.1}$$

The 6 particles, which are quark bilinears, would look something like




So, by a similar argument to that found on P.g.174, since there is only one reduced matrix element, the matrix element is proportional to $T_8$ and thus the hypercharge $Y$, thus the mass spacings are equal

$$M_A - M_B = M_B - M_C. \tag{11.2}$$

## 11.B Calculate Prob($\pi^0 P \to \Delta^+$) with Prob($K^- P \to \Sigma^{*0}$) assuming $\mathfrak{su}(3)$ symmetry of the S-matrix.

Both the $\Sigma^{*0}$ and $\Delta^+$ Baryons $\in$ 10 (decuplet) while $\pi^0 P$, $K^- P \in 8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27$ which clearly contains the 10 only once. The S-Matrix couples to the $\overline{10}$ of the decuplet. So, in the tensor notation we may write the Meson-Baryon states $\in$ the 10 as $M^i_l B^j_m \epsilon^{klm} \left|_{ijk}\right\rangle$ and the decuplet states as $D^{ijk} \left|_{ijk}\right\rangle$. Then, in $\left\langle \Delta^+|S|\pi^0 P\right\rangle$ and $\left\langle \Sigma^{*0}|S|K^- P\right\rangle$, because the S-matrix is taken to the $\mathfrak{su}(3)$ invariant, similarly to Problem (10.E.b), by Schur's Lemma $S \propto \mathbb{I} = \lambda\mathbb{I}$, with $\lambda$ a coupling constant.

Because we are only interested in $P, K^-, \pi^0$ states be can take the matrices $B, M$ to be (equation (11.15) in Georgi)

$$B = \begin{pmatrix} 0 & 0 & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\pi^0}{\sqrt{2}} & 0 \\ K^- & 0 & 0 \end{pmatrix} \tag{11.3}$$

So, the matrix elements are given by

$$\begin{aligned}
\left\langle \Delta^+ | S | \pi^0 P \right\rangle =& \lambda \left\langle {}^{abc} \right| \overline{D}_{abc} M^i_l B^j_m \epsilon^{klm} \left|_{ijk} \right\rangle & (11.4)\\
=& \delta^a_i \delta^b_j \delta^c_k \lambda \overline{D}_{abc} M^i_l B^j_m \epsilon^{klm} & (11.5)\\
=& \lambda \overline{D}_{ijk} M^i_l B^j_m \epsilon^{klm} & (11.6)\\
=& \lambda \overline{D}_{11k} M^1_1 B^1_3 \epsilon^{k13} + \lambda \overline{D}_{21k} M^2_2 B^1_3 \epsilon^{k23} & (11.7)\\
=& -\lambda \overline{D}_{112} M^1_1 B^1_3 + \lambda \overline{D}_{211} M^2_2 B^1_3 & (11.8)
\end{aligned}$$

we have $\overline{D}_{112} = \overline{D}_{121} = \overline{D}_{211} = \frac{\overline{\Delta^+}}{\sqrt{3}}$. so

$$\left\langle \Delta^+ | S | \pi^0 P \right\rangle = -2N\lambda \left( \frac{\overline{\Delta^+}\pi^0 P}{\sqrt{6}} \right) \tag{11.9}$$

so that

$$\text{Prob}(\pi^0 P \to \Delta^+) = -\frac{4}{6}\lambda^2 \tag{11.10}$$

similarly

$$\begin{aligned}
\left\langle \Sigma^{*0} | S | K^- P \right\rangle =& \lambda \overline{D}_{31k} M^3_1 B^1_3 \epsilon^{kl3} & (11.11)\\
=& -\lambda \overline{D}_{312} M^3_1 B^1_3. & (11.12)
\end{aligned}$$

and $\overline{D}_{123} = \text{perms} = \frac{\overline{\Sigma^{*0}}}{\sqrt{6}}$ so

$$\left\langle \Sigma^{*0} | S | K^- P \right\rangle = -\lambda \frac{\overline{\Sigma^{*0}} K^- P}{\sqrt{6}} \tag{11.13}$$

then the probability is

$$\text{Prob}(K^- P \to \overline{\Sigma^{*0}}) = -\frac{1}{6}\lambda^2. \tag{11.14}$$

So the ratios are

$$\frac{\text{Prob}(\pi^0 P \to \Delta^+)}{\text{Prob}(K^- P \to \overline{\Sigma^{*0}})} = 4. \tag{11.15}$$

## 11.C In $\mathfrak{su}(3)$ repeat the calculation by Coleman & Glashow, predicting the spin 1/2 baryon magnetic moments in terms of $\mu_P$ and $\mu_N$

The specifics of the magnetic moment operator is unknown but we know it is $\propto Q = \text{diag}(2/3, -1/3, -1/3)$, the charge operator. Therefore the only possibilities for the magnetic moment operator is $\mu(B) = \alpha \,\text{Tr}\,(B^\dagger Q B) + \beta \,\text{Tr}\,(B^\dagger B Q)$. This allows us to compute the magnetic moments to a good accuracy in terms of the proton and neutron magnetic moments $\mu_P$, $\mu_N$. We need $B$ which is given by (11.15) in Georgi.

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & P \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & N \\ \Xi^- & \Xi^0 & \frac{-2\Lambda}{\sqrt{6}} \end{pmatrix} \tag{11.16}$$

$$B^\dagger Q B = \frac{1}{3}\begin{pmatrix} \frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Sigma^-} & \overline{\Xi^-} \\ \overline{\Sigma^+} & \frac{-\overline{\Sigma^0}}{\sqrt{2}} - \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Xi^0} \\ \overline{P} & \overline{N} & \frac{-2\overline{\Lambda}}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{2\Sigma^0}{\sqrt{2}} + \frac{2\Lambda}{\sqrt{6}} & 2\Sigma^+ & 2P \\ -\Sigma^- & \frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}} & -N \\ -\Xi^- & -\Xi^0 & \frac{2\Lambda}{\sqrt{6}} \end{pmatrix}, \tag{11.17}$$

$$B^\dagger B Q = \frac{1}{3}\begin{pmatrix} \frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Sigma^-} & \overline{\Xi^-} \\ \overline{\Sigma^+} & \frac{-\overline{\Sigma^0}}{\sqrt{2}} - \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Xi^0} \\ \overline{P} & \overline{N} & \frac{-2\overline{\Lambda}}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{2\Sigma^0}{\sqrt{2}} + \frac{2\Lambda}{\sqrt{6}} & -\Sigma^+ & -P \\ 2\Sigma^- & \frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}} & -N \\ 2\Xi^- & -\Xi^0 & \frac{2\Lambda}{\sqrt{6}} \end{pmatrix}. \tag{11.18}$$

Then we compute the traces

$$\begin{aligned} \alpha\,\mathrm{Tr}\,(B^\dagger Q B) =& \frac{\alpha}{3}\big(2(\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}})(\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) - \Sigma^-\overline{\Sigma^-} - \Xi^-\overline{\Xi^-} + 2\Sigma^+\overline{\Sigma^+} \\ &+ (\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}})(\frac{-\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) - \Xi^0\overline{\Xi^0} + 2P\overline{P} - N\overline{N} - \frac{2\Lambda\overline{\Lambda}}{3}\big), \end{aligned}$$

$$\begin{aligned} \beta\,\mathrm{Tr}\,(B^\dagger B Q) =& \frac{\beta}{3}\big(2(\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}})(\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) + 2\Sigma^-\overline{\Sigma^-} + 2\Xi^-\overline{\Xi^-} - \Sigma^+\overline{\Sigma^+} \\ &+ (\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}})(\frac{-\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) - \Xi^0\overline{\Xi^0} + -P\overline{P} - N\overline{N} - \frac{2\Lambda\overline{\Lambda}}{3}\big). \end{aligned}$$

So, putting (11.19) and (11.19) together we have

$$\begin{aligned} \mu(B) =& \frac{\alpha}{3}\big(2(\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}})(\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) - \Sigma^-\overline{\Sigma^-} - \Xi^-\overline{\Xi^-} + 2\Sigma^+\overline{\Sigma^+} \\ &+ (\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}})(\frac{-\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) - \Xi^0\overline{\Xi^0} + 2P\overline{P} - N\overline{N} - \frac{2\Lambda\overline{\Lambda}}{3}\big) \\ &+ \frac{\beta}{3}\big(2(\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}})(\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) + 2\Sigma^-\overline{\Sigma^-} + 2\Xi^-\overline{\Xi^-} - \Sigma^+\overline{\Sigma^+} \\ &+ (\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}})(\frac{-\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}) - \Xi^0\overline{\Xi^0} + -P\overline{P} - N\overline{N} - \frac{2\Lambda\overline{\Lambda}}{3}\big). \end{aligned}$$

Thus we may read of the magnetic moments by picking out the coefficients,

$$\mu_P = \frac{2\alpha}{3} - \frac{\beta}{3} = \frac{1}{3}(2\alpha - \beta) \tag{11.19}$$

$$\mu_{\Sigma^0} = \frac{1}{6}(\alpha + \beta) \tag{11.20}$$

$$\mu_{\Lambda\Sigma^0} = \sqrt{3}(\alpha + \beta) \tag{11.21}$$

$$\mu_{\Sigma^-} = \frac{1}{3}(-\alpha + 2\beta) \tag{11.22}$$

$$\mu_{\Xi^-} = \frac{1}{3}(-\alpha + 2\beta) \tag{11.23}$$

$$\mu_{\Sigma^+} = \frac{1}{3}(2\alpha - \beta) \tag{11.24}$$

$$\mu_{\Lambda} = -\frac{1}{6}(\alpha + \beta) \tag{11.25}$$

$$\mu_N = -\frac{1}{3}(\alpha + \beta) \tag{11.26}$$

$$\mu_{\Xi^0} = -\frac{1}{3}(\alpha + \beta). \tag{11.27}$$

Writing $\alpha$ and $\beta$ in terms of $\mu_P$ & $\mu_N$. $\alpha = \mu_P - \mu_N$, $\beta = -\mu_P - 2\mu_N$. So in the end we have

$$\mu_{\Sigma^0} = -\frac{1}{2}\mu_N \tag{11.28}$$

$$\mu_{\Lambda\Sigma^0} = -\frac{\sqrt{3}}{2}\mu_N \tag{11.29}$$

$$\mu_{\Sigma^-} = -\mu_N - \mu_P \tag{11.30}$$

$$\mu_{\Xi^-} = -\mu_P - \mu_N \tag{11.31}$$

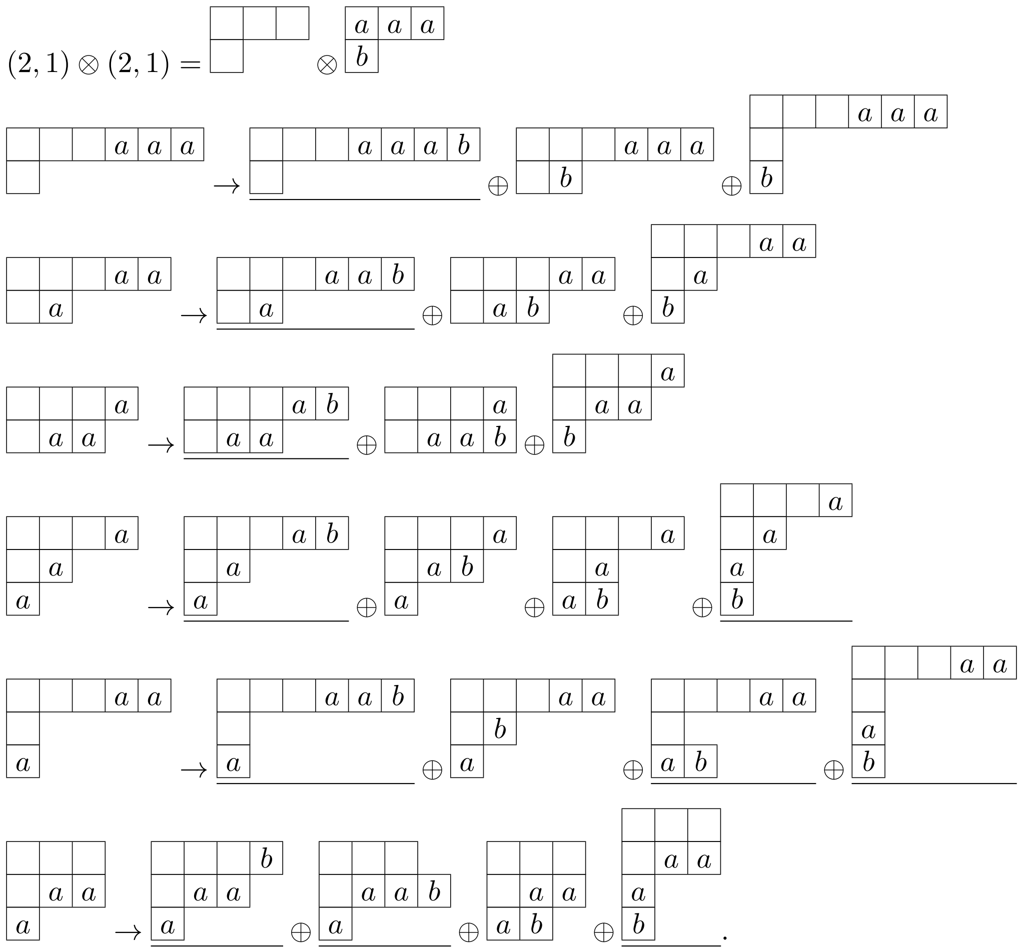
$$\mu_{\Sigma^+} = \mu_P \tag{11.32}$$

$$\mu_{\Lambda} = \frac{1}{2}\mu_N \tag{11.33}$$

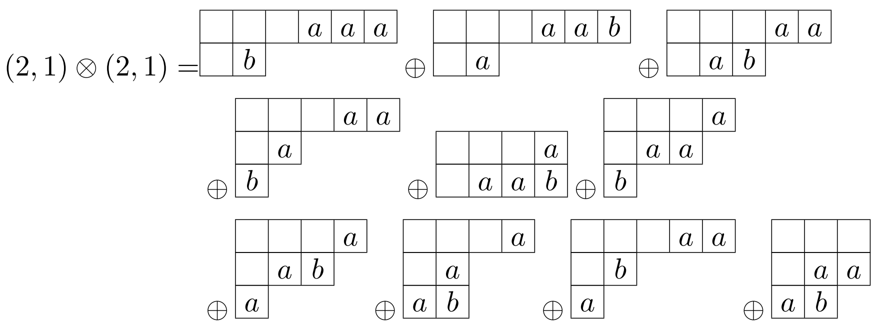
$$\mu_{\Xi^0} = \mu_N. \tag{11.34}$$

## 12 Chapter 12 - Young Tableux

### 12.A Find $(2,1)\otimes(2,1)$, which reps appear antisymmetrically? which symmetrically?



So together



(12.1)

$$
\begin{aligned}
&=(4,2)\oplus(5,0)\oplus(2,3)\oplus(3,1)\oplus(0,4)\oplus(1,2)\oplus(1,2)\\
&\quad\oplus(2,0)\oplus(3,1)\oplus(0,1) && (12.2)\\
&=60\oplus 21\oplus 42\oplus 24\oplus 15\oplus 24\oplus 6\oplus 15\oplus 24\oplus 3. && (12.3)
\end{aligned}
$$

To determine which reps appear antisymmetrically/symmetrically recall that in the Young's Tableaux a horizontal exchange corresponds to a symmetric interchange of indices, while a vertical exchange corresponds to an antisymmetric exchange. To see this label the boxes in the tensor product as follows

$$(2,1)\otimes(2,1) = \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} \otimes \begin{ytableau} 1' & 2' & 3' \\ 4' \end{ytableau} \tag{12.4}$$

Then we look at the interchanges $1 \leftrightarrow 1'$, $2 \leftrightarrow 2'$, etc. Then we see that the only symmetric representation is $(4,2)$, while all the others appear antisymmetrically

### 12.B Find $10 \otimes 8$.

$10 = (3,0) = A^{i_1 i_2 i_3}$, $8 = (1,1) = A^{i_1}_{j_1}$

$$10\otimes 8 = \begin{ytableau} \; & \; & \; \end{ytableau} \otimes \begin{ytableau} a & a \\ b \end{ytableau} \tag{12.5}$$

$$= \begin{ytableau} \; & \; & \; & a & a \end{ytableau} \to \underline{\begin{ytableau} \; & \; & \; & a & a & b \end{ytableau}} \oplus \begin{ytableau} \; & \; & \; & a & a \\ b \end{ytableau}$$

$$\begin{ytableau} \; & \; & \; & a \\ a \end{ytableau} \to \underline{\begin{ytableau} \; & \; & \; & a & b \\ a \end{ytableau}} \oplus \begin{ytableau} \; & \; & \; & a \\ a & b \end{ytableau} \oplus \begin{ytableau} \; & \; & \; & a \\ a \\ b \end{ytableau}$$

$$\begin{ytableau} \; & \; & \; \\ a & a \end{ytableau} \to \underline{\begin{ytableau} \; & \; & \; & b \\ a & a \end{ytableau}} \oplus \underline{\begin{ytableau} \; & \; & \; \\ a & a & b \end{ytableau}} \oplus \begin{ytableau} \; & \; & \; \\ a & a \\ b \end{ytableau} \tag{12.6}$$

$$\underline{\begin{ytableau} \; & \; & \; \\ a \\ a \end{ytableau}} \tag{12.7}$$

$$\begin{aligned} &= (4,1)\oplus(2,2)\oplus(3,0)\oplus(1,1) \\ &= 35\oplus 27\oplus 10\oplus 8. \end{aligned} \tag{12.8}$$

### 12.C Show that for any non-trivial representation $D$ of $\mathfrak{su}(3)$ then $D\otimes 8 \ni D$ using Young's Tableaux.

This can be understood because the tensor product will always contain a term such as $D\otimes(1,1) = D\otimes D_{adj} = D^{i_1\dots i_n}_{j_1\dots j_m} D_{adj}{}^{l}_{k} = \dots\oplus \delta^k_l D^{i_1\dots i_n}_{j_1\dots j_m} D_{adj}{}^{l}_{k} \oplus \dots$. More concretely, for an arbitrary representation $D=(n,m)$ the Young's tableaux for the product is given by

$$(n,m)\otimes(1,1) = \begin{array}{|c|c|c|c|c|c|}\hline k_1 & \dots & k_m & i_1 & \dots & i_n \\ \hline l_1 & \dots & l_m \\ \cline{1-3}\end{array} \otimes \begin{array}{|c|c|}\hline a & a \\ \hline b \\ \cline{1-1}\end{array} \tag{12.9}$$

$$= \dots \oplus \begin{array}{|c|c|c|c|c|c|c|}\hline k_1 & \dots & k_m & i_1 & \dots & i_n & a \\ \hline l_1 & \dots & l_m & a \\ \cline{1-4} b \\ \cline{1-1}\end{array} \oplus \dots \tag{12.10}$$

$$= \dots \oplus \begin{array}{|c|c|c|c|c|c|c|}\hline k_2 & \dots & k_m & i_1 & \dots & i_n & a \\ \hline l_2 & \dots & l_m & a \\ \cline{1-4}\end{array} \oplus \dots \tag{12.11}$$

$$= \dots \oplus (n,m) \oplus \dots . \tag{12.12}$$

Therefore $D \in D \otimes 8$.

# 13 Chapter 13 - $\mathfrak{su}(N)$

## 13.A Show that $\mathfrak{su}(N)$ has a $\mathfrak{su}(N-1)$ subalgebra. How do the fundamental representations of $\mathfrak{su}(N)$ decompose into $\mathfrak{su}(N-1)$ reps?

Look for $\mathfrak{su}(N-1) \in \mathfrak{su}(N)$. The simple roots of $\mathfrak{su}(N)$ are $\alpha^i = \nu^i - \nu^{i+1}$, $i = 1 \dots N-1$. Which form a basis for a $\mathbb{R}^{N-1}$ root space, from which we may construct the full $\mathfrak{su}(N)$ algebra. If we limit $i = 1 \dots N-2$ then the simple roots form a basis for a $\mathfrak{su}(N-1)$ algebra, therefore $\mathfrak{su}(N-1) \in \mathfrak{su}(N)$.

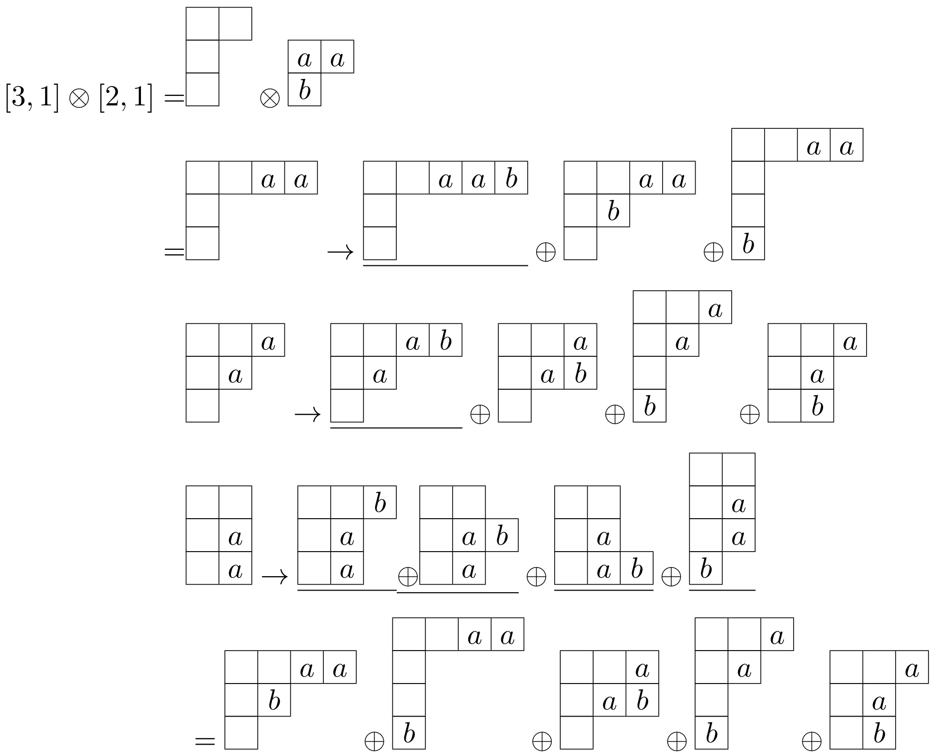
For $\mathfrak{su}(N)$ the weights in the fundamental representation are $\mu^j = \sum_{k=1}^{j} \nu^k$ for $j \leq N-1$ and for $\mathfrak{su}(N-1)$ the weights in the defining representation are $\mu^j = \sum_{k=1}^{j} \nu^k$ for $j \leq N-2$. Then the raising and lowering operators $E_i$ of $\mathfrak{su}(N-1)$ move between the $N-2$ weights $\nu^1 \dots \nu^{N-2}$ and the $\nu^{N-1}$ state cannot be reached and is an invariant subspace under $\mathfrak{su}(N-1)$. Thus $N = N-1 \oplus 1$

## 13.B Find $[3] \otimes [1]$ in $\mathfrak{su}(5)$ and check the dimensions

$$\begin{array}{|c|}\hline \phantom{a} \\ \hline \phantom{a} \\ \hline \phantom{a} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline a \\ \hline\end{array} = \begin{array}{|c|c|}\hline \phantom{a} & a \\ \hline \phantom{a} \\ \cline{1-1} \phantom{a} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \phantom{a} \\ \hline \phantom{a} \\ \hline \phantom{a} \\ \hline a \\ \hline\end{array} = [3,1] \oplus [4]. \tag{13.1}$$

The dimensions are $\dim([3]\otimes[1]) = \frac{5.4.3}{3.2}.5 = 50$, $\dim([3,1]) = \frac{5.6.4.3}{2.4} = 45$, $\dim([4]) = \frac{5.4.3.2}{2.3.4} = 5$. Therefore the dimensions work out.

## 13.C Find $[3,1] \otimes [2,1]$



(13.2)

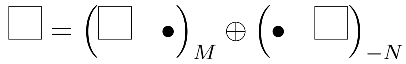
(13.3)

(13.4)

$$= [3,2,1,1] \oplus [4,1,1,1] \oplus [3,2,2] \oplus [4,2,1] \oplus [3,3,1] \tag{13.5}$$

## 13.D Under the subalgebra $\mathfrak{su}(N) \otimes \mathfrak{su}(M) \otimes \mathfrak{u}(1) \in \mathfrak{su}(N+M)$ how do the fundamental representations and the adjoint representations of $\mathfrak{su}(N+M)$ transform?

The fundamental/defining representations transforms as

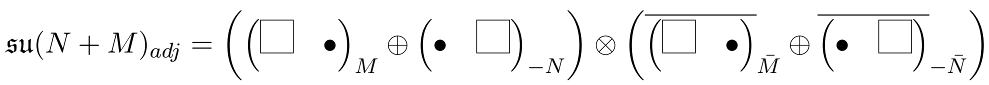


(13.6)

The adjoint representation

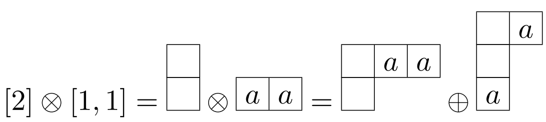
$$\mathfrak{su}(N+M)_{adj} \oplus 1 = (N+M) \otimes (N \bar{+} M) \tag{13.7}$$

Where $(N+M)$ is the defining rep of $\mathfrak{su}(N+M)$. The product contains two $\mathfrak{u}(1)$ factors, one of which is equal to the 1 on the LHS of (13.7) while the other contributes the $\mathfrak{u}(1)$ factor towards $\mathfrak{su}(N) \otimes \mathfrak{su}(M) \otimes \mathfrak{u}(1)$. So



(13.8)

## 13.E Find $[2] \otimes [1,1]$ in $\mathfrak{su}(N)$



$= [2,1,1] \oplus [3,1]$ (13.9)

$$\dim([2]\otimes[1,1]) = \frac{N(N-1)}{2}\frac{N(N+1)}{2} = \frac{N^2(N-1)(N+1)}{4} \tag{13.10}$$

$$\dim([2,1,1]\oplus[3,1]) = \frac{N(N+1)(N+2)(N-1)}{8} \oplus \frac{N(N+1)(N-2)(N-1)}{8} \tag{13.11}$$

$$= \frac{N^2(N+1)(N-1)}{4} = \dim([2]\otimes[1,1]) \tag{13.12}$$

# 14 Chapter 14 - 3-D Harmonic Oscillator

## 14.A Show that the operators $O^k_{ij} = a^\dagger_i a^\dagger_j a^k - \frac{1}{4}(\delta^k_i a^\dagger_l a^\dagger_j a^l + \delta^k_j a^\dagger_l a^\dagger_i a^l)$ transform like a tensor operator in the $(2,1)$ representation.

$$[Q_\alpha, O^k_{ij}] = [Q_\alpha, a^\dagger_i a^\dagger_j a^k - \frac{1}{4}(\delta^k_i a^\dagger_l a^\dagger_j a^l + \delta^k_j a^\dagger_l a^\dagger_i a^l)] \tag{14.1}$$

$$\begin{aligned} =&[Q_\alpha, a^\dagger_i]a^\dagger_j a^k + a^\dagger_i[Q_\alpha, a^\dagger_j]a^k + a^\dagger_i a^\dagger_j[Q_\alpha, a^k] \\ &- \frac{1}{4}\delta^k_i\left([Q_\alpha, a^\dagger_l]a^\dagger_j a^l + a^\dagger_j[Q_\alpha, a^\dagger_j]a^l + a^\dagger_l a^\dagger_j[Q_\alpha, a^l]\right) \\ &- \frac{1}{4}\delta^k_j\left([Q_\alpha, a^\dagger_l]a^\dagger_i a^l + a^\dagger_i[Q_\alpha, a^\dagger_i]a^l + a^\dagger_l a^\dagger_j[Q_\alpha, a^l]\right) \end{aligned} \tag{14.2}$$

$$\begin{aligned} =&a^\dagger_l[T_\alpha]^l_i a^\dagger_j a^k + a^\dagger_i a^\dagger_l[T_\alpha]^l_j a^k - a^\dagger_i a^\dagger_j a^l[T^*_\alpha]^k_l - \frac{1}{4}a^\dagger_p[T_\alpha]^p_l(\delta^k_j a^\dagger_i + \delta^k_i a^\dagger_j)a^l \\ &- \frac{1}{4}a^\dagger_p a^\dagger_l(\delta^k_j[T_\alpha]^p_i + \delta^k_i[T_\alpha]^p_j)a^l + \frac{1}{4}a^\dagger_l[T^*_\alpha]^l_p(\delta^k_j a^\dagger_i + \delta^k_i a^\dagger_j)a^p. \end{aligned} \tag{14.3}$$

so the operator transforms like a $(2,1)$ tensor operator.

## 14.B Calculate non-zero matrix elements of the operator $O^3_{11}$.

### 14.B.a $\langle 0|(a_k b_k)a_i O^3_{11} a^\dagger_i(a^\dagger_l b^\dagger_l)|0\rangle$ (no sum over $i$)

For $i = 1$

$$\begin{aligned} \langle 0|(a_k b_k)a_1 O^3_{11} a^\dagger_1(a^\dagger_l b^\dagger_l)|0\rangle &= \langle 0|(a_k b_k)a_1(a^\dagger_1)^3 a_3(a^\dagger_l b^\dagger_l)|0\rangle && (14.4)\\ &= \langle 0|(a_k b_k)a_1(a^\dagger_1)^3 b^\dagger_3|0\rangle && (14.5)\\ &= \langle 0|a_1([a_k b_k, (a^\dagger_1)^3] + (a^\dagger_1)^3 a_k b_k)b^\dagger_3|0\rangle && (14.6)\\ &= \langle 0|3a_1(a^\dagger_1)^2 b_1 b^\dagger_3|0\rangle = 0 && (14.7) \end{aligned}$$

For $i = 2$

$$\begin{aligned} \langle 0|(a_k b_k)a_2 O^3_{11} a^\dagger_2(a^\dagger_l b^\dagger_l)|0\rangle &= \langle 0|(a_k b_k)(a^\dagger_1)^2 a_3(1 + a^\dagger_2 a_2)(a^\dagger_l b^\dagger_l)|0\rangle && (14.8)\\ &= \langle 0|(a_k b_k)(a^\dagger_1)^2 b^\dagger_3|0\rangle && (14.9)\\ &= \langle 0|b_k([a_k, (a^\dagger_1)^2] + (a^\dagger_1)^2 a_k)b^\dagger_3|0\rangle && (14.10)\\ &= \langle 0|2b_2 b^\dagger_3|0\rangle = 0 && (14.11) \end{aligned}$$

**14.B.b** $a_i^\dagger a_j^\dagger (a_l^\dagger b_l^\dagger)\left|0\right\rangle$

## 14.C Show that Eqn (14.13) generates standard angular momentum $\underline{r}\cdot\underline{p}$.

$L_3 = 2Q_2$, $L_1 = 2Q_7$, $L_2 = -2Q_5$, with $Q_a = a_k^\dagger [T_a]_{kl} a_l$ To check this we simply need to verify that the operators $L_i$ have the same commutation relations of those of angular momentum $[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k$.

$$L_3 = a_k^\dagger [\lambda_2]_{kl} a_l = \mathrm{i}(a_2^\dagger a_1 - a_1^\dagger a_2) \tag{14.12}$$

$$\begin{aligned}
[L_1, L_2] &= -4[Q_7, Q_5] && (14.13)\\
&= -\frac{4}{4}\left[a_k^\dagger[\lambda_2]_{kl}a_l, a_{k'}^\dagger[\lambda_5]_{k'l'}a_{l'}\right] && (14.14)\\
&= -\left[\begin{pmatrix} a_1^\dagger a_2^\dagger a_3^\dagger \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{pmatrix}, \begin{pmatrix} a_1^\dagger & a_2^\dagger & a_3^\dagger \end{pmatrix}\begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{pmatrix}\right] && (14.15)\\
&= -\left[\mathrm{i}(a_3^\dagger a_2 - a_2^\dagger a_3), \mathrm{i}(a_3^\dagger a_1 - a_1^\dagger a_3)\right] && (14.16)\\
&= -[a_3^\dagger, a_3]a_2 a_1^\dagger - [a_3, a_3^\dagger]a_2^\dagger a_1 && (14.17)\\
&= a_1^\dagger a_2 - a_2^\dagger a_1 = \mathrm{i}L_3 && (14.18)
\end{aligned}$$

Where the second from last line is given by the commutation relations $[a_k, a_j] = 0$, $[a_j, a_k^\dagger] = \delta_{kj}$. The other relations can be found in a similar fashion giving,

$$\begin{aligned}
[L_1, L_3] &= -\mathrm{i}L_2 && (14.19)\\
[L_2, L_3] &= \mathrm{i}L_1 && (14.20)
\end{aligned}$$

so the operators generate the standard angular momentum.

## 14.D Show $[Q_\alpha, a_k b_k] = 0$

$$\begin{aligned}
[Q_\alpha, a_k b_k] &= \frac{1}{4}\left[a_{k'}^\dagger[\lambda_\alpha]_{k'l'}a_{l'} - b_{k'}^\dagger[\lambda_\alpha^*]_{k'l'}b_{l'}, a_k b_k\right] && (14.21)\\
&= \frac{1}{4}\left([\lambda_\alpha]_{k'l'}[a_{k'}^\dagger, a_k]b_k a_{l'} - [\lambda_\alpha^*]_{k'l'}a_k[b_{k'}^\dagger, b_k]b_{l'}\right) && (14.22)\\
&= \frac{1}{4}\left([\lambda_\alpha]_{kl'}b_k a_{l'} - [\lambda_\alpha^*]_{kl'}a_k b_{l'}\right). && (14.23)
\end{aligned}$$

But $\lambda_\alpha^* = (\lambda_\alpha^\dagger)^T$ but because $\lambda_\alpha$'s Hermitian then $[\lambda_\alpha^*]_{kl} = [\lambda_\alpha^T]_{kl} = [\lambda_\alpha]_{lk}$

$$[Q_\alpha, a_k b_k] = \frac{1}{4}\left([\lambda_\alpha]_{kl'}b_k a_{l'} - [\lambda_\alpha]_{l'k}a_k b_{l'}\right) = 0 \tag{14.24}$$

after relabelling.

## 15 Chapter 15 - $\mathfrak{su}(6)$ and the Quark Model

### 15.A Find the $\mathfrak{su}(6)$ wave functions for all spin $1/2$ Baryons

$$\begin{aligned}|\Lambda, 1/2\rangle = \frac{\sqrt{3}}{6}\big[&(|uds\rangle - |dus\rangle)(|+-+\rangle - |-++\rangle)\\ &+ (|sud\rangle - |sdu\rangle)(|++-\rangle - |+-+\rangle)\\ &+ (|dsu\rangle - |usd\rangle)(|-++\rangle - |++-\rangle)\big].\end{aligned} \tag{15.1}$$

$$\begin{aligned}|P, 1/2\rangle = \frac{\sqrt{2}}{6}\big[&\,|uud\rangle\,(2\,|++-\rangle - |+-+\rangle - |-++\rangle)\\ &+ |udu\rangle\,(2\,|+-+\rangle - |-++\rangle - |++-\rangle)\\ &+ |duu\rangle\,(2\,|-++\rangle - |++-\rangle - |+-+\rangle)\big].\end{aligned} \tag{15.2}$$

$$\begin{aligned}|N, 1/2\rangle = \frac{\sqrt{2}}{6}\big[&\,|udd\rangle\,(2\,|-++\rangle - |+-+\rangle - |++-\rangle)\\ &+ |dud\rangle\,(2\,|+-+\rangle - |++-\rangle - |-++\rangle)\\ &+ |ddu\rangle\,(2\,|++-\rangle - |-++\rangle - |+-+\rangle)\big].\end{aligned} \tag{15.3}$$

$$\begin{aligned}\left|\Sigma^+, 1/2\right\rangle = \frac{\sqrt{2}}{6}\big[&\,|uus\rangle\,(2\,|++-\rangle - |+-+\rangle - |++-\rangle)\\ &+ |usu\rangle\,(2\,|+-+\rangle - |++-\rangle - |-++\rangle)\\ &+ |suu\rangle\,(2\,|-++\rangle - |++-\rangle - |+-+\rangle)\big].\end{aligned} \tag{15.4}$$

The total wavefunction must be symmetric under exchange of flavour-spin. Whereas $\Lambda$ is an isospin singlet state and is anti-symmetric in both flavour and spin, $\Sigma^0$ is the orthogonal state which is symmetric in both flavour and spin. So,

$$\begin{aligned}\left|\Sigma^0, 1/2\right\rangle = \frac{\sqrt{3}}{6}\big[&(|uds\rangle + |dus\rangle)(|+-+\rangle + |-++\rangle)\\ &+ (|sud\rangle + |sdu\rangle)(|++-\rangle + |+-+\rangle)\\ &+ (|dsu\rangle + |usd\rangle)(|-++\rangle + |++-\rangle)\big].\end{aligned} \tag{15.5}$$

$$\begin{aligned}\left|\Sigma^-, 1/2\right\rangle = \frac{\sqrt{2}}{6}\big[&\,|dds\rangle\,(2\,|++-\rangle - |+-+\rangle - |-++\rangle)\\ &+ |dsd\rangle\,(2\,|+-+\rangle - |++-\rangle - |-++\rangle)\\ &+ |sdd\rangle\,(2\,|-++\rangle - |++-\rangle - |+-+\rangle)\big].\end{aligned} \tag{15.6}$$

$$
\begin{aligned}
\left|\Xi^0, 1/2\right\rangle = \frac{\sqrt{2}}{6}\big[&(\left|ssu\right\rangle (2\left|++-\right\rangle - \left|+-+\right\rangle - \left|-++\right\rangle) \\
&+ \left|sus\right\rangle (2\left|+-+\right\rangle - \left|++-\right\rangle - \left|-++\right\rangle) \\
&+ \left|uss\right\rangle (2\left|-++\right\rangle - \left|++-\right\rangle - \left|+-+\right\rangle)\big].
\end{aligned} \tag{15.7}
$$

$$
\begin{aligned}
\left|\Xi^-, 1/2\right\rangle = \frac{\sqrt{2}}{6}\big[&\left|ssd\right\rangle (2\left|++-\right\rangle - \left|+-+\right\rangle - \left|-++\right\rangle) \\
&+ \left|sds\right\rangle (2\left|+-+\right\rangle - \left|++-\right\rangle - \left|-++\right\rangle) \\
&+ \left|dss\right\rangle (2\left|-++\right\rangle - \left|++-\right\rangle - \left|+-+\right\rangle)\big].
\end{aligned} \tag{15.8}
$$

## 15.B Calculate the magnetic moments for the spin $1/2$ Baryons

### 15.B.a In the $\mathfrak{su}(6)$ limit calculate the ratios to $\mu_P \propto 1$

The magnetic moments are given by

$$
\mu_{56} \propto \langle 56|Q\sigma_3|56\rangle \tag{15.9}
$$

The $N$ & $P$ magnetic moments are given by equations (15.21) & (15.20) respectively $\mu_P \propto \langle N, 1/2|Q\sigma_3|N, 1/2\rangle = \frac{-2}{3}$, & $\mu_P \propto \langle P, 1/2|Q\sigma_3|P, 1/2\rangle = 1$.

$$
\begin{aligned}
\mu_\Lambda &\propto \langle \Lambda, 1/2|Q\sigma_3|\Lambda, 1/2\rangle \\
&= \langle \Lambda, 1/2|\,\frac{\sqrt{3}}{18}\Big[(2\left|uds\right\rangle + \left|dus\right\rangle)(\left|+-+\right\rangle - \left|-++\right\rangle) \\
&\quad + (-\left|uds\right\rangle - 2\left|dus\right\rangle)(-\left|+-+\right\rangle - \left|-++\right\rangle) \\
&\quad + (-\left|uds\right\rangle + \left|dus\right\rangle)(\left|+-+\right\rangle - \left|-++\right\rangle) + \text{perms.}\Big] \qquad (15.10) \\
&= -\frac{1}{3}. \qquad (15.11)
\end{aligned}
$$

$$
\begin{aligned}
\mu_{\Sigma^+} &\propto \left\langle \Sigma^+, 1/2|Q\sigma_3|\Sigma^+, 1/2\right\rangle \\
&= \left\langle \Sigma^+, 1/2\right|\,\frac{\sqrt{2}}{18}\Big[2\left|uus\right\rangle (2\left|++-\right\rangle - \left|+-+\right\rangle + \left|++-\right\rangle) \\
&\quad + 2\left|uus\right\rangle (2\left|++-\right\rangle - \left|+-+\right\rangle - \left|++-\right\rangle) \\
&\quad - \left|uus\right\rangle (-2\left|++-\right\rangle - \left|+-+\right\rangle - \left|++-\right\rangle) + \text{perms.}\Big] \qquad (15.12) \\
&=1. \qquad (15.13)
\end{aligned}
$$

The rest of the magnetic moments are

$$\mu_{\Sigma^0} \propto \left\langle \Sigma^0, 1/2 | Q\sigma_3 | \Sigma^0, 1/2 \right\rangle = \frac{1}{3} \tag{15.14}$$

$$\mu_{\Sigma^-} \propto \left\langle \Sigma^-, 1/2 | Q\sigma_3 | \Sigma^-, 1/2 \right\rangle = -\frac{1}{3} \tag{15.15}$$

$$\mu_{\Xi^0} \propto \left\langle \Xi^0, 1/2 | Q\sigma_3 | \Xi^0, 1/2 \right\rangle = -\frac{2}{3} \tag{15.16}$$

$$\mu_{\Xi^-} \propto \left\langle \Xi^-, 1/2 | Q\sigma_3 | \Xi^-, 1/2 \right\rangle = -\frac{1}{3}. \tag{15.17}$$

The ratios to $\mu_P$ are:

$$\frac{\mu_N}{\mu_P} = -\frac{2}{3} \tag{15.18}$$

$$\frac{\mu_\Lambda}{\mu_P} = -\frac{1}{3} \tag{15.19}$$

$$\frac{\mu_{\Sigma^+}}{\mu_P} = 1 \tag{15.20}$$

$$\frac{\mu_{\Sigma^0}}{\mu_P} = \frac{1}{3} \tag{15.21}$$

$$\frac{\mu_{\Sigma^-}}{\mu_P} = -\frac{1}{3} \tag{15.22}$$

$$\frac{\mu_{\Xi^0}}{\mu_P} = -\frac{2}{3} \tag{15.23}$$

$$\frac{\mu_{\Xi^-}}{\mu_P} = -\frac{1}{3}. \tag{15.24}$$

Which matches the results found in Problem (11.C).

### 15.B.b Put in $\mathfrak{su}(3)$ symmetry breaking by including $m_s \neq m_{u,d}$.

Assume $m \simeq \frac{1}{3}m_p \simeq \frac{1}{3}m_d \simeq 310\text{MeV}$ and $m_s \simeq 490\text{MeV}$. Using $\frac{\mu_x}{m_x} = \left\langle x | \frac{1}{m_{quark}} Q\sigma_3 | x \right\rangle / \langle x|\sigma_3|x\rangle$. For example

$$\frac{\mu_P}{m_P} = \frac{\left\langle P | \sum_{quarks} \frac{Q\sigma_3}{m_{quark}} | P \right\rangle}{\langle P|\sigma_3|P\rangle} \tag{15.25}$$

$$\begin{aligned} &= \langle P| \frac{\sqrt{2}}{6} \Big[ \frac{2}{3m_u} |uud\rangle (2\,|{+}{+}{-}\rangle - |{+}{-}{+}\rangle + |{-}{+}{+}\rangle) \\ &\quad - \frac{1}{3m_d} |udu\rangle (-2\,|{+}{-}{+}\rangle - |{-}{+}{+}\rangle - |{+}{+}{-}\rangle) \\ &\quad + \frac{2}{3m_u} |duu\rangle (2\,|{-}{+}{+}\rangle + |{+}{+}{-}\rangle - |{+}{-}{+}\rangle) \Big] \end{aligned} \tag{15.26}$$

$$= \frac{1}{m} \tag{15.27}$$

$$\Longrightarrow \mu_P = \frac{3m}{m} = 3. \tag{15.28}$$

Repeating, in the same fashion, for the others we find

$$\mu_N = m_N \frac{-2}{3m} = -2 \tag{15.29}$$

$$\mu_\Lambda = \frac{-m_\Lambda}{3m_s} \simeq \frac{-m_s - 2m}{3m_s} \simeq -0.75 \tag{15.30}$$

$$\mu_{\Sigma^+} = -\frac{m_\Sigma}{6}(\frac{16}{3m} + \frac{2}{3m_s}) \simeq (\frac{8}{9m} + \frac{1}{9m_s})(2m + m_s) \simeq 3.4 \tag{15.31}$$

$$\mu_{\Sigma^0} = \frac{m_{\Sigma^0}}{3m_s} \simeq \frac{m_s + 2m}{3m_s} \simeq 0.75 \tag{15.32}$$

$$\mu_{\Sigma^-} = \frac{m_{\Sigma^-}}{6}(\frac{-8}{3m} + \frac{2}{3m_s}) \simeq (\frac{1}{m_s} - \frac{4}{9m})(2m + m_s) \simeq -1.33 \tag{15.33}$$

$$\mu_{\Xi^0} = \frac{m_{\Xi^0}}{6}(-\frac{8}{3m_s} - \frac{4}{3m}) \simeq -(\frac{4}{9m_s} + \frac{2}{9m})(2m_s + m) \simeq -2.11 \tag{15.34}$$

$$\mu_{\Xi^-} = \frac{m_{\Xi^-}}{6}(\frac{-8}{3m_s} + \frac{2}{3m}) \simeq (\frac{1}{9m} - \frac{4}{9m_s})(2m_s + m) \simeq -0.71. \tag{15.35}$$

## 15.C Show that $|\Lambda, 1/2\rangle$ is an isospin singlet

To show this we must first show that the state has isospin 0 and is also annihilated by the raising and lowering operators of the isospin algebra. The isospin operator on $\mathfrak{su}(6)$ is $I_3 \otimes \mathbb{I} = \frac{1}{2}\lambda_3 \otimes \mathbb{I}$.

$$I_3 |u\rangle = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \frac{1}{2} |u\rangle \tag{15.36}$$

Similarly, $I_3 |d\rangle = -\frac{1}{2} |d\rangle$, $I_3 |s\rangle = 0 |s\rangle = 0$. So,

$$I_3 \otimes \mathbb{I} |\Lambda, 1/2\rangle = \frac{\sqrt{3}}{6}\Big((\frac{1}{2} - \frac{1}{2} + 0) |uds\rangle - (-\frac{1}{2} + \frac{1}{2} + 0) |dus\rangle\Big) \tag{15.37}$$

$$\otimes (|+-+\rangle - |-++\rangle) + \text{perms} \tag{15.38}$$

$$= 0 \otimes (|+-+\rangle - |-++\rangle) + 0 \otimes \text{perms}. \tag{15.39}$$

Then, by applying the ladder operators $E_{\pm 1,0} = \frac{1}{2}(\lambda_1 \pm \mathrm{i}\lambda_2)$, $E_{\pm 1/2, \pm\sqrt{3}/2} = \frac{1}{2}(\lambda_4 \pm \mathrm{i}\lambda_5)$, $E_{\mp 1/2, \pm\sqrt{3}/2} = \frac{1}{2}(\lambda_6 \pm \mathrm{i}\lambda_7)$ Applying these operators to the flavour states, the nonvanishing possibilities are

$$E_{-1,0} |u\rangle = |d\rangle \tag{15.40}$$

$$E_{+1,0} |d\rangle = |u\rangle \tag{15.41}$$

$$E_{-1/2,-\sqrt{3}/2} |u\rangle = |s\rangle \tag{15.42}$$

$$E_{+1/2,+\sqrt{3}/2} |s\rangle = |u\rangle \tag{15.43}$$

$$E_{-1/2,+\sqrt{3}/2} |d\rangle = |s\rangle \tag{15.44}$$

$$E_{+1/2,-\sqrt{3}/2} |s\rangle = |d\rangle . \tag{15.45}$$

So,

$$E_{+1,0} \otimes \mathbb{I} \left|\Lambda, 1/2\right\rangle = \frac{\sqrt{3}}{6}\Big( |0ds\rangle - |uus\rangle + |uus\rangle - |d0s\rangle + |ud0\rangle - |du0\rangle \Big) \tag{15.46}$$
$$\otimes \left(|+-+\rangle - |-++\rangle\right) + \text{perms} \tag{15.47}$$
$$= 0 \otimes \left(|+-+\rangle - |-++\rangle\right) + 0 \otimes \text{perms}. \tag{15.48}$$

Similarly all other raising/lowering operators annihilate the flavour part of the state as the raising/lowering ops will always annihilate 4 of the states and the two remaining states will cancel because of the antisymmetry (compare this with the flavour symmetrical $\Sigma^0$ where the states do not cancel but add.) To see they cancel, for example look at

$$E_{\mp 1/2, \pm\sqrt{3}/2} \left|\Lambda, 1/2\right\rangle = \frac{\sqrt{3}}{6}\Big( |0ds\rangle - |0us\rangle + |u0s\rangle - |d0s\rangle + |udd\rangle - |dud\rangle \Big) \tag{15.49}$$
$$\otimes \left(|+-+\rangle - |-++\rangle\right) + \text{perms} \tag{15.50}$$
$$= 0 \tag{15.51}$$

because the state is symmetrical under the interchange of a pair of flavours $|udd\rangle = |dud\rangle$.

# 16 Chapter 16 - Color

## 16.A Find a relation between the sum of the products of color charges in the color singlet $q\bar{q}$ meson state and the $qq$ pair in a baryon.

The meson in the color singlet in $3 \otimes \bar{3} = 8 \oplus 1$, $q\bar{q} = q^i \bar{q}_i \in 1$. $|q\bar{q}\rangle = \frac{1}{\sqrt{3}}(|g\bar{g}\rangle + |b\bar{b}\rangle + |r\bar{r}\rangle)$. The $qq$ pair makes up part of the color singlet baryon state $\epsilon_{ijk} q^i q^j q^k$ or $3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3})$ and therefore $q^i q^j \in \bar{3}$, because $3 \otimes \bar{3} \ni 1$ but $6 \otimes \bar{3} \not\ni 1$. Also we know that the $qq$ state must be antisymmetric in the exchange of a single color, therefore $|qq\rangle = \frac{1}{\sqrt{6}}(|r\bar{b}\rangle - |b\bar{r}\rangle + |b\bar{g}\rangle - |g\bar{b}\rangle + |g\bar{r}\rangle - |r\bar{g}\rangle)$. To calculate products of charges we must calculate the operator $T_a^A T_a^B = \frac{1}{2}(T_a^2 - T_a^{A^2} - T_a^{B^2})$ on the states. The Casimir operator $T_a^2 = 0$ for the singlet and $T_a^2 = \frac{4}{3}$ for the $\bar{3}$. The color states are

$$|r\rangle = \left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rangle, \quad |b\rangle = \left|\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle, \quad |g\rangle = \left|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle. \tag{16.1}$$

The $\bar{3}$ generators are given by $T_a^*$.

$$-\frac{1}{2}(T_a^{A^2} + T_a^{B^2})\,|g\bar{g}\rangle = -\frac{1}{4}(1+1+1+1+\frac{4}{3})\,|g\bar{g}\rangle = -\frac{4}{3}\,|g\bar{g}\rangle \tag{16.2}$$
$$-\frac{1}{2}(T_a^{A^2} + T_a^{B^2})\,|b\bar{b}\rangle = -\frac{4}{3}\,|b\bar{b}\rangle \tag{16.3}$$
$$-\frac{1}{2}(T_a^{A^2} + T_a^{B^2})\,|r\bar{r}\rangle = -\frac{4}{3}\,|r\bar{r}\rangle\,. \tag{16.4}$$

So,

$$T_a^A T_a^B\,|q\bar{q}\rangle = \frac{1}{\sqrt{3}} \cdot 3 \cdot \frac{-4}{3} = \frac{-4}{\sqrt{3}}. \tag{16.5}$$

For the $qq$ state, we have

$$-\frac{1}{2}(T_a^{A2} + T_a^{B2})\,|qq\rangle = 0. \tag{16.6}$$

So,

$$T_a^2 T_b^2\,|qq\rangle = \frac{1}{2}\cdot\frac{1}{\sqrt{6}}\cdot 3\cdot\frac{4}{3} = \frac{2}{\sqrt{6}}. \tag{16.7}$$

## 16.B Suppose ∃ 'Quix' $Q \in 6$ of color $\mathfrak{su}(3)$, what is the spectrum of bound states? How do they transform under Gell-Mann's flavour $\mathfrak{su}(3)$?

The Quix is a flavour singlet. We need to find possible color singlets which can be built from a quix and a number of quarks and antiquarks, i.e. $6\otimes ? = 1 \oplus ....$ The possible ways to built color singlets from a 6, 3's and $\bar{3}$'s are

$$Q\bar{q}\bar{q} = 6\otimes\bar{3}\otimes\bar{3} = \begin{array}{|c|c|}\hline \ & \ \\ \hline a & c \\ \hline b & d \\ \hline\end{array} \oplus ... = 1 + ... \tag{16.8}$$

$$Qqq\bar{q} = 6\otimes 3\otimes 3\otimes\bar{3} = \begin{array}{|c|c|}\hline \ & \ \\ \hline a & c \\ \hline b & d \\ \hline\end{array} \oplus ... = 1 + ... \tag{16.9}$$

$$Qqqqq = 6\otimes 3\otimes 3\otimes 3\otimes 3 = \begin{array}{|c|c|}\hline \ & \ \\ \hline a & b \\ \hline c & d \\ \hline\end{array} \oplus ... = 1 + ... \tag{16.10}$$

To find out how these states transform under flavour $\mathfrak{su}(3)$ we must find out how the quarks $\in \mathfrak{su}(18) \to \mathfrak{su}(6)_{fs}\otimes\mathfrak{su}(3)_c$ decompose into irreps of $\mathfrak{su}(3)$ flavour. We must look for the color $\bar{6}$ quark state which couples to the color 6 Quix state.

In $Q\bar{q}\bar{q}$ look for the anti-symmetric 2-quark state $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ in $\bar{1}\bar{8}\otimes\bar{1}\bar{8} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} + ... = \frac{18!}{2!(18-2)!} = 1\bar{5}\bar{3}$. Then under $\in \mathfrak{su}(18) \to \mathfrak{su}(6)_{fs}\otimes\mathfrak{su}(3)_c$

$$1\bar{5}\bar{3} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \to \left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}, \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}, \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\right) \oplus ... \tag{16.11}$$

$$= (1\bar{5}, \bar{6}) \oplus (2\bar{1}\oplus\bar{3}) \tag{16.12}$$

We want the $\bar{6}$ of $\mathfrak{su}(3)_c$, therefore we pick the 15 of $\mathfrak{su}(6)_{fs}$. Then, under $\mathfrak{su}(6)_{fs} \to \mathfrak{su}(3)_f\otimes\mathfrak{su}(2)_s$

$$1\bar{5} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \to \left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}, \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}, \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\right) \tag{16.13}$$

$$= (\bar{3}, 3) \oplus (\bar{6}\oplus 1) \tag{16.14}$$

Thus, under flavour $\mathfrak{su}(3)$ $Q\bar{q}\bar{q}$ transforms as $\bar{3}\oplus\bar{6}$.

For the $Qqqqq$ state the antisymmetric part of $qqqq = 18 \otimes 18 \otimes 18 \otimes 18$ is $(1,1,1,1) = \ydiagram{1,1,1,1}$. Under $\in \mathfrak{su}(18) \to \mathfrak{su}(6)_{fs} \otimes \mathfrak{su}(3)_c$.

$$\ydiagram{1,1,1,1} \to (\ydiagram{2,2}, \ydiagram{2,2}) \oplus (\ydiagram{1,1,1,1}, \ydiagram{4}) \oplus (\ydiagram{3,1}, \ydiagram{2,1,1}) \oplus (\ydiagram{2,1,1}, \ydiagram{3,1}). \tag{16.15}$$

The $\bar{6}$ of $\mathfrak{su}(3)_c$ is $\ydiagram{2,2}$. So we pick $\ydiagram{2,2}$ of $\mathfrak{su}(6)_{fs}$. Under $\mathfrak{su}(6)_{fs} \to \mathfrak{su}(3)_f \otimes \mathfrak{su}(2)_s$.

$$\ydiagram{2,2} \to (\ydiagram{2,2}, \ydiagram{2,2}) \oplus (\ydiagram{2,1,1}, \ydiagram{3,1}) \oplus (\ydiagram{2,2}, \ydiagram{3,1}) \oplus (\ydiagram{3,1}, \ydiagram{2,2}) \oplus (\ydiagram{3,1}, \ydiagram{3,1}) \tag{16.16}$$

$$= (\bar{6},1) \oplus (3,3) \oplus (\bar{6},3) \oplus (15,1) \oplus (15,3). \tag{16.17}$$

Thus, $Qqqqq$ transforms as a $\bar{6} \oplus 3 \oplus \bar{6} \oplus 15 \oplus 15$ under flavour $\mathfrak{su}(3)$.

For the $Q\bar{q}qq$ state

$$\bar{1}8 \otimes 18 \otimes 18 = \ytableau{\, & \cdots & \,} \otimes (\ytableau{a \\ b} \oplus \ytableau{a & a}) \tag{16.18}$$

$$= \ytableau{\, & \cdots & \, & a \\ a} \oplus \ytableau{\, & \cdots & \, \\ a & a} \oplus \ytableau{\, & \cdots & \, & a \\ b} \tag{16.19}$$

as we are only interested in the antisymmetric parts and here $\ytableau{\cdots}$ stands for a youngs tableux with 1 row containing 15 boxes. Under $\mathfrak{su}(18) \to \mathfrak{su}(6)_{fs} \otimes \mathfrak{su}(3)_c$ the $\bar{6}$ of $\mathfrak{su}(3)_c$ with 19 boxes is given by $[3,3,3,3,3,2,2]$ where each number represents the number of boxes per column. So ...........

### 16.C Prove (a) & (b) and calculate $C(8)$, $C(10)$ & $C(6)$.

We are told that $C(D) = {T_a}^2$ and

$$\text{Tr}\,({T_a}^2) = \dim(D)C(D) = \sum_a \text{Tr}\,(T_a T_b) = \sum_a \delta_{aa} k_D = 8k_D. \tag{16.20}$$

**16.C.a** $k_{D_1 \oplus D_2} = k_{D_1} + k_{D_2}$

$$\dim(D_1 \oplus D_2)C(D_1 \oplus D_2) = \sum_a \text{Tr}\,(T^2_{aD_1 \oplus D_2}) = 8k_{D_1 \oplus D_2}. \tag{16.21}$$

But, because $D_1 \oplus D_2 = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$. Thus the generators of $D_1 \oplus D_2$ may be decomposed into independent generators with dimension $\dim(D_1 \oplus D_2)$ act trivially on their respective invariant subspaces. So that

$$\dim(D_1 \oplus D_2)C(D_1 \oplus D_2) = \sum_a \text{Tr}\,(T^2_{aD_1} \oplus T^2_{aD_2}) = 8k_{D_1} + 8k_{D_2}. \tag{16.22}$$

Therefore $k_{D_1 \oplus D_2} = k_{D_1} + k_{D_2}$.

### 16.C.b $k_{D_1 \otimes D_2} = \mathbf{dim}(D_2) k_{D_1} + \mathbf{dim}(D_1) k_{D_2}$

$$\dim(D_1 \otimes D_2) C(D_1 \otimes D_2) = \sum_a \mathrm{Tr}\,(T^2_{aD_1 \otimes D_2}) = 8 k_{D_1 \otimes D_2}. \tag{16.23}$$

But $D_1 \otimes D_2 = D_1 \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes D_2$ and $\dim(D_1 \otimes D_2) = \dim(D_1)\dim(D_2)$. Therefore

$$\sum_a \mathrm{Tr}\,(T^2_{aD_1 \otimes D_2}) = \sum_a (\mathrm{Tr}\,(T^2_{aD_1} \otimes \mathbb{I}_2) + \mathrm{Tr}\,(\mathbb{I}_1 \otimes T^2_{aD_2})) \tag{16.24}$$

$$= \dim(D_2)(\sum_a \delta_{aa} k_{D_1}) + \dim(D_1)(\sum_a \delta_{aa} k_{D_2}) \tag{16.25}$$

$$= 8\dim(D_2) k_{D_1} + 8\dim(D_1) k_{D_2} \tag{16.26}$$

So $k_{D_1 \otimes D_2} = \dim(D_2) k_{D_1} + \dim(D_1) k_{D_2}$.

For the Casimir operators, for $D = 3, \bar{3}$, $C(3) = \frac{4}{3}$. For $D = 1$, $C(1) = 0$. We will also need the $\mathfrak{su}(3)$ decompositions $3 \otimes \bar{3} = 8 \oplus 1$, $3 \otimes 3 = 6 \oplus \bar{3}$, $3 \otimes 6 = 8 \oplus 10$.

$$k_{3 \otimes \bar{3}} = \dim(3) k_{\bar{3}} + \dim(\bar{3}) k_3 = 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 3 \tag{16.27}$$

$$= k_{8 \oplus 1} = k_8 + k_1 = k_8. \tag{16.28}$$

So $C(8) = \frac{8}{8} \times 3 = 3$

$$k_{3 \otimes 3} = 3 = k_{6 \otimes \bar{3}} = k_6 + k_{\bar{3}} = k_6 + \frac{1}{2}. \tag{16.29}$$

So, $k_6 = \frac{5}{2}$ and $C(6) = \frac{8}{6} \times \frac{5}{2} = \frac{10}{3}$.

$$k_{6 \otimes 3} = 6 k_3 + 3 k_6 = \frac{6}{2} + \frac{3 \times 5}{2} = \frac{21}{2} \tag{16.30}$$

$$= k_{8 \oplus 10} = k_8 + k_{10} = 3 + k_{10}. \tag{16.31}$$

So $k_{10} = \frac{21}{2} - 3 = \frac{15}{2}$, thus $C(10) = \frac{8}{10} \times \frac{15}{2} = 6$.

# 17 Chapter 17 - Constituent Quarks

### 17.A

### 17.B Estimate $m_{u,d}/m_s$ by comparing $\rho - \pi$ mass splitting with the $K^* - K$ mass splitting. Make an independent estimate of the ratio using combinations of the $\Sigma^* - \Sigma$ and $\Sigma - \Lambda$ splittings.

The particle masses are given by the quark masses + a contribution from the strong force (often called the chromomagnetic mass splitting)

$$H_{CM} = k \sum_{\text{pairs } i,j} \frac{\underline{\sigma}_i \cdot \underline{\sigma}_j}{m_i m_j} \tag{17.1}$$

For mesons we can write the contribution to the Hamiltonian as

$$M = \sum_i m_i + H_{CM} \tag{17.2}$$

$$= m_1 + m_2 + \frac{k}{2m_1m_2}\left(\sum_{i,j} \underline{\sigma}_i \cdot \underline{\sigma}_j - \sum_j \underline{\sigma}_j \cdot \underline{\sigma}_j\right) \tag{17.3}$$

$$= m_1 + m_2 + \frac{k}{2m_1m_2}\left(\underline{S}^2 - \underline{S_1}^2 - \underline{S_2}^2\right) \tag{17.4}$$

$$= m_1 + m_2 + \frac{k}{2m_1m_2}\left(\underline{S}^2 - \underline{S_1}^2 - \underline{S_2}^2\right) \tag{17.5}$$

$$= m_1 + m_2 + \frac{k}{2m_1m_2}\left(S(S+1) - S_1(S_1+1) - S_2(S_2+1)\right) \tag{17.6}$$

where $S_i = 1/2$ are the spins of the individual quarks because $\underline{S}^2 \left|l,m\right\rangle = l(l+1)\left|l,m\right\rangle$ and $S = 1, 0$ depending on whether the spins are aligned or anti-aligned

The $\pi$ mesons are spin-0 singlets so that $S = 0$ they consist of a $u$ and $d$ quark-antiquark pair where the spins are anti-aligned.

$$\left|\pi^+\right\rangle = \frac{1}{\sqrt{2}}\left|u\bar{d}\right\rangle\left(\left|+-\right\rangle + \left|-+\right\rangle\right) \tag{17.7}$$

$$m_\pi = \left\langle\pi^+\right| M \left|\pi^+\right\rangle = m_u + m_d - \frac{3k}{4m_um_d} = 2m - \frac{3k}{4m^2}. \tag{17.8}$$

where we take $m_u \simeq m_d = m$.

The $\rho$ mesons are spin-1 particles so that $S = 1$ they consist of a $u$ and $d$ quark-antiquark pair where the spins are aligned.

$$\left|\rho^+\right\rangle = \frac{1}{\sqrt{2}}\left|u\bar{d}\right\rangle\left(\left|++\right\rangle + \left|--\right\rangle\right) \tag{17.9}$$

$$m_\rho = \left\langle\rho^+\right| M \left|\rho^+\right\rangle = m_u + m_d + \frac{k}{4m_um_d} = 2m + \frac{k}{4m^2} \tag{17.10}$$

so

$$m_\rho - m_\pi = \frac{k}{m^2} \tag{17.11}$$

Similarly, the Kaon's consist of a $u/d - s$ quark-antiquark pair. The $K$ are spin-0 particles while $K^*$ are spin-1 particles where the spins are aligned so that

$$m_K = m + m_s - \frac{3}{4mm_s} \tag{17.12}$$

$$m_{K^*} = m + m_s + \frac{1}{4mm_s} \tag{17.13}$$

Therefore

$$m_{K^*} - m_K = \frac{1}{mm_s}. \tag{17.14}$$

So an estimate for $\frac{m_s}{m}$ is

$$\frac{m_s}{m} = \frac{m_\rho - m_\pi}{m_{K^*} - m_K}. \tag{17.15}$$

experimentally

$$m_\rho - m_\pi \simeq 770\text{MeV} - 138\text{MeV} = 632\text{MeV} \tag{17.16}$$

$$m_{K^*} - m_K \simeq 894\text{MeV} - 496\text{MeV} = 398\text{MeV} \tag{17.17}$$

so

$$\frac{m_s}{m} = \frac{m_\rho - m_\pi}{m_{K^*} - m_K} \simeq 1.58. \tag{17.18}$$

experimentally $m_s \simeq 483\text{MeV}$, $m = m_u = m_d \simeq 310\text{MeV}$ so that $\frac{m_s}{m} \simeq 1.55$ which is pretty close.

Both the $\Sigma$ and $\Lambda$ are spin 1/2 states made up of $u/d - u/d - s$ quarks, however the $\Sigma$ is an isospin 1 state and is flavour symmetric in $u$ and $d$ quarks and so the spins are aligned. The $\Lambda$ is an isospin 0 state is antisymmetric in $u$ and $d$ flavour and the spins are anti-aligned. The $\Sigma^*$ is a spin 3/2 state $u/d - u/d - s$ where the spins of the 3 quarks are all aligned. for the Baryons

$$\begin{aligned} M =& m_1 + m_2 + m_3 + \frac{k}{2m_1m_2}\left(\underline{S}_{12}^2 - \underline{S}_1^2 - \underline{S}_2^2\right) + \frac{k}{2m_1m_3}\left(\underline{S}_{13}^2 - \underline{S}_1^2 - \underline{S}_3^2\right) \\ &+ \frac{k}{2m_2m_3}\left(\underline{S}_{23}^2 - \underline{S}_2^2 - \underline{S}_3^2\right) \end{aligned} \tag{17.19}$$

$$\begin{aligned} =& m_1 + m_2 + m_3 + \frac{k}{2m_1m_2}\left(\underline{S}_{12}(\underline{S}_{12}+1) - \underline{S}_1(\underline{S}_1+1) - \underline{S}_2(\underline{S}_2+1)\right) \\ &+ \frac{k}{2m_1m_3}\left(\underline{S}_{13}(\underline{S}_{13}+1) - \underline{S}_1(\underline{S}_1+1) - \underline{S}_3(\underline{S}_3)\right) \\ &+ \frac{k}{2m_2m_3}\left(\underline{S}_{23}(\underline{S}_{23}+1) - \underline{S}_2(\underline{S}_2+1) - \underline{S}_3(\underline{S}_3+1)\right) \end{aligned} \tag{17.20}$$

The wavefunctions are given by

$$\left|\Sigma^0\right\rangle = \frac{1}{\sqrt{2}}\left|uds\right\rangle\left(\left|++-\right\rangle + \left|--+\right\rangle\right) \tag{17.21}$$

$$\left|\Lambda^0\right\rangle = \frac{1}{\sqrt{4}}\left|uds\right\rangle\left(\left|+-+\right\rangle + \left|-++\right\rangle + \left|+--\right\rangle + \left|+--\right\rangle\right) \tag{17.22}$$

$$\left|\Sigma^{*0}\right\rangle = \frac{1}{\sqrt{2}}\left|uds\right\rangle\left(\left|+++\right\rangle + \left|---\right\rangle\right). \tag{17.23}$$

Therefore,

$$m_\Sigma = \left\langle\Sigma^0\right| M \left|\Sigma^0\right\rangle = 2m + m_s + \frac{k}{4m^2} - \frac{6k}{4mm_s} \tag{17.24}$$

$$m_\Lambda = \left\langle\Lambda^0\right| M \left|\Lambda^0\right\rangle = 2m + m_s - \frac{3k}{4m^2} + \frac{k}{4mm_s} - \frac{3k}{4mm_s} \tag{17.25}$$

$$m_{\Sigma^*} = \left\langle\Sigma^{*0}\right| M \left|\Sigma^{*0}\right\rangle = 2m + m_s + \frac{k}{4m^2} + \frac{2k}{4mm_s}. \tag{17.26}$$

So the mass splittings are

$$m_{\Sigma^*} - m_\Sigma = \frac{k}{mm_s} \tag{17.27}$$

$$m_\Sigma - m_\Lambda = \frac{k}{m^2} - \frac{k}{mm_s}. \tag{17.28}$$

To estimate $\frac{m_s}{m}$ we take

$$\frac{m_\Sigma - m_\Lambda + m_{\Sigma^*} - m_\Sigma}{m_{\Sigma^*} - m_\Sigma} = \frac{m_{\Sigma^*} - m_\Lambda}{m_{\Sigma^*} - m_\Sigma} = \frac{m_s}{m} \tag{17.29}$$

experimentally

$$\frac{m_s}{m} = \frac{m_{\Sigma^*} - m_\Lambda}{m_{\Sigma^*} - m_\Sigma} \simeq \frac{1385\text{MeV} - 1116\text{MeV}}{1385\text{MeV} - 1193\text{MeV}} = \frac{269}{192} \simeq 1.40. \tag{17.30}$$

### 17.C

## 18 Chapter 18 - $\mathfrak{su}(5)$

### 18.A Check explicitly that mass terms for the $e^-$ and $u, d$ quarks are allowed in the $\mathfrak{su}(2) \times \mathfrak{u}(1)$ Higgs model

$[R_a, \phi_r^\dagger] = \phi_s^\dagger [\sigma_a]_{sr}/2$, $[S, \phi_r^\dagger] = \phi_r^\dagger/2$. Higgs can "produce" a mass via SSB provided the tensor product of a RH particle and a RH anti-particle $\ni$ Higgs rep $(1,2)_{1/2}$ or its conjugate rep. For a RH electron

$$(1,1)_{-1} \otimes (1,2)_{1/2} = (1,2)_{-1/2} \tag{18.1}$$

But, as $2 = \overline{2}$ for $\mathfrak{su}(2)$, so that $(1,2)_{-1/2} = (1,\overline{2})_{-1/2} = \overline{(1,2)_{1/2}}$ which is the conjugate Higgs rep.

For $u$ quark, using $\overline{3} \otimes 3 = 8 \oplus 1$

$$(3,1)_{2/3} \otimes (\overline{3},2)_{-1/6} = (8,2)_{1/2} \oplus (1,2)_{1/2} \tag{18.2}$$

which therefore contains the Higgs rep.

For the $d$ quarks

$$(3,1)_{-1/3} \otimes (\overline{3},2)_{-1/6} = (8,2)_{-1/2} \oplus (1,2)_{-1/2} = (8,2)_{-1/2} \oplus \overline{(1,2)_{1/2}} \tag{18.3}$$

Which therefore contains the conjugate Higgs rep.

### 18.B Find the symmetric tensor product $\big((3,1)_{-1/3} \oplus (1,2)_{1/2}\big)$ with itself

$$\big((3,1)_{-1/3} \oplus (1,2)_{1/2}\big) \oplus_{sym} \big((3,1)_{-1/3} \oplus (1,2)_{1/2}\big) \tag{18.4}$$

In $\mathfrak{su}(3)$

$$3 \otimes 3 = 6 \oplus \overline{3} \tag{18.5}$$

$$= \ydiagram{2} \oplus \ydiagram{1,1} \tag{18.6}$$

$$= \text{Sym} \oplus \text{AntiSym}. \tag{18.7}$$

$$3 \otimes \overline{3} = 1 \oplus 8 \tag{18.8}$$

$$= \ydiagram{1,1,1} \oplus \ydiagram{2,1} \tag{18.9}$$

$$= \text{AntiSym} \oplus \text{AntiSym}. \tag{18.10}$$

In $\mathfrak{su}(2)$

$$2 \otimes 2 = 3 \oplus 1 \tag{18.11}$$
$$= \ydiagram{2} \oplus \ydiagram{1,1} \tag{18.12}$$
$$= \text{Sym} \oplus \text{AntiSym}. \tag{18.13}$$

So

$$\big((3,1)_{-1/3} \oplus (1,2)_{1/2}\big) \otimes_{sym} \big((3,1)_{-1/3} \oplus (1,2)_{1/2}\big) = (6,1)_{-2/3} + (3,2)_{1/6} + (1,3)_{1} \tag{18.14}$$

**18.C Find the symmetric tensor product $\big((3,1)_{2/3} \oplus (1,1)_{-1} \oplus (\overline{3},2)_{-1/6}\big)$ with itself**

$$\begin{aligned}&\big((3,1)_{2/3} \oplus (1,1)_{-1} \oplus (\overline{3},2)_{-1/6}\big) \otimes_{sym} \big((3,1)_{2/3} \oplus (1,1)_{-1} \oplus (\overline{3},2)_{-1/6}\big)\\ &= (6,1)_{4/3} \oplus (1,1)_{-2} \oplus (3,1)_{-1/3} \oplus (\overline{3},2)_{-7/6} \oplus (\overline{6},1)_{-1/3} \oplus (3,3)_{-1/3}.\end{aligned} \tag{18.15}$$

**18.D Show that if the operator $O = \overline{e}^{\dagger}\epsilon^{abc}u_a u_b d_c$ appears in the Hamiltonian it has the correct charge and color properties to allow $P \to \pi^0 e^+$ decay.**

$$|P\rangle \to \left|\pi^0 e^+\right\rangle \tag{18.16}$$
$$|uud\rangle \to \frac{1}{\sqrt{2}}\left(|u\overline{u}\rangle - \left|d\overline{d}\right\rangle\right) \otimes \left|e^+\right\rangle \tag{18.17}$$

$O$ transforms under the $(1,1)_{-1}$ representation. It is therefore colorless transforming trivially under $\mathfrak{su}(3)_c$ and it can therefore couple to the colorless proton, likewise the $\left|\pi^0 e^+\right\rangle$ state is also colorless. Applying the charge operator $Q$ to the states

$$Q\,|P\rangle = +1\,|P\rangle \tag{18.18}$$
$$Q\left|\pi^0 e^+\right\rangle = +1\left|\pi^0 e^+\right\rangle \tag{18.19}$$
$$QO = +1 - \frac{2}{3} - \frac{2}{3} + \frac{1}{3} = 0. \tag{18.20}$$

Thus $O$ is chargeless as required then $O\,|P\rangle \to \left|\pi^0 e^+\right\rangle$ is allowed because in $\mathfrak{su}(5)$, $u, d \in$ the same rep. as $e^+$, which is not true in the standard model $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ theory.

**18.E How do the 45 and 50 of $\mathfrak{su}(5)$ transform under $\mathfrak{su}(3) \otimes \mathfrak{su}(2) \otimes \mathfrak{u}(1)$ subgroup?**

It is easier to work with the conjugate representations and then conjugate again at the end. Under $\mathfrak{su}(5) \to \mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$

$$\begin{aligned}\overline{45} = \ydiagram{2,1,1} &\to \left(\ydiagram{2,1,1}\,,\bullet\right) \oplus \left(\ydiagram{2,1}\,,\ydiagram{1}\right) \oplus \left(\ydiagram{1,1,1}\,,\ydiagram{1}\right) \oplus \left(\ydiagram{1,1}\,,\ydiagram{1,1}\right)\\ &\oplus \left(\ydiagram{1,1}\,,\ydiagram{2}\right) \oplus \left(\ydiagram{2}\,,\ydiagram{1,1}\right) \oplus \left(\ydiagram{1}\,,\ydiagram{2,1}\right).\end{aligned} \tag{18.21}$$

Taking the conjugate of (18.21) gives

$$45 = \ydiagram{2,2,1,1} \to (\overline{3},1) \oplus (8,2) \oplus (1,2) \oplus (3,1) \oplus (3,3) \oplus (\overline{6},1) \oplus (\overline{3},2). \tag{18.22}$$

For the 50

$$\begin{aligned}\overline{50} = \ydiagram{2,2} \to & (\ydiagram{2,2}, \bullet) \oplus (\ydiagram{2,1}, \ydiagram{1}) \oplus (\ydiagram{1}, \ydiagram{2,1}) \oplus (\ydiagram{1,1}, \ydiagram{1,1}) \\ & \oplus (\ydiagram{1,1}, \ydiagram{2}) \oplus (\bullet, \ydiagram{2,2}).\end{aligned} \tag{18.23}$$

Taking the conjugate gives

$$50 \to (6,1) \oplus (8,2) \oplus (\overline{3},1) \oplus (3,1) \oplus (3,3) \oplus (1,1). \tag{18.24}$$

# 19 Chapter 19 - Classical Groups

## 19.A Show that the 36 matrices $\sigma_a \otimes \mathbb{I} \otimes \mathbb{I} \equiv \sigma_a$, $\mathbb{I} \otimes \tau_a \otimes \mathbb{I} \equiv \tau_a$, $\mathbb{I} \otimes \mathbb{I} \otimes \eta_a \equiv \eta_a$, $\sigma_a \otimes \tau_b \otimes \eta_c \equiv \sigma_a \tau_b \eta_c$ form a Lie algebra. Find the roots, simple roots and the Dynkin diagram. What is the algebra?

First, construct the commutators

$$[\sigma_a, \sigma_b] = 2\mathrm{i}\epsilon_{abc}\sigma_c \tag{19.1}$$

$$[\tau_a, \tau_b] = 2\mathrm{i}\epsilon_{abc}\tau_c \tag{19.2}$$

$$[\eta_a, \eta_b] = 2\mathrm{i}\epsilon_{abc}\eta_c \tag{19.3}$$

$$[\sigma_a, \tau_b] = [\sigma_a, \eta_b] = [\tau_a, \eta_b] = 0 \tag{19.4}$$

$$[\sigma_a, \sigma_b\tau_c\eta_d] = 2\mathrm{i}\epsilon_{abe}\sigma_e\tau_b\eta_c \tag{19.5}$$

$$[\tau_a, \sigma_b\tau_c\eta_d] = 2\mathrm{i}\epsilon_{ace}\sigma_a\tau_e\eta_c \tag{19.6}$$

$$[\sigma_a, \sigma_b\tau_c\eta_d] = 2\mathrm{i}\epsilon_{ace}\sigma_a\tau_b\eta_e \tag{19.7}$$

$$\begin{aligned}[\sigma_a\tau_b\eta_c, \sigma_d\tau_e\eta_f] =& \sigma_a\sigma_d \otimes \left(\delta_{be}\delta_{cf}\mathbb{I}\otimes\mathbb{I} + \mathrm{i}\epsilon_{bei}\delta_{cf}\tau_i\otimes\mathbb{I} + \mathrm{i}\epsilon_{cfi}\delta_{be}\mathbb{I}\otimes\eta_i - \epsilon_{bei}\epsilon_{cfj}\tau_i\otimes\eta_j\right) \\ & - \sigma_d\sigma_a \otimes \left(\delta_{be}\delta_{cf}\mathbb{I}\otimes\mathbb{I} - \mathrm{i}\epsilon_{bei}\delta_{cf}\tau_i\otimes\mathbb{I} - \mathrm{i}\epsilon_{cfi}\delta_{be}\mathbb{I}\otimes\eta_i - \epsilon_{bei}\epsilon_{cfj}\tau_i\otimes\eta_j\right) \end{aligned} \tag{19.8}$$

$$\begin{aligned} =& 2\mathrm{i}\epsilon_{adi}\delta_{be}\delta_{cf}\sigma_i\otimes\mathbb{I}\otimes\mathbb{I} + 2\mathrm{i}\epsilon_{bei}\delta_{cf}\delta_{ad}\mathbb{I}\otimes\tau_i\otimes\mathbb{I} + 2\mathrm{i}\epsilon_{cfi}\delta_{be}\delta_{ad}\mathbb{I}\otimes\mathbb{I}\otimes\eta_i \\ & - 2\mathrm{i}\epsilon_{adk}\epsilon_{bei}\epsilon_{cfj}\sigma_k\otimes\tau_i\otimes\eta_j \end{aligned} \tag{19.9}$$

$$= 2\mathrm{i}\left(\epsilon_{adi}\delta_{be}\delta_{cf}\sigma_i + \epsilon_{bei}\delta_{cf}\delta_{ad}\tau_i + \epsilon_{cfi}\delta_{be}\delta_{ad}\eta_i - \epsilon_{adk}\epsilon_{bei}\epsilon_{cfj}\sigma_k\tau_i\eta_j\right). \tag{19.10}$$

Thus the commutators close and the matrices form a Lie algebra.

We pick the Cartan subalgebra as

$$H_1 = \sigma_3 = \mathrm{diag}\Big(1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1,\Big) \tag{19.11}$$

$$H_2 = \tau_3 = \mathrm{diag}\Big(1,\ 1,\ -1,\ -1,\ 1,\ 1,\ -1,\ -1\Big) \tag{19.12}$$

$$H_3 = \eta_3 = \mathrm{diag}\Big(1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1\Big) \tag{19.13}$$

$$H_4 = \sigma_3\tau_3\eta_3 = \mathrm{diag}\Big(1,\ -1,\ -1,\ 1,\ -1,\ 1,\ 1,\ -1\Big)\,. \tag{19.14}$$

The $H_m$ all have eigenvalues $\lambda = \pm 1$ and the eigenvectors are given by the eight dimensional unit vectors which we may write in component form as $[e^k]_m = \delta^k_m$, $k, m = 1...8$ e.g.

$$e^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,. \tag{19.15}$$

Then, the weights are given by

$$\mu_1 = (1, 1, 1, 1) = -\mu_8 \tag{19.16}$$

$$\mu_2 = (1, 1, -1, -1) = -\mu_7 \tag{19.17}$$

$$\mu_3 = (1, -1, 1, -1) = -\mu_6 \tag{19.18}$$

$$\mu_4 = (1, -1, -1, 1) = -\mu_5. \tag{19.19}$$

Then we expect $36 - 4 = 32$ roots for the 32 remaining generators. We can construct $4 \cdot 3 \cdot 2 = 24$ of the roots by considering just $\mu_k$ for $k = 1..4$, since $\mu_1 = -\mu_8$, etc. The generators connect the weight vectors so that 24 of the roots are given by the difference in the weights $\pm\mu_i \pm \mu_j$, $i, j = 1..4$, $i \neq j$. The remaining 8 roots must just be the weights themselves $\pm\mu_i$, $i = 1...4$.

Defining positivity to be from left to right the normalised simple roots are given by the lowest positive state

$$\alpha_1 = \mu_1 - \mu_2 = \frac{1}{\sqrt{8}}(0, 0, 2, 2) \tag{19.20}$$

$$\alpha_2 = \mu_2 - \mu_3 = \frac{1}{\sqrt{8}}(0, 2, -2, 0) \tag{19.21}$$

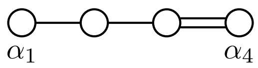
$$\alpha_3 = \mu_3 - \mu_4 = \frac{1}{\sqrt{8}}(0, 0, 2, -2) \tag{19.22}$$

$$\alpha_4 = \mu_4 = \frac{1}{2}(1, -1, -1, 1). \tag{19.23}$$

To obtain the Dynkin diagram we use the formula $\theta_{\alpha_i\alpha_j} = \arccos\left(\frac{\alpha_i\cdot\alpha_j}{\sqrt{\alpha_i^2\alpha_j^2}}\right)$. So that

$$\theta_{\alpha_1\alpha_2} = \frac{2}{3}\pi \tag{19.24}$$
$$\theta_{\alpha_2\alpha_3} = \frac{2}{3}\pi \tag{19.25}$$
$$\theta_{\alpha_3\alpha_4} = \frac{3}{4}\pi \tag{19.26}$$

So the Dynkin diagram is




which is $B_4$ or $\mathfrak{so}(9)$.

**19.B Show that the 28 matrices $\sigma_a \otimes \mathbb{I} \otimes \mathbb{I} \equiv \sigma_a$, $\mathbb{I} \otimes \tau_a \otimes \mathbb{I} \equiv \tau_a$, $\mathbb{I} \otimes \mathbb{I} \otimes \eta_3 \equiv \eta_3$, $\sigma_a \otimes \eta_1 \otimes \mathbb{I} \equiv \sigma_a\eta_1$, $\sigma_a \otimes \eta_2 \otimes \mathbb{I} \equiv \sigma_a\eta_2$, $\mathbb{I} \otimes \tau_a \otimes \eta_1 \equiv \tau_a\eta_1$, $\mathbb{I} \otimes \tau_a \otimes \eta_2 \equiv \tau_a\eta_2$, $\sigma_a \otimes \tau_b \otimes \eta_3 \equiv \sigma_a\tau_b\eta_3$ form a Lie algebra. Find the roots, simple roots and the Dynkin diagram. What is the algebra?**

To show the matrices form a Lie algebra we can, rather long-windedly, evaluate the commutators, letting Greek indices $\mu, \nu$ run from 1 to 2,

$$[\sigma_a, \sigma_b] = 2\mathrm{i}\epsilon_{abc}\sigma_c \tag{19.27}$$
$$[\sigma_a, \sigma_b\eta_\mu] = 2\mathrm{i}\epsilon_{abc}\sigma_c\eta_\mu \tag{19.28}$$
$$[\sigma_a, \tau_\mu] = [\sigma_a, \tau_b\eta_\mu] = 0 \tag{19.29}$$
$$[\sigma_a, \sigma_b\tau_c\eta_3] = 2\mathrm{i}\epsilon_{abd}\sigma_d\tau_c\eta_3 \tag{19.30}$$
$$[\tau_a, \tau_b] = 2\mathrm{i}\epsilon_{abc}\tau_c \tag{19.31}$$
$$[\tau_a, \tau_b\eta_\mu] = 2\mathrm{i}\epsilon_{abc}\tau_c\eta_\mu \tag{19.32}$$
$$[\tau_a, \sigma_b\tau_c\eta_3] = 2\mathrm{i}\epsilon_{acd}\sigma_b\tau_d\eta_3 \tag{19.33}$$
$$[\tau_a, \eta_3] = [\tau_a, \sigma_b\eta_\mu] = 0 \tag{19.34}$$
$$[\eta_3, \eta_3] = [\eta_3, \sigma_a\tau_b\eta_3] = 0 \tag{19.35}$$
$$[\eta_3, \sigma_a\eta_\mu] = 2\mathrm{i}\epsilon_{3\mu\nu}\sigma_a\eta_\nu \tag{19.36}$$
$$[\eta_3, \tau_a\eta_\mu] = 2\mathrm{i}\tau_a\eta_\mu \tag{19.37}$$
$$[\sigma_a\eta_\nu, \sigma_b\eta_\mu] = 2\mathrm{i}\epsilon_{abc}\sigma_c(\delta_{\nu\mu}\mathbb{I} + \mathrm{i}\epsilon_{\mu\nu3}\eta_3) \tag{19.38}$$
$$[\sigma_a\eta_\mu, \tau_b\eta_\nu] = 2\mathrm{i}\epsilon_{\mu\nu3}\sigma_a\tau_b\eta_3 = 0 \tag{19.39}$$
$$[\sigma_a\eta_\mu, \sigma_b\tau_c\eta_3] = 2\epsilon_{abd}\sigma_d\tau_c(\delta_{\nu\mu}\mathbb{I} + \mathrm{i}\epsilon_{\mu\nu3}\eta_3) \tag{19.40}$$
$$[\tau_a\eta_\mu, \tau_b\eta_\nu] = 2\mathrm{i}\epsilon_{abc}\tau_c(\delta_{\nu\mu}\mathbb{I} + \mathrm{i}\epsilon_{\mu\nu3}\eta_3) \tag{19.41}$$
$$[\sigma_a\tau_b\eta_3, \sigma_c\tau_d\eta_3] = 2\mathrm{i}\epsilon_{ace}\delta_{bd}\sigma_e + 2\mathrm{i}\epsilon_{bde}\delta_{ac}\tau_e. \tag{19.42}$$

and thus the algebra closes under commutation so the matrices form a Lie algebra.

The algebra is also of rank 4, and we may pick the Cartan generators to be the same as those in Problem(19.A) given by (19.11), therefore the weight are the same also, the difference now is that we have only $28 - 4 = 24$ roots given by $\pm\mu_i \pm \mu_j$, $i \neq j$, $i, j = 1...4$, now the normalised simple roots are just

$$\alpha_1 = \mu_1 - \mu_2 = \frac{1}{\sqrt{8}}(0, 0, 2, 2) \tag{19.43}$$

$$\alpha_2 = \mu_2 - \mu_3 = \frac{1}{\sqrt{8}}(0, 2, -2, 0) \tag{19.44}$$

$$\alpha_3 = \mu_3 - \mu_4 = \frac{1}{\sqrt{8}}(0, 0, 2, -2) \tag{19.45}$$

$$\alpha_4 = \mu_3 + \mu_4 = \frac{1}{\sqrt{8}}(2, -2, 0, 0). \tag{19.46}$$

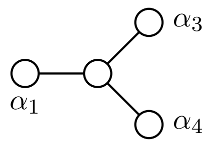
with the angles between the simple roots given by

$$\theta_{\alpha_1\alpha_2} = \frac{2}{3}\pi \tag{19.47}$$

$$\theta_{\alpha_2\alpha_3} = \frac{2}{3}\pi \tag{19.48}$$

$$\theta_{\alpha_2\alpha_4} = \frac{2}{3}\pi \tag{19.49}$$

and the Dynkin diagram is therefore




Which is $D_4$ or $\mathfrak{so}(8)$.

# 20 Chapter 20 - Classification Theorem

## 20.A Prove that decomposable Π-Systems yield decomposable root systems

A decomposable Π-system may be written as the union of indecomposable Π-systems $\Pi = \Pi_1 \cup \Pi_2$ s. t. $\Pi_1, \Pi_1$ indecomposable with $\alpha_i \in \Pi_1$, $\beta_i \in \Pi_2$ simple roots. The Π-system is decomposable if $\alpha_i \cdot \beta_j = 0 \forall i, j$. If Π decomposable its Dynkin diagram may be spilt into decomposed systems without breaking any line, i.e. there is a $\frac{\pi}{2}$ angle between $\Pi_1$ and $\Pi_2$.

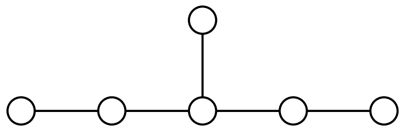
$$\boxed{\Pi} = \boxed{\Pi_1} \quad \boxed{\Pi_2}$$

With each box representing a full Dynkin diagram. Then $\exists$ a pair of roots which we may choose to be $\alpha_1$ & $\beta_1$ s. t. $\frac{\alpha_1 \cdot \beta_1}{\sqrt{{\alpha_1}^2{\beta_1}^2}} = \cos\left(\frac{\pi}{2}\right) = 0$, and as ${\alpha_1}^2{\beta_1}^2 \neq 0$ then $\alpha_1 \cdot \beta_1 = 0$. But as $\Pi_1$, $\Pi_2$ indecomposable $\exists \alpha_i$ s. t. $\alpha_i \cdot \alpha_j \neq 0 \forall j$ and $\exists \beta_i$ s. t. $\beta_i \cdot \beta_j \neq 0 \forall j$. Thus, if $\alpha_1 \perp \beta_1$ and $\alpha_1 \not\perp \alpha_i$ then $\alpha_i \perp \beta_1$, similarly $\beta_i \perp \alpha_1$, $\implies \alpha_i \cdot \beta_j = 0 \forall i, j$.

## 20.B Find the regular maximal subalgebras of $E_6$

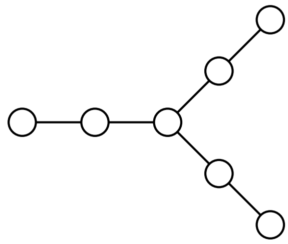
$E_6$ is

$E_6$



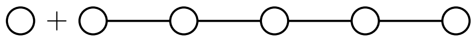
So to find $E_6'$ and then remove single circles.

$E_6'$



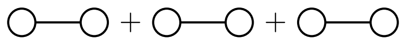
The trivial regular maximal subalgebra ($E_6$) itself is given by removing a circle from any end.

Removing the circle second from the end gives



$$\begin{aligned} &= A_1 + A_5 && (20.1)\\ &= \mathfrak{su}(2) + \mathfrak{su}(6) && (20.2) \end{aligned}$$

Removing the middle circle gives



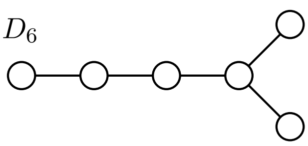
$$\begin{aligned} &= A_2 + A_2 + A_2 && (20.3)\\ &= \mathfrak{su}(3) + \mathfrak{su}(3) + \mathfrak{su}(3) && (20.4) \end{aligned}$$

These are the only possible regular maximal subalgebras of $E_6$ because $A_5, A_2, A_1$ have no non-trivial regular maximal subalgebras.

## 20.C Find the regular maximal subalgebras of $\mathfrak{so}(12)$.

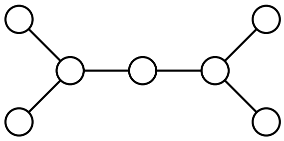
$\mathfrak{so}(12) = D_6$ is

$D_6$



So that $D_6'$ is given by

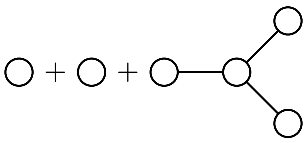
$D_6'$



Removing a circle from $D_6'$ gives back the trivial regular maximal subalgebra, $D_6$ itself. Removing the middle circle from $D_6'$ gives

$$\begin{aligned} &= A_3 + A_3 && (20.5) \\ &= \mathfrak{su}(4) + \mathfrak{su}(4) && (20.6) \end{aligned}$$

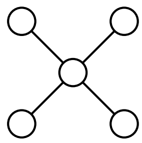
Removing the second circle from either end from $D_6'$ gives



$$\begin{aligned} &= A_1 + A_1 + D_4 && (20.7) \\ &= \mathfrak{su}(2) + \mathfrak{su}(2) + \mathfrak{su}(8) && (20.8) \end{aligned}$$

But $D_4$ contains a regular maximal subalgebra. $D_4'$ is given by

$D_4'$



The trivial regular subalgebra ($D_4$ itself) is given by removing one circle from any of the four legs of $D_4'$. Removing the middle circle from $D_4'$ gives

$$\begin{aligned} &= A_1 + A_1 + A_1 + A_1 && (20.9) \\ &= \mathfrak{su}(2) + \mathfrak{su}(2) + \mathfrak{su}(2) + \mathfrak{su}(2) && (20.10) \end{aligned}$$

So that the regular non-trivial maximal subalgebras of $\mathfrak{so}(12)$ are $2A_3$, $4A_1$ and $D_4 + 2A_1$.

## 21 Chapter 21 - $\mathfrak{so}(2n+1)$ & Spinors

### 21.A Show that the 10 matrices $\frac{1}{2}\sigma_a \otimes \mathbb{I}$, $\frac{1}{2}\sigma_a \otimes \tau_1$, $\frac{1}{2}\sigma_a \otimes \tau_3$, $\frac{1}{2}\mathbb{I} \otimes \tau_2$ generate the spinor representation of $\mathfrak{so}(5)$. Find the matrix $R = R^{-1}$ s. t. $T_a = -RT_a^* R$.

Take the Cartan subalgebra to be

$$H_1 = \frac{1}{2}\sigma_3 \otimes \mathbb{I} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \frac{1}{2}\sigma_3 \otimes \tau_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21.1}$$

which have eigenvalues $\lambda = \pm\frac{1}{2}$. The eigenvectors and associated weights are

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mu_1 = (\frac{1}{2}, \frac{1}{2}) \tag{21.2}$$

$$x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mu_2 = (\frac{1}{2}, -\frac{1}{2}) \tag{21.3}$$

$$x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mu_3 = (-\frac{1}{2}, -\frac{1}{2}) = -\mu_1 \tag{21.4}$$

$$x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \mu_4 = (-\frac{1}{2}, \frac{1}{2}) = -\mu_2. \tag{21.5}$$

The spinor rep of $\mathfrak{so}(5)$ is $\left|\pm\frac{e^1}{2} \pm \frac{e^2}{2}\right\rangle$ with $e^1 = (1,1) = -(-1,-1)$, $e^2 = (1,-1) = -(-1,1)$. Thus the matrices generate the spinor rep of $\mathfrak{so}(5)$ with $\pm\mu_1 = \pm\frac{1}{2}e^1$, $\pm\mu_2 = \pm\frac{1}{2}e^2$.

Then to find the matrix $R$, when we know that from the Hermitian generators $\sigma_3 \otimes \tau_2$, $\sigma_3 \otimes \tau_1$ we can construct all of the other generators via commutation, thus we may find the $R$ for which $T_a = -RT_a^* R^{-1}$. Let $R = r^1 \otimes r^2$. So, using $\sigma_3^* = \sigma_3$, $\sigma_1 = \sigma_1$ & $\sigma_2^* = -\sigma_2$.

$$\sigma_3 \otimes \tau_2 = - r^1\sigma_3^* {r^1}^{-1} \otimes r^2\tau_2^* {r^2}^{-1} \tag{21.6}$$

$$= r^1\sigma_3 {r^1}^{-1} \otimes r^2\tau_2 {r^2}^{-1} \tag{21.7}$$

and

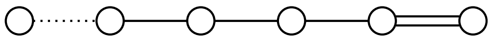
$$\sigma_3 \otimes \tau_1 = - r^1\sigma_3^* {r^1}^{-1} \otimes r^2\tau_1^* {r^2}^{-1} \tag{21.8}$$

$$= - r^1\sigma_3 {r^1}^{-1} \otimes r^2\tau_1 {r^2}^{-1}. \tag{21.9}$$

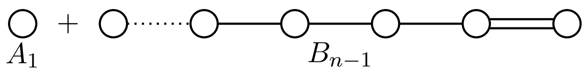
Pick $r^1 = \sigma_2$ and $r^2 = \tau_1$ which gives the desired properties, so $R = \sigma_2 \otimes \tau_1$.

### 21.B Determine how the spinor rep of $\mathfrak{so}(2n+1)$ transforms under the $\mathfrak{so}(2n-1)$ subalgebra.

$\mathfrak{so}(2n-1)$ can be generated by the products of of $n-1$ Pauli matrices $\sigma_a^1 \otimes \sigma_a^2 \otimes ... \otimes \sigma_a^{n-1}$ with the raising/lowering ops given by $E_{\pm e^j} = \frac{1}{2}\sigma_3^1 \otimes ... \otimes \sigma_3^{j-1} \otimes \sigma_1^j \otimes ... \otimes \mathbb{I}^{n-1}$. Acting with the $\mathfrak{so}(2n-1)$ operators on the full $\mathfrak{so}(2n+1)$ spinor rep $\left|\pm\frac{e^1}{2} \pm \frac{e^2}{2} ... \pm \frac{e^n}{2}\right\rangle$ produces the same transformation on the first $\pm e^1/2 ... \pm e^{n-1/2}$ terms as under the full $\mathfrak{so}(2n+1)$ operators, but the $\pm e^n/2$ state is invariant under $\mathfrak{so}(2n-1)$ thus the spinor rep of $\mathfrak{so}(2n+1)$ is not an irrep under the $\mathfrak{so}(2n-1)$ sub-algebra. This can be seen from the Dynkin diagram

$B_n$ 

has a

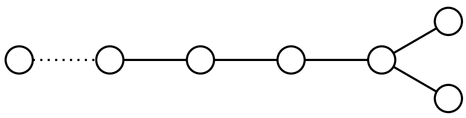



$= \mathfrak{su}(2) + \mathfrak{so}(2n-1)$ subalgebra.

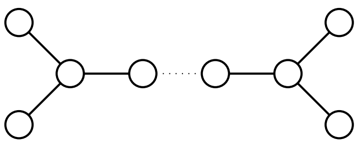
## 22 Chapter 22 - $\mathfrak{so}(2n+2)$ Spinors

### 22.A Show that $\mathfrak{so}(2n)$ has a regular maximal subalgebra $\mathfrak{so}(2m) \times \mathfrak{so}(2n-2m)$. How does the spinor rep. of $\mathfrak{so}(2n)$ transform under the subalgebra?

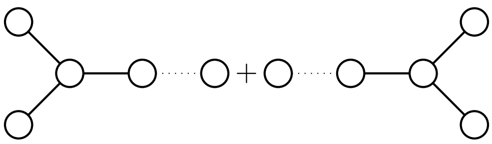
$\mathfrak{so}(2n) = D_n$

$D_n$ 

$D'_n$ is

$D'_n$ 

removing the $m^{\text{th}}$ circle from $D'_n$ gives

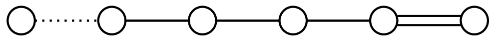



which is $D_{n-m} + D_m = \mathfrak{so}(2n-2m) + \mathfrak{so}(2m)$.

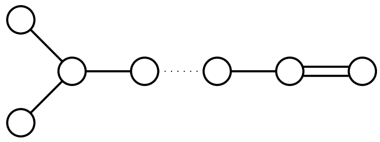
The $\mathfrak{so}(2n)$ spinor transforms on first $m$ components as a $\mathfrak{so}(2m)$ spinor and on the other $m+1...n-m$ components as a $\mathfrak{so}(2n-2m)$ spinor.

**22.B Show that $\mathfrak{so}(2n+1)$ has a regular maximal subalgebra $\mathfrak{so}(2m) \times \mathfrak{so}(2n-2m+1)$. How does the spinor rep. of $\mathfrak{so}(2n+1)$ transform under the subalgebra?**

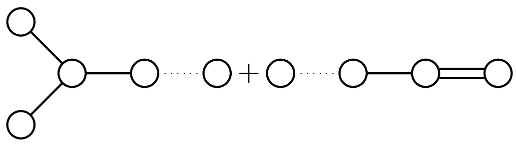
$\mathfrak{so}(2n+1) = B_n$

$B_n$ 

So that $B'_n$ is

$B'_n$ 

Removing the $m^{\text{th}}$ circle from $B'_n$ gives



which is $D_m + B_{n-m} = \mathfrak{so}(2m) + \mathfrak{so}(2n-2m+1)$.

The spinor rep. of $\mathfrak{so}(2n+1)$ transforms under the subalgebra as a $\mathfrak{so}(2m)$ component spinor on the first $m$ components and as a $\mathfrak{so}(2n-2m+1)$ spinor on the remaining $m+1 ... n-m$ components.

**22.C Show that $\mathfrak{so}(4)$ has the same algebra as $\mathfrak{su}(2) \times \mathfrak{su}(2)$ and thus is not simple, however the spinor arguments still apply, explain how.**

To show we can show that $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ by verifying that the two algebras have the same commutation relations and that there exists a one-to-one mapping between the generators of the two algebras.

$\mathfrak{so}(4)$ has $\frac{n(n-1)}{2} = \frac{4 \cdot 3}{2} = 6$ generators and is of rank 2 with three hermitian generators

$$h_3 = M_{56} = -\frac{1}{2}\sigma_3 \otimes \sigma_3 \equiv T_{23} \tag{22.1}$$

$$h_1 = M_{16} = -\frac{1}{2}\sigma_3 \otimes \sigma_1 \equiv T_{21} \tag{22.2}$$

$$h_2 = M_{26} = -\frac{1}{2}\sigma_3 \otimes \sigma_2 \equiv T_{22}. \tag{22.3}$$

The other 3 generators are given by commutation

$$[h1, h3] = \frac{\mathrm{i}}{2}\mathbb{I} \otimes \sigma_2 \equiv T_{12} \tag{22.4}$$

$$[h2, h3] = \frac{\mathrm{i}}{2}\mathbb{I} \otimes \sigma_1 \equiv T_{11} \tag{22.5}$$

$$[h1, h2] = \frac{\mathrm{i}}{2}\mathbb{I} \otimes \sigma_3 \equiv T_{13}. \tag{22.6}$$

Then, constructing the commutators

$$[T_{1a}, T_{1b}] = 2\mathrm{i}\epsilon_{abc}T_{1c} \tag{22.7}$$
$$[T_{2a}, T_{2b}] = 2\mathrm{i}\epsilon_{abc}T_{1c} \tag{22.8}$$
$$[T_{1a}, T_{2b}] = 2\mathrm{i}\epsilon_{abc}T_{2c}. \tag{22.9}$$

We know from Problem(8.B) which describes the algebra $\mathfrak{su}(2)\times\mathfrak{su}(2)$ has the commutation relations

$$[\sigma_a \otimes \mathbb{I}, \sigma_b \otimes \mathbb{I}] = 2\mathrm{i}\epsilon_{abc}\sigma_c \otimes \mathbb{I} \tag{22.10}$$
$$[\sigma_a \otimes \eta_1, \sigma_b \otimes \eta_1] = 2\mathrm{i}\epsilon_{abc}\sigma_c \otimes \mathbb{I} \tag{22.11}$$
$$[\sigma_a \otimes \mathbb{I}, \sigma_b \otimes \eta_1] = 2\mathrm{i}\epsilon_{abc}\sigma_c \otimes \eta_1. \tag{22.12}$$

Thus we can see that an appropriate one-to-one mapping is

$$T_{1a} \to \sigma_a \otimes \mathbb{I} \tag{22.13}$$
$$T_{2a} \to \sigma_a \otimes \eta_1 \tag{22.14}$$

and therefore, $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$. The two simple roots in $\mathfrak{so}(4)$ are $\alpha^1 = (e^1 - e^2)$ and $\alpha^2 = (e^1 + e^2)$ so $\alpha^1 \cdot \alpha^2 = 0$ just as, for arbitrary $n$ in $\mathfrak{so}(2n)$, $\alpha^n \cdot \alpha^{n+1} = 0$ and the two simple roots in $\mathfrak{so}(4)$ are connected in the same way as the roots $\alpha^n$, $\alpha^{n+1} = 0$ in $\mathfrak{so}(2n)$.

**22.D Show that the $\mathfrak{so}(6)$ algebra and the $\mathfrak{su}(4)$ algebra are equivalent with the 4 of $\mathfrak{su}(4)$ correspondig to a spinor rep of $\mathfrak{so}(6)$. In $\mathfrak{su}(4)$, $4 \otimes 4 = 6 \oplus 10$. The 6 is the vector rep. of $\mathfrak{so}(6)$, what is the 10?**

Both $\mathfrak{so}(6)$ and $\mathfrak{su}(4)$ have 15 generators.

The 15 generators of $\mathfrak{su}(4)$ may be constructed by the 15 independent, traceless, hermitian matrices $T_{ab} = \sigma_a \otimes \sigma_b$, $T_{0a} = \mathbb{I} \otimes \sigma_a$ and $T_{a0} = \sigma_a \otimes \mathbb{I}$ which form a basis for the $\mathfrak{su}(4)$ Lie algebra with commutators

$$[T_{ab}, T_{mn}] = \sigma_a\sigma_m \otimes \sigma_b\sigma_n - \sigma_m\sigma_a \otimes \sigma_n\sigma_b \tag{22.15}$$
$$= (\mathrm{i}\epsilon_{ami}\sigma_i + \delta_{am}\mathbb{I}) \otimes (\mathrm{i}\epsilon_{bnj}\sigma_j + \delta_{bn}\mathbb{I}) - (\mathrm{i}\epsilon_{mai}\sigma_i + \delta_{ma}\mathbb{I}) \otimes (\mathrm{i}\epsilon_{nbj}\sigma_j + \delta_{nb}\mathbb{I}) \tag{22.16}$$
$$= 2\mathrm{i}\epsilon_{ami}\delta_{bn}\sigma_i \otimes \mathbb{I} + 2\mathrm{i}\epsilon_{bnj}\delta_{am}\mathbb{I} \otimes \sigma_j \tag{22.17}$$
$$= 2\mathrm{i}\epsilon_{ami}\delta_{bn}T_{i0} + 2\mathrm{i}\epsilon_{bnj}\delta_{am}T_{0j} \tag{22.18}$$
$$[T_{ab}, T_{0m}] = 2\mathrm{i}\epsilon_{bmj}T_{aj} \tag{22.19}$$
$$[T_{ab}, T_{m0}] = 2\mathrm{i}\epsilon_{amj}T_{jb} \tag{22.20}$$
$$[T_{0a}, T_{m0}] = 0. \tag{22.21}$$

The $\mathfrak{so}(6)$ generators may be obtained from the following four Hermitian matrices

$$M_{56} = \frac{1}{2}\sigma_3 \otimes \sigma_3 \equiv X_{33} \tag{22.22}$$
$$M_{16} = \frac{1}{2}\sigma_1 \otimes \mathbb{I} \equiv X_{10} \tag{22.23}$$
$$M_{36} = \frac{1}{2}\sigma_3 \otimes \sigma_1 \equiv X_{31} \tag{22.24}$$
$$M_{46} = \frac{1}{2}\sigma_3 \otimes \sigma_2 \equiv X_{32}. \tag{22.25}$$

The other generators can be constructed via commutation

$$[X_{3i}, X_{3j}] = \frac{1}{2}\mathbb{I} \otimes \mathrm{i}\epsilon_{ijk}\sigma_k \equiv X_{0k} \tag{22.26}$$
$$[X_{3i}, X_{10}] = \frac{1}{2}\sigma_2 \otimes \mathrm{i}\sigma_i \equiv X_{2i} \tag{22.27}$$
$$[X_{3i}, [X_{3j}, X_{10}]] = [X_{3i}, X_{2j}] = \frac{1}{2}\sigma_1 \otimes \mathrm{i}\epsilon_{ijk}\sigma_k \equiv X_{1k} \tag{22.28}$$
$$[X_{3i}, X_{0i}] = \frac{1}{2}\sigma_3 \otimes \mathbb{I} \equiv X_{30} \tag{22.29}$$
$$[X_{3i}, X_{3j}] = \frac{1}{2}\sigma_2 \otimes \mathbb{I} \equiv X_{20} \tag{22.30}$$

So the algebra closes. All together,

$$X_{ij} = \frac{1}{2}\sigma_i \otimes \sigma_j \tag{22.31}$$
$$X_{j0} = \frac{1}{2}\sigma_j \otimes \mathbb{I} \tag{22.32}$$
$$X_{0j} = \frac{1}{2}\mathbb{I} \otimes \sigma_j. \tag{22.33}$$

Thus $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ with the one-to-one mapping $X_{\mu\nu} \to \frac{1}{2}T_{\mu\nu}$, $\mu, \nu = 0...4$, $\mu+\nu \neq 0$.

# 23 Chapter 23 - $\mathfrak{su}(n) \subset \mathfrak{so}(2n)$

## 23.A Use the binomial theorem to show the dimensions work out.

We make use of the following formulas

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{23.1}$$
$$\binom{n}{k} = \binom{n}{n-k} \tag{23.2}$$
$$\binom{n}{k} = \binom{n-1}{n} + \binom{n-1}{k-1}. \tag{23.3}$$

**23.A.a** $\quad D^{2n+1} = \sum_{j=0}^{n}[2j+1]$ **and** $D^{2n} = \sum_{j=0}^{n}[2j]$

For the embedding of $\mathfrak{su}(2n+1)$ $\mathfrak{so}(4n+2)$

$$\dim(D^{2n+1}) = \dim(D^{2n}) = 2^{2n}. \tag{23.4}$$

To check the dimension of $D^{2n+1}$

$$\dim\left(\sum_{j=0}^{n}[2j+1]\right) = \sum_{j=0}^{n}\binom{2n+1}{2j+1} \tag{23.5}$$

But because $\binom{2n+1}{2j+1} = \binom{2n+1}{2n+1-(2j+1)} = \binom{2n+1}{2n-2j}$ so $\sum_{j=0}^{n}\binom{2n+1}{2j+1} = \sum_{j=0}^{n}\binom{2n+1}{2n-2j}$

$$\implies \sum_{j=0}^{n}\binom{2n+1}{2j+1} = \left(\sum_{j=0}^{n}\binom{2n+1}{2j+1} + \sum_{j=0}^{n}\binom{2n+1}{2n-2j}\right)/2 \tag{23.6}$$

but this simply sums over all odd+even binomial coeff's so this is equivalent to

$$\sum_{j=0}^{n}\binom{2n+1}{2j+1} = \frac{\sum_{j=0}^{2n+1}\binom{2n+1}{j}}{2} = \frac{2^{2n+1}}{2} = 2^{2n} = \dim(D^{2n+1}). \tag{23.7}$$

thus the dimension works out.

Now to check the dimension of $D^{2n}$

$$\begin{aligned}\dim\left(\sum_{j=0}^{n}[2j]\right) &= \sum_{j=0}^{n}\binom{2n+1}{2j} && (23.8)\\ &= \sum_{j=0}^{n}\left[\binom{2n}{2j} + \binom{2n}{2j-1}\right] && (23.9)\\ &= \sum_{j=0}^{2n}\binom{2n}{j} && (23.10)\\ &= 2^{2n} = \dim(D^{2n}). && (23.11)\end{aligned}$$

**23.A.b** $\quad D^{2n} = \sum_{j=0}^{n}[2j]$ **and** $D^{2n-1} = \sum_{j=0}^{n-1}[2j]$

$$\begin{aligned}\dim(D^{2n}) &= 2^{2n-1} && (23.12)\\ &= \sum_{j=0}^{n}\binom{2n}{2j} && (23.13)\\ &= \sum_{j=0}^{n}\left[\binom{2n-1}{2j} + \binom{2n-1}{2j-1}\right] && (23.14)\\ &= \sum_{j=0}^{2n-1}\binom{2n-1}{j} = 2^{2n-1} && (23.15)\end{aligned}$$

$$
\begin{aligned}
\dim(D^{2n-1}) =& 2^{2n-1} && (23.16)\\
=& \sum_{j=0}^{n-1} \binom{2n}{2j-1} && (23.17)\\
=& \sum_{j=0}^{n-1} \left[\binom{2n-1}{2j} + \binom{2n-1}{2j-1}\right] && (23.18)\\
=& \sum_{j=0}^{2n-1} \binom{2n-1}{j} = 2^{2n-1} && (23.19)
\end{aligned}
$$

**23.B Determine how the vector representation $D^1$ of $\mathfrak{so}(2n)$ transforms under $\mathfrak{su}(n)$.**

**23.C Let $u^{jkl}$ be a completely antisymmetric tensor in $\mathfrak{so}(6)$ a self-duality condition is $u^{jkl} = \lambda \epsilon^{jklabc} u^{abc}$. What are the possible values of $\lambda$?**

For $\mathfrak{so}(4m+2)$: $D^{2m+1} \otimes D^{2m+1}$, $D^{2m+1} \otimes D^{2m}$, $D^{2m} \otimes D^{2m}$. $m = 1$ and the rank of $\mathfrak{so}(6)$ is $n = 3$ so in $\mathfrak{so}(6)$: $\overline{D}^2 = D^3$ and $\overline{D}^3 = D^2$.

$D^3 \otimes D^2 = D^3 \otimes \overline{D}^3$, $D^3 \otimes D^3 = D^3 \otimes \overline{D}^2$ and $D^2 \otimes D^2 = D^2 \otimes \overline{D}^3$

Thus the relation is self-dual and complex so $\lambda = \frac{\mathrm{i}}{3!}$.

**23.D How does the spinor representation of $\mathfrak{so}(14)$ transform under the following subgroups:**

**23.D.a $\mathfrak{su}(7)$?**

**23.D.b $\mathfrak{so}(4) \times \mathfrak{su}(5)$?**

# 24 Chapter 24 - $\mathfrak{so}(10)$

**24.A Show that the matrices generate a spinor representation of $\mathfrak{so}(10)$. Find an $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$ subgroup**

The matrices are

$$
\frac{1}{2}\sigma_i, \quad \frac{1}{2}\tau_i, \quad \frac{1}{2}\eta_i, \quad \frac{1}{2}\sigma_i\rho_1, \quad \frac{1}{2}\tau_i\rho_2, \quad \frac{1}{2}\eta_i\rho_3, \quad \frac{1}{2}\sigma_i\tau_j\rho_3, \quad \frac{1}{2}\tau_i\eta_j\rho_1, \quad \frac{1}{2}\sigma_i\eta_j\rho_2. \tag{24.1}
$$

Take the Cartan subalgebra to be

$$
\begin{aligned}
H_1 =& \frac{1}{2}\sigma_3, && (24.2)\\
H_2 =& \frac{1}{2}\tau_3, && (24.3)\\
H_3 =& \frac{1}{2}\eta_3, && (24.4)\\
H_4 =& \frac{1}{2}\eta_3\rho_3, && (24.5)\\
H_5 =& \frac{1}{2}\sigma_3\tau_3\rho_3. && (24.6)
\end{aligned}
$$

which clearly all have eigenvalues $\lambda = \pm 1/2$ and the eigenvectors are given by the unit vectors $[e^k]_m = e^k_m$. Thus, as on page 238 in Georgi, the simple roots are $\alpha^j = e^j - e^{j+1}$ for $j = 1...4$ and $\alpha^5 = e^4 + e^5$. So that the fundamental weights, satisfying (8.1) are given by

$$\mu^1 = (1,0,0,0,0) \tag{24.7}$$
$$\mu^2 = (1,1,0,0,0) \tag{24.8}$$
$$\mu^3 = (1,1,1,0,0) \tag{24.9}$$
$$\mu^4 = \frac{1}{2}(1,1,1,1,-1) \tag{24.10}$$
$$\mu^5 = \frac{1}{2}(1,1,1,1,1). \tag{24.11}$$

For the $\mathfrak{su}(4) \times \mathfrak{su}(2) \times \mathfrak{su}(2)'$ subgroup, the $\mathfrak{su}(2)'$ may be generated by the subset $\eta_i(1+\rho_3)/4 = \frac{1}{2}\mathbb{I} \otimes \mathbb{I} \otimes \eta_i \otimes P_1$ with $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This has the commutation relations of $\mathfrak{su}(2)$

$$\frac{1}{4}[\eta_i P_1, \eta_j P_1] = \frac{1}{4}[\eta_i, \eta_j] P_1 = \frac{\mathrm{i}}{2}\epsilon_{ijk}\eta_k P_1. \tag{24.12}$$

The other $\mathfrak{su}(2)$ factor may be taken to be generated by $\eta_i(1-\rho_3)/4$.

The $\mathfrak{su}(4)$ factor is generated by the $4^2 - 1 = 15$ matrices which we can take to be $\frac{1}{2}\sigma_i$, $\frac{1}{2}\tau_i$, $\frac{1}{2}\sigma_i\tau_j\rho_3$, these have the commutation relations

$$\frac{1}{4}[\sigma_i, \sigma_j] = \frac{\mathrm{i}}{2}\epsilon_{ijk}\sigma_k \tag{24.13}$$
$$\frac{1}{4}[\tau_i, \tau_j] = \frac{\mathrm{i}}{2}\epsilon_{ijk}\tau_k \tag{24.14}$$
$$\frac{1}{4}[\sigma_i, \tau_j] = 0 \tag{24.15}$$
$$\frac{1}{4}[\sigma_i, \sigma_j\tau_k\rho_3] = \frac{\mathrm{i}}{2}\epsilon_{ijl}\sigma_l\tau_k\rho_3 \tag{24.16}$$
$$\frac{1}{4}[\tau_i, \sigma_j\tau_k\rho_3] = \frac{\mathrm{i}}{2}\epsilon_{ikl}\sigma_j\tau_l\rho_3 \tag{24.17}$$
$$\frac{1}{4}[\sigma_i\tau_k\rho_3, \sigma_j\tau_l\rho_3] = \frac{\mathrm{i}}{2}\epsilon_{ijm}\delta_{kl} + \frac{\mathrm{i}}{2}\epsilon_{klm}\delta_{ij}\tau_m \tag{24.18}$$

which are the same commutations of those of the $\mathfrak{su}(4)$ found in problem (22.D)

### 24.B What is the dimension of the $\mathfrak{so}(10)$ representation with highest weight $2\mu^5$

We can use the Weyl dimension (character) formula [4]

$$\dim(R) = \prod_{\alpha>0} \frac{\sum_i k^i_\alpha(\Lambda_i + 1)\alpha^i \cdot \alpha^i}{\sum_i k^i_\alpha \alpha^i \cdot \alpha^i} \tag{24.19}$$

where $\alpha = \sum_i k^i_\alpha \alpha^i$ are positive roots and $\Lambda_i$'s are the Dynkin coefficients of the highest weight states.

Because we are interested in $\mathfrak{so}(10) = D_5$ the $\alpha^i \cdot \alpha^i$ factors cancel because all the roots have the same length.

The simple roots of $\mathfrak{so}(10)$ are given by $\alpha^j = e^j - e^{j+1}$ for $j = 1...4$ and $\alpha^5 = e^4 + e^5 = (0,0,0,1,1)$. The 20 positive roots are given by $e^j \pm e^k$ for $j < k$. So we associate with the fundamental weight $2\mu^5$ the vector $(0,0,0,0,2) = \Lambda$.

Denoting the positive roots in terms of the simple roots as $\alpha^1 \equiv (1)$,..., $\alpha^5 \equiv (5)$, so, for example $\alpha^1 + \alpha^2 = (1)(2)$. The characters which are not equal to unity, for example $(34) := \frac{1(0+1)+1(0+1)}{2} = 1$, are given for the positive roots

$$(5): \quad \frac{\sum_i k^i_\alpha (\Lambda_i + 1)}{\sum_i k^i_\alpha} = \frac{1(2+1)}{1} = 3 \tag{24.20}$$

$$(3)(5): \quad \frac{1(2+1)+1(0+1)}{2} = 2 \tag{24.21}$$

$$(2)(3)(5): \quad \frac{1(2+1)+1(0+1)+1(0+1)}{3} = \frac{5}{3} \tag{24.22}$$

$$(3)(4)(5): \quad \frac{1(2+1)+1(0+1)+1(0+1)}{3} = \frac{5}{3} \tag{24.23}$$

$$(1)(2)(3)(5): \quad \frac{3+1+1+1}{4} = \frac{6}{4} \tag{24.24}$$

$$(2)(3)(4)(5): \quad \frac{3+1+1+1}{4} = \frac{6}{4} \tag{24.25}$$

$$(1)(2)(3)(4)(5): \quad \frac{3+1+1+1+1}{5} = \frac{7}{5} \tag{24.26}$$

$$(2)(3^2)(5): \quad \frac{3+1+1+1+1}{5} = \frac{7}{5} \tag{24.27}$$

$$(1)(2)(3^2)(4)(5): \quad \frac{3+1+1+1+1+1}{6} = \frac{8}{6} \tag{24.28}$$

$$(1)(2^2)(3^2)(4)(5): \quad \frac{3+1+1+1+1+1+1}{7} = \frac{9}{7}. \tag{24.29}$$

Then it is just a case of multiplying all the characters together which gives dim $(2\mu^5) = 126$

## 25 Chapter 25 - Automorphisms

### 25.A $\mathfrak{so}(8)$ Determine which of the representations $(2,2,1,1) \oplus (1,1,2,2)$ and $(2,1,1,2) \oplus (1,2,2,1)$ corresponds to $D^1$ and which corresponds to $D^4$.

Repeating from the book, $\mathfrak{so}(8)$ has a $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(2)$ subalgebra with the four orthogonal roots $\alpha^0 = -e^1 - e^2, \alpha^1 = e^1 - e^2, \alpha^3 = e^3 - e^4, \alpha^4 = e^3 + e^4$.

Under the subalgebra on the spinor representation $D^3$ with weights

$$\eta_j e^j/2; \qquad \prod_j \eta_j = -1. \tag{25.1}$$

under the subalgebra the 8 weights break into two sets transforming irreducibly under the subalgebra. The set

$$\begin{aligned} &\frac{1}{2}(e^1 + e^2 + e^3 - e^4), \quad \frac{1}{2}(e^1 + e^2 - e^3 + e^4), \\ &\frac{1}{2}(-e^1 - e^2 + e^3 - e^4), \quad \frac{1}{2}(-e^1 - e^2 - e^3 + e^4). \end{aligned} \tag{25.2}$$

is orthogonal to $\alpha^1$ and $\alpha^4$ and so transforms like a singlet under the two $\mathfrak{su}(2)$'s associated to $\alpha^1$ and $\alpha^4$ but transforms like a doublet under the $\mathfrak{su}(2)$'s associated to $\alpha^0$ and $\alpha^3$. Similarly the set

$$\begin{aligned} \frac{1}{2}(e^1 - e^2 + e^3 + e^4), \quad & \frac{1}{2}(e^1 - e^2 - e^3 - e^4), \\ \frac{1}{2}(-e^1 + e^2 + e^3 + e^4), \quad & \frac{1}{2}(-e^1 + e^2 - e^3 - e^4) \end{aligned} \tag{25.3}$$

is orthogonal to $\alpha^0$ and $\alpha^3$ and transforms trivially under the two $\mathfrak{su}(2)$'s and transforms like a doublet under the $\mathfrak{su}(2)$'s associated with $\alpha^1$ and $\alpha^4$ and so under the subalgebra the representation $D^3$ transforms as $(2,1,2,1) \oplus (1,2,1,2)$

In a similar fashion we can deduce how $D^4$ transforms. The spinor representation $D^4$ has weights

$$\eta_j e^j/2; \qquad \prod_j \eta_j = 1. \tag{25.4}$$

which break up into two sets

$$\begin{aligned} \frac{1}{2}(e^1 + e^2 + e^3 + e^4), \quad & \frac{1}{2}(-e^1 - e^2 - e^3 - e^4), \\ \frac{1}{2}(+e^1 + e^2 - e^3 - e^4), \quad & \frac{1}{2}(-e^1 - e^2 + e^3 + e^4) \end{aligned} \tag{25.5}$$

is orthogonal to $\alpha^1$ and $\alpha^3$ so the spinor transforms trivially under the $\mathfrak{su}(2)$'s associated to $\alpha^1$ and $\alpha^3$ and transforms like a doublet under the $\mathfrak{su}(2)$'s associated to $\alpha^0$ and $\alpha^4$. Likewise, the set

$$\begin{aligned} \frac{1}{2}(e^1 - e^2 + e^3 - e^4), \quad & \frac{1}{2}(-e^1 + e^2 - e^3 + e^4), \\ \frac{1}{2}(e^1 - e^2 - e^3 + e^4), \quad & \frac{1}{2}(-e^1 + e^2 + e^3 - e^4) \end{aligned} \tag{25.6}$$

is orthogonal to $\alpha^0$ and $\alpha^4$ so the spinor transforms trivially under the $\mathfrak{su}(2)$'s associated to $\alpha^0$ and $\alpha^4$ and transforms like a doublet under the $\mathfrak{su}(2)$'s associated to $\alpha^1$ and $\alpha^3$. Therefore, under the $\mathfrak{su}(2)^0 \times \mathfrak{su}(2)^1 \times \mathfrak{su}(2)^3 \times \mathfrak{su}(2)^4$ subalgebra $D^4$ transforms as

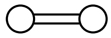
$$(2,1,1,2) \oplus (1,2,2,1). \tag{25.7}$$

The vector representation $D^1$ has weights $\pm e^j$ and so splits into two sets. The weights $\pm e^1$ and $\pm e^2$ form one set orthogonal to $\alpha^3$ and $\alpha^4$. While the other set is composed of $\pm e^3$ and $\pm e^4$ orthogonal to $\alpha^0$ and $\alpha^1$ and therefore, by the same arguments as before, the vector representation $D^1$ transforms as

$$(2,2,1,1) \oplus (1,1,2,2). \tag{25.8}$$

### 25.B Does $SO(8)$ have a $SO(5)$ subgroup under which one spinor ($D^4$) transforms like two $SO(5)$ spinors while the other spinor ($D^3$) transforms like an $SO(5)$ vector and three singlets? To DO!

The Dynkin diagram of $\mathfrak{so}(5)$ or $B_2$ is



The two roots have an angle $\theta_{\alpha\beta} = \frac{3\pi}{4}$. Therefore we choose the two $\mathfrak{so}(8)$ roots corresponding to the $\mathfrak{so}(5)$ roots as $\alpha^1 = e^1 - e^2$ and $\alpha^2 = e^2 - e^3$ constrained to the $2-d$ space.

# 26 Chapter 26 - $\mathfrak{sp}(2n)$

## 26.A

## 26.B

# 27 Chapter 27 - Exceptional Algebras and Anomalies

## 27.A Calculate $A(6)$ and $A(10)$ in $\mathfrak{su}(3)$

We use the following relations

$$A(\overline{D}) = -A(D) \tag{27.1}$$
$$A(D_1 \oplus D_2) = A(D_1) + A(D_2) \tag{27.2}$$
$$A(D_1 \otimes D_2) = \dim(D_1)A(D_2) + \dim(D_2)A(D_1) \tag{27.3}$$

and

$$3 \otimes 3 = 6 \oplus \overline{3} \tag{27.4}$$
$$6 \otimes 3 = 10 \oplus 8 \tag{27.5}$$
$$3 \otimes \overline{3} = 8 \oplus 1. \tag{27.6}$$

$$\mathrm{Tr}\left[\{T^3_a, T^3_b\}T^3_c\right] \equiv A(3)d_{abc} \tag{27.7}$$

So that $A(3) = 1$. Then,

$$A(6) = A(3 \otimes 3) - A(\overline{3}) \tag{27.8}$$
$$= A(3 \otimes 3) + A(3) \tag{27.9}$$
$$= 3A(3) + 3A(3) + A(3) = 7 \tag{27.10}$$
$$\tag{27.11}$$

For the 10,

$$A(10) = A(6 \otimes 3) - A(8) \tag{27.12}$$
$$= 6A(3) + 3A(6) - A(8) \tag{27.13}$$
$$= 6 + 21 - A(8) \tag{27.14}$$

but the 8 is anomaly-free ($A(8) = 0$) because $8 = \overline{8}$ so

$$A(8) = A(\overline{8}) = -A(8) \implies A(8) = 0. \tag{27.15}$$

So

$$A(10) = 27. \tag{27.16}$$

### 27.B Show that the $A(10)_{\mathfrak{su}(5)} = A(5)_{\mathfrak{su}(5)}$ in $\mathfrak{su}(5)$

The anomaly of the fundamental representation of $\mathfrak{su}(N)$ may be calculated by calculating the $\mathfrak{su}(3)$ subalgebra of $\mathfrak{su}(N)$ under which the $N = [1]$ transforms like a single 3 and $N - 3$ singlets.

Under the $\mathfrak{su}(3)$ subalgebra the $5 = [1]$ of $\mathfrak{su}(5)$

$$\square \to \square \oplus \bullet \oplus \bullet = 3 \oplus 1 \oplus 1 \tag{27.17}$$

which gives the anomaly $A(5)_{\mathfrak{su}(5)} = A(3) + A(1) + A(1) = 1 + 0 + 0 = 1$.

Under the $\mathfrak{su}(3)$ subalgebra the $10 = [2]$ of $\mathfrak{su}(5)$

$$\begin{array}{c}\square\\ \square\end{array} \to \square \oplus \square \oplus \begin{array}{c}\square\\ \square\end{array} \oplus \bullet = 3 \oplus 3 \oplus \overline{3} \oplus 1. \tag{27.18}$$

which gives the anomaly $A(10)_{\mathfrak{su}(5)} = 2A(3) + A(\overline{3}) + A(1) = 2 - 1 + 0 = 1 = A(5)_{\mathfrak{su}(5)}$.

### 27.C Prove $A(D_1 \oplus D_2) = A(D_1) + A(D_2)$ and $A(D_1 \otimes D_2) = \dim(D_1)A(D_1) + \dim(D_2)A(D_1)$

$$\mathrm{Tr}\left[\{T_a^{D_1\oplus D_2}, T_b^{D_1\oplus D_2}\}T_c^{D_1\oplus D_2}\right] \equiv A(D_1 \oplus D_2)d_{abc} \tag{27.19}$$

But, because $D_1 \oplus D_2 = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$. Thus the generators of $D_1 \oplus D_2$ may be decomposed into independent generators $T_a^{D_1'}, T_a^{D_2'}$ with dimension $\dim(D_1 \oplus D_2)$ which act trivially on their respective invariant subspaces. So $D_1' = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_2' = \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix}$ so that $T_a^{D_1'} \cdot T_b^{D_2'} = 0$

$$\begin{aligned}
\mathrm{Tr}\left[\{T_a^{D_1\oplus D_2}, T_b^{D_1\oplus D_2}\}T_c^{D_1\oplus D_2}\right] &= \mathrm{Tr}\left[\{T_a^{D_1'} + T_a^{D_2'}, T_b^{D_1'} + T_b^{D_2'}\}(T_c^{D_1'} + T_c^{D_2'})\right] && (27.20)\\
&= \mathrm{Tr}\left[\{T_a^{D_1'}, T_b^{D_1'}\}T_c^{D_1'}\right] + \mathrm{Tr}\left[\{T_a^{D_12'}, T_b^{D_2'}\}T_c^{D_2'}\right] && (27.21)\\
&= A(D_1)d_{abc} + A(D_2)d_{abc} && (27.22)
\end{aligned}$$

because $\mathrm{Tr}[D_1'] = \mathrm{Tr}[D_1]$ etc. So,

$$A(D_1 \oplus D_2) = A(D_1) + A(D_2) \tag{27.23}$$

To prove $A(D_1 \otimes D_2) = \dim(D_1)A(D_2) + \dim(D_2)A(D_1)$ we use the fact that $\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\,\mathrm{Tr}[B]$.

We can decompose the generators

$$T_a^{D_1\otimes D_2} = T_a^{D_1} \otimes \mathbb{I}^{D_2} + \mathbb{I}^{D_1} \otimes T_a^{D_2} \tag{27.24}$$

So, writing in the shorthand form: $T_a^{D_1} \otimes \mathbb{I}^{D_2} = T_a^{D_1}\mathbb{I}^{D_2}$,

$$\begin{aligned}
&\mathrm{Tr}\left[\{T_a^{D_1\otimes D_2}, T_b^{D_1\otimes D_2}\}T_c^{D_1\otimes D_2}\right] \\
&= \mathrm{Tr}\left(\{T_a^{D_1}\mathbb{I}^{D_2} + \mathbb{I}^{D_1}T_a^{D_2}, T_b^{D_1}\mathbb{I}^{D_2} + \mathbb{I}^{D_1}T_b^{D_2}\}(T_c^{D_1}\mathbb{I}^{D_2} + \mathbb{I}^{D_1}T_c^{D_2})\right] && (27.25) \\
&= \mathrm{Tr}\left[\left(\{T_a^{D_1}\mathbb{I}^{D_2}, T_b^{D_1}\mathbb{I}^{D_2}\} + \{T_a^{D_1}\mathbb{I}^{D_2}, \mathbb{I}^{D_1}T_b^{D_2}\} + \{\mathbb{I}^{D_1}T_a^{D_2}, T_b^{D_1}\mathbb{I}^{D_2}\} + \{\mathbb{I}^{D_1}T_a^{D_2}, \mathbb{I}^{D_1}T_b^{D_2}\}\right)\right. \\
&\quad \left.\cdot (T_c^{D_1}\mathbb{I}^{D_2} + \mathbb{I}^{D_1}T_c^{D_2})\right] && (27.26) \\
&= \mathrm{Tr}\left[\{T_a^{D_1}\mathbb{I}^{D_2}, T_b^{D_1}\mathbb{I}^{D_2}\}T_c^{D_1}\mathbb{I}^{D_2} + \{\mathbb{I}^{D_1}T_a^{D_2}, \mathbb{I}^{D_1}T_b^{D_2}\}\mathbb{I}^{D_1}T_c^{D_2}\right] && (27.27) \\
&= \mathrm{Tr}\,[\mathbb{I}^{D_2}]\,\mathrm{Tr}\left[\{T_a^{D_1}, T_b^{D_1}\}T_c^{D_1}\right] + \mathrm{Tr}\,[\mathbb{I}^{D_1}]\,\mathrm{Tr}\left[\{T_a^{D_2}, T_b^{D_2}\}T_c^{D_2}\right] && (27.28) \\
&= \dim(D_2)A(D_1)d_{abc} + \dim(D_1)A(D_2)d_{abc} && (27.29)
\end{aligned}$$

where the second from last line follows because terms such as

$$\mathrm{Tr}\,[\{T_a^{D_1}\mathbb{I}^{D_2}, \mathbb{I}^{D_1}T_b^{D_2}\}T_c^{D_1}\mathbb{I}^{D_2}] = 2\,\mathrm{Tr}\,[T_a^{D_1}T_c^{D_1}T_b^{D_2}] = 2\,\mathrm{Tr}\,[T_a^{D_1}T_c^{D_1}]\,\mathrm{Tr}\,[T_b^{D_2}] = 0 \quad (27.30)$$

because of the fact that the generators are traceless $\mathrm{Tr}\,[T_a^D] = 0$.

Therefore,

$$A(D_1 \otimes D_2) = \dim(D_1)A(D_2) + \dim(D_2)A(D_1). \quad (27.31)$$

## References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