0.1 Preserved Superconformal Algebra

0.1.1 Even Subalgebra

The even subalgebra of $\mathfrak{psu}(2,2|4)$ is $\mathfrak{b} = \mathfrak{so}(5,1) \oplus \mathfrak{su}(4)$ which we take to be generated by $M^{\mu\nu}, K_\mu, P^\mu, E$ with $\mu,\nu = 1,2,3,4$ and R_I^J, $I,J = 1,2,3,4$. The Cartans of $\mathfrak{su}(4)$ are $R_i = R_i^i - R_{i+1}^{i+1}$ with $i = 1,2,3$. We wish to discuss which generators are preserved by the S-folding/discrete gauging procedure. Recall that $SL(2,\mathbb{Z})$ transformations can be defined such that they commute with the generators of $\mathfrak{b}$ [?]. In particular $[s_k, \mathfrak{b}] = 0$. Hence s_k acts non-trivially only on the fermionic subalgebra which we will discuss momentarily. Hence the subalegbra of $\mathfrak{b}$ preserved by the S-folding/discrete gauging is simply the centraliser of $r_k = \frac{R_1}{2} + R_2 + \frac{3R_3}{2} = \frac{1}{2}\sum_{i=1}^{3} R_i^i - \frac{3}{2}R_4^4$ modulo k in $\mathfrak{b}$. Clearly $[r_k, \mathfrak{so}(5,1)] = 0$. On the other hand, using $\left[R_I^J, R_Q^P\right] = \delta_Q^J R_I^P - \delta_I^P R_Q^J$ it can be shown that

$$[r_k, R_I^J] = \begin{cases} 0 & I,J \in \{1,2,3\}\,, \\ 0 & I = J = 4\,, \\ 2R_I^4 & I \in \{1,2,3\}\,, J = 4\,, \\ -2R_4^J & I = 4\,, J \in \{1,2,3\}\,. \end{cases} \tag{1}$$

Therefore, the subalgebra of $\mathfrak{su}(4)$ preserved by $r_{k\geqslant 3}$ are given by the R_I^J with $I,J = 1,2,3$ and R_4^4. These generators span a $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ algebra. Note however that, since we quotient by $e^{\frac{2\pi \mathrm{i}}{k} r_k + s_k}$, when $k = 1,2$ the full $\mathfrak{su}(4)$ is preserved.

0.1.2 Odd Subalgebra

The odd subalgebra of $\mathfrak{psu}(2,2|4)$ is spanned by nilpotent generators (supercharges) which sit in representations of the bosonic subalgebra $\mathfrak{b}$. Any representation of $\mathfrak{b}$ can be decomposed into representations of a maximal compact subalgebra $\mathfrak{u}(1)_E \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \mathfrak{su}(4)$. The supercharges are then given by

$$\mathcal{Q}_\alpha^I \in \left(\frac{1}{2}, \mathbf{2}, \mathbf{1}, \mathbf{4}\right)\,, \quad \widetilde{\mathcal{Q}}_{\dot\alpha I} \in \left(\frac{1}{2}, \mathbf{1}, \mathbf{2}, \overline{\mathbf{4}}\right)\,, \tag{2}$$

$$\mathcal{S}_I^\alpha \in \left(-\frac{1}{2}, \overline{\mathbf{2}}, \mathbf{1}, \overline{\mathbf{4}}\right)\,, \quad \widetilde{\mathcal{S}}^{\dot\alpha I} \in \left(-\frac{1}{2}, \mathbf{1}, \overline{\mathbf{2}}, \mathbf{4}\right)\,. \tag{3}$$

The action on the supercharges is then given by

$$[r_k, \mathcal{Q}_\alpha^I] = \begin{cases} \mathcal{Q}_\alpha^I & I = 1,2,3 \\ -3\mathcal{Q}_\alpha^4 & I = 4 \end{cases}\,, \quad [r_k, \widetilde{\mathcal{Q}}_{\dot\alpha I}] = \begin{cases} -\widetilde{\mathcal{Q}}_{\dot\alpha I} & I = 1,2,3 \\ 3\widetilde{\mathcal{Q}}_{\dot\alpha 4} & I = 4 \end{cases}\,, \tag{4}$$

$$[r_k, \mathcal{S}_I^\alpha] = \begin{cases} -\mathcal{S}_\alpha^I & I = 1,2,3 \\ 3\mathcal{S}_\alpha^4 & I = 4 \end{cases}\,, \quad [r_k, \widetilde{\mathcal{S}}^{\dot\alpha I}] = \begin{cases} \widetilde{\mathcal{S}}_{\dot\alpha I} & I = 1,2,3 \\ -3\widetilde{\mathcal{S}}_{\dot\alpha 4} & I = 4 \end{cases}\,, \tag{5}$$

On the other hand, s_k acts on the supercharges by [?, ?, ?]

$$[s_k, \mathcal{Q}^I_\alpha] = -\mathcal{Q}^I_\alpha\,, \quad [s_k, \widetilde{\mathcal{Q}}_{\dot\alpha I}] = \widetilde{\mathcal{Q}}_{\dot\alpha I}\,, \tag{6}$$

$$[s_k, \mathcal{S}^\alpha_I] = \mathcal{S}^\alpha_I\,, \quad [r_k, \widetilde{\mathcal{S}}^{\dot\alpha I}] = -\widetilde{\mathcal{S}}_{\dot\alpha I}\,. \tag{7}$$

Therefore, for $k \geqslant 3$, quotienting by $e^{\frac{2\pi i}{k}(r_k+s_k)} \in \mathbb{Z}_k$ preserves 12 Poincaré supercharges and 12 conformal supercharges giving rise to $\mathcal{N}=3$ superconformal symmetry in four dimensions. All in all, for $k \geqslant 3$, a full $\mathfrak{su}(2,2|3) \subset \mathfrak{psu}(2,2|4)$ superconformal algebra is preserved.

0.2 Indices of $\mathfrak{su}(2,2|2)$ Multiplets

Long multiplets $\mathcal{A}^E_{R,r,(j_1,j_2)}$ are generic, unitary, modules of the $\mathfrak{su}(2,2|2)$ superconformal algebra. The multiplets are labelled by the values of the highest weight state (superconformal primary) (E, R, r, j_1, j_2) under the maximal bosonic subalgebra (??). When the some of representation labels take on certain values the superconformal primary is annihilated by (linear combinations of) some of the supercharges $\mathcal{Q}^I_\alpha$, $\widetilde{\mathcal{Q}}_{\dot\alpha I}$ and the multiplet is said to be shortened. The superconformal index (??) counts short multiplets modulo those that can recombine into long multiplets. The recombination rules are given by [?]

$$\mathcal{A}^{2R+r+2j_1+2}_{R,r,(j_1,j_2)} \cong \mathcal{C}_{R,r,(j_1,j_2)} \oplus \mathcal{C}_{R+\frac12,r+\frac12,(j_1-\frac12,j_2)}\,, \tag{8}$$

$$\mathcal{A}^{2R-r+2j_2+2}_{R,r,(j_1,j_2)} \cong \overline{\mathcal{C}}_{R,r,(j_1,j_2)} \oplus \overline{\mathcal{C}}_{R+\frac12,r-\frac12,(j_1,j_2-\frac12)}\,, \tag{9}$$

$$\begin{aligned}\mathcal{A}^{2R+j_1+j_2+2}_{R,j_1-j_2,(j_1,j_2)} \cong \hat{\mathcal{C}}_{R,(j_1,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac12,(j_1-\frac12,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac12,(j_1,j_2-\frac12)} \\ \oplus \hat{\mathcal{C}}_{R+1,(j_1-\frac12,j_2-\frac12)}\end{aligned}\,. \tag{10}$$

By formally allowing the j_1, j_2 to take on the value $-\frac12$ we can write

$$\mathcal{C}_{R,r,(-\frac12,j_2)} \cong \mathcal{B}_{R+\frac12,r+\frac12,(0,j_2)}\,, \quad \overline{\mathcal{C}}_{R,r,(j_1,-\frac12)} \cong \overline{\mathcal{B}}_{R+\frac12,r-\frac12,(j_1,0)}\,, \tag{11}$$

$$\hat{\mathcal{C}}_{R,(-\frac12,j_2)} \cong \mathcal{D}_{R+\frac12,(0,j_2)}\,, \quad \hat{\mathcal{C}}_{R,(j_1,-\frac12)} \cong \overline{\mathcal{D}}_{R+\frac12,(j_1,0)}\,, \tag{12}$$

$$\hat{\mathcal{C}}_{R,(-\frac12,-\frac12)} \cong \mathcal{D}_{R+\frac12,(0,-\frac12)} \cong \overline{\mathcal{D}}_{R+\frac12,(-\frac12,0)} \cong \hat{\mathcal{B}}_{R+1}\,, \tag{13}$$

for $R \geqslant 0$. Equations (8)-(13) constitute the most general recombination rules for any unitary $\mathcal{N}=2$ SCFT. We have that

$$\mathcal{I}_{\mathcal{E}_{r,(0,j_2)}} = t^{2r}(pq)^r \frac{1-t(pq)^{-1}\chi_1(y)+t^2(pq)^{-2}}{(-1)^{2j_2}(1-t^3y)(1-t^3y^{-1})}\chi_{2j_2}(y) \quad r \geqslant 2\,, \tag{14}$$

$$\mathcal{I}_{\mathcal{D}_{0,(0,j_2)}} = \frac{pqt^2\chi_{2j_2}(y) - t^3\chi_{2j_2+1}(y) - t^5pq\chi_{2j_2-1}(y) + t^6\chi_{2j_2}(y)}{(-1)^{2j_2}(1-t^3y)(1-t^3y^{-1})}\,, \tag{15}$$

Shortening Conditions				Multiplet
$\mathcal{B}_1$	$\mathcal{Q}_{1\alpha}\lvert R,r\rangle^{h.w.}=0$	$j_1=0$	$E=2R+r$	$\mathcal{B}_{R,r(0,j_2)}$
$\overline{\mathcal{B}}_2$	$\widetilde{\mathcal{Q}}_{2\dot{\alpha}}\lvert R,r\rangle^{h.w}=0$	$j_2=0$	$E=2R-r$	$\overline{\mathcal{B}}_{R,r(j_1,0)}$
$\mathcal{E}$	$\mathcal{B}_1\cap\mathcal{B}_2$	$R=0$	$E=r$	$\mathcal{E}_{r(0,j_2)}$
$\overline{\mathcal{E}}$	$\overline{\mathcal{B}}_1\cap\overline{\mathcal{B}}_2$	$R=0$	$E=-r$	$\overline{\mathcal{E}}_{r(j_1,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B}_1\cap\overline{\mathcal{B}}_2$	$r=0,j_1,j_2=0$	$E=2R$	$\hat{\mathcal{B}}_R$
$\mathcal{C}_1$	$\epsilon^{\alpha\beta}\mathcal{Q}_{1\beta}\lvert R,r\rangle^{h.w.}_{\alpha}=0$		$E=2+2j_1+2R+r$	$\mathcal{C}_{R,r(j_1,j_2)}$
	$(\mathcal{Q}_1)^2\lvert R,r\rangle^{h.w.}=0$ for $j_1=0$		$E=2+2R+r$	$\mathcal{C}_{R,r(0,j_2)}$
$\overline{\mathcal{C}}_2$	$\epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{2\dot{\beta}}\lvert R,r\rangle^{h.w.}_{\dot{\alpha}}=0$		$E=2+2j_2+2R-r$	$\overline{\mathcal{C}}_{R,r(j_1,j_2)}$
	$(\widetilde{\mathcal{Q}}_2)^2\lvert R,r\rangle^{h.w.}=0$ for $j_2=0$		$E=2+2R-r$	$\overline{\mathcal{C}}_{R,r(j_1,0)}$
	$\mathcal{C}_1\cap\mathcal{C}_2$	$R=0$	$E=2+2j_1+r$	$\mathcal{C}_{0,r(j_1,j_2)}$
	$\overline{\mathcal{C}}_1\cap\overline{\mathcal{C}}_2$	$R=0$	$E=2+2j_2-r$	$\overline{\mathcal{C}}_{0,r(j_1,j_2)}$
$\hat{\mathcal{C}}$	$\mathcal{C}_1\cap\overline{\mathcal{C}}_2$	$r=j_2-j_1$	$E=2+2R+j_1+j_2$	$\hat{\mathcal{C}}_{R(j_1,j_2)}$
	$\mathcal{C}_1\cap\mathcal{C}_2\cap\overline{\mathcal{C}}_1\cap\overline{\mathcal{C}}_2$	$R=0,r=j_2-j_1$	$E=2+j_1+j_2$	$\hat{\mathcal{C}}_{0(j_1,j_2)}$
$\mathcal{D}$	$\mathcal{B}_1\cap\overline{\mathcal{C}}_2$	$r=j_2+1$	$E=1+2R+j_2$	$\mathcal{D}_{R(0,j_2)}$
$\overline{\mathcal{D}}$	$\overline{\mathcal{B}}_2\cap\mathcal{C}_1$	$-r=j_1+1$	$E=1+2R+j_1$	$\overline{\mathcal{D}}_{R(j_1,0)}$
	$\mathcal{E}\cap\overline{\mathcal{C}}_2$	$r=j_2+1,R=0$	$E=r=1+j_2$	$D_{0,(0,j_2)}$
	$\overline{\mathcal{E}}\cap\mathcal{C}_1$	$-r=j_1+1,R=0$	$E=-r=1+j_1$	$\overline{\mathcal{D}}_{0,(j_1,0)}$

Table 1: *Shortening conditions and short multiplets* $\mathfrak{su}(2,2|2)$.

$$\mathcal{I}_{\overline{\mathcal{D}}_{0,(j_1,0)}} = \frac{t^{4j_1+4}}{(pq)^{j_1+1}} \frac{1-(pq)t^2}{(-1)^{2j_1+1}(1-t^3y)(1-t^3y^{-1})}\,, \tag{16}$$

$$\mathcal{I}_{\mathcal{C}_{R,r(j_1,j_2)}} = \frac{t^{4+4R+6j_1+2r}}{(pq)^{R+1-r}} \frac{\left(1-t^2pq\right)\left(t^2pq-t^3\chi_1(y)+\frac{t^4}{pq}\right)}{(-1)^{2j_1+2j_2+1}\left(1-t^3y\right)\left(1-t^3y^{-1}\right)}\chi_{2j_2}(y)\,, \tag{17}$$

$$\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} = \frac{t^{6+4R+4j_1+2j_2}}{(pq)^{R+j_1-j_2}} \frac{\left(1-t^2pq\right)\left(\frac{t}{pq}\chi_{2j_2+1}(y)-\chi_{2j_2}(y)\right)}{(-1)^{2j_1+2j_2}(1-t^3y)(1-t^3y^{-1})}\,, \tag{18}$$

$$\mathcal{I}_{\overline{\mathcal{E}}_{r,(j_1,0)}} = \mathcal{I}_{\mathcal{E}_{0,(0,0)}} = \mathcal{I}_{\overline{\mathcal{C}}_{R,r(j_1,j_2)}} = \mathcal{I}_{\mathcal{A}^E_{R,r(j_1,j_2)}} = 0\,. \tag{19}$$

These may be obtained from [?] by conjugation (exchanging $r \to -r$, $j_1 \leftrightarrow j_2$) and setting $\tau = t^2(pq)^{-1/2}$, $\sigma = ty(pq)^{1/2}$, $\rho = ty^{-1}(pq)^{1/2}$. By applying (??)-(??) in combination with (11)-(19) one can compute the contribution to the index of the $\mathfrak{su}(2,2|3)$ multiplets of $\hat{\mathcal{B}}_{[R_1,R_2]}$.

0.3 Reduction to Three Dimensions.

Let us define $\mathfrak{q} = e^{-\beta}$ where β is the radii of the $\mathbb{S}^1$ factor. Following [?, ?, ?] let us write

$$t = \mathfrak{q}^{1/3}\,, \quad y = \mathfrak{q}^{\eta}\,, \quad p = \mathfrak{q}^{\rho}\,, \quad q = \mathfrak{q}^{\gamma}\,. \tag{20}$$

We can rewrite the Elliptic Gamma functions as

$$\Gamma\left(\mathfrak{q}^{\alpha};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right) = \prod_{n,m=0}^{\infty} \frac{\left[-\alpha+2+n(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}}{\left[\alpha+n(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}} \tag{21}$$

where $[n]_{\mathfrak{q}} = (1-\mathfrak{q}^n)/(1-\mathfrak{q})$ is the q-number. The q-number satisfies $\lim_{\mathfrak{q}\to 1}[n]_{\mathfrak{q}} = n$. The $\beta \to 0$ limit corresponds to $\mathfrak{q} \to 1$. Therefore

$$\lim_{\mathfrak{q}\to 1}\Gamma\left(\mathfrak{q}^{\alpha};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right) = \prod_{n,m=0}^{\infty} \frac{-\alpha+2+n(1+\eta)+m(1-\eta)}{\alpha+n(1+\eta)+m(1-\eta)}\,. \tag{22}$$

We define $\eta = (1-b^2)/(1+b^2)$. We then have

$$\lim_{\mathfrak{q}\to 1}\Gamma\left(\mathfrak{q}^{\alpha};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right) = s_b\left(\frac{\mathrm{i}Q}{2}(1-\alpha)\right)\,. \tag{23}$$

where $Q = b + b^{-1}$ and $s_b(x)$ is the double sine function. Let us now discuss the limit applied to the index (??). We may rewrite (??) as

$$\begin{aligned}
\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k}(t,y,p,q) &= \frac{1}{k}\sum_{l=0}^{k-1}\Bigg\{\Gamma\left(\mathfrak{q}^{2/3+\rho+\gamma-\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\\
&\times\Gamma\left(\mathfrak{q}^{2/3+\gamma-\rho-\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\Gamma\left(\mathfrak{q}^{2/3-2\gamma+\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\\
&\times\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi i l}{k\beta}+(n+1)(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}}{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}}\\
&\times\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+(m+1)(1-\eta)\right]_{\mathfrak{q}}}{\left[\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}}\Bigg\}.
\end{aligned} \tag{24}$$

It is useful to consider splitting the sum over $l = 0, 1, \ldots, k-1$ in order to isolate the $l = 0$ term as follows

$$\begin{aligned}
\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k}(t,y,p,q) &= \frac{1}{k}\sum_{l=1}^{k-1}\Bigg\{\Gamma\left(\mathfrak{q}^{2/3+\gamma-\rho-\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\\
&\times\frac{\Gamma\left(\mathfrak{q}^{2/3-2\gamma+\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\Gamma\left(\mathfrak{q}^{2/3+\rho+\gamma-\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)}{\prod_{n=0}^{\infty}\left[-\frac{2\pi i l}{k\beta}+n(1-\eta)\right]_{\mathfrak{q}}\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)\right]_{\mathfrak{q}}}\\
&\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}}{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}}\\
&\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}}{\left[\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}}\Bigg\}\\
&+\frac{1}{k}\frac{\Gamma\left(\mathfrak{q}^{2/3+\gamma-\rho};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\Gamma\left(\mathfrak{q}^{2/3-2\gamma};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)}{\Gamma\left(\mathfrak{q}^{4/3-\rho-\gamma};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\prod_{n=0}^{\infty}\left[n(1-\eta)\right]_{\mathfrak{q}}\left[n(1+\eta)\right]_{\mathfrak{q}}}.
\end{aligned} \tag{25}$$

Due the form of the denominator in the second line it is clear that in the $\mathfrak{q} \to 1$ limit the factors with $l \neq 0$ vanish. Moreover, when $l \neq 0$ no regularisation is required. On the other hand the product from $l = 0$ requires regularisation. The usual prescription is simply to drop the overall infinite contribution [?], rendering the limit finite. This regularisation is of course independent of k. Moreover, following the presciprition of [?], we identify

$$\eta = \frac{1-b^2}{1+b^2}\,,\quad \gamma = \frac{1}{12}+\frac{\sigma}{\mathrm{i}Q}\,,\quad \rho = \frac{1}{4}-\frac{\sigma}{\mathrm{i}Q}\,. \tag{26}$$

Applying (23) we therefore have that

$$\lim_{\mathfrak{q}\to 1} \mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k}(t,y,p,q) = \frac{1}{k} s_b\left(\frac{\mathrm{i}Q}{4}+\sigma\right) s_b\left(\frac{\mathrm{i}Q}{4}-\sigma\right) . \tag{27}$$

0.4 Vanishing of $\mathbb{Z}_n$ Anomalies

One possible obstruction to the ideas that we have discussed in this paper is the potential that the $\mathbb{Z}_n \subset SL(2,\mathbb{Z})$ has 't Hooft-anomaly. Since the symmetry is only emergent at strong coupling checking the $\mathbb{Z}_n$-anomalies is a non-trivial. In [?] Vafa and Witten studied the S-duality conjecture in topologically twisted $\mathcal{N}=4$ SYM with gauge group G with $Lie(G)=\mathfrak{g}$ simply laced on a four-manifold $\mathcal{M}$. The partition function for the topologically twisted theory is given by [?]

$$Z_G(\tau) = |Z(G)|^{b_1(\mathcal{M})-1} \sum_{v\in H^2(\mathcal{M},\pi_1(G))} Z_v(\tau) , \tag{28}$$

with $Z(G)$ the center of G and

$$Z_v(\tau) = e^{-2\pi \mathrm{i}\tau s} \sum_{K\in\mathbb{Z}-\frac{1}{2}\langle v,v\rangle} \chi\left(\mathbf{M}_{K,v}\right) e^{2\pi\mathrm{i}\tau K} , \quad \hat{Z}_v(\tau) := \eta(\tau)^{-w} Z_v(\tau) , \tag{29}$$

where $v = w_2(P) \in H^2(\mathcal{M},\pi_1(G))$ is the second Stiefel-Whitney class of the G-bundle P over $\mathcal{M}$, $\langle\cdot,\cdot\rangle$ the intersection form on $H^2(\mathcal{M},\pi_1(G))$ and $\mathbf{M}_{K,v}$ is the moduli space of rank K anti-self-dual instantons on $\mathcal{M}$. Additionally [?, ?, ?, ?]

$$s = (\mathrm{rank}\,\mathfrak{g}+1)\chi(\mathcal{M})/4 , \quad w = -\chi(\mathcal{M}) . \tag{30}$$

Under modular transformations (**??**) the partition function transforms as

$$\hat{Z}_v(\tau+1) = e^{-\pi\mathrm{i}(2s+w/12+\langle v,v\rangle)}\hat{Z}_v(\tau) , \tag{31}$$

$$\hat{Z}_v\left(\frac{-1}{\tau}\right) = \pm\frac{1}{|Z(G)|^{b_2(\mathcal{M})/2}} \sum_{u\in H^2(\mathcal{M},Z(G))} e^{2\pi\mathrm{i}\langle v,u\rangle}\hat{Z}_u(\tau) . \tag{32}$$

Let $\mathcal{M} = \mathbb{S}^1\times\mathbb{S}^3$. Its Poincaré polynomial is given by $P_{\mathbb{S}^1\times\mathbb{S}^3}(x) = 1+x+x^3+x^4$ and therefore $b_2(\mathbb{S}^1\times\mathbb{S}^3) = \chi(\mathbb{S}^1\times\mathbb{S}^3) = 0$. By Poincaré duality $H^2(\mathbb{S}^1\times\mathbb{S}^3)\cong H_2(\mathbb{S}^1\times\mathbb{S}^3) = \{1\}$ in particular this fixes $v=0$. Therefore, on $\mathcal{M}=\mathbb{S}^1\times\mathbb{S}^3$, the partition function (28) satisfies

$$Z_G(\tau+1) = Z_G\left(\frac{-1}{\tau}\right) = Z_G(\tau) = Z_{{}^LG}(\tau) . \tag{33}$$

Since the partition function (28) is fully $SL(2,\mathbb{Z})$ invariant, following the arguments of [?], we can conclude that on $\mathbb{S}^1\times\mathbb{S}^3$ the $\mathbb{Z}_n \subset SL(2,\mathbb{Z})$ symmetries at τ fixed as in (**??**) of (twisted) $\mathcal{N}=4$ SYM have vanishing 't Hooft anomaly. Therefore we expect that they can be consistently gauged.

0.5 S-Fold $\stackrel{?}{\equiv}$ Discrete Gauging

In a few cases some of the theories that can be obtained from $\mathbb{Z}_n$ discrete gauging of $\mathcal{N}=4$ SYM are equivalent to some of the theories $S^N_{k,\ell,p'}$. However, as we will now show, in most cases this is a possibility only when the parent S-fold theory $S^N_{k,\ell}$ has enhanced $\mathcal{N}=4$ supersymmetry. The strategy is simply to compute the possibilities which allow for (**??**) to be equal to (**??**). A more refined strategy, via the comparison of the $\frac{1}{8}$-BPS partition functions has been employed in [**?**].

In the following we limit the discussion to only the connected part of the gauge group of the parent theories.

$\mathfrak{g}=\mathfrak{u}(N)$ Equating (**??**) with (**??**) we have

$$N^2 = kN^2 + (2\ell - k - 1)N\,. \tag{34}$$

This has solution only for $k=\ell=1$ with N arbitrary ($S^N_{1,1}$) and $N=\ell=1$ with k arbitrary ($S^1_{k,1}$). However, in both cases, the S-fold parent theory has an operator of dimension 1 and therefore supersymmetry is automatically enhanced to $\mathcal{N}=4$ [**?**, **?**]. Therefore, performing a $\mathbb{Z}_{p'}$ gauging to the S-fold parent is automatically equivalent to making a $\mathbb{Z}_n=\mathbb{Z}_{p'}$ discrete gauging to $\mathcal{N}=4$ SYM with gauge algebra $\mathfrak{u}(1)$ since they are the same theory! In both cases, after the discrete gauging, we have the theory $S^1_{k,1,p'}$.

$\mathfrak{g}=\mathfrak{su}(N+1)=A_N$ Equating (**??**) with (**??**) we have

$$N^2 + 2N = kN^2 + (2\ell - k - 1)N\,. \tag{35}$$

The only solutions are $N=1$ $\ell=2$ with k arbitrary, $N=2$ $k=3$ $\ell=1$ and $N=3$ $k=2$ $\ell=1$. The parent S-fold theory $S^1_{k,2}$ does not exist as an S-fold for $k\neq 2$ [**?**]. $S^2_{3,1}$ and $S^3_{2,1}$ do but they automatically enhance to $\mathcal{N}=4$ supersymmetry and are conjectured to be equivalent to the $\mathfrak{g}=\mathfrak{su}(3)$ and $\mathfrak{g}=\mathfrak{so}(6)$ $\mathcal{N}=4$ theories with gauge coupling $\tau=e^{\pi \mathrm{i}/3}$ and $\tau=$ any [**?**].

$\mathfrak{g}=\mathfrak{so}(2N)=D_N$ Equating (**??**) with (**??**) gives

$$2N^2 - N = kN^2 + (2\ell - k - 1)N\,. \tag{36}$$

Again we have solution for $N=\ell=1$, k arbitrary as well as for $k=2$, $\ell=1$ with N arbitrary. The first case is the same as for $\mathfrak{g}=\mathfrak{u}(1)$. For $k=2$ we always have enhancement to $\mathcal{N}=4$, in our language these theories are $S^N_{2,1}$.

$\mathfrak{g} = \mathfrak{so}(2N+1) = B_N$ Equating (**??**) with (**??**)

$$2N^2 + N = kN^2 + (2\ell - k - 1)N\,. \tag{37}$$

Which has solution only for $k = \ell = 2$ with N arbitrary, $N = 1$ $\ell = 2$ with k arbitrary, $k = 4$ $\ell = 1$ $N = 2$ and $k = 3$ $\ell = 1$ $N = 3$. In the first three cases $S^N_{2,2}$, $S^1_{k,2}$ and $S^2_{4,1}$ always have enhancement to $\mathcal{N} = 4$ [**?**]. They have the correct spectrum of Coulomb branch operators to be equivalent to the $\mathfrak{g} = B_N$ or C_N, B_1 and $B_2 \cong C_2$ $\mathcal{N} = 4$ theories respectively. In the final case we find that the S-fold $S^3_{3,1}$ has the same central charges as the $\mathcal{N} = 4$ $\mathfrak{so}(7)$ theory. Clearly they are not the same theory, however due to the matching of central charges we cannot rule out the possibility that the discrete gauging $S^3_{3,1,p'}$ may yield the same theory as a $\mathbb{Z}_n = \mathbb{Z}_{p'}$ gauging of $\mathcal{N} = 4$ $\mathfrak{so}(7)$ theory. Our analysis (**??**) would seem to imply that this is infact not the case however, since the $S^3_{3,1}$ theory has a discrete $\mathbb{Z}_3$ global symmetry while the $\mathfrak{so}(7)$ theory can only have $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_8$ discrete symmetry groups.

$\mathfrak{g} = \mathfrak{sp}(N) = C_N$ Since the degree of the Casimir invariants for $\mathfrak{sp}(N)$ are the same as for $\mathfrak{so}(2N+1)$ the discussion is the same as above.

$\mathfrak{g} = E_6$

$$78 = 36k + 6(2\ell - k - 1)\,. \tag{38}$$

There is solution for $k = \ell = 2$, the corresponding $S^6_{2,2}$ has a dimension 2 Coulomb branch operator and therefore $\mathcal{N} = 4$ enhancement this S-fold is the standard $\mathfrak{g} = B_6$ or C_6 perturbative orientifold.

$\mathfrak{g} = E_7$

$$133 = 49k + 7(2\ell - k - 1)\,, \tag{39}$$

There is solution only for $k = 3$, $\ell = 1$. There is no $\mathcal{N} = 4$ enhancement of the corresponding $S^7_{3,1}$ theory.

$\mathfrak{g} = E_8$

$$248 = 64k + 8(2\ell - k - 1)\,, \tag{40}$$

There is solution only for $k = 4$, $\ell = 2$. There is no $\mathcal{N} = 4$ enhancement however $S^8_{4,2}$ does not exist as an S-fold [**?**].

$\mathfrak{g} = F_4$

$$52 = 16k + 4(2\ell - k - 1)\,, \tag{41}$$

There is solution only for $k = 4$, $\ell = 1$. There is no $\mathcal{N} = 4$ enhancement.

$\mathfrak{g} = G_2$

$$14 = 4k + 2(2\ell - k - 1)\,, \tag{42}$$

There is solution only for $k = 6$ $\ell = 1$ and $k = 4$, $\ell = 2$. In the first case $S^2_{6,1}$ exists as an S-fold but there is $\mathcal{N} = 4$ enhancement and it is believed to be equal to the G_2 $\mathcal{N} = 4$ SYM theory with fixed gauge coupling. In the second case the S-folds of type $S^2_{4,2}$ do not fall into the classification of [**?**] and are believed to not exist as S-folds.