0.1 Instantons

In this appendix we review some basic facts about instanton solutions, instanton moduli space and the ADHM construction. A mathematical review can be found in [?] while more physics based reviews can be found in [?, ?, ?, ?]. Consider pure Yang-Mills theory in $d = 4$ dimensions based on a Lie algebra $\mathfrak{g}$ on a orientable Riemannian manifold $\mathcal{M}$ with metric g and volume form $\omega \in \Omega^d(\mathcal{M})$. This is a theory of vector bundles $V \to \mathcal{M}$ equipped with connection ∇ with curvature $F = \mathrm{ad}_\nabla \nabla \in \mathfrak{g} \otimes \Omega^2(\mathcal{M})$ associated to principal G-bundles $P \to \mathcal{M}$ where G is a connected Lie group of rank r with Lie algebra $\mathfrak{g}$. This theory has action

$$S = -\frac{1}{2g^2} \int_{\mathcal{M}} \mathrm{tr}\, F \wedge \star F = -\frac{1}{2g^2} \int_{\mathcal{M}} \mathrm{tr}\langle F, F \rangle\, \omega\,. \tag{1}$$

$\star$ denotes the Hodge star on $\mathcal{M}$. Recalling that $\star^2 = (-1)^{p(d-p)}$ on p-forms we can write

$$\mathrm{tr}\langle F, F\rangle = \frac{1}{2}\,\mathrm{tr}\langle F \pm \star F, F \pm \star F\rangle \mp \mathrm{tr}\langle F, \star F\rangle \geqslant \mp\, \mathrm{tr}\langle F, \star F\rangle\,, \tag{2}$$

we therefore have that the action minimising configurations satisfy $F = \pm \star F$.

In a local trivialisation of V we write $\nabla = \mathrm{d} + A$, $A \in \mathfrak{g} \otimes \Omega^1(\mathcal{M})$. We define a G-framed Yang-Mills instanton as a solution to the (anti-)self-dual Yang-Mills equations

$$F = \pm \star F \tag{3}$$

such that A approaches pure gauge at infinity. In a given instanton bundle such solutions admit a topological invariant - the instanton number

$$k = \int_{\mathcal{M}} c_1(F)^2 = \frac{1}{16\pi^2} \int_{\mathcal{M}} \mathrm{tr}\, F \wedge F = \frac{1}{16\pi^2} \int_{\mathcal{M}} \mathrm{tr}\langle F, \star F\rangle\, \omega \in \mathbb{Z}\,. \tag{4}$$

Here the trace is always normalised such that k is integral, $c_1^2(F) \in H^4(\mathcal{M}, \pi_3(G))$ is the square of the first Chern class and $\pi_3(G) \cong \mathbb{Z}$ for any connected compact simple Lie group. For generic $G, \mathcal{M}$ there may be other topological invariants besides (4), for instance when the second Steifel-Whitney class $w_2(V) \in H^2(\mathcal{M}, \mathbb{Z}_2)$ is non-trivial there may be G bundles which do not lift to $\tilde{G}$ bundles where $\tilde{G}$ denotes the universal cover of G [?]. For example it was shown in [?] that when $G = SO(n)$ and $\mathcal{M}$ is any 4-complex the associated bundles V are classified by a pair $(w_2(V), c_2(F))$. Unless otherwise stated we will specialise to $G = SU(N)$ and $\mathcal{M} = \mathbb{R}^4$ (possibly with Ω-deformation parameters ϵ_1, ϵ_2 turned on). Since $H^2(\mathbb{R}^4, \mathbb{Z})$ (and also $H^2(\mathbb{S}^4, \mathbb{Z})$) is trivial such bundles are classified by (4) alone.

0.2 Instanton Moduli Space

We identify solutions to the instanton equations (3) if they are related by gauge transformations $g(x) \in G$. We therefore have a moduli space $\mathbf{M}$ of solutions called *instanton moduli space*. Because of the topologically invariant instanton number the instanton moduli space formally admits a decomposition

$$\mathbf{M} \cong \bigoplus_{k=1}^{\infty} q^k \mathbf{M}_k \,. \tag{5}$$

$\mathbf{M}_k$ is called the *k-instanton moduli space*. To compute its dimension we can consider the case when the k-instanton solution looks like k 1-instanton solutions whos centres are well-separated and then superimpose them while adding corrections to satisfy (3)(assuming that the dimension does not change). For $\mathcal{M} = \mathbb{R}^4$ the instanton moduli space $\mathbf{M}_{k=1}$ is parametrised by collective coordinates:

- four parameters labelling the center of the instanton $(X^1, X^2, X^3, X^4) \in \mathbb{R}^4$
- one parameter for the size of the instanton $\rho^2 \in \mathbb{R}_{\geqslant 0}$
- $4h^\vee(\mathfrak{g}) - 5$ embedding parameters $\mathfrak{su}(2) \to \mathfrak{g}$ of the 1-instanton $\mathfrak{su}(2)$ solution into $\mathfrak{g}$

Therefore, the dimension of the instanton moduli space is $\dim_{\mathbb{R}} \mathbf{M}_k = 4kh^\vee(\mathfrak{g})$. For the case $G = SU(N)$ and $\mathcal{M}$ compact [**?**]

$$\dim_{\mathbb{R}} \mathbf{M}_k = 4Nk - (N^2 - 1)\frac{\chi(\mathcal{M}) + \sigma(\mathcal{M})}{2} \,. \tag{6}$$

Choosing local coordinates on x^μ on $\mathbb{R}^4$ the solution $A = A(X^\mu, \rho, g, x^\mu)$ to (3) for $G = SU(2)$ is

$$A_\mu \mathrm{d}x^\mu = \sum_{i=1}^{3} \frac{\rho^2 (x - X)^\nu \overline{\eta}^{i\nu}_\mu}{(x - X)^2((x - X)^2 + \rho^2)} (g(x)\sigma^i g(x)^\dagger)\mathrm{d}x^\mu \,. \tag{7}$$

Solutions for general N are harder to explicitly write down and we will come back to these in the next section.

Denoting the collective coordinates by $\{X^a\}$, $a = 1, \ldots, \dim_{\mathbb{R}} \mathbf{M}_k$ the and a solution $A = A(X^a, x^\mu)$ to (3). The moduli space inherits a metric h given by

$$h = \left(\int_{\mathcal{M}} \operatorname{tr} \delta_a A \wedge \star \delta_b A \right) dX^a \otimes dX^b \tag{8}$$

where $\delta_a A = \partial A / \partial X^a$. One can also show that h defined this way is *hyperKähler*.

0.3 ADHM Construction

Due to Atiyah, Drinfeld, Hitchin and Manin [**?**] there is a powerful method to solve the self-dual Yang-Mills equations for $G = SU(N)$ on $\mathbb{R}^4$. The idea is to realise $\mathbf{M}_k$ as a hyperKähler quotient of a complex vector space. In order to specify a solution one gives the so-called ADHM data. This is made up of a pair of complex vector spaces V, W with $\dim_{\mathbb{C}} V = k, \dim_{\mathbb{C}} W = N$ and a set of linear maps

$$B_1, B_2 : V \to V, \quad I : V \to W, \quad J : W \to V. \tag{9}$$

Subject to the ADHM (moment map) equations

$$\mu_{\mathbb{C}} = [B_1, B_2] + JI = 0, \quad \mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + JJ^\dagger - I^\dagger I = 0. \tag{10}$$

Moreover there are natural $U(k) : V \to V$, $U(N) : W \to W$ actions which act by

$$(B_1, B_2, I, J) \mapsto (gB_1 g^\dagger, gB_2 g^\dagger, Ig^\dagger, gJ)\,, \quad g \in U(k) \tag{11}$$

$$(B_1, B_2, I, J) \mapsto (B_1, B_2, \tau I, J\tau^\dagger)\,, \quad \tau \in U(N) \tag{12}$$

which respects the hyperKähler structure. The moduli space is

$$\mathbf{M}_{k,\zeta} := \{B_1, B_2, I, J \,|\, \mu_{\mathbb{C}} = 0\,, \mu_{\mathbb{R}} = \zeta \mathbb{I}_k\} / U(k)\,. \tag{13}$$

Note that here we have added a non-zero FI-term ζ. Mathematically; allowing for non-zero ζ, means are actually considering the space obtained from the original instanton moduli space by a series of blowups (resolutions of singularities) which are smooth and we consider not only bundles but torsion free sheaves [**?**, **?**, **?**, **?**, **?**, **?**]. Physicially this corresponds to studying instantons on non-commutative $\mathbb{R}^4$.

In practical applications it is often more convenient to instead work with an equivalent form where we drop the real constraint and quotient out by complexified gauge transformations

$$\widetilde{\mathbf{M}}_{k,\zeta} := \{B_1, B_2, I, J \,|\, \mu_{\mathbb{C}} = 0, \ \text{stability}(\zeta)\} / GL(k, \mathbb{C})\,. \tag{14}$$

The main point of [**?**] is that they were able to prove that

$$\mathbf{M}_k \cong \mathbf{M}_{k,\zeta=0} \cong \widetilde{\mathbf{M}}_{k,\zeta=0}\,. \tag{15}$$

To construct the self-dual solutions explicitly we consider the $(N + 2k) \times 2k$ matrix

$$Y = \begin{pmatrix} J^\dagger & I \\ B_2^\dagger + z_2^\dagger & -B_1 - z_1 \\ B_1^\dagger + z_1^\dagger & B_2 + z_2 \end{pmatrix} \tag{16}$$

where $(z_1, z_2) = (x^1 + \mathrm{i}x^2, x^3 + \mathrm{i}x^4) \in \mathbb{C}^2 \cong \mathbb{R}^4$. We then have

$$Y^\dagger Y = \begin{pmatrix} f^{-1} & \mu_{\mathbb{C}} \\ \mu_{\mathbb{C}}^\dagger & -\mu_{\mathbb{R}} + f^{-1} \end{pmatrix} = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix} \tag{17}$$

with $f^{-1} := JJ^\dagger + B_2 B_2^\dagger + B_2 z_2^\dagger + B_2^\dagger z_2 + B_1^\dagger z_1 + B_1 z_1^\dagger + z_1^\dagger z_1 + z_2^\dagger z_2$ and therefore Y factorises and is invertible (the inverse is simply $Y^{-1} = fY^\dagger$). Now, the basis vectors for the null-space of $Y^\dagger$ can be assembled into a $(N + 2k) \times N$ matrix $U = U(x)$

$$Y^\dagger U = 0\,, \quad U^\dagger U = 1_N\,. \tag{18}$$

The k-instanton gauge potential is then given by

$$A_\mu \mathrm{d}x^\mu = U^\dagger \frac{\partial}{\partial x^\mu} U \mathrm{d}x^\mu\,. \tag{19}$$

0.4 ADHM Construction from Type-II String Theory

For physical purposes one of the most natural ways to view the ADHM construction is within string theory. String theory can be a powerful tool in understanding the dynamics of gauge theories. Indeed one of the most important tools that we have used in Chapter **??** of this thesis is the relationship between the ADHM construction of instantons and D$(p-4)$-branes [**?**, **?**, **?**, **?**, **?**].

Let us suppose that we have a gauge theory in $p+1$ dimensions which can be embedded within Type II string theory on $\mathbb{R}^{10}$ as a theory living on the worldvolume of a stack of N Dp-branes with coordinates $X^1, \ldots, X^{p+1}$. We now further consider inserting a stack of k D$(p-4)$-branes.

The type-II string action contains terms of the form

$$\sum_l \int_{\mathbb{R}^{10}} C_l \wedge J_{10-l} \tag{20}$$

with $l = 0, 2, 4, 6, 8, 10$ for IIB and $l = 1, 3, 5, 7, 9$ for IIA. Here C_l is the RR l-form that couples (electrically) to D$(l-1)$-branes and the $(10-l)$-form J_{10-l}. The D$(9-l)$-brane is a source for C_l.

If we take the $l = p-3$ term while taking C_{p-3} constant over the $\mathbb{R}^{10-(p-4)}$ transverse to the D$(p-4)$-branes we can write

$$\int_{\mathbb{R}^{p+1}} C_{p-3} \wedge \left(\int_{\mathbb{R}^{9-p}} J_{13-p} \right) = \frac{1}{2g^2} \int_{\mathbb{R}^{p+1}} C_{p-3} \wedge F \wedge F = \frac{8k\pi^2}{g^2} \int_{\mathbb{R}^{p-3}} C_{p-3}\,. \tag{21}$$

In other words an instanton with topological charge k gives rise to a source for the RR-form. This is nothing other than the source induced by a D$(p-4)$-brane. Therefore we have that

$$|k| \text{ (A)SD instantons in a D}p\text{-brane} \equiv |k| \text{ (anti-)D}(p-4)\text{-branes}\,. \tag{22}$$

Demanding that C_{p-3} constant over the transverse $\mathbb{R}^{9-p}$ is equivalent to the statement that the D$(p-4)$-branes are confined to live within the Dp-branes. This is the Higgs-branch of the theory on the D$(p-4)$-branes. Therefore, the moduli space of k-instantons for the gauge theory living on the Dp-branes, $\mathbf{M}_k$, is isomorphic to the Higgs branch of the theory living on the D$(p-4)$-branes

$$\mathbf{M}_k \cong \mathbf{HB}_{k\ \mathrm{D}(p\,-\,4)} = \left\{ \{\phi\} | V_{p-3} = 0\,, X^{p+2} = X^{p+3} = \cdots = X^{10} \right\} / U(k) \tag{23}$$

where $\{\phi\}$ collectively denotes all of the scalars of theory on the Higgs branch and $V_{p-3} = \sum |F|^2 + \frac{1}{2} D^2$ is the scalar potential of the $(p-3)$-dimensional theory living on the worldvolume of the D$(p-4)$-branes. The vanishing of the potential $F = D = 0$ translate into the ADHM constraints [**?**, **?**] as we will explicitly demonstrate in the next section. When supersymmetry is present the Higgs branch is protected from quantum corrections and the fluctuation determinants in the instanton measure cancel. The action of the theory on the D$(p-4)$-branes is the equivalent to the instanton action, hence the partition function of the theory of k D$(p-4)$-branes is then nothing else but the partition function of k instantons for the theory living on the Dp-branes

$$Z_k(a, \epsilon_i, \dots) = \int_{\mathbf{M}_k} e^{-S^{(k)}_{\text{inst}}(a,\dots)} \tag{24}$$

$$= \mathrm{Tr}_{\mathcal{H}_k} (-1)^F e^{a^A h_A + \sum_i \epsilon_i j_i} \tag{25}$$

$$= Z_{\text{extra}}(\epsilon_i)\, Z_{\text{Higgs}}(a, \epsilon_i, \dots)\,, \tag{26}$$

where $\mathcal{H}_k$ denotes the Hilbert space of the worldvolume theory on k D$(p-4)$-branes. The factor Z_{extra} is often present due to the fact that the theory on the D$(p-4)$-branes provides a UV completion of the ADHM sigma model [**?**, **?**] and therefore it may contain extra degrees of freedom which do not appear in the ADHM construction (such as a non-trivial Coulomb branch) which should be excluded from the instanton calculus. Those extra degrees of freedom generally decouple from the the ADHM degrees of freedom and the partition function factorises as above.

0.4.1 Example: ADHM for $\mathcal{N} = 4$ SYM

Let us now demonstrate the power of the string-theoretic realisation of the ADHM construction. We consider $p = 3$ with a stack of N D3-branes along X^1, X^2, X^3, X^4. In the low energy limit quantisation of open D3–D3 strings gives rise to the familiar $G = U(N)$ $\mathcal{N} = 4$ SYM theory. The self-dual instantons are realised as $D(-1)$-branes. The number of supercharges preserved by this brane configuration is $32/(2\cdot 2) = 8$ as required for $\frac{1}{2}$-BPS instantons of $\mathcal{N} = 4$ SYM.

Quantisation of open D(-1)–D(-1) strings gives rise to the reduction to zero dimensions of 4d $\mathcal{N} = 4$ SYM with gauge group $U(k)$. This has on-shell bosonic field content as a 4d theory of $A_{\mu=7,8,9,10}$, $z = X^1 + \mathrm{i}X^2$ $\widetilde{z} = X^3 + \mathrm{i}X^4$, $\phi = X^5 + \mathrm{i}X^6$ all in the adjoint representation of $\mathfrak{u}(k)$.

D(-1)–D3 strings gives rise to the dimensional reduction of 4d $\mathcal{N} = 2$ hypermultiplets $(q, \widetilde{q})$ in the $\mathbf{K} \times \overline{\mathbf{N}}$ representation of $U(K) \otimes U(N)$. The coupling of these bifundamental hypers to the maximally supersymmetric $U(k)$ theory is fixed by demanding $\mathcal{N} = 2$ supersymmetry. As a 4d theory the superpotential is

$$W = q\phi\widetilde{q} + \mathrm{tr}\left([\phi, z]\widetilde{z} + W_\alpha W^\alpha\right)\,. \tag{27}$$

The bosonic part of the action reduced to zero dimensions is

$$\begin{aligned}\mathcal{L}_{bos} =& \frac{1}{2}\,\mathrm{tr}\left([X_\mu, X_\nu]^2 + |[X_\mu, \phi]|^2 + |[X_\mu, z]|^2 + |[X_\mu, \widetilde{z}]|^2\right) + q^\dagger X^\mu X_\mu q \\ &+ \widetilde{q}^\dagger X^\mu X_\mu \widetilde{q} + \left(\sum_{f\in\{q,\widetilde{q},z,\widetilde{z},\phi\}} \mathrm{tr}\left(\frac{\partial W}{\partial f} F_f + F_f^2\right) + h.c.\right) \\ &+ \mathrm{tr}\left(\frac{1}{2}D^2 + D\left([\phi, \phi^\dagger] + [\widetilde{z}, \widetilde{z}^\dagger] + [z, z^\dagger] + qq^\dagger - \widetilde{q}^\dagger\widetilde{q} - \zeta\right)\right)\,.\end{aligned} \tag{28}$$

The F terms are

$$F_\phi = -q\widetilde{q} - [z, \widetilde{z}]\,, \quad F_q = -\phi\widetilde{q}\,, \quad F_{\widetilde{q}} = -q\phi\,, \quad F_z = [\phi, \widetilde{z}]\,, \quad F_{\widetilde{z}} = [\phi, z]\,, \tag{29}$$

and the D-term is

$$D = -qq^\dagger + \widetilde{q}^\dagger\widetilde{q} - [z, z^\dagger] - [\widetilde{z}, \widetilde{z}^\dagger] - [\phi, \phi^\dagger] + \zeta\mathbb{I}_k = 0\,. \tag{30}$$

As we mentioned in the previous section, to reach the correspondence with instantons we must move to the Higgs branch of this theory. The Higgs branch is reached by setting $\phi = X_{\mu=7,8,9,10} = 0$ while enforcing that the scalar potential vanishes $D = 0$, and $F_f = 0$

$$\mathbf{HB}_{k\ \mathrm{D}(-1)} = \{q, \widetilde{q}, z, \widetilde{z} | F_\phi = 0\,, D = 0\}\,/U(k) \tag{31}$$

$$F_\phi = -q\widetilde{q} - [z, \widetilde{z}] = 0\,, \quad D = -qq^\dagger + \widetilde{q}^\dagger\widetilde{q} - [z, z^\dagger] - [\widetilde{z}, \widetilde{z}^\dagger] - \zeta\mathbb{I}_k = 0 \tag{32}$$

These are precisely the ADHM equations (10)

$$\mu_{\mathbb{C}} = -F_\phi = 0\,, \quad \mu_{\mathbb{R}} - \zeta\mathbb{I}_k = -D = 0 \tag{33}$$

with $z = B_1$, $\widetilde{z} = B_2$, $q = J$ and $\widetilde{q} = I$. And therefore we have

$$\mathbf{HB}_{k\ \mathrm{D}(-1)} \cong \mathbf{M}^{\mathrm{ADHM}}_{k,\zeta}\,. \tag{34}$$