Given a compact Riemann surface $\mathcal{X}$ of genus $g$. We can consider the space of closed holomorphic 1-forms $\Omega^{1,0}(\mathcal{X})$ [?]. By the maximum principle all exact holomorphic one forms are identically zero. One can show that $\dim \Omega^{1,0}(\mathcal{X}) = \dim \Omega^{0,1}(\mathcal{X}) = g$ where $\Omega^{0,1}(\mathcal{X})$ is the space of antiholomorphic closed one forms, $\Omega^{1,0}(\mathcal{X}) \cap \Omega^{0,1}(\mathcal{X}) = 0$ and consequently that $H^1(\mathcal{X}, \mathbb{C}) = \Omega^{1,0}(\mathcal{X}) \oplus \Omega^{0,1}(\mathcal{X})$

One can also consider the meromorphic 1-forms on $\mathcal{X}$, for any non-zero meromorphic 1-form $\lambda$ we have that $\sum_{P\in\mathcal{X}} \mathrm{ord}_P(\lambda) = 2g-2$ as a consequence of the Riemann-Hurwitz formula, also, $\sum_{P\in\mathcal{X}} \mathrm{Res}_P(\omega) = 0$.

Choosing a basis for the homology as $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$, because the is a natural isomorphism $H^1(\mathcal{X}, \mathbb{C}) = \Omega^{1,0}(\mathcal{X}) \oplus \Omega^{0,1}(\mathcal{X}) \cong \mathrm{Hom}(H_1\mathcal{X}, \mathbb{C})$ provided by integration over one-cycles, there then exists a unique, linearly independent, basis $\{\omega_1, \ldots, \omega_g\}$ for $\Omega^{1,0}(\mathcal{X})$ such that

$$\int_{A_j} \omega_i = \delta_{ji} \tag{1}$$

$$\int_{B_j} \omega_i = \tau_{ji} \,. \tag{2}$$

$\tau_{ji} = \tau_{ij}$ is symmetric and obeys $\Im\tau > 0$ (by Riemann's bilinear relations). Explicitly, for a hyperelliptic curve $y^2 = F(x, \ldots)$ where, considered as a polynomial in $x$, $\deg F = 2N = 2g+2$. An explicit basis is given by $\omega_i = x^{i-1}dx/y$.

Now define a mapping $f : H_1\mathcal{X} \hookrightarrow \Lambda \subset \mathbb{C}^g$ given by $\gamma \mapsto (\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g)$ which is an embedding of $H_1\mathcal{X}$ into the lattice $\Lambda$.

We now focus our attention to the case $g = 1$. Then $\mathcal{X} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and the pair of cycles $a_1 = [0, 1]$ and $b_1 = [0, \tau]$ give a basis for $H_1\mathcal{X}$. Such a Riemann surface can be thought of as a two-sheeted covering of $\mathbb{S}^2$ with 4-branch points, by a Möbius transformation these branch points can be taken to be at $x = 0, 1, \infty, q$. We can then take the curve $y^2 = x(x-1)(x-q)$ then $\omega_1 = \omega = \frac{dx}{y}$ is the unique (up to a constant multiple) holomorphic 1-form on $\mathcal{X}$ and hence satisfies all of the above discussions. How is $\lambda$ related to the complex structure $\tau$ of the torus? To answer this question we rewrite the curve as $y^2 = \prod_{i=1}^{3}(x - e_i)$ because the discriminant $\Delta \neq 0$ then $e_1 + e_2 + e_3 = 0$. Then $q = \frac{e_2 - e_3}{e_1 - e_3}$. After computing the periods (1) & (2) and inverting one finds that

$$q = \lambda(\tau) = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4} \,. \tag{3}$$

$\lambda(\tau)$ is called the modular lambda function.

For hyperelliptic curves, which can be realised as two-sheeted branch coverings of $\mathbb{S}^2$, they have $2g+2$ branch points and explicit computations of the period matrix are harder to perform.

**Relation to gauge theory** In the Seiberg-Witten theory of the $\mathcal{N} = 2$ gauge theory we have

- The SW curve $\mathfrak{X} = \mathcal{X}_{\{u_i\}}$ which is a rank $\mathfrak{g}$ cover of a Riemann surface $\mathcal{X}_{\{u_i\}} \xrightarrow{\text{rank}\,\mathfrak{g}:1} \mathcal{C}$. The period matrix of $\mathcal{X}_{\{u_i\}}$, in turn, is identified with the IR $U(1)$ gauge couplings and therefore computes the pre-potential $\mathcal{F}$. For a $SU(N)$ $\mathcal{N}=2$ gauge theory, we have the following relations
$$\frac{\partial a_i^D}{\partial a^j} = \tau_{ij} \tag{4}$$
we then define $a_i$ and ${a^D}_i$ as integrals of the $A_i$ and $B_i$ cycles respectively over a certain meromorphic differential $\lambda_{\text{SW}} \sim \lambda_{\text{SW}} + \mathrm{d}f$ since the addition of an exact form $\mathrm{d}f$ does not affect the integration under closed one-cycles.
$$a^i = \int_{A_i} \lambda\,, \quad a_i^D = \int_{B_i} \lambda_{\text{SW}}\,. \tag{5}$$
and we demand
$$\frac{\partial \lambda_{\text{SW}}}{\partial u_i}|_{\text{fixed } x} = \omega_i + \mathrm{d}f_i \tag{6}$$
furthermore we require that for all $p \in X$ $\text{Res}_p\lambda_{\text{SW}}$ is at most linear in the quark bare masses and that $\sum_{p\in X}\text{Res}_p\lambda_{\text{SW}} = 0$.
- For theories of class $\mathcal{S}$ the space of UV gauge couplings is identified with the space of complex structure deformations $\mathcal{E}$ of the underlying surface $\mathcal{C}$. We have an isomorphism
$$\mathcal{E} \cong Teich(\mathcal{C})/MCG(\mathcal{C}) \tag{7}$$
where $Teich(\mathcal{C})$ is the Teichmüller space of $\mathcal{C}$ and is parametrised by the same cross ratios $q$ appearing in the SW curve. $MCG(\mathcal{C})$ is the mapping class group of $\mathcal{C}$, in physics terms this is the 'generalised S-duality group'.

We begin with the SW-curve for the pure $\mathfrak{su}(2)$ $\mathcal{N}=2$ theory
$$y^2 = (x-\Lambda^2)(x+\Lambda^2)(x-u)\,. \tag{8}$$
The curve becomes singular at $u = -\Lambda^2, +\Lambda^2, \infty$. The $B$-cycle is taken to enclose $-\Lambda^2, +\Lambda^2$ while the $A$-cycle is taken from $+\Lambda^2, u$. We first change coordinates by $x = \Lambda^2(2x'+1)$ and $y = (2\Lambda^2)^{3/2}y'$
$$y'^2 = x'(x'-1)\left(x' - \frac{\Lambda^2+u}{2\Lambda^2}\right) \tag{9}$$
Defining $q = \frac{\Lambda^2+u}{2\Lambda^2}$ we arrive at the desired form
$$y^2 = x(x-1)(x-q)\,. \tag{10}$$
In these coordinates the singular points of the curve are mapped to $q = 0, 1, \infty$. So we have to compute integrals of the form
$$p(\gamma) = \oint_\gamma \omega = \oint_\gamma \frac{dx}{\sqrt{x(x-1)(x-q)}}\,. \tag{11}$$

The we deform the cycle $A$ to enclose $1, q$ and $B$ to enclose $0, 1$. So we can write

$$p(A) = 2\int_1^q \omega + \oint_{C_1} \omega + \oint_{C_q} \omega\,, \quad p(B) = 2\int_0^1 \omega + \oint_{C_0} \omega + \oint_{C_1} \omega \tag{12}$$

where $C_{x_0}$ denotes sphere centred around the point $x = x_0$ with infinitesimal radius $\epsilon$. However, one can immediately see that

$$\oint_{C_p} \omega \sim \int_{-\pi}^{\pi} \frac{i\sqrt{\epsilon}e^{i\theta/2}d\theta}{\sqrt{\prod_{p' \in \{0,1,q\}\backslash\{p\}}(p-p')}} = \frac{4i\sqrt{\epsilon}}{\sqrt{\prod_{p' \in \{0,1,q\}\backslash\{p\}}(p-p')}} \to 0 \tag{13}$$

for any $p \in \{0, 1, q\}$. Writing $x = w^2$

$$\int_0^p \frac{dx}{\sqrt{x(x-1)(x-q)}} = \int_0^{\sqrt{p}} \frac{2dw}{\sqrt{(w^2-1)(w^2-q)}} = \frac{2}{\sqrt{q}}F(\sqrt{p}; 1/\sqrt{q}) \tag{14}$$

where $F(x; k)$ is the incomplete elliptic integral of the first kind in Jacobi form. We can therefore write

$$p(A) = \frac{4}{\sqrt{q}}\left(F(\sqrt{q}; 1/\sqrt{q}) - F(1; 1/\sqrt{q})\right)\,, \quad p(B) = \frac{4}{\sqrt{q}}F(1; 1/\sqrt{q})\,. \tag{15}$$

Therefore the period matrix

$$\tau = \frac{p(A)}{p(B)} = \frac{F(\sqrt{q}; 1/\sqrt{q})}{F(1; 1/\sqrt{q})} - 1 \tag{16}$$