Exact Results for $\mathcal{N} = 1$ Theories of Class $\mathcal{S}_k$

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Outline

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Motivation

Ambition: Understand the non-perturbative dynamics of QFTs

This is a **hard** problem (QCD confinement, Yang-Mills mass-gap, etc) $\implies$ consider simplified models $\implies$ supersymmetry

- For $\mathcal{N} \geq 2$ a lot is understood $\implies$ $\boxed{\text{Exact results!}}$
- But, $\mathcal{N} \geq 2$ is far away from 'real-world physics'

$\mathcal{N} = 1$ is a lot closer e.g. $\mathcal{N} = 1$ SQCD exhibits [Seiberg '94]:

- Color confinement
- Chiral symmetry breaking
- Conformal phase

$\mathcal{N} = 1$ SUSY $\implies$ more control over theory than $\mathcal{N} = 0$ - still potentially constraining enough to understand the dynamics non-perturbatively $\implies$ exact results?.

Exact Results

By 'exact result' we mean any quantity that we compute that is valid for any value of the parameter space. They give us a window into the non-perturbative dynamics of a theory. Some examples for $\mathcal{N} = 2$

- Moduli space **M** of SUSY vacua
  - Coulomb branch **CB** - Seiberg-Witten Theory: Exact computation of low-energy effective action
  - Higgs branch **HB** - Quantum mechanically exact
- Exact evaluation (localisation) of path integral on various manifolds: $\mathbb{S}^4$, $\mathbb{S}^1 \times \mathbb{S}^3$ [Pestun '07] [Romelsberger '05] [...]
- Nekrasov's instanton counting [Nekrasov '02]: path integral $Z_{\text{inst}} = \sum_{K>0} q^K Z_K$ over instanton moduli space
- 2d/4d correspondences: AGT, TQFT-Index, ... [Alday, Gaiotto, Tachikawa '10] [Gadde, Pomoni, Rastelli, Razamat '10]

$$\boxed{\text{What about exact results for } \mathcal{N} = 1\text{?}}$$

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Class $\mathcal{S}$ Theories

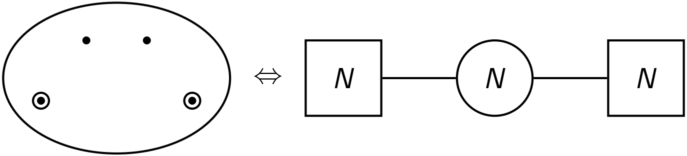
A large class of 4d $\mathcal{N}=2$ theories: [Gaiotto '09] [Gaiotto, Moore, Neitzke '09]

- 6d $\mathcal{N}=(2,0)$ SCFT (stack of N M5 branes) on $\mathbb{R}^4 \times \mathcal{C}$ where $\mathcal{C}$ is a compact Riemann surface

	$\mathbb{C}^2$				$\mathcal{C}$		$\mathbb{C}^2_\perp$				$\mathbb{S}^1$
N M5	–	–	–	–	–	–	·	·	·	·	·

- We take Area$(\mathcal{C}) \to 0 \implies$ 4d $\mathcal{N}=2$ theory on $\mathbb{R}^4$

$\mathcal{C}$ = four punctured sphere $\Leftrightarrow$ $\mathcal{N}=2$ SQCD



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Class $\mathcal{S}_k$

Natural generalisation of Class $\mathcal{S}$

Instead: Compactify a 6d $\mathcal{N} = (1,0)$ theory on $\mathcal{C}$.

- The 6d $\mathcal{N} = (1,0)$ theory we use is a $\mathbb{Z}_k$ orbifold of the 6d $\mathcal{N} = (2,0)$ theory [Gaiotto, Razamat '15]

- The $\mathbb{Z}_k$ breaks supersymmetry by 1/2, leaving $\mathcal{N} = 1$ in 4d

	$\mathbb{C}^2$				$\mathcal{C}$		$\mathbb{C}^2_\perp$				$\mathbb{S}^1$
N M5	–	–	–	–	–	–	·	·	·	·	·
$\mathbb{Z}_k$	·	·	·	·	·	·	$\times$	$\times$	$\times$	$\times$	·

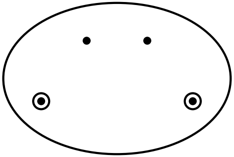
$$\mathbb{Z}_k : (v_\perp, w_\perp) \mapsto (e^{2\pi i/k} v_\perp, e^{-2\pi i/k} w_\perp)$$

- Class of 'nice' 4d $\mathcal{N} = 1$ theories classified by punctured Riemann surfaces $\mathcal{C}$

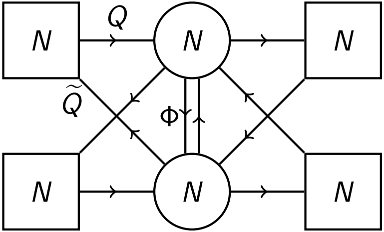
Example

$\mathcal{C}$ = four punctured sphere

- For $k = 2$ - $\mathcal{N} = 1$ superconformal quiver



$\Leftrightarrow$



- $W \simeq \left(Q \Phi \widetilde{Q} - \widetilde{Q} \Phi Q \right) - \frac{i\tau}{8\pi^2} \operatorname{tr} W^{\alpha} W_{\alpha}$

- $SU(2)_{R_{\mathcal{N}=2}} \times U(1)_{r_{\mathcal{N}=2}} \xrightarrow{\mathbb{Z}_k} U(1)_r \times U(1)_t$

- Global Symmetry:
  $SU(2,2|1) \times U(1)_t \times U(1)_{\gamma}^{k-1} \times U(1)_{\beta}^{k-1} \times SU(N)^{2k} \times U(1)_{\alpha}^{2}$

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Moduli space

$\mathcal{N} = 1$ theories can have a *moduli space* of susy vacua

$$\mathbf{M} = \left\{ \Phi, Q, \widetilde{Q} \middle| V(\Phi, Q, \widetilde{Q}) = 0 \right\} / (\text{gauge transformations})$$

Kähler manifold **M** param'd by top components of $\frac{1}{2}$-BPS multiplets $\overline{\mathcal{B}}_{r(0,0)}$ of $SU(2,2|1)$.
For class $\mathcal{S}_k$ we have **distinct** Higgs and Coulomb branches [TB, A. Pini, E. Pomoni (to appear)].

$\boxed{\text{For generic } \mathcal{N} = 1 \text{ theories such distinction is not possible!}}$

The non-anomalous $U(1)_r \times U(1)_t$ global symmetries means these branches are separate and cannot mix.

Higgs Branch

$U(1)_r$ is the $\mathcal{N}=1$ R-symmetry. $U(1)_t$ inherited from the broken $\mathcal{N}=2$ R-Symmetry group

Field	E	$U(1)_r$	$U(1)_t$
Q	1	$2/3$	$1/2$
$\widetilde{Q}$	1	$2/3$	$1/2$
Φ	1	$2/3$	-1

$$\boxed{\mathbf{HB} = \mathbf{M}|_{\Phi=0}\,, \qquad E = \frac{3}{2}r = 2q_t}$$

- Gauge invariants: $Q_i\widetilde{Q}_j$, $\det Q_i$, $\det \widetilde{Q}_i$
- As with $\mathcal{N}=2$ SUSY **HB** is *exact*
- On the other hand only Kähler (rather than hyperKähler).

Coulomb Branch

$$\boxed{\mathbf{CB} = \mathbf{M}\big|_{Q=\tilde{Q}=0}\,, \quad E = \frac{3}{2}r = -q_t}$$

- Gauge invariants: $u_n = \mathrm{tr}(\Phi_1 \ldots \Phi_k)^n$ and $B_i = \det \Phi_i$
- As with $\mathcal{N}=2$ Coulomb branch superpotential has quantum corrections encoded by Seiberg-Witten curve $\Sigma \hookrightarrow T^*\mathcal{C}$ fibred over **CB**

Curve can be defined for $\mathcal{N}=1$ theories [Intrilgator, Seiberg '94]

$$\Sigma: \quad z^{kN} + \sum_{l=1}^{N} z^{k(N-l)} \phi_{kl}(t; u_n, B_i) = 0$$

[Coman, Pomoni, Taki, Yagi '15], [TB, Pomoni (to appear)]

$$W_{eff} = \frac{i\tau_{eff}}{8\pi} W_\alpha W^\alpha\,, \quad \tau_{eff} = \frac{\oint_B \lambda}{\oint_A \lambda}\,, \quad \lambda = \frac{v}{t}dt$$

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Supersymmetric Index ($\mathbb{S}^3 \times \mathbb{S}^1$ partition function)

Pick $\mathcal{Q} = \widetilde{\mathcal{Q}}_{\dot{-}} \implies \boxed{2\{\mathcal{Q}, \mathcal{Q}^\dagger\} = E - 2j_2 - 3r/2 = 0}$

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{j_1+j_2+\frac{r}{2}-\frac{2}{3}q_t} q^{-j_1+j_2+\frac{r}{2}-\frac{2}{3}q_t} t^{q_t} e^{-2\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}$$

Witten index graded by fugacities for maximal commutant in $SU(2,2|1) \times U(1)_t \times H$

- Counts all short multiplets modulo recombination to long ones

$$\mathcal{I}(\mathcal{S}) = \chi_{\mathcal{S}}\,, \quad \mathcal{I}(\mathcal{A}) = \mathcal{I}(\mathcal{S}) + \mathcal{I}(\mathcal{S}') = 0$$

Use localisation ($k = N = 2$)

$$\mathcal{I} = (p;p)^2(q;q)^2 \int \frac{dz_1 dz_2}{(4\pi i)^2} \prod_{i=1}^{k=2} \frac{\Gamma_e(\sqrt{t} z_i^{\pm 1})^2\, \Gamma_e(\frac{pq}{t} z_i^{\pm 1} z_{i-1}^{\pm 1})}{\Gamma_e(z_i^{\pm 2})}$$

2d/4d relation: Equal to 2d TQFT correlator on $\mathcal{C}$ [Gaiotto, Razamat '15] $\implies$ comes 'for free' from 6d construction

Supersymmetric Localisation

Localisation principle: [Witten '88]

We would like to compute observables:

$$\langle \mathcal{O}[\phi] \rangle = \int\limits_{\mathbf{C}(\mathcal{M})} [\mathcal{D}\phi]\, e^{-S[\phi]}\, \mathcal{O}[\phi]$$

Assume $\exists\, \mathfrak{Q}$ such that $\mathfrak{Q}S = \mathfrak{Q}\mathcal{O} = 0$ & $\mathfrak{Q}^2 = 0$. We can then deform $S \to S + t\mathfrak{Q}V$

$$\langle \mathcal{O}[\phi] \rangle_t = \int\limits_{\mathbf{C}(\mathcal{M})} [\mathcal{D}\phi]\, e^{-S[\phi] - t\mathfrak{Q}V[\phi]}\, \mathcal{O}[\phi]\,, \quad t \in \mathbb{R}$$

$\boxed{\text{The answer is independent of } t!}$ we can take $t \to \infty$ $\mathfrak{Q}V = 0$

$$\langle \mathcal{O}[\phi] \rangle_t = \int\limits_{\mathbf{C}_{BPS}} da \frac{e^{-S[a]}\, \mathcal{O}[a]}{\mathrm{SDet}\left[\frac{\delta^2 \mathfrak{Q}V[a]}{\delta a^2}\right]}\,, \quad \mathbf{C}_{BPS} = \{a \in \mathbf{C}(\mathcal{M}) | \mathfrak{Q}V = 0\}$$

Limits of the Index

First defined for **all** $\mathcal{N}=2$ SCFTs [Gadde, Rastelli, Razamat, Yan '11]. The limits can be defined for '*all*' class $\mathcal{S}_k$ theories [TB, Pini, Pomoni (to appear)]

- Hall-Littlewood $p, q \to 0$, t fixed $(2q_t = E + j_2\,, j_1 = 0)$

$$\mathrm{HL} = \mathrm{Tr}_{HL}(-1)^F t^{q_t} = \int \frac{dz_1 dz_2}{(4\pi i)^2} \prod_{i=1}^{k=2} \frac{(1-z_i^{\pm 2})(1-t z_{i-1}^{\pm 1} z_i^{\pm 1})}{(1-\sqrt{t} z_i^{\pm 1})^2}$$

- Coulomb $t, p, q \to 0$, $T = pq/t$, $V = p/q$ fixed $(E + 2j_2 + \frac{r}{2} + \frac{4q_t}{3} = 0)$

$$\mathcal{I}^C = \mathrm{Tr}_C(-1)^F T^{E+j_2} V^{j_1} = \int \frac{dz_1 dz_2}{(4\pi i)^2} \prod_{i=1}^{k=2} \frac{(1-z_i^{\pm 2})}{(1-T z_i^{\pm 1} z_{i-1}^{\pm 1})}$$

Can also define Macdonald $p/\sqrt{t} \to 0$, q fixed and Schur $q = t$ limits. $\boxed{\text{Such limits **do not** exist for generic } \mathcal{N}=1 \text{ theories!}}$

Outline

Instantons for Class $\mathcal{S}_k$

Partition function over Instantons

- Nekrasov instanton partition function: $Z_{\text{inst}} = \sum_n q^K Z_K$ for $\mathcal{N} = 2$ theories. $Z_{\mathbb{S}^4} \sim \int da Z_{\text{pert}} |Z_{\text{inst}}|^2$
- Instantons in susy gauge theories embed in string theory $\Longrightarrow$ can compute Z_K's for class $\mathcal{S}_k$

$Z_K \sim \int_{\mathbf{M}_{\text{inst}}} e^{-S[A_{\text{inst}}]}$ ['t Hooft '76]

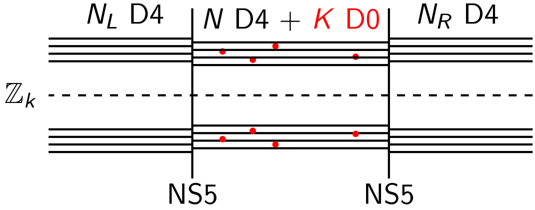
String theory construction

- Embed in Type-IIA [Witten '97]

	$\mathbb{C}^2$				$\mathbb{R}^2$		$\mathbb{S}^1$	$\mathbb{R}^3_\perp$		
N D4	–	–	–	–	·	·	–	·	·	·
NS5	–	–	–	–	–	–	·	·	·	·
$\mathbb{Z}_k$	·	·	·	·	×	×	·	×	×	·
K D0	·	·	·	·	·	·	–	·	·	·

ADHM Construction

$$\boxed{K \text{ instantons in a D}p\text{-brane} \equiv K\, D(\text{p-4})-branes}$$



- D0 Higgs branch = ADHM moduli space of instantons

$$\mathbf{HB}^{\text{D0}} \cong \mathbf{M}^{\text{D4}}_{\text{inst}}$$

  [Douglas '95 '96] [Witten '95]

- 0d susy matrix model with two supercharges $\mathcal{Q}_+, \widetilde{\mathcal{Q}}_+$

Computation of Z_{inst}

- Partition function Z_{D0} for sigma model living on K D0-branes equals Z_K
- $2\times$ T-duality D0 $\to$ D2 and we compute $\mathbb{S}^1 \times \mathbb{S}^1$ partition function (supersymmetric index) of the 2d theory

$$Z_K = Z_{\text{D0}} = \lim_{\beta_1,\beta_2\to 0} \mathcal{I}_{\text{D2}}$$

- $\mathcal{Q}_+$ is preserved by the $\mathbb{Z}_k$ orbifold

$$\mathcal{I}_{\text{D2}} = \text{Tr}(-1)^F q^{H_-} v^{2j_1} t^{2j_D} \mathbf{x}^{\mathbf{f}}$$

Count gauge & $\mathbb{Z}_k$ invariant operators which have $\{\mathcal{Q}_+, \mathcal{Q}_+^\dagger\} = 0$ [TB, Pomoni '17]

Orbifolding to Class $\mathcal{S}_k$

$$\boxed{\mathcal{I}_{\mathrm{D2}} = \int [d\mu_G(\mathbf{z})]\,\mathrm{PE}\left[\sum_{\{\text{multiplets}\}} i^{\text{orb}}_{multiplet}(q,\mathbf{z},\dots)\right]}$$

- Take the limit

$$\lim_{\beta_1,\beta_2\to 0}\mathcal{I}_{\mathrm{D2}} \propto \prod_{i=1}^{k}\int\prod_{I=1}^{K_i} du_{i,I}\prod_{I=1}^{K_i}\frac{u'_{ii,IJ}\prod_{j\neq i}\prod_{J=1}^{K_j}\left(u_{ij,IJ}-2\epsilon_+\right)}{\prod_{j=1}^{k}\prod_{J=1}^{K_j}\left(u_{ij,IJ}+\epsilon_1\right)\left(u_{ij,IJ}+\epsilon_2\right)}$$
$$\times\prod_{j=1}^{k}\prod_{I=1}^{K_i}\prod_{A=1}^{N}\frac{\left(u_{i,I}-\widetilde{m}_{L,j,A}\right)\left(u_{i,I}-\widetilde{m}_{R,j,A}\right)}{\left(u_{i,I}-\widetilde{a}_{j,A}-\epsilon_+\right)\left(u_{i,I}-\widetilde{a}_{j,A}+\epsilon_+\right)}$$

- We expect this to be equal to the integration over the moduli space of $\{K_1,K_2,\dots,K_k\}$ instantons for this $\mathcal{N}=1$ theory

Conclusions and Future Directions

Class $\mathcal{S}_k$ provides a host of 'non-generic' $\mathcal{N}=1$ theories which lend themselves to computing exact results

- They have **distinct** Higgs and Coulomb branches
- Several 'nice' simple limits of the index $\mathcal{I}$ & TQFT structure
- Via orbifold projection $\implies$ Z_{inst} instanton partition function

Future Directions:

- Localisation on other manifolds e.g. $\mathbb{S}^2 \times T^2$ [Closset, Shamir '14]. $\mathbb{S}^4$ partition function? and relation with deconstruction of $\mathcal{N}=(1,1)$ LST [Hayling, Panerai, Papageorgakis '18]
- Z_{inst} equal to $\mathcal{W}_{kN}$ conformal blocks - Hints towards AGT for class $\mathcal{S}_k$? [Mitev, Pomoni '17]
- $\mathcal{S}_k$ Index $\mathcal{I} =$ (TQFT Correlator) but explicit construction of TQFT currently unknown
- Gauging of discrete symmetries $\implies$ new 'exotic' $\mathcal{N}=1,2,3$ theories: $\mathcal{I}$, Hilbert series, ... [TB, Pini, Pomoni '18]

Appendix: Limits of the Index and Hilbert Series

HL index: counts $Q, \widetilde{Q}, \widetilde{\mathcal{Q}}_{\dot{+}}\overline{\Phi} = \overline{\lambda} \implies$ these have $2q_t = E + j_2, j_1 = 0 \implies$ Higgs branch **HB** type operators (plus the fermions). For genus zero Lagrangian theories in class $\mathcal{S}_k$

$$\mathrm{HL} = \mathrm{Hilb}(t; \textbf{HB})$$

True for all genus zero in class $\mathcal{S}$ [Gadde, Rastelli, Razamat, Yan '11]

Coulomb index: counts $\Phi \implies$ these have $E + 2j_2 + \frac{r}{2} + \frac{4q_t}{3} = 0$ and are Coulomb branch **CB** type operators and we have

$$\mathcal{I}^C = \mathrm{Hilb}(T; \textbf{CB})$$

The Hilbert series of the variety **M** counts independent holomorphic functions

$$\mathrm{Hilb}(t; \mathbf{M}) = \mathrm{Tr}_{\mathbf{M}}\, t^E\,, \qquad \begin{aligned} \mathrm{Hilb}(t; \mathbb{C}^2) =& 1 + 2t + 3t^2 + \cdots = (1-t)^{-2} \\ & 1\,, [z_1, z_2]\,, [z_1^2, z_2^2, z_1 z_2] \end{aligned}$$