0.1 Elliptic Genus Computation

The Elliptic genus for 2d $\mathcal{N} = (0,2)$ theories was computed via localisation in [**?**, **?**]. Writing schematically it is given by

$$\mathrm{Ell}(q,a) = \mathrm{Tr}_{\mathrm{R}}\left[(-1)^F q^{H_-} \prod a^f\right] . \tag{1}$$

A $\mathcal{N} = (0,2)$ chiral multiplet Φ in representations R of $G \times F$ contributes to Ell

$$\mathrm{Ell}_{\Phi,R}(q,a) = \prod_{\rho \in R} \mathrm{i}\frac{\eta(q)}{\theta_1(a^\rho; q)} . \tag{2}$$

Similarly, a $\mathcal{N} = (0,2)$ Fermi multiplet Ψ contributes

$$\mathrm{Ell}_{\Psi,R}(q,a) = \prod_{\rho \in R} \mathrm{i}\frac{\theta_1(a^\rho; q)}{\eta(q)} . \tag{3}$$

Finally we present the formula for the $\mathcal{N} = (0,2)$ Vector multiplet V

$$\mathrm{Ell}_{V,G}(q,a) = (\mathrm{i}\eta(q))^{2\,\mathrm{rank}\,G} \prod_{\alpha \in G} \mathrm{i}\frac{\theta_1(a^\alpha; q)}{\eta(q)} , \tag{4}$$

which is that of a Fermi multiplet in the adjoint representation of G. The vector multiplets should be paired with the corresponding integration over the maximal torus of G

$$\frac{1}{|W(G)|} \oint_{T[G]} \prod_{n=1}^{\mathrm{rank}\,G} \frac{da_n}{2\pi \mathrm{i} a_n} , \tag{5}$$

which is essentially the Haar measure divided by the Vandermonde determinant. $W(G)$ is the Weyl group of G and the contour is taken over $|a| = 1$.

0.1.1 M-Strings Index without Defect

The partition functions corresponding to each $\mathcal{N} = (0,2)$ multiplet read

$$\mathrm{Ell}_Y = \prod_{n=1}^{M} \prod_{I=1}^{K_n} \prod_{J=1}^{K_{n+1}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{c}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{n,I}}{y_{n+1,J}}; Q_\tau\right)} , \tag{6}$$

$$\mathrm{Ell}_{\tilde{Y}} = \prod_{n=1}^{M} \prod_{I=1}^{K_n} \prod_{J=1}^{K_{n-1}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\frac{1}{\mathfrak{c}}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{n,I}}{y_{n-1,J}}; Q_\tau\right)} , \tag{7}$$

$$\mathrm{Ell}_\zeta = \prod_{n=1}^{M} \prod_{I,J=1}^{K_n} \mathrm{i}\frac{\theta_1\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{y_{n,I}}{y_{n,J}}; Q_\tau\right)}{\eta(Q_\tau)} , \tag{8}$$

$$\mathrm{Ell}_{\Upsilon} = \prod_{n=1}^{M} (\mathrm{i}\eta(Q_\tau))^{2K_n} \prod_{I\neq J} \mathrm{i}\frac{\theta_1\left(\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{9}$$

$$\mathrm{Ell}_{X} = \prod_{n=1}^{M}\prod_{I,J=1}^{K_n} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{q}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}\,, \tag{10}$$

$$\mathrm{Ell}_{\widetilde{X}} = \prod_{n=1}^{M}\prod_{I,J=1}^{K_n} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{t}^{-1}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}\,, \tag{11}$$

$$\mathrm{Ell}_{\lambda} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{J=1}^{K_{n+1}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}\sqrt{\mathfrak{q}\mathfrak{t}}\frac{y_{n,I}}{y_{n+1,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{12}$$

$$\mathrm{Ell}_{\widetilde{\lambda}} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{J=1}^{K_{n-1}} \mathrm{i}\frac{\theta_1\left(\frac{\sqrt{\mathfrak{q}\mathfrak{t}}}{\mathfrak{c}}\frac{y_{n,I}}{y_{n-1,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{13}$$

$$\mathrm{Ell}_{\phi} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{y_{n,I}}{x_{n,A}};Q_\tau\right)}\,, \tag{14}$$

$$\mathrm{Ell}_{\widetilde{\phi}} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{x_{n,A}}{y_{n,I}};Q_\tau\right)}\,, \tag{15}$$

$$\mathrm{Ell}_{\psi} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{y_{n,I}}{x_{n-1,A}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{16}$$

$$\mathrm{Ell}_{\widetilde{\psi}} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{x_{n+1,J}}{y_{n,I}};Q_\tau\right)}{\eta(Q_\tau)}\,. \tag{17}$$

0.1.2 M-Strings Index with Defects

We now list the result of performing the $\mathbb{Z}_k$ orbifold.

$$\mathrm{Ell}_{Y}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n+1)(i-1)}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{c}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{ni,I}}{y_{(n+1)(i-1),J}};Q_\tau\right)}\,, \tag{18}$$

$$\mathrm{Ell}_{\Upsilon}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k} (\mathrm{i}\eta(Q_\tau))^{2K_{ni}} \prod_{I\neq J} \mathrm{i}\frac{\theta_1\left(\frac{y_{ni,I}}{y_{ni,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{19}$$

$$\mathrm{Ell}_{\widetilde{Y}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n-1)i}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{c}^{-1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{ni,I}}{y_{(n-1)i,J}};Q_\tau\right)}\,, \tag{20}$$

$$\mathrm{Ell}_{\zeta}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{n(i+1)}} \mathrm{i}\frac{\theta_1\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{y_{ni,I}}{y_{n(i+1),J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{21}$$

$$\mathrm{Ell}_{X}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{n(i+1)}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{q}\frac{y_{ni,I}}{y_{n(i+1),J}};Q_\tau\right)}\,, \tag{22}$$

$$\mathrm{Ell}_{\widetilde{X}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I,J=1}^{K_{ni}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{t}^{-1}\frac{y_{ni,I}}{y_{ni,J}};Q_\tau\right)}\,, \tag{23}$$

$$\mathrm{Ell}_{\lambda}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n+1)i}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}\mathfrak{q}\mathfrak{t}\frac{y_{ni,I}}{y_{(n+1)i,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{24}$$

$$\mathrm{Ell}_{\widetilde{\lambda}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n-1)(i-1)}} \mathrm{i}\frac{\theta_1\left(\frac{\mathfrak{q}\mathfrak{t}}{\mathfrak{c}}\frac{y_{n,I}}{y_{(n-1)(i-1),J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{25}$$

$$\mathrm{Ell}_{\phi}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{ni}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{y_{ni,I}}{x_{ni,A}};Q_\tau\right)}\,, \tag{26}$$

$$\mathrm{Ell}_{\widetilde{\phi}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{n(i-1)}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{x_{n(i-1),A}}{y_{ni,I}};Q_\tau\right)}\,, \tag{27}$$

$$\mathrm{Ell}_{\psi}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{(n-1)i}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{y_{ni,I}}{x_{(n-1)i,A}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{28}$$

$$\mathrm{Ell}_{\widetilde{\psi}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{(n+1)(i-1)}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{x_{(n+1)(i-1),A}}{y_{ni,I}};Q_\tau\right)}{\eta(Q_\tau)}\,. \tag{29}$$

Clearly each $\mathrm{Ell}_P^{\mathbb{Z}_k}$ is invariant under the action (**??**).

0.2 Review of the 5d Index Localisation

We describe in more detail, following [**?**], the localisation of the superconformal index

$$Z\left(s,p,v,\mathbf{q}_A\right) = \mathrm{Tr}_{\mathcal{H}_{\mathbb{S}^4}}\left[(-1)^F e^{-\beta\delta} s^{-2J_R-2J_R^R} p^{-2J_L} v^{2J_L^R} \prod_{A=1}^{N}\mathbf{q}_A^{K_A}\right]\,. \tag{30}$$

It receives contributions only from those states which satisfy the BPS condition (**??**). After Wick rotation to Euclidean time $X^5 = \mathrm{i}\tau_E$ the index Z admits a path integral representation on $\mathbb{S}^1\times\mathbb{S}^4$

$$Z(s,p,v,\mathbf{q}_A) = \int_{\mathbf{C}(\mathbb{S}^1\times\mathbb{S}^4)}[\mathcal{D}\Phi]e^{-S_E[\Phi]}\,. \tag{31}$$

The insertion of chemical potentials results in twisted boundary conditions upon going around the $\mathbb{S}^1$

$$\Phi(\tau_E+\beta) = (-1)^F e^{-\beta(-2J_R-3J_R^R)} s^{-2J_R-2J_R^R} p^{-2J_L} v^{2J_L^R}\Phi(\tau_E)\,. \tag{32}$$

The twisted boundary conditions (30) may be taken into account by shifting time derivatives

$$\partial_{\tau_E} \to \hat{\partial}_{\tau_E} = \partial_{\tau_E} + (2-\mathrm{i}\epsilon_+)J_R + (3-\mathrm{i}\epsilon_+)J_R^R - \mathrm{i}\epsilon_- J_L + 2mJ_L^R \tag{33}$$

and giving periodic boundary conditions to all fermions to account for the insertion of $(-1)^F$. To perform the localisation we deform the Lagrangian by the $\widetilde{\mathtt{Q}}$ exact term

$$\mathcal{L} \to \mathcal{L} + t\left\{\widetilde{\mathtt{Q}}, V\right\}\,, \tag{34}$$

t may then be taken to infinity in which case the path integral localises around the set of saddle points of $\{\widetilde{\mathtt{Q}}, V\} = 0$. The supersymmetry transformation $\widetilde{\mathtt{Q}} + \widetilde{\mathtt{S}}$ is parametrised by the Killing spinor

$$\varepsilon = \varepsilon_q + \varepsilon_s = e^{\frac{1}{2}\theta_1\gamma^{51}} e^{\frac{1}{2}\theta_2\gamma^{12}} e^{\frac{1}{2}\theta_3\gamma^{23}} e^{\frac{1}{2}\theta_4\gamma^{34}} \left(\varepsilon_0^q + \gamma^5\varepsilon_0^s\right)\,. \tag{35}$$

where ε_0^q, ε_0^s are constant spinors corresponding to $\widetilde{\mathtt{Q}}$, $\widetilde{\mathtt{S}}$. ε satisfies the Killing spinor equation

$$\hat{\nabla}_\mu\varepsilon = \frac{1}{2}\gamma_\mu\gamma^5\tilde{\varepsilon} \tag{36}$$

where $\hat{\nabla}$ denotes the covariant derivative on $\mathbb{S}^4\times\mathbb{S}^1$ with the twisted time derivative (31) and $\tilde{\varepsilon} = -\varepsilon_q + \varepsilon_s$. The square of the supercharge $\widetilde{\mathtt{Q}} + \widetilde{\mathtt{S}}$ is then given by

$$\delta_\varepsilon^2 = -\mathrm{i}\mathcal{L}_{\frac{\partial}{\partial\tau_E}} + \mathrm{i}G - \epsilon_+ J_R - \epsilon_+ J_R^R - \epsilon_- J_L + 2mJ_L^R \tag{37}$$

where $\mathcal{L}_v$ denotes the Lie derivative and G denotes a gauge transformation. We then choose V such that

$$\{\widetilde{\mathtt{Q}}, V\} = \delta_\varepsilon\left((\delta_\varepsilon\lambda)^\dagger\lambda\right) \tag{38}$$

upon taking the $t\to\infty$ limit the path integral for the vector multiplets localises onto the critical points of the potential

$$\begin{aligned}(\delta_\varepsilon\lambda)^\dagger\lambda =& F_{\tau_E\mu}F^{\tau_E\mu} + \cos^2\frac{\theta_1}{2}\left(F_{ij}^- - \omega_{ij}^-\phi\right)^2 + \sin^2\frac{\theta_1}{2}\left(F_{ij}^+ - \omega_{ij}^+\right)^2 \\ &+ \left(\hat{\nabla}_\mu\phi\right)^2 - D^2\end{aligned} \tag{39}$$

which is positive semi-definite (recall that in Euclidean signature D is pure imaginary) $F_{ij}^\pm = \frac{1}{2}\left(F_{ij} \mp \star_4 F_{ij}\right)$ and

$$\omega_{ij}^+ = \frac{\mathrm{i}}{2\sin^2\frac{\theta_1}{2}}\bar{\tilde{\varepsilon}}^+\gamma^5\gamma^{ij}\varepsilon^+\,,\quad \omega_{ij}^- = \frac{\mathrm{i}}{2\cos^2\frac{\theta_1}{2}}\bar{\tilde{\varepsilon}}^-\gamma^5\gamma^{ij}\varepsilon^-\,, \tag{40}$$

$$\omega_{ij}^+\omega^{+ij} = \omega_{ij}^-\omega^{-ij} = 1\,. \tag{41}$$

Here $\varepsilon^{\pm} = \frac{1}{2}\left(1 \pm \gamma^5\right)\varepsilon$, $\tilde{\varepsilon}^{\pm} = \frac{1}{2}\left(1 \pm \gamma^5\right)\tilde{\varepsilon}$. The classical saddle points of the potential (37) are given by $F_{\tau_E \mu} = D^A = 0$ while ϕ is covariantly constant everywhere on $\mathbb{S}^4$. On the other hand, by the Bianchi identity, the second and third terms of (37) imply that away from the north and south poles we have $F_{ij} = \phi = 0$. Note also that (anti-)self-dual instantons ($F^- = 0$) $F^+ = 0$ can be localised at (north)south-pole. The index hence factorises as in (**??**)

$$Z = \int \prod_{A=1}^{N} [da_A]\, Z_{\text{south}}\left(a_{A,n}, \epsilon_1, \epsilon_2, m, \mathbf{q}_A\right) Z_{\text{north}}\left(a_{A,n}, \epsilon_1, \epsilon_2, m, \mathbf{q}_A^{-1}\right) \tag{42}$$

After the gauge fixing we have a BRST operator $\widetilde{\mathtt{Q}} + \widetilde{\mathtt{S}} \to \mathfrak{Q}$ and may make a cohomological formulation of the supercharge $\mathfrak{Q}$. The bosonic and fermionic fields may be regarded as differential forms on a supermanifold $\mathcal{X}$ such that they form a $\mathfrak{Q}$-complex

$$\mathfrak{Q}\Phi_{b,f} = \Phi'_{f,b}\,, \quad \mathfrak{Q}\Phi'_{f,b} = \mathfrak{Q}^2\Phi_{b,f} \tag{43}$$

and

$$\begin{aligned}\mathfrak{Q}^2 =& \mathcal{L}_{\frac{\partial}{\partial \tau_E}} - \frac{a}{\beta} - \mathrm{i}\epsilon_+(J_R + J_R^R) - \mathrm{i}\epsilon_- J_L + 2\mathrm{i}mJ_L^R \\ =& \mathcal{L}_{\frac{\partial}{\partial \tau_E}} - \frac{a}{\beta} + \mathrm{i}\epsilon_1(J_{12} - J_R^R) + \mathrm{i}\epsilon_2(J_{34} - J_R^R) + 2\mathrm{i}mJ_L^R\,.\end{aligned} \tag{44}$$

We now study the $\mathfrak{Q}^2$-equivariant cohomology of $\mathcal{X}$. After expanding the gauge fixed $\mathfrak{Q}$ invariant terms to quadratic order the Gaussian integrals may be evaluated. Due to cancelations due to pairing by the $\mathfrak{Q}$-complex (41) the 1-loop contributions take the form [**?**]

$$Z^{\text{1-loop}} = \sqrt{\frac{\det_{\operatorname{coker} D} \mathfrak{Q}^2|_f}{\det_{\ker D} \mathfrak{Q}^2|_b}} \tag{45}$$

where D is the quadratic operator in (36). $Z^{\text{1-loop}}$ may computed using the equivariant Atiyah-Singer index theorem. The fixed point of the torus action of $\mathfrak{Q}^2$ (42) are the north and south poles. In a neighbourhood of the north pole D is isomorphic to the anti-self-dual complex (d, d^-)

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^-} \Omega^{2-}\,, \tag{46}$$

while at the south pole D is isomorphic to the self-dual complex (d, d^+)

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+}\,, \tag{47}$$

where (Ω^{2-}) Ω^{2+} denotes the space of (anti-)self-dual 2-forms. For concreteness let us focus on the south pole. At the south pole we may choose local coordinates z_1, z_2 parametrising $\mathbb{C}^2$. The torus action acts on those coordinates by
$(z_1, z_2) \mapsto (e^{\mathrm{i}\epsilon_1} z_1, e^{\mathrm{i}\epsilon_2} z_2)$. Furthermore we expand the elements of the self-dual

complex in eigenmodes of the circle momenta $\Phi = \sum_{p\in\mathbb{Z}} \Phi_p e^{\frac{2\pi i p}{\beta}}$. The equiviarint index for the vectors multiplets is then given by

$$\mathrm{ind}_{\mathrm{vec}}\, D = -\frac{1+e^{\mathrm{i}\beta\epsilon_+}}{2(1-\epsilon^{\mathrm{i}\epsilon_1})(1-\epsilon^{\mathrm{i}\epsilon_2})} \sum_{A=1}^{N} \sum_{n\neq m} e^{\frac{\mathrm{i}}{\beta}(a_{A,n}-a_{A,m})} \sum_{p\in\mathbb{Z}} e^{\frac{2\pi \mathrm{i} p}{\beta}}\,. \tag{48}$$

One may compute the 1-loop determinants for the hypermultiplets by localisation. In that case the differential operator D for the hypermultiplet is isomorphic to a Dirac complex

$$\Omega^{\left(\frac{1}{2},0\right)} \xrightarrow{D_{\mathrm{Dirac}}} \Omega^{\left(0,\frac{1}{2}\right)}\,. \tag{49}$$

The equivariant index at the south pole reads

$$\mathrm{ind}_{\mathrm{hyp}} = \frac{e^{\mathrm{i}\epsilon_+/2}}{(1-e^{\mathrm{i}\epsilon_1})(1-e^{\mathrm{i}\epsilon_2})} \sum_{A=1}^{N} \sum_{m=1}^{M_A} \sum_{n=1}^{M_{A+1}} e^{\frac{\mathrm{i}}{\beta}(a_{A,n}-a_{A+1,m})+\mathrm{i}m} \sum_{p\in\mathbb{Z}} e^{\frac{2\pi\mathrm{i}}{\beta}p}\,. \tag{50}$$

0.3 Refined Topological String Computation

The partition function for M M5-branes on A_{N-1} is equivalent to the refined topological string partition function of certain Calabi-Yau 3-folds $X_{M,N}$

$$Z^{\mathrm{top}}_{\mathrm{refined}}(X_{M,N}) = Z^{\text{M-theory}}\left((A_{N-1}\times \underbrace{\mathbb{R}^4)\ltimes T^2}_{M\ \text{M5-branes}} \times\mathbb{R} \right) \tag{51}$$

$$= \left(\prod_{A=1}^{N} Z^{(A)}_{U(1)}\right) Z_M^{A_{N-1}}\,, \tag{52}$$

where $Z^{(A)}_{U(1)}$ is the partition function for a single M5-brane. The Calabi-Yau 3-folds arise as the M-theory lift of the dual (p,q)-brane web construction. The (p,q) web is obtained by compactifying the setup of Table **??** along X^6. This gives rise to the 5d quiver gauge theories $\mathcal{N}_{M,N}$ supported on the D4-branes with $\mathcal{N}=1$ supersymmetry. The defects become D4'-branes. T-dualising along the Taub-Nut circle X^8 results the setup of Table 1 After turning on mass deformation we end up with a (p,q)-brane web. We may now lift the IIB setup on $\mathbb{S}^1$ to M-theory on T^2. (p,q)-branes corresponds to the degeneration of the (p,q) cycle of the T^2 as we vary along the X^5, X^7 base. Hence the (p,q)-brane web lifts to M-theory on a non-compact, elliptically fibered CY_3 whose toric diagram is given by the (p,q)-web itself. We compute the refined A-model open string amplitude for the strip geometry using the refined topological vertex formalism [**?**, **?**]. The refined topological vertex is labelled by three Young diagrams and is given by

$$C_{\lambda\mu\nu}(\mathbf{t},\mathbf{q}) =$$

$$\mathfrak{t}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}} \mathfrak{q}^{\frac{||\mu||^2+||\nu||^2}{2}} \widetilde{Z}_\nu(\mathfrak{t},\mathfrak{q}) \sum_\eta \left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{|\lambda|+|\eta|-|\mu|}{2}} s_{\lambda^{\mathrm{T}}/\eta}(\mathfrak{t}^{-\rho}\mathfrak{q}^{-\nu}) s_{\mu/\eta}(\mathfrak{t}^{-\nu^{\mathrm{T}}}\mathfrak{q}^{-\rho}) \tag{53}$$

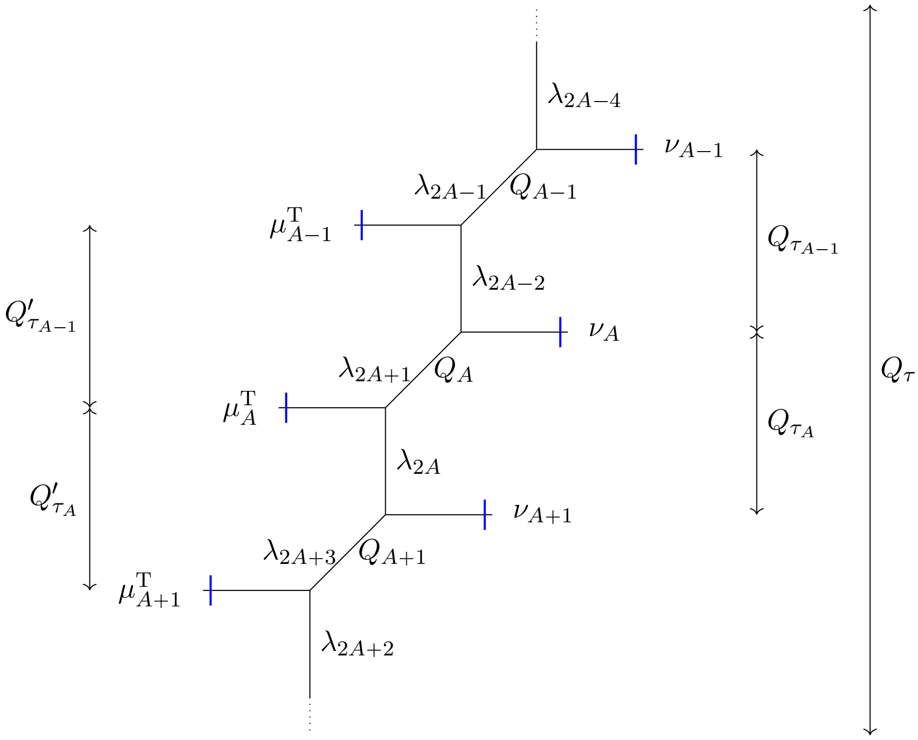

Figure 1: *Strip geometry which builds the partition function of for the A_{N-1} geometry. Blue lines denote the direction of the refined topological vertex. The dotted lines indicate the fact that we are dealing with the partial compactification of the strip geometry.*

	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}_{\epsilon_2}$		$\mathbb{S}^1$	$\mathbb{R}$	$\mathbb{S}^1$	$\mathbb{R}^3$		
	X^1	X^2	X^3	X^4	X^5	X^7	X^8	X^9	X^{10}	X^{11}
N $NS5$	–	–	–	–	–	–	·	·	·	·
M $D5$	–	–	–	–	–	·	–	·	·	·
K $F1$	·	·	·	·	–	–	·	·	·	·
k $D3'$	·	·	–	–	–	·	·	–	·	·

Table 1: *Type-IIB (p,q)-brane web.*

where $\rho = \{-1/2, -3/2, -5/2, \dots\}$, $s_{\lambda/\eta}(x)$ is the skew Schur function and

$$\widetilde{Z}_\nu(\mathfrak{t},\mathfrak{q}) = \prod_{(l,p)\in\nu} \frac{1}{1-\mathfrak{q}^{\nu_l - p}\mathfrak{t}^{\nu_p^{\mathrm{T}} - l + 1}} \,. \tag{54}$$

$\mathfrak{q}, \mathfrak{t}$ are related torus action $(z_1, z_2) \mapsto (e^{2\pi \mathrm{i}\epsilon_1} z_1, e^{2\pi\mathrm{i}\epsilon_2} z_2)$ on $\mathbb{C}^2$ by

$$\mathfrak{q} = e^{2\pi\mathrm{i}\epsilon_1}\,, \quad \mathfrak{t} = e^{-2\pi\mathrm{i}\epsilon_2}\,. \tag{55}$$

The framing factors are

$$f_\nu(\mathfrak{t},\mathfrak{q}) = (-1)^{|\nu|}\mathfrak{t}^{\frac{||\nu^{\mathrm{T}}||^2}{2}}\mathfrak{q}^{\frac{-||\nu||^2}{2}}\,, \quad \widetilde{f}_\nu(\mathfrak{t},\mathfrak{q}) = \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{|\nu|}{2}} f_\nu(\mathfrak{t},\mathfrak{q})\,. \tag{56}$$

0.3.1 Without Defect - M-Strings Review

Applying the standard rules of the refined topological vertex formalism [?] the partition function for the strip geometry Figure 1 is

$$\begin{aligned} Z^{\mu_1,\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A};\mathfrak{t},\mathfrak{q}) = \sum_{\{\lambda\}}\prod_{A=1}^N \Big\{ &(-Q_A)^{|\lambda_{2A+1}|}\left(-Q_A^{-1}Q_{\tau_A}\right)^{|\lambda_{2A}|} \\ &\times C_{\lambda^{\mathrm{T}}_{2A}\lambda^{\mathrm{T}}_{2A+1}\mu^{\mathrm{T}}_A}\left(\mathfrak{t}^{-1},\mathfrak{q}^{-1}\right) C_{\lambda_{2A+2}\lambda_{2A+1}\nu_A}\left(\mathfrak{q}^{-1},\mathfrak{t}^{-1}\right)\Big\}\,, \end{aligned} \tag{57}$$

note that, since we partially compactify the strip geometry (dotted lines in the figure) we identify the indices $A \sim A+N$. $Q_\tau = \prod_{A=1}^N Q_{\tau_A} = \prod_{A=1}^N Q'_{\tau_A}$. Inserting the explicit expression for the vertex we have

$$\begin{aligned} &Z^{\mu_1,\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A};\mathfrak{t},\mathfrak{q}) = \prod_{A=1}^N \Bigg\{ \mathfrak{q}^{-\frac{||\mu_A^{\mathrm{T}}||^2}{2}}\mathfrak{t}^{-\frac{||\nu_A||^2}{2}}\widetilde{Z}_{\mu_A^{\mathrm{T}}}\left(\mathfrak{t}^{-1},\mathfrak{q}^{-1}\right) \\ &\widetilde{Z}_{\nu_A}\left(\mathfrak{q}^{-1},\mathfrak{t}^{-1}\right)\sum_{\{\lambda\},\{\sigma\}}\Bigg[(-Q_A)^{|\lambda_{2A+1}|}\left(-\frac{Q_{\tau_A}}{Q_A}\right)^{|\lambda_{2A}|} s_{\lambda_{2A}/\sigma_{2A}}\left(\mathfrak{t}^\rho\mathfrak{q}^{\mu_A^{\mathrm{T}}}\right) \\ &s_{\lambda^{\mathrm{T}}_{2A+1}/\sigma_{2A}}\left(\mathfrak{q}^{\rho+\frac{1}{2}}\mathfrak{t}^{\mu_A-\frac{1}{2}}\right) s_{\lambda^{\mathrm{T}}_{2A+2}/\sigma_{2A+1}}\left(\mathfrak{q}^\rho\mathfrak{t}^{\nu_A}\right) \\ &s_{\lambda_{2A+1}/\sigma_{2A+1}}\left(\mathfrak{q}^{\nu_A^{\mathrm{T}}-\frac{1}{2}}\mathfrak{t}^{\rho+\frac{1}{2}}\right)\Bigg]\Bigg\}\,. \end{aligned} \tag{58}$$

The method for simplifying this product was given in [?, ?]. Consider

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) &:= \prod_{A=1}^{N} \Bigg((-Q_A)^{|\lambda_{2A+1}|} \left(\frac{-Q_{\tau_A}}{Q_A} \right)^{|\lambda_{2A}|} \\
&\times s_{\lambda_{2A}/\sigma_{2A}}(X_A)\, s_{\lambda^{\mathrm{T}}_{2A+2}/\sigma_{2A+1}}(Y_A)\, s_{\lambda^{\mathrm{T}}_{2A+1}/\sigma_{2A}}(Z_A)\, s_{\lambda_{2A+1}/\sigma_{2A+1}}(W_A) \Bigg)
\end{aligned}
\tag{59}
$$

with the products over A are defined modulo N. We now apply repeatedly the identities (??), (??) and (??). We have

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) &= \prod_{A=1}^{N} \left(-Q_A^{-1} Q_{\tau_A}\right)^{|\sigma_{2A}|} (-Q_A)^{|\sigma_{2A+1}|} \\
&\times s_{\sigma^{\mathrm{T}}_{2A}/\lambda_{2A}}(Y_{A-1})\, s_{\sigma^{\mathrm{T}}_{2A-1}/\lambda^{\mathrm{T}}_{2A}}\left(-Q_A^{-1} Q_{\tau_A} X_A\right) s_{\sigma^{\mathrm{T}}_{2A}/\lambda^{\mathrm{T}}_{2A+1}}(-Q_A W_A) \\
&\times s_{\sigma^{\mathrm{T}}_{2A+1}/\lambda_{2A+1}}(Z_A) \prod_{l,p=1}^{\infty} \left(1 - Q_A^{-1} Q_{\tau_A} X_{A;l} Y_{A-1;p}\right) \left(1 - Q_A Z_{A;l} W_{A;p}\right)
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) &= \prod_{A=1}^{N} (-Q_{\tau_A})^{|\lambda_{2A+1}|} \\
&\times \prod_{l,p=1}^{\infty} \frac{\left(1 - Q_A^{-1} Q_{\tau_A} X_{A;l} Y_{A-1;p}\right) \left(1 - Q_A Z_{A;l} W_{A;p}\right)}{\left(1 - Q_{\tau_A} W_{A;l} Y_{A-1;p}\right) \left(1 - Q_A Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1;l} Z_{A;p}\right)} \\
&\times s_{\lambda^{\mathrm{T}}_{2A+1}/\sigma^{\mathrm{T}}_{2A}}(Y_{A-1})\, s_{\lambda_{2A}/\sigma^{\mathrm{T}}_{2A}}(-Q_{\tau_A} W_A)\, s_{\lambda^{\mathrm{T}}_{2A}/\sigma^{\mathrm{T}}_{2A-1}}(-Q_{A-1} Z_{A-1}) \\
&\times s_{\lambda_{2A+1}/\sigma^{\mathrm{T}}_{2A+1}}\left(-Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1}\right)
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) &= \prod_{A=1}^{N} (-Q_{\tau_A})^{|\sigma_{2A}|} \\
&\times \prod_{l,p=1}^{\infty} \Bigg\{ \frac{\left(1 - Q_A^{-1} Q_{\tau_A} X_{A;l} Y_{A-1;p}\right) \left(1 - Q_A Z_{A;l} W_{A;p}\right)}{\left(1 - Q_{\tau_A} W_{A;l} Y_{A-1;p}\right) \left(1 - Q_A Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1;l} Z_{A;p}\right)} \\
&\times \left(1 - Q_{A-1} Z_{A-1} Q_{\tau_A} W_A\right) \left(1 - Q_{A+1}^{-1} Q_{\tau_A} Q_{\tau_{A+1}} X_{A+1} Y_{A-1}\right) \Bigg\} \\
&\times s_{\sigma_{2A}/\lambda^{\mathrm{T}}_{2A+1}}\left(-Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1}\right) s_{\sigma_{2A}/\lambda_{2A}}\left(-Q_{\tau_{A-1}} Z_{A-1}\right) \\
&\times s_{\sigma_{2A-1}/\lambda^{\mathrm{T}}_{2A}}(Q_{\tau_A} W_A)\, s_{\sigma_{2A+1}/\lambda_{2A+1}}(Q_{\tau_A} Y_{A-1})
\end{aligned}
\tag{62}
$$

$$G^{(N)}(X_A,Y_A,Z_A,W_A;Q_A,Q_{\tau_A}) = \prod_{A=1}^{N}\left(-Q_A^{-1}Q_{\tau_A}\right)^{|\lambda_{2A}|}(-Q_A)^{|\lambda_{2A+1}|}$$
$$s_{\lambda^{\mathrm{T}}_{2A+1}/\sigma_{2A}}\left(Q_{A-1}Q_A^{-1}Q_{\tau_A}Z_{A-1}\right)s_{\lambda_{2A}/\sigma_{2A}}\left(Q_{A+1}^{-1}Q_{\tau_{A+1}}Q_AX_{A+1}\right)$$
$$s_{\lambda_{2A+1}/\sigma_{2A+1}}\left(Q_{\tau_{A+1}}W_{A+1}\right)s_{\lambda^{\mathrm{T}}_{2A+2}/\sigma_{2A+1}}\left(Q_{\tau_A}Y_{A-1}\right)$$
$$\prod_{l,p=1}^{\infty}\left\{\frac{\left(1-Q_A^{-1}Q_{\tau_A}X_{A;l}Y_{A-1;p}\right)(1-Q_AZ_{A;l}W_{A;p})}{(1-Q_{\tau_A}W_{A;l}Y_{A-1;p})\left(1-Q_{A+1}^{-1}Q_{\tau_{A+1}}Q_AX_{A+1;l}Z_{A;p}\right)}\right.$$
$$\frac{\left(1-Q_{A+1}^{-1}Q_{\tau_A}Q_{\tau_{A+1}}Y_{A-1;l}X_{A+1;p}\right)}{\left(1-Q_{\tau_A}Q_{A-1}Q_{A+1}^{-1}Q_{\tau_{A+1}}X_{A+1;l}Z_{A-1};p\right)}$$
$$\left.\frac{(1-Q_{\tau_A}Q_{A-1}W_{A;l}Z_{A-1;p})}{\left(1-Q_{\tau_{A+1}}Q_{\tau_A}Y_{A-1;l}W_{A+1;p}\right)}\right\}. \tag{63}$$

Finally, this can be written as

$$\begin{aligned}&G^{(N)}(X_A,Y_A,Z_A,W_A;Q_A,Q_{\tau_A}) = \\ &\prod_{A=1}^{N}\prod_{l,p=1}^{\infty}\left\{\frac{\left(1-Q_A^{-1}Q_{\tau_A}X_{A;l}Y_{A-1;p}\right)(1-Q_AZ_{A;l}W_{A;p})}{(1-Q_{\tau_A}W_{A;l}Y_{A-1;p})\left(1-Q_{A+1}^{-1}Q_{\tau_{A+1}}Q_AX_{A+1;l}Z_{A;p}\right)}\right.\\ &\times\left.\frac{\left(1-Q_{A+1}^{-1}Q_{\tau_A}Q_{\tau_{A+1}}Y_{A-1;l}X_{A+1;p}\right)(1-Q_{\tau_A}Q_{A-1}W_{A;l}Z_{A-1;p})}{\left(1-Q_{\tau_A}Q_{A-1}Q_{A+1}^{-1}Q_{\tau_{A+1}}X_{A+1;l}Z_{A-1};p\right)\left(1-Q_{\tau_{A+1}}Q_{\tau_A}Y_{A-1;l}W_{A+1;p}\right)}\right\}\\ &G^{(N)}\left(\frac{Q_{\tau_{A+1}}Q_AX_{A+1}}{Q_{A+1}},Q_{\tau_A}Y_{A-1},\frac{Q_{A-1}Q_{\tau_A}Z_{A-1}}{Q_A},Q_{\tau_{A+1}}W_{A+1};Q_A,Q_{\tau_A}\right)\end{aligned} \tag{64}$$

The steps (58)-(61) may then be iterated $N-1$ more times until one finds

$$\begin{aligned}&G^{(N)}(X_A,Y_A,Z_A,W_A;Q_A,Q_{\tau_A}) = \\ &G^{(N)}\left(Q_\tau X_A,Q_\tau Y_A,Q_\tau Z_A,Q_\tau W_A;Q_A,Q_{\tau_A}\right)\\ &\times\prod_{A,B=1}^{N}\prod_{l,p=1}^{\infty}\left\{\frac{\left(1-Q_\tau Q_{AB}^{-1}X_{A;l}Y_{B;p}\right)\left(1-Q_\tau^2Q_{AB}^{-1}X_{A;l}Y_{B;p}\right)}{\left(1-\widetilde{Q}'_{AB}Z_{A;l}X_{B;p}\right)\left(1-Q_\tau\widetilde{Q}'_{AB}Z_{A;l}X_{B;p}\right)}\right.\\ &\left.\frac{(1-Q_{AB}Z_{A;l}W_{B;p})(1-Q_\tau Q_{AB}Z_{A;l}W_{B;p})}{\left(1-\widetilde{Q}_{AB}Y_{A;l}W_{B;p}\right)\left(1-Q_\tau\widetilde{Q}_{AB}Y_{A;l}W_{B;p}\right)}\right\}\end{aligned} \tag{65}$$

here $Q_\tau = \prod_{A=1}^{N}Q_{\tau_A}$. Now we perform (58)-(63) an infinite number of times and use the fact that

$$\lim_{r\to\infty}G^{(N)}\left(Q_\tau^rX_A,Q_\tau^rY_A,Q_\tau^rZ_A,Q_\tau^rW_A;Q_A,Q_{\tau_A}\right) = \prod_{r=1}^{\infty}\frac{1}{1-Q_\tau^r}, \tag{66}$$

provided $|Q_\tau| < 1$. Hence

$$
\begin{aligned}
&G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) = \prod_{r,l,p=1}^{\infty} (1 - Q_\tau^r)^{-N^2} \\
&\prod_{A,B=1}^{N} \prod_{r,l,p=1}^{\infty} \frac{\left(1 - Q_\tau^r Q_{AB}^{-1} X_{A;l} Y_{B;p}\right)\left(1 - Q_\tau^{r-1} Q_{AB} Z_{A;l} W_{B;p}\right)}{\left(1 - Q_\tau^{r-1} \widetilde{Q}'_{AB} Z_{A;l} X_{B;p}\right)\left(1 - Q_\tau^{r-1} \widetilde{Q}_{AB} Y_{A;l} W_{B;p}\right)} .
\end{aligned}
\tag{67}
$$

All in all, the partition function for the strip geometry reads

$$
\begin{aligned}
&Z^{\mu_1 \dots \mu_N}_{\nu_1 \dots \nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) = \\
&\prod_{A=1}^{N} \mathfrak{q}^{-\frac{||\mu_A^{\mathrm{T}}||^2}{2}} \mathfrak{t}^{\frac{-||\nu_A||^2}{2}} \widetilde{Z}_{\mu_A^{\mathrm{T}}}(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}) \widetilde{Z}_{\nu_A}(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}) \\
&\times \prod_{A,B=1}^{N} \prod_{l,p,r=1}^{\infty} \frac{\left(1 - Q_\tau^{r-1} Q_{AB} \mathfrak{t}^{\mu_{A;l} - p + 1/2} \mathfrak{q}^{\nu^{\mathrm{T}}_{B;p} - l + 1/2}\right)}{(1 - Q_\tau^r)\left(1 - Q_\tau^{r-1} \widetilde{Q}_{BA} \mathfrak{t}^{\nu_{B;l} - p + 1} \mathfrak{q}^{\nu^{\mathrm{T}}_{A;p} - l}\right)} \\
&\times \prod_{A,B=1}^{N} \prod_{l,p,r=1}^{\infty} \frac{\left(1 - Q_\tau^r Q_{AB}^{-1} \mathfrak{t}^{\nu_{B;l} - p + 1/2} \mathfrak{q}^{\mu^{\mathrm{T}}_{A;p} - l + 1/2}\right)}{\left(1 - Q_\tau^{r-1} \widetilde{Q}'_{AB} \mathfrak{t}^{\mu_{A;l} - p} \mathfrak{q}^{\mu^{\mathrm{T}}_{B;p} - l + 1}\right)}
\end{aligned}
\tag{68}
$$

where we define $Q_\tau := \prod_{A=1}^{N} Q_{\tau_A}$. We may then define the domain wall partition function

$$
D^{\mu_1 \dots \mu_N}_{\nu_1 \dots \nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) := \frac{Z^{\mu_1 \dots \mu_N}_{\nu_1 \dots \nu_N}(Q_A, Q_{\tau_A}; \mathfrak{t}, \mathfrak{q})}{Z^{\varnothing \dots \varnothing}_{\varnothing \dots \varnothing}(Q_A, Q_{\tau_A}; \mathfrak{t}, \mathfrak{q})}
\tag{69}
$$

which may be expressed in terms of $\mathcal{N}_{\nu,\mu}(Q; \mathfrak{q}, \mathfrak{t})$ (**??**)

$$
\begin{aligned}
&D^{\mu_1 \dots \mu_N}_{\nu_1 \dots \nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) = \\
&\prod_{A=1}^{N} \mathfrak{q}^{-\frac{||\mu_A^{\mathrm{T}}||^2}{2}} \mathfrak{t}^{\frac{-||\nu_A||^2}{2}} \widetilde{Z}_{\mu_A^{\mathrm{T}}}(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}) \widetilde{Z}_{\nu_A}(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}) \\
&\times \prod_{A,B=1}^{N} \prod_{r=1}^{\infty} \frac{\mathcal{N}_{\mu_A \nu_B}\left(Q_\tau^{r-1} Q_{AB} \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}; \mathfrak{t}, \mathfrak{q}\right) \mathcal{N}_{\nu_B \mu_A}\left(Q_\tau^r Q_{AB}^{-1} \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}; \mathfrak{t}, \mathfrak{q}\right)}{\mathcal{N}_{\mu_A \mu_B}\left(Q_\tau^{r-1} \widetilde{Q}'_{AB}; \mathfrak{t}, \mathfrak{q}\right) \mathcal{N}_{\nu_A \nu_B}\left(Q_\tau^{r-1} \widetilde{Q}_{AB} \frac{\mathfrak{t}}{\mathfrak{q}}; \mathfrak{t}, \mathfrak{q}\right)}
\end{aligned}
\tag{70}
$$

where

$$
Q_{AB} = Q_A \prod_{i=1}^{A-1} Q_{\tau_i} \prod_{j=B}^{N} Q_{\tau_j} \bmod Q_\tau \,,
\tag{71}
$$

$$
\widetilde{Q}_{AB} = \begin{cases} \prod_{i=B}^{A-1} Q_{\tau_i} & A > B \,, \\ Q_\tau & A = B \,, \\ Q_\tau / \prod_{i=A}^{B-1} Q_{\tau_i} & A < B \,, \end{cases}
\tag{72}
$$

$$
\widetilde{Q}'_{AB} = \frac{Q_A}{Q_B} \widetilde{Q}_{AB} \,.
\tag{73}
$$

The M5-brane partition function on A_{N-1} singularity is then given by

$$Z_{\text{M5}}\left(Q_{n,A},Q_{\tau_{n,A}},Q_{f,n,A};\mathfrak{t},\mathfrak{q}\right)=Z_{\text{rel.}}Z_M^{A_{N-1}}\left(Q_{n,A},Q_{\tau_{n,A}},Q_{f,n,A};\mathfrak{t},\mathfrak{q}\right) \tag{74}$$

where

$$\begin{aligned}
&Z_M^{A_{N-1}}\left(Q_{n,A},Q_{\tau_{n,A}},Q_{f,n,A};\mathfrak{t},\mathfrak{q}\right)=\\
&\sum_{\{\vec{\mu}_n\}}\left(\prod_{n=1}^{M-1}\prod_{A=1}^{N}\left(-Q_{f,n,A}\right)^{|\mu_{n,A}|}\right)Z^{A_{N-1}}_{M,\{\vec{\mu}_n\}}\left(Q_{n,A},Q_{\tau_{n,A}};\mathfrak{t},\mathfrak{q}\right)\,,
\end{aligned} \tag{75}$$

$$Z_{U(1)}^{(n)}:=Z^{\varnothing\ldots\varnothing}_{\varnothing\ldots\varnothing}\left(Q_{n,A},Q_{\tau_{n,A}},Q'_{\tau_{n,A}};\mathfrak{t},\mathfrak{q}\right)\,, \tag{76}$$

$$Z_{\text{rel.}}=\prod_{n=1}^{M}Z^{(n)}_{U(1)}\,, \tag{77}$$

and

$$\begin{aligned}
&Z^{A_{N-1}}_{M,\vec{\mu}_n}\left(Q_{n,A},Q_{\tau_{n,A}};\mathfrak{t},\mathfrak{q}\right)=D^{\varnothing\ldots\varnothing}_{\mu_{1,1}\ldots\mu_{1,N}}\left(Q_{1,A},Q_{\tau_{1,A}},Q'_{\tau_{1,A}};\mathfrak{t},\mathfrak{q}\right)\\
&\times D^{\mu_{1,1}\ldots\mu_{1,N}}_{\mu_{2,1}\ldots\mu_{2,N}}\left(Q_{2,A},Q_{\tau_{2,A}},Q'_{\tau_{2,A}};\mathfrak{t},\mathfrak{q}\right)\\
&\times\cdots\times D^{\mu_{M-1,1}\ldots\mu_{M-1,N}}_{\varnothing\ldots\varnothing}\left(Q_{M,A},Q_{\tau_{M,A}},Q'_{\tau_{M,A}};\mathfrak{t},\mathfrak{q}\right)\,.
\end{aligned} \tag{78}$$

Note that the gluing requires

$$Q'_{\tau_{n+1,A}}=Q_{\tau_{n,A}}\implies\widetilde{Q}'_{n+1,AB}=\widetilde{Q}_{n,AB}\,. \tag{79}$$

It may be shown that [?] (76) may be written as

$$\begin{aligned}
&Z_M^{A_{N-1}}=\\
&\sum_{\{\mu_{n,A}\}}\prod_{n=1}^{M-1}\prod_{A=1}^{N}\left(\overline{Q}^{|\mu_{n,A}|}_{f,n,A}\right)\prod_{(l,p)\in\mu_{n,A}}\prod_{B=1}^{N}\frac{\theta_1\left(z_{n,AB}(l,p)|\tau\right)\theta_1\left(w_{n,AB}(l,p)|\tau\right)}{\theta_1\left(u_{n,AB}(l,p)|\tau\right)\theta_1\left(v_{n,AB}(l,p)|\tau\right)}
\end{aligned} \tag{80}$$

where

$$e^{2\pi\mathrm{i}z_{n,AB}(l,p)}=Q^{-1}_{n+1,AB}\mathfrak{t}^{-\mu_{n,A;l}+p-1/2}\mathfrak{q}^{\mu^{\mathrm{T}}_{n+1,B;p}+l-1/2} \tag{81}$$

$$e^{2\pi\mathrm{i}w_{n,AB}(l,p)}=Q^{-1}_{n,BA}\mathfrak{t}^{\mu_{n,A;l}-p+1/2}\mathfrak{q}^{\mu^{\mathrm{T}}_{n-1,B;p}-l+1/2} \tag{82}$$

$$e^{2\pi\mathrm{i}u_{n,AB}(l,p)}=\widehat{Q}^{-1}_{n,BA}\mathfrak{t}^{\mu_{n,A;l}-p}\mathfrak{q}^{\mu^{\mathrm{T}}_{n,B;p}-l+1} \tag{83}$$

$$e^{2\pi\mathrm{i}v_{n,AB}(l,p)}=\widehat{Q}^{-1}_{n,AB}\mathfrak{t}^{\mu_{n,A;l}+p-1}\mathfrak{q}^{-\mu^{\mathrm{T}}_{n,B;p}+l} \tag{84}$$

and

$$\overline{Q}_{f,n,A}=\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{N-1}{2}}Q_{f,n,A}\left(\prod_{A=1}^{N}Q_{n,A}\right)\,,\quad\widehat{Q}_{n,AB}=\begin{cases}1 & A=B\,,\\ \widetilde{Q}_{n,AB} & A\neq B\,.\end{cases} \tag{85}$$

The authors of [?] were, remarkably, able to show

$$Z_M^{A_{N-1}}=Z_{\text{string}} \tag{86}$$

where Z_{string} is the same as in (??).

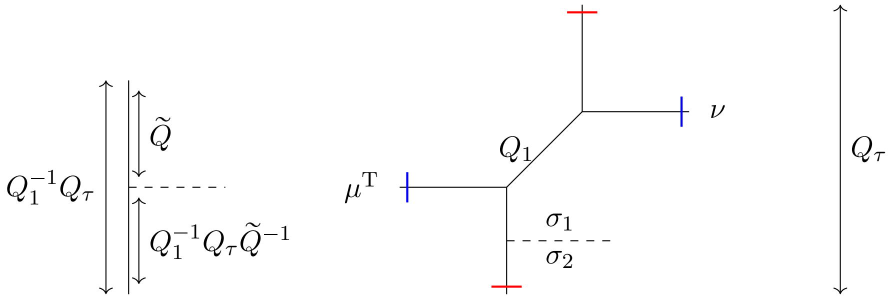

Figure 2: Left: *Assignment of Kähler parameters for the Lagrangian brane* Right: *Strip geometry for the* A_0 *singularity with a single Lagrangian brane corresponding to the defect. The blue lines denote the preferred direction of the refined topological vertex. The red lines denote the direction periodic identification.*

0.3.2 With Minimal Defect

Let us consider the minimal type defects in the A_0 theory. By minimal we mean the defects of type

$$\rho = [(M-k), \underbrace{1,\ldots,1}_{k \text{ times}}]\,. \tag{87}$$

The relation between the string Elliptic genus and refined topological partition function in the presence of a defect of type $2 = 1+1$ has been studied in [**?**]. We can compute the domain wall partition function (68) in the presence of a D3'-brane ending on D5-brane. The effect of the Lagrangian brane is to insert the factor

$$\mathrm{tr}_{\sigma_1^{\mathrm{T}}}(X)\,\mathrm{tr}_{\sigma_2^{\mathrm{T}}}\left(X^{-1}\right) = s_{\sigma_1^{\mathrm{T}}}(x)\,s_{\sigma_2^{\mathrm{T}}}\left(x^{-1}\right)\,. \tag{88}$$

Where we assume that the brane has framing factor exponents $p = 1$. The refined open topological string amplitude for the strip geometry with $N = 1$ with a single Lagrangian brane

$$\begin{aligned}
&\widehat{Z}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) = \\
&\sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2} \Big\{ (-Q_1)^{|\lambda_2|} \left(-Q_1^{-1}Q_\tau\right)^{|\lambda_1|} \left(-\widetilde{Q}^{-1}Q_1^{-1}Q_\tau\right)^{|\sigma_1|} \left(-\widetilde{Q}\right)^{|\sigma_2|} \\
&\times s_{\sigma_1^{\mathrm{T}}}(x)\,s_{\sigma_2^{\mathrm{T}}}\left(x^{-1}\right) C_{(\lambda_1^{\mathrm{T}}\otimes\sigma_1)\lambda_2^{\mathrm{T}}\mu^{\mathrm{T}}}(\mathfrak{t}^{-1},\mathfrak{q}^{-1}) C_{(\lambda_1\otimes\sigma_2)\lambda_2\nu}(\mathfrak{q}^{-1},\mathfrak{t}^{-1}) \Big\}\,.
\end{aligned} \tag{89}$$

From (87) after expanding out the topological vertex we have

$$\widehat{Z}^{\mu}_{\nu}(Q_1,Q_{\tau_1},\widetilde{Q};\mathfrak{t},\mathfrak{q}) = \mathfrak{t}^{-\frac{||\nu||^2}{2}}\mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}}\widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1},\mathfrak{q}^{-1})\widetilde{Z}_{\nu}(\mathfrak{q}^{-1},\mathfrak{t}^{-1})$$
$$\sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2,\eta_1,\eta_2}\Bigg\{\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{|\eta_1|-|\eta_2|}{2}}(-Q_1)^{|\lambda_2|}\left(-Q_1^{-1}Q_{\tau_1}\right)^{|\lambda_1|}$$
$$\times\, s_{\sigma_1^{\mathrm{T}}}\left(\widetilde{Q}^{-1}Q_1^{-1}Q_{\tau_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}x\right)s_{\sigma_2^{\mathrm{T}}}\left(\widetilde{Q}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}x^{-1}\right)s_{(\lambda_1\otimes\sigma_1^{\mathrm{T}})/\eta_2}\left(\mathfrak{t}^{\rho}\mathfrak{q}^{\mu^{\mathrm{T}}}\right) \tag{90}$$
$$\times\, s_{\lambda_2^{\mathrm{T}}/\eta_2}\left(\mathfrak{t}^{\mu}\mathfrak{q}^{\rho}\right)s_{(\lambda_1^{\mathrm{T}}\otimes\sigma_2^{\mathrm{T}})/\eta_1}\left(\mathfrak{q}^{\rho}\mathfrak{t}^{\nu}\right)s_{\lambda_2/\eta_1}\left(\mathfrak{q}^{\nu^{\mathrm{T}}}\mathfrak{t}^{\rho}\right)\Bigg\}\,.$$

Using the identity (**??**) we have

$$\widehat{Z}^{\mu}_{\nu}(Q_1,Q_{\tau_1},\widetilde{Q};\mathfrak{t},\mathfrak{q}) = \mathfrak{t}^{-\frac{||\nu||^2}{2}}\mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}}\widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1},\mathfrak{q}^{-1})\widetilde{Z}_{\nu}(\mathfrak{q}^{-1},\mathfrak{t}^{-1})$$
$$\sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2}\Bigg\{\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{|\eta_1|-|\eta_2|}{2}}(-Q_1)^{|\lambda_2|}\left(-Q_1^{-1}Q_{\tau_1}\right)^{|\lambda_1|}c^{\gamma_1}_{\lambda_1\sigma_1^{\mathrm{T}}}c^{\gamma_2^{\mathrm{T}}}_{\lambda_1^{\mathrm{T}}\sigma_2^{\mathrm{T}}}$$
$$\times\, s_{\sigma_1^{\mathrm{T}}}\left(\widetilde{Q}^{-1}Q_1^{-1}Q_{\tau_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}x\right)s_{\sigma_2^{\mathrm{T}}}\left(\widetilde{Q}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}x^{-1}\right)s_{\gamma_1/\eta_2}\left(\mathfrak{t}^{\rho}\mathfrak{q}^{\mu^{\mathrm{T}}}\right) \tag{91}$$
$$\times s_{\lambda_2^{\mathrm{T}}/\eta_2}\left(\mathfrak{t}^{\mu}\mathfrak{q}^{\rho}\right)s_{\gamma_2^{\mathrm{T}}/\eta_1}\left(\mathfrak{q}^{\rho}\mathfrak{t}^{\nu}\right)s_{\lambda_2/\eta_1}\left(\mathfrak{q}^{\nu^{\mathrm{T}}}\mathfrak{t}^{\rho}\right)\Bigg\}\,.$$

Now apply the identity (**??**) to obtain

$$\widehat{Z}^{\mu}_{\nu}(Q_1,Q_{\tau_1},\widetilde{Q};\mathfrak{t},\mathfrak{q}) = \mathfrak{t}^{-\frac{||\nu||^2}{2}}\mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}}\widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1},\mathfrak{q}^{-1})\widetilde{Z}_{\nu}(\mathfrak{q}^{-1},\mathfrak{t}^{-1})$$
$$\sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2}\Bigg\{(-Q_1)^{|\lambda_2|}\left(\frac{-Q_{\tau_1}}{Q_1}\right)^{|\lambda_1|}s_{\gamma_1/\lambda_1}\left(\frac{Q_{\tau_1}}{\widetilde{Q}Q_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}x\right)s_{\gamma_1/\eta_2}\left(\mathfrak{t}^{\rho}\mathfrak{q}^{\mu^{\mathrm{T}}}\right) \tag{92}$$
$$s_{\gamma_2^{\mathrm{T}}/\lambda_1^{\mathrm{T}}}\left(\widetilde{Q}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}x^{-1}\right)s_{\lambda_2^{\mathrm{T}}/\eta_2}\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\mathfrak{t}^{\mu}\mathfrak{q}^{\rho}\right)s_{\gamma_2^{\mathrm{T}}/\eta_1}\left(\mathfrak{q}^{\rho}\mathfrak{t}^{\nu}\right)s_{\lambda_2/\eta_1}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathfrak{q}^{\nu^{\mathrm{T}}}\mathfrak{t}^{\rho}\right)\Bigg\}\,.$$

Therefore let us consider

$$G(X,Y,Z,W,A,B;a,b) = \sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2}\Big\{(a)^{|\lambda_1|}(b)^{|\lambda_2|}s_{\gamma_1/\lambda_1}(A)$$
$$s_{\gamma_2^{\mathrm{T}}/\lambda_1^{\mathrm{T}}}(B)\,s_{\gamma_1/\eta_2}(X)\,s_{\lambda_2^{\mathrm{T}}/\eta_2}(Y)\,s_{\gamma_2^{\mathrm{T}}/\eta_1}(Z)\,s_{\lambda_2/\eta_1}(W)\Big\} \tag{93}$$

$$G(X,Y,Z,W,U,A,B;a,b) =$$
$$\sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2}\Big\{(a)^{|\gamma_1|}(b)^{|\gamma_1|}s_{\lambda_1/\gamma_1}(aX)\,s_{\lambda_1^{\mathrm{T}}/\gamma_2}(Z)\,s_{\eta_2^{\mathrm{T}}/\lambda_2^{\mathrm{T}}}(W) \tag{94}$$
$$s_{\eta_1^{\mathrm{T}}/\lambda_2}(bY)\,s_{\eta_1/\gamma_2}(B)\,s_{\eta_2/\gamma_1}(bA)\Big\}\prod_{l,p=1}^{\infty}\frac{(1+bY_lW_p)}{(1-A_lX_p)\,(1-B_lZ_p)}$$

$$
\begin{aligned}
G(X,Y,Z,W,U,A,B;a,b) = &\sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2} \Big\{(a)^{|\eta_2|}(b)^{|\lambda_1|} s_{\gamma_1^{\mathrm{T}}/\lambda_1}(bZ) \\
&\times s_{\gamma_2^{\mathrm{T}}/\lambda_1^{\mathrm{T}}}(aX)\, s_{\lambda_2/\eta_2^{\mathrm{T}}}(bA)\, s_{\gamma_1^{\mathrm{T}}/\eta_2}(aW)\, s_{\lambda_2^{\mathrm{T}}/\eta_1^{\mathrm{T}}}(B)\, s_{\gamma_2^{\mathrm{T}}/\eta_1}(bY)\Big\} \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)}{(1-A_lX_p)(1-B_lZ_p)}
\end{aligned}
\tag{95}
$$

$$
\begin{aligned}
G(X,Y,Z,W,U,A,B;a,b) = &\sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2} \Big\{(a)^{|\lambda_2|}(b)^{|\lambda_1|} s_{\lambda_1/\gamma_1^{\mathrm{T}}}(aW) \\
&\times s_{\lambda_1^{\mathrm{T}}/\gamma_2^{\mathrm{T}}}(bY)\, s_{\eta_2/\lambda_2}(aB)\, s_{\eta_2/\gamma_1^{\mathrm{T}}}(bZ)\, s_{\eta_1/\lambda_2^{\mathrm{T}}}(bA)\, s_{\eta_1/\gamma_2^{\mathrm{T}}}(aX)\Big\} \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)(1+bB_lA_p)}{(1-A_lX_p)(1-B_lZ_p)(1-abW_lZ_p)(1-abY_lX_p)}
\end{aligned}
\tag{96}
$$

Which is, again, rather similar to our original expressions:

$$
\begin{aligned}
G(X,Y,Z,W,A,B;a,b) &= G(bZ,aW,aX,bY,aB,bA;a,b) \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)(1+bB_lA_p)}{(1-A_lX_p)(1-B_lZ_p)(1-abW_lZ_p)(1-abY_lX_p)}
\end{aligned}
\tag{97}
$$

So, repeating steps (92) to (94) again we have that

$$
\begin{aligned}
G(X,Y,Z,W,A,B;a,b) &= G(Q_\tau X,Q_\tau Y,Q_\tau Z,Q_\tau W,Q_\tau A,Q_\tau B;a,b) \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)(1+bB_lA_p)}{(1-A_lX_p)(1-B_lZ_p)(1-abW_lZ_p)(1-abY_lX_p)} \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+Q_\tau bY_lW_p)(1+Q_\tau aX_lZ_p)(1+Q_\tau bA_lW_p)}{(1-Q_\tau A_lX_p)(1-Q_\tau B_lZ_p)(1-Q_\tau^2 W_lZ_p)} \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+bQ_\tau Y_lB_p)(1+Q_\tau bB_lA_p)}{(1-Q_\tau^2 Y_lX_p)}
\end{aligned}
\tag{98}
$$

where $Q_\tau := ab$ so again iterating the steps (92) to (96) an infinite number of times and using

$$
\lim_{r\to\infty} G(Q_\tau^r X,Q_\tau^r Y,Q_\tau^r Z,Q_\tau^r W,Q_\tau^r A,Q_\tau^r B;a,b) = \prod_{r=1}^{\infty} \frac{1}{1-Q_\tau^r}
\tag{99}
$$

we arrive at

$$
\begin{aligned}
G(X,Y,Z,W,A,B;a,b) = &\prod_{r,l,p=1}^{\infty} \left\{ \frac{\left(1+Q_\tau^{r-1}bY_lW_p\right)\left(1+Q_\tau^{r-1}aX_lZ_p\right)}{\left(1-Q_\tau^r\right)\left(1-Q_\tau^{r-1}A_lX_p\right)} \right. \\
&\left. \times \frac{\left(1+Q_\tau^{r-1}bA_lW_p\right)\left(1+bQ_\tau^{r-1}Y_lB_p\right)\left(1+Q_\tau^{r-1}bB_lA_p\right)}{\left(1-Q_\tau^{r-1}B_lZ_p\right)(1-Q_\tau^r W_lZ_p)(1-Q_\tau^r Y_lX_p)} \right\}
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
&\widehat{Z}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) = \mathfrak{t}^{-\frac{||\nu||^2}{2}} \mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}} \widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}) \widetilde{Z}_{\nu}(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}) \\
&\times \prod_{r,l,p=1}^{\infty} \left\{ \frac{\left(1 - Q_1^{-1} Q_\tau^r \mathfrak{q}^{\mu_l^{\mathrm{T}} - p + 1/2} \mathfrak{t}^{\nu_p - l + 1/2}\right)}{\left(1 - Q_\tau^r \mathfrak{t}^{\mu_l - p} \mathfrak{q}^{\mu_p^{\mathrm{T}} - l + 1}\right)\left(1 - Q_\tau^r \mathfrak{q}^{\nu_l^{\mathrm{T}} - p} \mathfrak{t}^{\nu_p - l + 1}\right)} \right. \\
&\frac{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{q}^{\nu_l^{\mathrm{T}}} \mathfrak{t}^{-l+1/2} x_p\right)\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{t}^{\mu_l} \mathfrak{q}^{-l+1/2} x_p^{-1}\right)}{\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{\mu_l^{\mathrm{T}} + 1/2} \mathfrak{t}^{-l} x_p\right)\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{\nu_l + 1/2} x_p^{-1}\right)} \\
&\left. \frac{\left(1 - Q_\tau^r x_l x_p^{-1}\right)\left(1 - Q_1 Q_{\tau_1}^{r-1} \mathfrak{t}^{\mu_l - p + 1/2} \mathfrak{q}^{\nu_p^{\mathrm{T}} - l + 1/2}\right)}{(1 - Q_\tau^r)} \right\} .
\end{aligned}
\tag{101}
$$

As before, we define the domain wall partition function; it turns out that it factorises in the following fashion

$$
\begin{aligned}
\widehat{D}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) &:= \frac{\widehat{Z}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}; \mathfrak{t}, \mathfrak{q})}{\widehat{Z}^{\varnothing}_{\varnothing}(Q_1, Q_\tau, \widetilde{Q}; \mathfrak{t}, \mathfrak{q})} \\
&= D^{\mu}_{\nu}(Q_1, Q_\tau; \mathfrak{t}, \mathfrak{q}) \widehat{d}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) .
\end{aligned}
\tag{102}
$$

Where D is given by (68) and

$$
\begin{aligned}
&\widehat{d}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) \\
&= \prod_{r,p=1}^{\infty} \left\{ \prod_{l=1}^{\ell(\nu^{\mathrm{T}})} \frac{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{q}^{\nu_l^{\mathrm{T}}} \mathfrak{t}^{-l+1/2} x_p\right)}{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{t}^{-l+1/2} x_p\right)} \prod_{l=1}^{\ell(\nu)} \frac{\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{1/2} x_p^{-1}\right)}{\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{\nu_l + 1/2} x_p^{-1}\right)} \right. \\
&\left. \prod_{l=1}^{\ell(\mu)} \frac{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{t}^{\mu_l} \mathfrak{q}^{-l+1/2} x_p^{-1}\right)}{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{q}^{-l+1/2} x_p^{-1}\right)} \prod_{l=1}^{\ell(\mu^{\mathrm{T}})} \frac{\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{1/2} \mathfrak{t}^{-l} x_p\right)}{\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{\mu_l^{\mathrm{T}} + 1/2} \mathfrak{t}^{-l} x_p\right)} \right\}
\end{aligned}
\tag{103}
$$

$$
\begin{aligned}
&= \prod_{r,p=1}^{\infty} \left\{ \prod_{(l,q)\in\nu} \frac{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{q}^{l} \mathfrak{t}^{-q+1/2} x_p\right)\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{q-1/2} x_p^{-1}\right)}{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{q}^{l-1} \mathfrak{t}^{-q+1/2} x_p\right)\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{q+1/2} x_p^{-1}\right)} \right. \\
&\left. \prod_{(l,q)\in\mu} \frac{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{t}^{q} \mathfrak{q}^{-l+1/2} x_p^{-1}\right)\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{l-1/2} \mathfrak{t}^{-q} x_p\right)}{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{t}^{q-1} \mathfrak{q}^{-l+1/2} x_p^{-1}\right)\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{l+1/2} \mathfrak{t}^{-q} x_p\right)} \right\} .
\end{aligned}
\tag{104}
$$

is the contribution of the defect to the partition function. It does not quite assemble into a nice form in terms of θ_1 functions. However, in the unrefined $\mathfrak{q} = \mathfrak{t}$ limit:

$$
\begin{aligned}
\widehat{d}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{q}, \mathfrak{q}) = \prod_{p=1}^{\infty} \Bigg\{ \mathfrak{q}^{\frac{|\mu|-|\nu|}{2}} \prod_{(l,q)\in\nu} \frac{\theta_1\left(\widetilde{Q}^{-1}\mathfrak{q}^{l-q+1/2}x_p; Q_\tau\right)}{\theta_1\left(\widetilde{Q}^{-1}\mathfrak{q}^{l-q-1/2}x_p; Q_\tau\right)} \\
\times \prod_{(l,q)\in\mu} \frac{\theta_1\left(Q_1^{-1}\widetilde{Q}^{-1}\mathfrak{q}^{l-1/2-q}x_p; Q_\tau\right)}{\theta_1\left(Q_1^{-1}\widetilde{Q}^{-1}\mathfrak{q}^{l+1/2-q}x_p; Q_\tau\right)} \Bigg\}
\end{aligned}
\tag{105}
$$

We may then compute the M5-brane partition function in the presence of the defect labelled by the partition (85) by gluing together k domain wall partitions of type (100) with $M-k$ of type (68). This builds the theory labelled by partition $\rho = [M-k, \underbrace{1, \ldots, 1}_{k}]$. Hence we write

$$
Z_{\mathrm{M5},\rho} = \left(\prod_{n=1}^{M-k} Z^{(n)}_{U(1)}\right)\left(\prod_{n=M-k+1}^{M} Z^{(n)}_{U(1),\rho}\right) Z^{A_0}_{\rho}
\tag{106}
$$

where,

$$
Z^{(n)}_{U(1),\rho}(Q_{n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q}) = \widehat{Z}^{\varnothing}_{\varnothing}(Q_{n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q})
\tag{107}
$$

$$
\begin{aligned}
Z^{A_0}_{\rho}\left(Q_{f,n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q}\right) &:= \sum_{\{\mu_n\}} \left(\prod_{n=1}^{M-1} (-Q_{f,n,1})^{|\mu_n|}\right) \\
&\times D^{\varnothing}_{\mu_1}(Q_{1,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \times D^{\mu_1}_{\mu_2}(Q_{2,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \\
&\times \cdots \times D^{\mu_{M-k-1}}_{\mu_{M-k}}(Q_{M-k,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \\
&\times \widehat{D}^{\mu_{M-k}}_{\mu_{M-k+1}}(Q_{M-k+1,1}, Q_\tau, \widetilde{Q}_{M-k+1}, x_{M-k+1}; \mathfrak{t}, \mathfrak{q}) \\
&\times \cdots \times \widehat{D}^{\mu_{M-1}}_{\varnothing}(Q_{M,1}, Q_\tau, \widetilde{Q}_M, x_M; \mathfrak{t}, \mathfrak{q})\,.
\end{aligned}
\tag{108}
$$

By the factorisation (101) it is clear that

$$
\begin{aligned}
Z^{A_0}_{\rho}\left(Q_{f,n,1}, Q_{n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q}\right) &:= \sum_{\{\mu_n\}} \left(\prod_{n=1}^{M-1} (-Q_{f,n,1})^{|\mu_n|}\right) \\
&\times Z^{A_0}_{M,\{\vec{\mu}_n\}}(Q_{n,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \times \widehat{d}^{\mu_{M-k}}_{\mu_{M-k+1}}(Q_{M-k+1,1}, Q_\tau, \widetilde{Q}_{M-k+1}, x_{M-k+1}; \mathfrak{t}, \mathfrak{q}) \\
&\times \cdots \times \widehat{d}^{\mu_{M-1}}_{\varnothing}(Q_{M,1}, Q_\tau, \widetilde{Q}_M, x_M; \mathfrak{t}, \mathfrak{q})\,.
\end{aligned}
\tag{109}
$$