In this Appendix we discuss the relationship between the Hall-Littlewood limit of the index and the Higgs-branch Hilbert series. For $\mathfrak{g} = A_{N-1}$ Class $\mathcal{S}$ theories associated to genus $g = 0$ theories it is conjectured that these two quantities are equal. For $g \geqslant 1$ it is known to no longer hold.

Here we will compare these two quantities, restricting most of our attention to $\mathfrak{g} = A_{N-1}$ class $\mathcal{S}$ SCFTs associated to $g = 1$ Riemann surfaces with a collection of minimal punctures.

0.1 The Higgs Branch of Class $\mathcal{S}$ Theories

For $\mathcal{N} \geqslant 2$ SCFTs the Higgs branch is reached by giving zero vev to operators with $r \neq 0$ while allowing vevs for those operators with $r = j_1 = j_2 = 0$. For theories with $\mathcal{N} > 2$ this depends on a choice of embedding $\mathfrak{su}(2,2|2) \hookrightarrow \mathfrak{su}(2,2|\mathcal{N})$. The Higgs branch is protected from quantum corrections and thus, when the theory has a Lagrangian description can be described as a purely classical object. The coordinate ring of the Higgs-branch is known as the Higgs-branch chiral ring. By abuse of notation we will identify the Higgs-branch as a complex affine variety with it's chiral ring. The Higgs branch (chiral ring) is given by

$$HB = \{\mathcal{O}_i | \widetilde{\mathcal{Q}}^I_{\dot\alpha} \mathcal{O}_i = 0 \,, M_{\mu\nu}\mathcal{O}_i = 0 \,, r\mathcal{O}_i = 0\} \,. \tag{1}$$

For superconformal theories the Higgs branch is parametrised by the top components of $\hat{\mathcal{B}}_R$ multiplets which have $E = 2R$ and $r = j_1 = j_2 = 0$, where R is the Cartan of the $\mathfrak{su}(2)_R$ R-symmetry of the $\mathcal{N} = 2$ superconformal algebra. There is no recombination rule (**??**)-(**??**) involving only $\hat{\mathcal{B}}_R$ operators. For gauge theories based on a gauge group G with a collection of hypermultiplets whose scalars are collectively denoted by $Q, \widetilde{Q}$ the ring HB has a rather simple description. Firstly, one constructs the coordinate ring associated to the *master space* (restricted to the Higgs branch) which is

$$F_H = R/I \,, \quad R := \mathbb{C}[Q, \widetilde{Q}] \,, \quad I := \langle \partial_\Phi W \rangle \tag{2}$$

here W denotes the superpotential of the theory, and Φ collectively denotes the vector multiplet scalars. Finally, to obtain HB one takes the G-invariant part of F_H

$$HB = (F_H)^G \,. \tag{3}$$

The Hilbert series counts gauge invariant chiral operators graded by their charges under a maximally commuting subalgebra of the global symmetry algebra. It is given by

$$\mathrm{HS}(\tau, u_i; HB) \equiv \mathrm{HS}(\tau, u_i) := \mathrm{Tr}_{HB}\, \tau^{2R} \prod_i u_i^{f_i} \,. \tag{4}$$

For the case of $\mathcal{N} = 2$ gauge theories the Hilbert series for the Higgs branch takes the form

$$\mathrm{HS}(\tau, u_i) = \int d\mu_G(\mathbf{z}) \mathrm{HS}_F(\tau, u_i, \mathbf{z}) \,, \tag{5}$$

where $\mathrm{HS}_F(\tau, u_i, \mathbf{z})$ denotes the Hilbert-series for $F = R/I$, defined as

$$\mathrm{HS}_F(\tau, u_i, \mathbf{z}) = \mathrm{HS}(\tau, u_i, \mathbf{z}; F) = \mathrm{Tr}_{R/I}\, \tau^{2R} \prod_i u_i^{f_i} \prod_{a=1}^{\mathrm{rank}\,\mathfrak{g}} z_a^{g_a} \,. \tag{6}$$

In the case of genus $g = 0$ class $\mathcal{S}$ theories gauge theories one can show that the set of F-terms generating I form a regular sequence. This implies that the affine variety $\mathbf{F}$ whose coordinate ring is $F = R/I$ is a complete intersection, which further means that it's Hilbert series can be written as $\mathrm{HS}_F = \mathrm{PE}[p(\tau, u_i, z_a)]$ with p a polynomial in τ. This implies that for those theories one can use letter counting in order to compute HS_F.

For genus $g \geqslant 1$ this fails to be the case and letter counting, in general, cannot be used. In that case one must use an algebraic geometry package such as *Macaulay2* [**?**]. By inputting the ring of polynomials R and the ideal I *Macaulay2* can compute the Hilbert series for $F = R/I$.

0.2 The Hall-Littlewood Index

The superconformal index for a class $\mathcal{S}$ theory is defined as [**?**, **?**]

$$\mathcal{I}\left(\rho, \sigma, \tau, u_i\right) = \mathrm{Tr}_{\mathbb{S}^3} (-1)^F \rho^{\frac{1}{2}\delta_{1-}} \sigma^{\frac{1}{2}\delta_{1+}} \tau^{\frac{1}{2}\tilde{\delta}_{2\dot{+}}} \prod u_i^{f_i} \,. \tag{7}$$

The trace is taken over the Hilbert space of the theory in the radial quantisation. The index (7) receives contributions only from those states satisfying

$$\delta = \tilde{\delta}_{1\dot{-}} := 2\left\{ \tilde{\mathcal{Q}}_{1\dot{-}}, \tilde{\mathcal{S}}^{1\dot{-}} \right\} = E - 2j_2 - 2R + r = 0 \,. \tag{8}$$

We also have

$$\delta_{1\pm} = E \pm 2j_1 - 2R - r \,, \quad \tilde{\delta}_{\dot{\pm}} = E \pm 2j_2 + 2R + r \,. \tag{9}$$

The superconformal index is independent under continuous deformation of the corresponding QFT. That means that, if the theory admits a free-field limit, (7) may be computed in the free theory by enumerating all of the free fields that satisfy $\delta = 0$ and then projecting onto gauge invariants. The projection onto gauge invariants is implemented by integration over the gauge group G. The index (7) for a gauge theory then takes the form

$$\mathcal{I}(\rho, \sigma, \tau, u_i) = \int d\mu_G(\mathbf{z})\, \mathrm{PE}\left[i(\rho, \sigma, \tau, u_i, \mathbf{z})\right] \,, \tag{10}$$

$d\mu_G$ denotes the Haar measure of the gauge group G and $\mathrm{PE}\left[f(x)\right]$ denotes the Plethystic exponential of a function $f(x)$, defined in (**??**). The single letter index i may be computed by enumerating all free field 'letters' with $\delta = 0$. The more powerful statement, however, is that, for class $\mathcal{S}$ theories, the index (7) has the interpretation as a partition function of a 2d TQFT living on the Riemann surface C. Given a pair of pants decomposition of C into a collection of three-punctured spheres (where each puncture carries an associated representation of A_n) and tubes. The index of any class $\mathcal{S}$ theory can then be written in terms of the indices of the elementary three point functions (three-punctured sphere indices), expanded in a basis

$$\mathcal{I}_{\mathbf{abc}} = \sum_{\alpha,\beta,\chi} C_{\alpha\beta\gamma} f^{\alpha}(\mathbf{a}) f^{\beta}(\mathbf{b}) f^{\chi}(\mathbf{c})\,, \tag{11}$$

and propagators (indicies of tubes)

$$\mathcal{I}_{\mathbf{ab}} = \eta_{\mathbf{ab}} = \int \mu_{SU(N)}(\mathbf{a}) \Delta(\mathbf{a}) \mathcal{I}^{V}(\mathbf{a}) \delta(\mathbf{a}, \mathbf{b}^{-1})\,. \tag{12}$$

The index (7) counts short representations of the $\mathfrak{su}(2,2|\mathcal{N})$ superconformal algebra, modulo recombination. Recombination happens when a long multiplet hits the unitary bound and decomposes into semi-direct sums of short representations. We list the possible $\mathcal{N}=2$ recombination rules in equations (**??**)-(**??**).

The Hall-Littlewood index is defined as

$$\mathrm{HL}(\tau, u_i) = \lim_{\rho,\sigma\to 0} \mathcal{I}\left(\rho, \sigma, \tau, u_i\right) = \mathrm{Tr}_{\mathbb{S}^3|_{\delta_{1\pm}=0}} (-1)^F \tau^{2R+2j_2} \prod_i u_i^{f_i}\,. \tag{13}$$

This limit is always well defined since superconformal symmetry implies $\delta_{1\pm} \geqslant 0$. This limit of the index counts a restricted number of operators, namely those with

$$j_1 = 0\,, \quad j_2 = r\,, \quad E = 2R + j_2\,. \tag{14}$$

Note that the only superconformal multiplets contributing to the index in this limit are

$$\mathrm{HL}_{\hat{\mathcal{B}}_R}(\tau) = \tau^{2R}\,, \quad \mathrm{HL}_{\mathcal{D}_{R(0,j_2)}}(\tau) = (-1)^{2j_2+1} \tau^{2+2R+2j_2}\,. \tag{15}$$

We notice also that the Higgs branch chiral ring HB is contained as a subset of Hall-littlewood operators. For genus zero theories it is conjectured that these two rings are equal.

We plan to consider the quantity

$$\frac{\mathrm{HL}(\tau, u_i)}{\mathrm{HS}(\tau, u_i)} = \begin{array}{l}\text{Partition function of operators}\\ \text{with } j_1 = 0,\, j_2 = r \geqslant \frac{1}{2},\, E = 2R + j_2\end{array}\,. \tag{16}$$

Equivalently the ratio HL/HS has an expansion in terms of $\mathcal{D}_{R(0,j_2)}$ multiplet indices

$$\frac{\mathrm{HL}(\tau, u_i)}{\mathrm{HS}(\tau, u_i)} = \sum_{R, j_2 \in \mathbb{N}/2} p_{R,j_2}(u_i) \mathrm{HL}_{\mathcal{D}_{R(0,j_2)}}(\tau)\,, \tag{17}$$

with p_{R,j_2} K-symmetric polynomials in the u_i with positive integer coefficients, where K is the global symmetry group of the theory. Note however that the Hall-Littlewood index can distinguish only equivalence classes of multiplets, namely

$$[\tilde{R}]_+ = \hat{\mathcal{B}}_{\tilde{R}} \cup \{\mathcal{D}_{\tilde{R}-j_2-1(0,j_2)} | \tilde{R} - j_2 - 1 \geqslant 0, 2j_2 \in 2\mathbb{N} + 1 \in\} \tag{18}$$

$$[\tilde{R}]_- = \{\mathcal{D}_{\tilde{R}-j_2-1(0,j_2)} | \tilde{R} - j_2 - 1 \geqslant 0, 2j_2 \in 2\mathbb{N}\} \tag{19}$$

and

$$\mathrm{HL}_{[\tilde{R}]_+} = -\mathrm{HL}_{[\tilde{R}]_-} = \tau^{2\tilde{R}} \,. \tag{20}$$

Note that the following multiplets contain only a single representative: $[1/2]_+ = \hat{\mathcal{B}}_{\frac{1}{2}}$, $[1]_+ = \hat{\mathcal{B}}_1$, $[1]_- = \mathcal{D}_{0,(0,0)}$, $[3/2]_- = \mathcal{D}_{1/2,(0,0)}$ these correspond to free half-hypers, moment map operator, free vector multiplet (chiral piece) and super-symmetry current. Note that the multiplets $\mathcal{D}_{0(j_1,0)}$ contain free fields. As we just mentioned when $j_1 = 0$ this is a free vector multiplet, when $j_1 \geqslant \frac{1}{2}$ these contain higher-spin free fields.

We will also use the fact that the Plethystic Logarithm counts all single trace operators, in other words

$$\mathrm{PLog}\left[\mathrm{HL}(\tau, u_i)\right] = \begin{array}{l}\text{Partition function of single trace operators}\\ \text{with } j_1 = 0,\, j_2 = r,\, E = 2R + j_2\end{array} \,. \tag{21}$$

0.3 $\mathcal{N} = 4$ SYM Theories

From now we will label quantities by the Class $\mathcal{S}$ data, i.e. type $\mathfrak{g}$ and the Riemann surface data of genus g and n punctures. We will focus much of our attention to the class $\mathcal{S}$ theory associated to a torus with a single puncture, $n = 1$, $g = 1$. This yields the $G = SU(N), U(N)$ MSYM theory (depending on whether we choose to gauge the c.o.m. degree of freedom). This example is particularly tractable because we can compute the Hilbert series for any N. Viewed as an $\mathcal{N} = 2$ theory $\mathcal{N} = 4$ SYM has a $U(1)$ flavour symmetry associated to the puncture which we give fugacity u for, this enhances to $SU(2)$ on the Higgs-branch. As an affine variety, the Higgs branch of this theory is given by

$$\mathbf{HB} = \mathbb{C}^{2r}/W(G) \tag{22}$$

where $W(G)$ denotes the Weyl group of G and $r = \mathrm{rank}\, G$. The chiral ring is obviously just therefore $HB = (\mathbb{C}[x_1, x_2, \ldots, x_{2r}])^{W(G)}$, although this description can be rather cumbersome to work with in practise. The Hilbert series is then simply the Molien series

$$\mathrm{HS}^G_{1,1}(\tau, u) = M(\tau, u; \mathbb{C}^{2r}/W(G)) \,. \tag{23}$$

The Hall-Littlewood index is expressed as the matrix integral

$$\mathrm{HL}^G_{1,1} = \oint d\mu_G\, \mathrm{PE}\left[h(\tau, u)\chi_G^{(adj.)}\right] , \tag{24}$$

$$h(\tau, u) = \chi_1(u)\tau - \tau^2 = \chi_1(u)\mathrm{HL}_{\hat{\mathcal{B}}_{1/2}} + \mathrm{HL}_{\mathcal{D}_{0(0,0)}} \,, \tag{25}$$

here $\chi_k(u) \equiv \chi_k = \sum_{i=0}^{k} u^{k-2i}$ is the character of the spin-$k/2$ $SU(2)$ representation. The Hall-Littlewood letters are the top components of half-hypers $X = Q$, $Y = \widetilde{Q}$ which transform in the $\mathbf{2}$ under the enhanced $SU(2)$ flavour symmetry and $\overline{\lambda} = \overline{\lambda}_{1\dot{+}}$

in the **1** under the $SU(2)$. All the letters are in the adjoint representation of G. Operators appearing in the expansion of the PLog of the Hall-Littlewood index are of the form

$$\operatorname{tr} X^n Y^m \overline{\lambda}^k \in \begin{cases} \hat{\mathcal{B}}_{\frac{m+n}{2}} & \text{if } k=0 \\ \mathcal{D}_{\frac{m+n+k-1}{2}(0,\frac{k-1}{2})} & \text{if } k \geqslant 1 \end{cases} \tag{26}$$

due to $SU(2)$ flavour symmetry these must occur symmetrically under $m \leftrightarrow n$.

0.3.1 $G = U(N)$

The Hilbert series for this theory was computed already in Chapter **??** and reads

$$\mathrm{HS}_{1,1}^{U(N)} = \frac{1}{N!} \frac{\partial^N}{\partial \nu^N} \mathrm{PE}\left[\frac{\nu}{(1-u\tau)(1-u^{-1}\tau)}\right]\Bigg|_{\nu=0} . \tag{27}$$

In particular, the Higgs branch of this theory as a variety is $\mathrm{Sym}^N(\mathbb{C}^2)$.

$N=1$

We can immediately write down the ratio

$$\frac{\mathrm{HL}_{1,1}^{U(1)}}{\mathrm{HS}_{1,1}^{U(1)}} = \mathrm{PE}[-\tau^2] = \mathrm{PE}[\mathrm{HL}_{[1]_-}] = \mathrm{PE}[\mathrm{HL}_{\mathcal{D}_{0,(0,0)}}] . \tag{28}$$

I.e. the difference is simply an additional free vector multiplet.

$N=2$

The Hilbert series reads

$$\mathrm{HS}_{1,1}^{U(2)} = \mathrm{PE}\left[\chi_1\tau + \chi_2\tau^2 - \tau^4\right] . \tag{29}$$

The Hall-Littlewood index can be evaluated by means of residues and reads

$$\mathrm{HL}_{1,1}^{U(2)} = (1+\tau^2-\chi_1\tau^3)\,\mathrm{PE}\left[\chi_1\tau + \chi_2\tau^2 - 2\tau^2\right] . \tag{30}$$

The ratio is

$$\frac{\mathrm{HL}_{1,1}^{U(2)}}{\mathrm{HS}_{1,1}^{U(2)}} = \frac{(1+\tau^2-\chi_1\tau^3)(1-\tau^2)^2}{1-\tau^4} = \left(1 - \frac{\chi_1\tau^3}{1+\tau^2}\right)\mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] \tag{31}$$

$$= \left(1 + \chi_1 \sum_{n=1}^{\infty} (-1)^n \tau^{2n+1}\right)\mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] \tag{32}$$

$$= \left(1 + \chi_1 \sum_{m=1}^{\infty} \left(\mathrm{HL}_{[2m+1/2]_+} + \mathrm{HL}_{[2m-1/2]_-}\right)\right)\mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] \tag{33}$$

note that here we have factored out the contribution from a $\mathcal{D}_{0(0,0)}$ multiplet, as we know that this multiplet is always present for the $\mathfrak{u}(N)$ theory and corresponds to the free decoupled $\mathfrak{u}(1)$ in the decomposition $\mathfrak{u}(N) \cong \mathfrak{su}(N) \oplus \mathfrak{u}(1)$. The Plethystic logarithm (spectrum of single-trace operators) is

$$\mathrm{PLog}\left[\frac{\mathrm{HL}_{1,1}^{U(2)}}{\mathrm{HS}_{1,1}^{U(2)}}\right] = -\tau^2 - (u+u^{-1})\tau^3 + (u+u^{-1})\tau^5 + \mathcal{O}(\tau^6) \tag{34}$$

corresponding to $\operatorname{tr}\overline{\lambda}$, $\operatorname{tr}X\overline{\lambda}$, $\operatorname{tr}Y\overline{\lambda}$, $\operatorname{tr}X\overline{\lambda}^2$, $\operatorname{tr}Y\overline{\lambda}^2$, past this order it is no longer possible to uniquely determine the operators.

$N=3$

The Hilbert series in this case is

$$\mathrm{HS}_{1,1}^{U(3)} = \left(\chi_1\tau^3 + \sum_{n=0}^{3}\tau^{2n}\right)\mathrm{PE}\left[3\chi_1\tau + \chi_2\tau^2 - \tau^2 + (\chi_1 - \chi_3)\tau^3\right] \tag{35}$$

where we used the identity $\sum_{n=0}^{p} x^n = (1-x^{p+1})/(1-x) = \mathrm{PE}[x - x^{p+1}]$.

For $N=3$ it is still possible to compute using residues and, after using some identities we arrive at

$$\begin{aligned}\mathrm{HL}_{1,1}^{U(3)} = &\left(-\chi_2\tau^4 - \chi_1\tau^5 - (\chi_2-1)\tau^6 + (\chi_3-\chi_1)\tau^7 + \sum_{n=0}^{4}\tau^{2n}\right)\\ &\times \mathrm{PE}\left[3\chi_1\tau + \chi_2\tau^2 + \chi_1\tau^3 - 2\tau^2 - \chi_3\tau^3\right].\end{aligned} \tag{36}$$

The ratio is

$$\begin{aligned}&\frac{\mathrm{HL}_{1,1}^{U(3)}}{\mathrm{HS}_{1,1}^{U(3)}} = \mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right]\\ &\times\left(1 - \frac{\chi_1\tau^3 + \chi_2\tau^4 + \chi_1\tau^5 + (\chi_2-1)\tau^6 - (\chi_3-\chi_1)\tau^7 - \tau^8}{1+\tau^2+\chi_1\tau^3+\tau^4+\tau^6}\right).\end{aligned} \tag{37}$$

$N=\infty$

In Chapter **??** we wrote a simple formula for the $U(\infty)$ $\mathcal{N}=4$ Hilbert series, it reads

$$\mathrm{HS}_{1,1}^{U(\infty)} = \mathrm{PE}\left[\frac{1}{(1-u\tau)(1-u^{-1}\tau)}\right] = \mathrm{PE}\left[\sum_{k\geqslant 0}\chi_k(u)\tau^k\right]. \tag{38}$$

The spectrum of single trace Higgs-branch operators is a collection of $\hat{\mathcal{B}}_R$ multiplets in the spin R representation of the $SU(2)$ global symmetry. In the large N limit the

Hall-Littlewood index can easily be written down by appealing to AdS/CFT [?] it reads

$$\mathrm{HL}_{1,1}^{U(\infty)} = \mathrm{PE}\left[\mathrm{HL}^{KK}\right] , \quad \mathrm{HL}^{KK} = \frac{\chi_1(u)\tau - 2\tau^2}{(1-u\tau)(1-u^{-1}\tau)} . \tag{39}$$

So, the ratio is

$$\frac{\mathrm{HL}_{1,1}^{U(\infty)}}{\mathrm{HS}_{1,1}^{U(\infty)}} = \mathrm{PE}\left[\frac{-\tau^2}{(1-u\tau)(1-u^{-1}\tau)}\right] = \mathrm{PE}\left[\sum_{k\geqslant 0} \chi_k(u)\mathrm{HL}_{[1+k/2]_-}\right] \tag{40}$$

0.3.2 $G = SU(N)$

The Hilbert series for this theory was computed already in Chapter **??** and is given by

$$\mathrm{HS}_{1,1}^{SU(N)} = (1-u\tau)(1-u^{-1}\tau)\mathrm{HS}_{1,1}^{U(N)} = \frac{\mathrm{HS}_{1,1}^{U(N)}}{\mathrm{HS}_{1,1}^{U(1)}} . \tag{41}$$

The Higgs branch of this theory as a variety is $\mathbb{C}^{2N-2}/S_N \subset \mathrm{Sym}^N(\mathbb{C}^2)$, this is the subvariety of the $U(N)$ case defined by demanding tracelessness of the adjoint representation. We checked via series expansion for various low values of N that

$$\mathrm{HL}_{1,1}^{SU(N)} = \frac{\mathrm{HL}_{1,1}^{U(N)}}{\mathrm{HL}_{1,1}^{U(1)}} . \tag{42}$$

So, we can apply the results of the previous section using

$$\frac{\mathrm{HL}_{1,1}^{SU(N)}}{\mathrm{HS}_{1,1}^{SU(N)}} = \frac{\mathrm{HL}_{1,1}^{U(N)}}{\mathrm{HL}_{1,1}^{U(1)}}\frac{\mathrm{HS}_{1,1}^{U(1)}}{\mathrm{HS}_{1,1}^{U(N)}} = \mathrm{PE}[\mathrm{HL}_{\mathcal{D}_{0,(0,0)}}]\frac{\mathrm{HL}_{1,1}^{U(N)}}{\mathrm{HS}_{1,1}^{U(N)}} . \tag{43}$$

0.4 Elliptic Quiver Theories

We now allow for n minimal punctures. For $U(N)$ gauge groups this is the $\mathbb{Z}_n$ orbifold theory of $\mathcal{N}=4$ SYM.

0.4.1 $G = U(N)$

The Higgs branch of this theory as a variety is $\mathrm{Sym}^N(\mathbb{C}^2/\mathbb{Z}_n)$ and the Hilbert series is therefore

$$\mathrm{HS}_{1,n}^{U(N)} = \frac{1}{N!}\frac{\partial^N}{\partial\nu^N}\mathrm{PE}\left[\frac{\nu(1-\tau^{2n})}{(1-\tau^2)(1-u^n\tau^n)(1-u^{-n}\tau^n)}\right]\Bigg|_{\nu=0} . \tag{44}$$

The Hall-Littlewood index is

$$\mathrm{HL}_{1,n}^{U(N)} = \oint\prod_{i=1}^{n}\left(d\mu_{U(N)_i}\,\mathrm{PE}\left[\left(uf_i\overline{f}_{i+1} + u^{-1}\overline{f}_i f_{i+1}\right)\tau - \chi_{U(N)_i}^{(adj.)}\tau^2\right]\right) \tag{45}$$

where $f_i = \sum_{a=1}^N z_{i,a}$, $\overline{f}_i = \sum_{a=1}^N z_{i,a}^{-1}$, $\chi_i^{(adj.)} = \sum_{a,b=1}^N \frac{z_{i,a}}{z_{i,b}}$ we also take $z_{i+n,a} = z_{i,a}$.

$N = 1$

For $N = 1$ the results are rather simple

$$\mathrm{HS}_{1,n}^{U(1)} = \frac{1-\tau^{2n}}{(1-\tau^2)(1-u^n\tau^n)(1-u^{-n}\tau^n)} \tag{46}$$

and

$$\mathrm{HL}_{1,n}^{U(1)} = \frac{1-\tau^{2n}}{(1-u^n\tau^n)(1-u^{-n}\tau^n)} \tag{47}$$

we again have that the two quantities differ by a free vector multiplet

$$\frac{\mathrm{HL}_{1,n}^{\mathfrak{u}(1)}}{\mathrm{HS}_{1,n}^{\mathfrak{u}(1)}} = (1-\tau^2) = \mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] . \tag{48}$$

$N = n = 2$

The Hilbert series is

$$\mathrm{HS}_{1,n}^{U(2)} = \frac{(1-\tau^{2n})\left[(1+\tau^{2n})(1-\tau^{2+2n}) + (u^{-n}+u^n)(\tau^{2+n}-\tau^{3n})\right]}{(1-\tau^2)(1-\tau^4)(1-u^{\pm n}\tau^n)(1-u^{\pm 2n}\tau^{2n})} . \tag{49}$$

For $n = 2$ this simplifies to

$$\mathrm{HS}_{1,2}^{U(2)} = \left(\chi_2\tau^4 + \sum_{i=0}^{4}\tau^{2i}\right)\mathrm{PE}\left[(\chi_2-1)\tau^2 + (\chi_4-\chi_2)\tau^4\right] . \tag{50}$$

The Hall-Littlewood index is

$$\mathrm{HL}_{1,2}^{U(2)} = \left(1+\tau^4-(\chi_2-1)\tau^6\right)\mathrm{PE}\left[(\chi_2-1)\tau^2 + (\chi_4-\chi_2-1)\tau^4\right] . \tag{51}$$

The ratio is

$$\frac{\mathrm{HL}_{1,2}^{U(2)}}{\mathrm{HS}_{1,2}^{U(2)}} = \frac{1-\tau^8-(\chi_2-1)\tau^6(1-\tau^4)}{1+\chi_2\tau^4-\chi_2\tau^6-\tau^{10}}\,\mathrm{PE}[-\tau^2] \tag{52}$$

$$= \left(1+\chi_2\mathrm{HL}_{[2]_-} + \mathrm{HL}_{[3]_+} + (\chi_4+\chi_2)\mathrm{HL}_{[4]_+} + \mathcal{O}(\tau^{10})\right)\mathrm{PE}[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}] \tag{53}$$

0.4.2 Generic $\mathfrak{g} = A_1$ Class $\mathcal{S}$ Theories

The Hall-Littewood index for the A_1 theory associated to a genus g Riemann surface with n punctures is [**?**]

$$\mathrm{HL}_{g,n}^{SU(2)} = \frac{(1+\tau^2)^\chi}{(1-\tau^2)^{1-g}}\sum_{\lambda=0}^{\infty}\frac{1}{P_\lambda(\tau,\tau^{-1}|\tau)^\chi}\prod_{I=1}^{n}\frac{P_\lambda(a_I,a_I^{-1}|\tau)}{(1-a_I^2\tau^2)(1-a_I^{-2}\tau^2)} \tag{54}$$

where $\chi = 2g - 2 + n$ and the Hall-Littlewood polynomials are

$$P_\lambda(a, a^{-1}|\tau) = \begin{cases} \chi_\lambda(a) - \tau^2 \chi_{\lambda-2}(a) & \lambda \geqslant 1 \\ \sqrt{1+\tau^2} & \lambda = 0 \end{cases} \tag{55}$$

with $\chi_\lambda = (a^{1+\lambda} - a^{-1-\lambda})/(a - a^{-1})$ the $SU(2)$ characters. On the other hand, the Hilbert series for the same theory is given by [?]

$$\begin{aligned} \mathrm{HS}_{g,n}^{SU(2)} =& \frac{(1+\tau^2)^\chi}{(1-\tau^2)} \left((1+\tau^2)^{1-2g} \prod_{I=1}^{n} \frac{1}{(1-a_I^2\tau^2)(1-a_I^{-2}\tau^2)} \right. \\ & \left. + \sum_{\lambda=1}^{\infty} \frac{1}{P_\lambda(\tau, \tau^{-1}|\tau)^\chi} \prod_{I=1}^{n} \frac{P_\lambda(a_I, a_I^{-1}|\tau)}{(1-a_I^2\tau^2)(1-a_I^{-2}\tau^2)} \right) . \end{aligned} \tag{56}$$

It is immediate that, when $g = 0$ we have $\mathrm{HS}_{0,n}^{SU(2)} = \mathrm{HL}_{0,n}^{SU(2)}$. Let us consider the case of a theory associated to a genus $g \geqslant 1$ surface without punctures, in which case the sums can be performed explicitly

$$\mathrm{HL}_{g,0}^{SU(2)} = \frac{(1-\tau^2)^{\chi/2}(\tau^\chi + (1+\tau^2)^{\chi/2}(1-\tau^\chi))}{(1-\tau^\chi)} \tag{57}$$

while the corresponding Hilbert series becomes

$$\mathrm{HS}_{g,0}^{SU(2)} = \mathrm{PE}\left[\tau^4 + \tau^\chi + \tau^{\chi+2} - \tau^{2\chi+4}\right] . \tag{58}$$

The ratio then takes the form

$$\begin{aligned} \frac{\mathrm{HL}_{g,0}^{SU(2)}}{\mathrm{HS}_{g,0}^{SU(2)}} &= \left(\tau^{2g-2} + (1+\tau^2)^{g-1}(1-\tau^{2g-2})\right) \\ & \times \mathrm{PE}\left[-(g-1)\tau^2 - \tau^4 - \tau^{2g}(1-\tau^{2g})\right] . \end{aligned} \tag{59}$$