The purpose of this Appendix is to collect the various definitions and identities used through this thesis. Formulas and identities that are used only once will not be listed here, but rather in the relevant place where they are used. Conversely, identities used multiple times throughout the thesis will simply be stated in this Appendix and then referred to from the main text.

0.1 Identities and Special Functions

0.1.1 Plethystics

The *Plethystic Exponential* is given by

$$\mathrm{PE}\left[f(x_1,\dots,x_m)\right] := \exp\left(\sum_{n=1}^{\infty}\frac{f(x_1^n,\dots,x_m^n)-f(0,\dots,0)}{n}\right). \tag{1}$$

Its inverse is called the *Plethystic Logarithm*

$$\mathrm{PE}^{-1}[f(x_1,\dots,x_m)] := \mathrm{PLog}[f(x_1,\dots,x_m)] = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log f(x_1^n,\dots,x_m^n)\,, \tag{2}$$

where $\mu(n)$ is the Möbius μ function. Some basic identities are

$$\mathrm{PE}[t] = \frac{1}{1-t}\,,\quad \mathrm{PE}[-t] = 1-t\,,\quad \sum_{n=0}^{N-1} t^n = \frac{1-t^N}{1-t} = \mathrm{PE}[t-t^N]\,. \tag{3}$$

0.1.2 Modular Forms and Theta Functions

The Dedekind eta function is

$$\eta(q) = q^{1/24}\prod_{a=1}^{\infty}(1-q^a) \tag{4}$$

and

$$\begin{aligned}\theta_1(x;q) &= \mathrm{i}q^{1/8}(x^{-1/2}-x^{1/2})\prod_{r=1}^{\infty}(1-q^r)(1-xq^r)(1-x^{-1}q^r)\\ &= -\mathrm{i}q^{1/8}x^{1/2}\prod_{r=1}^{\infty}(1-q^r)(1-xq^r)(1-x^{-1}q^{r-1})\\ &= -\mathrm{i}q^{1/8}x^{1/2}\sum_{r=-\infty}^{\infty}(-1)^r(xq^{\frac{1}{2}})^r q^{\frac{r^2}{2}}\end{aligned} \tag{5}$$

is the Jacobi theta function. Clearly

$$\theta_1\left(x;q\right) = -\theta_1\left(x^{-1};q\right) \tag{6}$$

We often also write

$$\theta_1(z|\tau) \equiv \theta_1(x;q) \tag{7}$$

with $x := e^{2\pi\mathrm{i}z}$ and $q := e^{2\pi\mathrm{i}\tau}$. The Jacobi theta function satisfies the properties

$$\theta_1\left(xq^{a+b/\tau};q\right) = (-1)^{a+b}x^{-a}q^{-a^2/2}\theta_1(x;q)\ , \quad \theta_1(x;q) = -\theta_1(x^{-1};q)\,. \tag{8}$$

$\theta_1(x;q)$ has simple zeros for $x = q^{a+b/\tau}$ for $a,b \in \mathbb{Z}$ and no poles. Furthermore, to compute residues, note that

$$\frac{\partial}{\partial y}\theta_1(y;q)\,|_{y=1} = 2\pi\eta(q)^3 \tag{9}$$

hence the residue is given by

$$\oint_{y=q^{a+b/\tau}} \frac{dy}{2\pi\mathrm{i}y}\frac{1}{\theta(y;q)} = \frac{(-1)^{a+b}q^{a^2/2}}{2\pi\eta(q)^3}\,. \tag{10}$$

0.1.3 Other Special Functions

We will often use the shorthand notation

$$f(z^{\pm n}) \equiv f(z^n)f(z^{-n})\,. \tag{11}$$

We use the following notation for the q-Pochammer symbols

$$(a;q)_N = \prod_{n=0}^{N}(1-aq^n)\,, \quad (a;q) := (a;q)_\infty = \mathrm{PE}\left[\frac{a}{1-q}\right]\,. \tag{12}$$

The q-theta function is

$$\theta(x;q) := (x;q)\left(qx^{-1};q\right) = \mathrm{PE}\left[\frac{x+qx^{-1}}{1-q}\right]\,. \tag{13}$$

They are related to the Jacobi theta function

$$\theta_1(x;q) := \mathrm{i}q^{\frac{1}{12}}\eta(q)(x^{-\frac{1}{2}}-x^{\frac{1}{2}})\prod_{n=1}^{\infty}(1-xq^n)(1-x^{-1}q^n) = \mathrm{i}q^{\frac{1}{12}}\eta(q)x^{-\frac{1}{2}}\theta(x;q)\,. \tag{14}$$

An obvious but important identity that we often employ is

$$x\theta(qx;q) = -\theta(x;q)\ . \tag{15}$$

The function $\theta(y;q)$ has simple zeros for $y = q^{a+b/\tau}$ for $a,b \in \mathbb{Z}$ and no poles. Furthermore, to compute residues note that

$$\frac{\partial}{\partial y}\theta(y;q)\,|_{y=1} = -(q;q)^2\ . \tag{16}$$

Using the identity $x\theta\left(qx;q\right) = -\theta\left(x;q\right)$ the residue is given by

$$\oint\limits_{y=q^{a+b/\tau}} \frac{dy}{2\pi \mathrm{i} y} \frac{1}{\theta\left(y;q\right)} = (-1)^{a+1}\left(q;q\right)^{-2} q^{\frac{a}{2}(a-1)} \,. \tag{17}$$

We are often interested in the $q \to 1$ limit of the above q-series. To take the limit, first note that the ratio of q-theta function may be rewritten as

$$\frac{\theta\left(q^a;q\right)}{\theta\left(q^b;q\right)} = \frac{[a]_q}{[b]_q}\prod_{n=1}^{\infty}\frac{[n+a]_q\,[n-a]_q}{[n+b]_q\,[n-b]_q} \tag{18}$$

where $[n]_q := (1-q^n)/(1-q)$ is the q-number. The q-number has the property that

$$\lim_{q\to 1}[n]_q = n \tag{19}$$

and therefore, for q-independent a, b, we have

$$\lim_{q\to 1}\frac{\theta\left(q^a;q\right)}{\theta\left(q^b;q\right)} = \frac{\sinh \mathrm{i}\pi a}{\sinh \mathrm{i}\pi b}\,, \tag{20}$$

where we used the Euler infinite product representation for the sine function is

$$\sin(x) = x\prod_{t=1}^{\infty}\left(1-\frac{x^2}{\pi^2 t^2}\right)\,. \tag{21}$$

The Elliptic Gamma function is defined as

$$\Gamma_e(z) := \Gamma(z;p,q) = \prod_{i,j=0}^{\infty}\frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j} = \mathrm{PE}\left[\frac{z-\frac{pq}{z}}{(1-p)(1-q)}\right]\,. \tag{22}$$

An obvious, yet important, identity is

$$\Gamma_e(z)\Gamma_e(pq/z) = 1\,. \tag{23}$$

Multiple gamma functions are defined as the regularised infinite products

$$\Gamma_r(z|\vec{\omega}) \sim \prod_{n_1,n_2,\ldots,n_r=0}^{\infty}\left(\vec{n}\cdot\vec{\omega}+z\right)^{-1} \tag{24}$$

and $\vec{n}\cdot\vec{\omega} = n_1\omega_1 + n_2\omega_2 + n_3\omega_3$.

The multiple sine function is defined as the regularised product

$$\begin{aligned} S_r(z|\vec{\omega}) &= \Gamma_r(z|\vec{\omega})^{-1}\Gamma_r(|\vec{\omega}|-z|\vec{\omega})^{(-1)^r} \\ &\sim \prod_{n_1,n_2,\ldots,n_r=0}^{\infty}\left(\vec{n}\cdot\vec{\omega}+|\vec{\omega}|-z\right)\left(\vec{n}\cdot\vec{\omega}+z\right)^{(-1)^{r+1}}\,, \end{aligned} \tag{25}$$

with $|\vec{\omega}| = \omega_1 + \cdots + \omega_r$. It has the symmetry property

$$S_r(z|\vec{\omega}) = S_r(|\vec{\omega}|-z|\vec{\omega})^{(-1)^{r+1}}\,. \tag{26}$$

0.1.4 Young Diagrams

We use Greek letters η, μ, λ, ν to denote partitions of natural numbers. We denote the empty partition by $\varnothing$. A non-empty partition is a set of integers λ

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \lambda_l \geqslant \dots \geqslant \lambda_{\ell(\lambda)} > 0\,, \tag{27}$$

with $\lambda_A \in \mathbb{N}$ and $\ell(\lambda)$ the number of parts of λ. This definition is also extended to include $\lambda_{l>\ell(\lambda)} \equiv 0$. λ^{T} is the transpose. We denote

$$|\lambda| := \sum_{l=1}^{\ell(\lambda)} \lambda_l\,, \quad ||\lambda||^2 := \sum_{l=1}^{\ell(\lambda)} \lambda_l^2 = \sum_{(l,p)\in\lambda^{\mathrm{T}}} \lambda_l\,. \tag{28}$$

We give a box s in the Young diagram coordinates $s = (l,p)$ such that

$$\lambda = \{(l,p)|l = 1,\dots,\ell(\lambda); p = 1,\dots,\lambda_l\}\,. \tag{29}$$

We will also be interested in collections of Young diagrams in which case we will give them labels, for example, λ_A in which case we write $\lambda_{A;l}$ to denote the value of the l^{th} column of the diagram λ_A. By definition

$$\prod_{(l,p)\in\lambda} g(l,p) = \prod_{l=1}^{\ell(\lambda)} \prod_{p=1}^{\lambda_l} g(l,p)\,. \tag{30}$$

We will also be interested in N-tuples of Young diagrams

$$\vec{\mu} = \{\mu_A|I = 1,2,\dots,N\}\,. \tag{31}$$

We write $\mu_{I;i}$ to denote the number of boxes in the i^{th} column of the diagram μ_I. We will also make use the identity [**?**, **?**]

$$\begin{aligned}
&\sum_{(i,j)\in\mu} e^{i\epsilon_1+j\epsilon_2} + \sum_{(i',j')\in\mu'} e^{\epsilon_+-(i'\epsilon_1+j'\epsilon_2)} \\
&-(1-e^{\epsilon_1})(1-e^{\epsilon_2}) \sum_{(i,j)\in\mu} \sum_{(i',j')\in\mu'} e^{(i-j')\epsilon_1+(j-j')\epsilon_2} \\
&= \sum_{(i,j)\in\mu} e^{-(\mu'^{\mathrm{T}}_j-i)\epsilon_1+(\mu_i-j+1)\epsilon_2} + e^{2\epsilon_+} \sum_{(i',j')\in\mu'} e^{(\mu^{\mathrm{T}}_{j'}-i')\epsilon_1-(\mu'_{i'}-j'+1)\epsilon_2}\,.
\end{aligned} \tag{32}$$

with $\epsilon_\pm = \epsilon_1 \pm \epsilon_2$.

We also use the identity

$$\mathcal{N}_{\nu,\mu}(Q;\mathfrak{q},\mathfrak{t}) := \prod_{l,p=1}^{\infty} \frac{1-Q\mathfrak{q}^{\nu_l-p}\mathfrak{t}^{\mu_p^{\mathrm{T}}-l+1}}{1-Q\mathfrak{q}^{-p}\mathfrak{t}^{-l+1}} \tag{33}$$

$$= \prod_{(l,p)\in\nu} \left(1-Q\mathfrak{q}^{\nu_l-p}\mathfrak{t}^{\mu_p^{\mathrm{T}}-l+1}\right) \prod_{(l,p)\in\mu} \left(1-Q\mathfrak{q}^{-\mu_l+p-1}\mathfrak{t}^{-\nu_p^{\mathrm{T}}+l}\right) \tag{34}$$

$$= \mathcal{N}_{\nu^{\mathrm{T}},\mu^{\mathrm{T}}}(Q;\mathfrak{t},\mathfrak{q})\,. \tag{35}$$

0.1.5 Schur and Skew-Schur Functions

The Schur functions $s_\mu(x)$ are functions which depend on a given young diagram μ. They form an orthogonal basis for the ring of symmetric functions Λ. The basis is orthogonal with respect to the Hall inner product:

$$\langle s_\eta(x), s_\nu(x)\rangle = \delta_{\eta\nu}\,. \tag{36}$$

An explicit representation is given by

$$s_\lambda(\mathbf{x}) = \frac{\det_{AB} x_A^{\lambda_B+N-B}}{\det_{AB} x_A^{N-B}}\,, \tag{37}$$

which are orthogonal with respect to the measure

$$\langle s_\lambda, s_\mu\rangle := \oint d\mu(\mathbf{x}) s_\lambda(\mathbf{x}) s_\mu(\mathbf{x}) = \delta_{\lambda,\mu}\,, \tag{38}$$

$$\oint d\mu_{U(N)}(\mathbf{z}) = \oint d\mu(\mathbf{z}) = \frac{1}{N!}\oint_{|z_A|=1}\prod_{A=1}^{N}\frac{dz_A}{2\pi \mathrm{i} z_A}\prod_{A\neq B}\left(1-\frac{z_A}{z_B}\right)\,. \tag{39}$$

Furthermore

$$s_{\lambda\otimes\mu}(x) = s_\lambda(x)s_\mu(x) = \sum_\eta c^\eta_{\lambda\mu}s_\eta(x)\,. \tag{40}$$

where

$$c^\mu_{\nu\eta} = c^{\mu^{\mathrm{T}}}_{\nu^{\mathrm{T}}\eta^{\mathrm{T}}} \tag{41}$$

are Littlewood-Richardson coefficients. The skew Schur functions may then be defined by

$$\langle s_{\lambda/\mu}(x), s_\nu(x)\rangle = \langle s_\lambda(x), s_\mu(x)s_\nu(x)\rangle \tag{42}$$

and $s_\eta(x)\equiv s_{\eta/\varnothing}(x)$. From (40) we have the identity

$$s_{\lambda_1\otimes\cdots\otimes\lambda_n}(x) = \prod_{i=1}^n s_{\lambda_i}(x) = \sum_{\eta_1,\ldots,\eta_{n-1}} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2}c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right)s_{\eta_{n-1}}(x)\,. \tag{43}$$

Hence,

$$\left\langle s_{(\lambda_1\otimes\cdots\otimes\lambda_n)/\mu}(x), s_\nu(x)\right\rangle = \langle s_{\lambda_1\otimes\cdots\otimes\lambda_n}(x), s_\mu(x)s_\nu(x)\rangle \tag{44}$$

$$= \sum_{\eta_1,\ldots,\eta_{n-1},\sigma} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2}c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right)c^\sigma_{\mu\nu}\left\langle s_{\eta_{n-1}}(x), s_\sigma(x)\right\rangle \tag{45}$$

$$= \sum_{\eta_1,\ldots,\eta_{n-1}} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2}c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right)c^{\eta_{n-1}}_{\mu\nu} \tag{46}$$

$$= \sum_{\eta_1,\ldots,\eta_{n-1},\rho} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2}c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right)c^{\eta_{n-1}}_{\mu\rho}\langle s_\rho(x), s_\nu(x)\rangle\,, \tag{47}$$

where we understand $\eta_0 := \varnothing$ and $c^{\nu}_{\varnothing\mu} = 1$. By the non-degeneracy of $\langle\cdot,\cdot\rangle$ we have

$$s_{(\lambda_1\otimes\cdots\otimes\lambda_n)/\mu}(x) = \sum_{\eta_1,\ldots,\eta_{n-1},\rho} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) c^{\eta_{n-1}}_{\mu\rho} s_\rho(x) \tag{48}$$

The skew Schur function may be equivalently expressed as

$$s_{\mu/\nu}(x) = \sum_\eta c^{\mu}_{\nu\eta} s_\eta(x)\,. \tag{49}$$

Therefore (48) may be written as

$$s_{(\lambda_1\otimes\cdots\otimes\lambda_n)/\mu}(x) = \sum_{\eta_1,\ldots,\eta_{n-1}} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) s_{\eta_{n-1}/\mu}(x)\,. \tag{50}$$

The skew Schur functions satisfy the Cauchy identities

$$\sum_\eta s_{\eta/\lambda}(x)s_{\eta/\mu}(y) = \prod_{l,p=1}^{\infty}(1-x_ly_p)^{-1}\sum_\eta s_{\mu/\eta}(x)s_{\lambda/\eta}(y)\,, \tag{51}$$

$$\sum_\eta s_{\eta^{\mathrm{T}}/\lambda}(x)s_{\eta/\mu}(y) = \prod_{l,p=1}^{\infty}(1+x_ly_p)\sum_\eta s_{\mu^{\mathrm{T}}/\eta}(x)s_{\lambda^{\mathrm{T}}/\eta^{\mathrm{T}}}(y)\,, \tag{52}$$

$$s_{\mu/\nu}(Qx) = Q^{|\mu|-|\nu|}s_{\mu/\nu}(x)\,. \tag{53}$$

For Schur polynomials of type A_1 the Littlewood-Richardson coefficients are, of course, simply

$$s_{2j}s_{2j'} = \sum_{2j''=2|j-j'|}^{2j+2j'} s_{2j''}\,, \tag{54}$$

with $2j, 2j', 2j'' \in \mathbb{N}$.

0.1.6 Spiridonov-Warnaar Inversion Formula

Define

$$\delta(x,y;T) := \frac{\Gamma_e(Tx^{\pm1}y^{\pm1})}{\Gamma_e(T^2)\Gamma_e(x^{\pm2})}\,, \tag{55}$$

and consider the integral

$$f(z) = \kappa\oint\frac{dw}{4\pi \mathrm{i} w}\delta(w,z;T)\hat{f}(w)\,, \tag{56}$$

such that $\max\{|p|,|q|\} < |T|^2 < 1$. A consequence of the Spiridonov-Warnaar theorem is that [**?**, **?**]

$$\hat{f}(w) = \kappa\oint_{C_w}\frac{dz}{4\pi \mathrm{i} z}\delta(z,w;T^{-1})f(z)\,, \tag{57}$$

where C_w is a deformation of the unit circle that includes the poles at $z = T^{-1}w^{\pm 1}$ but excludes those at $Tw^{\pm 1}$. Note that, if $\lim_{p,q\to 0}(f, \hat{f}, T) := (\tilde{f}, \tilde{\hat{f}}, T)$ is smooth, (57) implies

$$\begin{aligned} \tilde{f}(z) &= \oint \frac{dw}{4\pi \mathrm{i} w} \frac{(1-T^2)(1-w^{\pm 2})}{(1-Tw^{\pm 1}z^{\pm 1})} \tilde{\hat{f}}(w) \\ \implies \tilde{\hat{f}}(w) &= \oint_{C_w} \frac{dz}{4\pi \mathrm{i} z} \frac{(1-T^{-2})(1-z^{\pm 2})}{(1-T^{-1}w^{\pm 1}z^{\pm 1})} \tilde{f}(z) \,. \end{aligned} \tag{58}$$

0.2 Lie Groups, Lie Algebras and Representations

0.2.1 $SU(N)$

Highest weight, irreducible representations $\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}$ of $SU(N)$ may be labelled by a Young diagram λ (27) of length $\ell(\lambda) = N$ with λ_N. The relations between the Dynkin labels of the representation and the Young diagram is

$$d_A = \lambda_A - \lambda_{A+1} \,, \quad \lambda_A = \sum_{i=A}^{N-1} d_i \,. \tag{59}$$

The conjugate representation $\overline{\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}} = \mathcal{R}_{(d_{N-1},d_{N-2},\ldots,d_1)}$ is therefore associated to the Young diagram $\overline{\lambda}$ with $\overline{\lambda}_A = \sum_{r=A}^{N-1} d_{N-r} = \sum_{r=A}^{N-1}(\lambda_{N-r} - \lambda_{N-r+1}) = \lambda_1 - \lambda_{N-A+1}$.

The characters for the representation $\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}$ are given by Schur polynomials (37) for the Young diagram λ that specifies the representation

$$\chi_{(d_1,d_2,\ldots,d_{N-1})}(\mathbf{x}) = s_\lambda(\mathbf{x}) \tag{60}$$

with $\lambda_N = 0$ and $\prod_{A=1}^N x_A = 1$. We also often abuse notation and denote these characters simply by their Dynkin labels $\chi_{(d_1,d_2,\ldots,d_{N-1})}(\mathbf{x}) \equiv [d_1, d_2, \ldots, d_{N-1}]$. The representation labelled by λ has dimension

$$|\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}| = s_\lambda(\mathbf{1}) = \prod_{1\leqslant A<B\leqslant N} \frac{\lambda_A - \lambda_B - A + B}{-A+B} \,. \tag{61}$$

The characters are orthogonal with respect to the Haar measure of $SU(N)$

$$\oint d\mu_{SU(N)}(\mathbf{z}) = \oint d\mu(\mathbf{z}) \delta\left(\prod_{A=1}^N z_A - 1\right) \,, \tag{62}$$

with $d\mu$ defined in (39). One fact that we will often use is, that for any class function $f : SU(N) \to \mathbb{C}$ that is also invariant under the Weyl group of $SU(N)$ we can write

$$\oint d\mu_{SU(N)}(\mathbf{z}) f(\mathbf{z}) = \oint_{|z_A|=1} \prod_{A=1}^{N-1} \frac{dz_A}{2\pi \mathrm{i} z_A} \prod_{1\leqslant A<B\leqslant N} \left(1 - \frac{z_A}{z_B}\right) f(\mathbf{z}) \,. \tag{63}$$

0.2.2 $SO(2N)$

We will also sporadically make use of $SO(2N)$ characters. These are given by

$$\hat{s}_\lambda(\mathbf{x}) = \frac{\det_{AB}\left(x_A^{\lambda_B+N-B} - x_A^{-\lambda_B-N+B}\right)}{\det_{AB}(x_A - x_A^{-1})^{N-B}}\,. \tag{64}$$

where the $\lambda_1 \geqslant \lambda_2 \geqslant, \ldots, \geqslant |\lambda_N| \geqslant 0$ is related to the Dynkin labels $(d_1, d_2, \ldots, d_N)$ by

$$\lambda_A = -d_{N-1}\delta_{A,N} + \frac{1}{2}(d_N + d_{N-1}) + \sum_{n=A}^{N-2} d_n\,. \tag{65}$$

0.3 Algebraic Geometry

The aim of this section is largely to present the relevant definitions that will be required in order to understand the Hilbert series and the relation to supersymmetric theories. Complete proofs and derivations are beyond the scope of this appendix but can be found in [**?**].

0.3.1 Definitions

Definition 1. *(Graded) Rings* A ring R is an abelian group with a multiplication operation $(a, b) \mapsto ab$ and an identity element 1 satisfying

- Associativity: $a(bc) = (ab)c$
- Identity: $1a = a1 = a$
- Distributivity : $(b + c)a = ba + ca$

for all $a, b, c \in R$. R is commutative if $ab = ba$ and for the remainder of the text we will always take R to be commutative.

A graded ring is a ring R together with a direct sum decomposition

$$R = \oplus_{i=0}^{\infty} R_i \quad \text{such that } R_i R_j \subset R_{i+j} \text{ for all } i, j\,. \tag{66}$$

Definition 2. *Invertible elements* An invertible element in a ring R is an element $a \in R$ such that $ab = 1$ with $b = a^{-1} \in R$ unique.

Definition 3. *Fields* A field $\mathbb{K}$ is a ring in which every nonzero element $a \in K$ is invertible. For example $\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Definition 4. *Polynomials* A polynomial f in $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$ is a finite linear combination of monomials of the form $f = \sum_\alpha f_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The set of all such polynomials is denoted by $\mathbb{K}[x_1, \ldots, x_n]$. In particular $\mathbb{K}[x_1, \ldots, x_n]$ forms a $\mathbb{Z}^n$-graded ring where the grading is provided by degree.

Definition 5. *Ideals* An ideal in a ring R is an additive subgroup $I \subset R$ such that $ap \in I$ for all $a \in R$ and $p \in I$. An ideal is said to be generated by a subset $S \subset R$ if every element $p \in I$ can be written as

$$p = \sum_i a_i s_i \tag{67}$$

for $a_i \in R$ and $s_i \in S$. In particular, for $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$

$$\langle f_1, \ldots, f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i | h_1, \ldots, h_s \in \mathbb{K}[x_1, \ldots, x_n] \right\} \tag{68}$$

is an ideal of $\mathbb{K}[x_1, \ldots, x_n]$ which we call the ideal generated by $f_1, \ldots, f_s$.

Definition 6. *Radical Ideals* An ideal I is radical if $f^m \in I$ if for some $m \in \mathbb{Z}^+$ implies $f \in I$.

Definition 7. *Radical of an Ideal* The radical $\sqrt{I}$ of an ideal $I \in \mathbb{K}[x_1, \ldots, x_n]$ is

$$\sqrt{I} = \{f | f^m \in I \, \text{for some} \, m \in \mathbb{Z}^+\} \,. \tag{69}$$

It follows $I \subset \sqrt{I}$ (since $f \in I$ implies that $f^1 \in I$). Moreover the radical of an ideal I is always a radical ideal.

Definition 8. *(Graded) Modules* Let R be a ring, an R-module M is an abelian group with an action of R, $R \times M \to M$ satisfying

- Associativity: $a(bm) = (ab)m$
- Identity: $1m = m1 = m$
- Distributivity : $(b + c)m = bm + cm$, $a(m + n) = am + an$

for all $a, b \in R$ and $m, n \in M$ Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring. Then a graded module over R is a module M with a decomposition

$$M = \bigoplus_{-\infty}^{\infty} M_i \quad \text{such that } R_i M_j \subset M_{i+j} \text{ for all } i, j \tag{70}$$

Definition 9. *Affine spaces* Let $\mathbb{K}$ be a field and $n \in \mathbb{Z}^+$. The n-dimensional affine space over $\mathbb{K}$ is

$$\mathbb{K}^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in \mathbb{K}\} \,. \tag{71}$$

Definition 10. *Affine Varieties* Let $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$ then the affine variety defined by $f_1, \ldots, f_s$ is

$$\mathfrak{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n | f_i(a_1, \ldots, a_n) = 0 \, \forall \, i \in \{1, \ldots, s\}\} \subset \mathbb{K}^n \tag{72}$$

Definition 11. *Irreducible Varieties* An affine variety $\mathbf{V} \subset \mathbb{K}^n$ is said to be *irreducible* if whenever $\mathbf{V}$ is written in the form $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$, where $\mathbf{V}_1$ and $\mathbf{V}_2$ are affine varieties, then either $\mathbf{V}_1 = \mathbf{V}$ or $\mathbf{V}_2 = \mathbf{V}$.

0.3.2 Algebra-Geometry Correspondence

Given an affine variety $\mathbf{V} \subset \mathbb{K}^n$ (this will play the role of the moduli space of supersymmetric vacua $\mathbf{M}$ (**??**)). To this variety we can associate to it an ideal $\mathfrak{I}(\mathbf{V}) \subset \mathbb{K}[x_1, ..., x_n]$ (this will play the role of an operator algebra, such as chiral ring, etc) using the following map

$$\mathfrak{I}(\mathbf{V}) = \{f \in \mathbb{K}[x_1, ..., x_n] \mid f(a_1, \dots, a_n) = 0 \ \forall \ (a_1, \dots, a_n) \in \mathbf{V}\}\,, \tag{73}$$

the proof that this indeed gives an ideal can be found in [**?**].

On the other hand, given an ideal $I \subset \mathbb{K}[x_1, ...x_n]$ we can associate to it a variety $\mathfrak{V}(I)$ (see Definition 10)

$$\mathfrak{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in \mathbb{K}^n | f_i(a_1, \dots, a_n) = 0 \,\forall\, i \in \{1, \dots, s\}\} \subset \mathbb{K}^n\,. \tag{74}$$

It can be shown that $\mathfrak{V}(\mathfrak{I}(\mathbf{V})) = \mathbf{V}$.

We therefore have two maps which provide a correspondence and affine varieties. However, in general, this correspondence is not one to one. For example let us consider the family of distinct ideals $\langle x^n \rangle$ (with $n \in \mathbb{N}$) in $\mathbb{C}[x]$, then it's easy to see that to the map (74) associates to each of them the same variety, namely $\mathfrak{V}(x^n) = \{0\}$.

Theorem 1. *The (Strong) Nullstellensatz Let* $\mathbb{K}$ *be algebraically closed and let* $I \subset \mathbb{K}[x_1, \dots, x_n]$ *be an ideal, then*

$$\mathfrak{I}(\mathfrak{V}(I)) = \sqrt{I}\,. \tag{75}$$

In particular this theorem tells us that the maps (73)-(74) provide us a one to one correspondence between affine varieties and radical ideals. This dictionary can be extended reformulating in algebraic terms geometrical problems, see Table 1. A particular useful class of ideals is provided by the so called *prime ideals*

Definition 12. *Prime Ideals* An ideal $I \subset \mathbb{K}[x_1, ..., x_n]$ is a *prime ideal* if $f, g \in \mathbb{K}[x_1, ...x_n]$ and $fg \in I$, implies either $f \in I$ or $g \in I$.

It's easy to prove that every prime ideal is also a radical ideal, and that moreover there is a one-to-one correspondence between *irreducible varieties* and prime ideals via the maps (73) & (74)[**?**].

Definition 13. *Noetherian Rings* A ring R is called *Noetherian* if there is no infinite ascending sequence of left (or right) ideals. Therefore, given any chain of left (or right) ideals,

$$I_1 \ \subseteq \ \dots \ I_{k-1} \ \subseteq \ I_k \ \subseteq \ I_{k+1} \ \dots\,, \tag{76}$$

there exists an n such that

$$I_n = I_{n+1} = \dots \tag{77}$$

Algebra	$\leftrightarrow$	Geometry
Radical ideal $I = \sqrt{I}$		Affine variety $\mathbf{V} = \mathfrak{V}(I)$
Addition of ideals $I + J$		Intersection of varieties $\mathfrak{V}(I) \cap \mathfrak{V}(J)$
Product of ideals IJ		Union of varieties $\mathfrak{V}(I) \cup \mathfrak{V}(J)$
Prime ideal I		Affine irreducible variety $\mathfrak{V}(I)$
Regular sequence		Complete intersection

Table 1: *Summary of the algebra-geometry correspondence.*

When R is a *Noetherian ring*, using the *Lasker-Noether theorem*, we can always decompose and ideal of R as an irredundant intersection of a finite set $\{J_i\}$ of primary ideals [?], this procedure is called *primary decomposition.* In particular we can take into account the ideal I and we get

$$I = \bigcap_i J_i \ . \tag{78}$$

We can then consider the different radical ideals $\sqrt{J_i}$ associated to each of the primary ideals in (78). When the J_i are all prime ideals we have a one to one correspondence between affine irreducible complex variates and the radical ideals $\sqrt{J_i}$. This implies that the corresponding complex variety, can be written as

$$\mathbf{F} = \bigcup_i \mathfrak{V}(\sqrt{J_i})\,, \tag{79}$$

in this way the algebraic approach turns out to be very powerful since it provides a systematic way to decompose the F-flat moduli space $\mathbf{F}$ of a theory into different irreducible branches.

Remarkably we can also establish when the space $\mathbf{F}$ is a complete intersection.

Definition 14. *Regular Sequence* A sequence of non-constant polynomials $P_1, P_2, ...P_r$ is said to be *regular* if for all $i = 1, ..., r$, P_i is not a zero divisor modulo the partial ideal $(P_1, ..., P_{r-i})$.

Given an ideal generated by a regular sequence the following theorem holds [?]

Theorem 2. *Given the ring of polynomials $R = \mathbb{C}[x_1, ...x_n]$ and the ideal $I \subset R$ then the algebraic variety associated to the quotient ring R/I is a complete intersection if and only if I is generated by a regular sequence of homogeneous polynomials.*

Therefore if the ideal is generated by a *regular sequence* of polynomials then we can use letter counting for the computation of the corresponding Hilbert series. For application of the above theorem in a different context see [?], we also refer the interested reader to Appendix A of that paper for a more detailed discussion related to this issue.

0.3.3 Hilbert Series

We are now in a position to define the Hilbert function and Hilbert series.

Definition 15. *Hilbert function & Hilbert series* Let M be a finitely generated graded module over $\mathbb{K}[x_1, \ldots, x_n]$ graded by degree $\deg x_i = 1$ for all $i = 1, \ldots, n$. The Hilbert function is defined to be

$$\mathrm{HF}_M(s) := \dim_{\mathbb{K}} M_s\,, \tag{80}$$

note that HF is finite in every degree. The Hilbert series is then defined in a formal power series as the generating function for the Hilbert function

$$\mathrm{HS}(\tau; M) := \sum_{s=0}^{\infty} \mathrm{HF}_M(s)\tau^s = \mathrm{Tr}_M\, \tau^E \tag{81}$$

and here $E : M_s \to \mathbb{Z}$ stands for the operator which computes the degree of an element $y \in M_s$ $E(y) = s$. If M is generated by d homogeneous elements of degrees $E_1, \ldots, E_d$ the Hilbert series takes the form

$$\mathrm{HS}(\tau; M) = \frac{P(\tau)}{\prod_{i=1}^{d} (1 - \tau^{E_i})} \tag{82}$$

where $P(\tau)$ is a polynomial in the formal parameter τ with integer coefficients.

For example, let $M = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring with degrees $E(x_i) = 1$. The Hilbert series is simply

$$\mathrm{HS}(\tau; M) = \frac{1}{(1-\tau)^n}\,. \tag{83}$$

The Hilbert function is therefore

$$\mathrm{HF}_M(s) = \frac{\prod_{i=0}^{s-1} (n+i)}{s!}\,. \tag{84}$$

The results (81) & (82) may also be generalised to the case of rings with $\mathbb{Z}^b$ grading $M = \bigoplus_{s_1, s_2, \ldots, s_b} M_{s_1 s_2 \ldots s_b}$ in which case the result takes the general form

$$\mathrm{HS}(\tau_1, \tau_2, \ldots, \tau_b; M) = \mathrm{Tr}_M \prod_{l=1}^{b} \tau_l^{E^{(l)}} = \frac{P(\tau_1, \ldots, \tau_b)}{\prod_{i=1}^{d} \left(1 - \prod_{l=1}^{b} \tau_l^{E_i^{(l)}}\right)}\,. \tag{85}$$

If we let $\mathbf{M} = \mathfrak{V}(I)$ be the algebraic variety defined as the set of zeros of the ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ then the Hilbert series of $R = \mathbb{K}[x_1, \ldots, x_n]/I$ (as written in the form (82)) allows us to compute the dimension of $\mathbf{M}$ and the degree of $\mathbf{M}$ (the number of intersection points of $\mathbf{M}$ with $d = \dim_{\mathbb{K}} \mathbf{M}$ generic hyperplanes) $\deg \mathbf{M}$ as

$$\dim_{\mathbb{K}} \mathbf{M} = \dim_{\mathbb{K}} M = \text{Order of pole at } \tau = 1 \text{ of } \mathrm{HS}(\tau; M) = d\,, \tag{86}$$

$$\deg \mathbf{M} = P(1)\,. \tag{87}$$

0.3.4 Example Computations with *Macaulay2*

Let us present some example code for *Macaulay2* [?]. We firstly focus on the computations of the Hilbert series performed in the introduction ??.

Note that *Macaulay2* represents elements of $\mathbb{C}$ via floating point approximations. $\mathbb{C}$ is an example of an *inexact field*. *Macaulay2* uses Gröbner bases and the algorithms it uses do not work over $\mathbb{C}$. Therefore when performing practical computations we must instead work over $\mathbb{Q}$ (or any other field of characteristic zero will do) while using *Macaulay2* before tensoring any final result with $\mathbb{C}$. Tensoring with $\mathbb{C}$ can destroy some properties which may hold over $\mathbb{Q}$. For example an ideal that is prime over $\mathbb{Q}$ may fail to be prime after tensoring with $\mathbb{C}$.[1] Hence the *Macaulay2* output should always be read with the above caveat in mind; nevertheless, our main object of interest, the Hilbert series is expected to be independent of the above.

The first example that we presented there was the most simple case of $\mathcal{N} = 4$ SYM with gauge group $G = U(1)$ (free theory). This computation can of course simply be performed by hand. Nevertheless one can also use *Macaulay2* by inputting $F = R/I$ with $R = \mathbb{C}[X, Y, Z]$ and I trivial.

```
i1 : R=QQ[X,Y,Z,Degrees=>{{1,0,0,0},{0,1,0,0},{0,0,1,0}}]

i2 : hilbertSeries(R)
```

This of course outputs the Hilbert series

```
                   1
o2 = ------------------------
     (1 - T )(1 - T )(1 - T )
           2       1       0
```

One then takes the output and sets $\tau_{1,2,3} = T_{0,1,2}$.

For the second case presented in the introduction of the $\mathcal{N} = 1$ $G = U(1)$ gauge theory with $F = R/I$, $R = \mathbb{C}[X, Y, Z]$ and $I = \langle XY, XZ, YZ\rangle$ an example *Macaulay2* input is

```
i1 : R=QQ[X,Y,Z,Degrees=>{{1,0,0,1},{0,1,0,-1},{0,0,1,0}}]

i2 : I=ideal(X*Y,X*Z,Y*Z)

i3 : hilbertSeries(R/I)

i4 : primaryDecomposition I

i5 : radical I

i6 : isPrime I
```

[1] See e.g. [?].

This outputs

```
                                          -1
     1 - T T  - T T T  - T T T   + 2T T T
          0 1    0 2 3    1 2 3      0 1 2
o3 = -------------------------------------
                                     -1
         (1 - T )(1 - T T  )(1 - T T )
               2       1 3        0 3

o4 = {ideal (X, Y), ideal (X, Z), ideal (Y, Z)}

o5 = monomialIdeal (X*Y, X*Z, Y*Z)

o6 = false
```

Setting $\tau_{1,2,3} = T_{0,1,2}$ and $z = T_3$ identifies $o3$ with (**??**).

We can also compute for more complicated non-abelian cases. Of course the computations become more and more complex with $\dim G$. Let us take the case of $\mathcal{N} = 4$ SYM with gauge group $G = SU(2)$. In this case $F = R/I$ with $R = \mathbb{C}[X, Y, Z]$ where now $X, Y, Z \in \mathfrak{su}(2)$

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & -Y_1 \end{pmatrix}, \quad Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1 \end{pmatrix} \tag{88}$$

The ideal is $I = \langle [X,Y], [X,Z], [Y,Z] \rangle$ coming from the superpotential $\operatorname{tr} X[Y,Z]$. We input

```
i1 : R=QQ[X1, X2, X3, Y1, Y2, Y3, Z1, Z2, Z3,
Degrees=>{{1, 0, 0, 0}, {1, 0, 0, -2}, {1, 0, 0, 2},
{0, 1, 0, 0}, {0, 1, 0, -2}, {0, 1, 0, 2},
{0, 0, 1, 0}, {0, 0, 1, -2}, {0, 0, 1, 2}}]

i2 : I=ideal(-*Y3*Z2 + *Y2*Z3, *Y3*Z1 - *Y1*Z3, -*Y2*Z1 + *Y1*Z2,
*X3*Z2 -*X2*Z3, -*X3*Z1 + *X1*Z3, *X2*Z1 - *X1*Z2,
 -*X3*Y2 + *X2*Y3, *X3*Y1 - *X1*Y3, -*X2*Y1 + *X1*Y2)

i3 : hilbertSeries(R/I)

i4 : primaryDecomposition I

i5 : radical I

i6 : isPrime I
```

This outputs (here we have suppressed the output for the Hilbert series $o3$ in the raw form to save on space, we present it in the simple form below)

```
o4 = {ideal (Y3*Z2 - Y2*Z3, X3*Z2 - X2*Z3, Y3*Z1 - Y1*Z3,
Y2*Z1 - Y1*Z2, X3*Z1 - X1*Z3, X2*Z1 - X1*Z2, X3*Y2 - X2*Y3,
X3*Y1 - X1*Y3, X2*Y1 - X1*Y2)}

o5 = ideal (Y3*Z2 - Y2*Z3, X3*Z2 - X2*Z3, Y3*Z1 - Y1*Z3,
Y2*Z1 - Y1*Z2, X3*Z1 - X1*Z3, X2*Z1 - X1*Z2, X3*Y2 - X2*Y3,
X3*Y1 - X1*Y3, X2*Y1 - X1*Y2)

o6 = true
```

After converting the raw output for the Hilbert series as before we have

$$\begin{aligned}\mathrm{HS}(\tau_1,\tau_2,\tau_3,z;F) = \mathrm{PE}[\chi_2(z)(\tau_1+\tau_2+\tau_3)]\Big\{&1-(\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3)\chi_2(z)\\ &+(\chi_4(z)+\chi_2(z))\tau_1\tau_2\tau_3+(\tau_1+\tau_2)(\tau_1+\tau_3)(\tau_2+\tau_3)\\ &-\chi_2(z)(\tau_1+\tau_2+\tau_3)\tau_1\tau_2\tau_3+\tau_1^2\tau_2^2\tau_3^2\Big\}\end{aligned} \tag{89}$$