Let us now discuss how to obtain the instanton partition function for the 5d $\mathcal{N}=1^*$ theory on $\mathbb{S}^1\times\mathbb{C}^2/\mathbb{Z}_p$. This is a more general result of the one computed in Section **??** where we take the more generic $\mathbb{Z}_p$ action on the coordinates (θ,z_1,z_2) of $\mathbb{S}^1\times\mathbb{C}^2$ to be

$$\mathbb{Z}_p:(\theta,z_1,z_2)\mapsto(\theta,\gamma^{q_1}z_1,\gamma^{q_2}z_2)\,, \tag{1}$$

with $\gamma^p=1$, $q_1,q_2,p\in\mathbb{Z}$ and $gcd(q_1,p)=gcd(q_2,p)=1$. This is generalisation of the results computed in [**?**] and of Section **??** which were made with the specialisation $q_1=0$, $q_2=1$, $p=k$. In the Nekrasov instanton counting the Ω-deformation forces the instantons to sit at the origin $z_1=z_2=0$. The instantons therefore sit at the fixed point of the action (1). Note that in order to preserve the supercharges used in the instanton localisation computation we have to turn on a background R-current. We *assume* that the projections do not change the localisation structure. Under that assumption the Nekrasov partition function takes the form of an 'orbifolded' Nekrasov partition function $\widetilde{Z}^{\rm orb}_{\rm nek}=Z^{\rm orb}_{\rm cl}Z^{\rm orb}_{\rm inst}$. We have to project onto states left invariant by the orbifold action

$$\alpha_{A,I}\mapsto\alpha_{A,I}-\frac{2\pi{\rm i}A}{p}\,,\quad m\mapsto m-\frac{2\pi{\rm i}(q_1+q_2)}{2p}\,, \tag{2}$$

$$\epsilon_1\mapsto\epsilon_1+\frac{2\pi{\rm i}q_1}{p}\,,\quad \epsilon_2\mapsto\epsilon_2+\frac{2\pi{\rm i}q_2}{p}\,. \tag{3}$$

Using the same conventions as those in Appendix **??** we decompose the vector spaces W, V (for a fixed momentum mode around the $\mathbb{S}^1$) with respect to their $\mathbb{Z}_p$ grading

$$W=\bigoplus_{A=1}^{p}W_A\,,\quad V=\bigoplus_{A=1}^{p}V_A\,, \tag{4}$$

of dimension $\dim_{\mathbb{C}}W_A=N_A$ and $\dim_{\mathbb{C}}V_A=k_A$. Moreover, we also take the index $A=1,\ldots,p$ modulo p. Under the $\mathbb{Z}_p$ action the ADHM data transforms as

$$B^{(l)}\mapsto\gamma^{q_l}B^{(l)}\,,\quad P\mapsto P\,,\quad Q\mapsto\gamma^{q_1+q_2}Q\,, \tag{5}$$

where $q_3:=-q_1-q_2$, $q_4:=0$. In order to have a non-trivial result, following [**?**], we also quotient by a $\mathbb{Z}_p\hookrightarrow U(k)$ corresponding to (**??**) with $g=\operatorname{diag}\left(\gamma\mathbb{I}_{k_1},\gamma^2\mathbb{I}_{k_2}\ldots,\right)\in U(k)$. This breaks $U(k)\to\prod_{A=1}^{p}U(k_A)$ with $k=\sum_{A=1}^{p}k_A$. The surviving components are

$$B^{(l)}_A\in\operatorname{Hom}\left(V_A,V_{A+q_l}\right)\,,\quad P_A\in\operatorname{Hom}\left(V_A,V_A\right)\,, \tag{6}$$

$$Q_A\in\operatorname{Hom}\left(V_A,V_{A+q_1+q_2}\right)\,. \tag{7}$$

The ADHM equations $\mu^{(i)}_{\mathbb{C},A}=\mu_{\mathbb{R},A}=0$ are given by performing the projections to (**??**) and (**??**). The ramified instanton moduli space is then given by

$$\mathbf{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}}=\left\{B^{(l)}_A\,,\,P_A\,,\,Q_A\middle|\mu^{(i)}_{\mathbb{C},A}=\mu_{\mathbb{R},A}=0\right\}\,. \tag{8}$$

The fixed points after the $\mathbb{Z}_p$ quotient are still labelled by N-tuples of Young diagrams $\vec{\mu}$ which we now label by $\vec{\mu} = \{\mu_{A,I}\}$. We choose bases

$$W_A = \mathrm{span}_{\mathbb{C}} \{w_{A,I} | I = 1, \ldots, N_A\} \tag{9}$$

$$V_{A+q_1 i+q_2 j} = \mathrm{span}_{\mathbb{C}} \left\{ v^{(i,j)}_{A+q_1 i+q_2 j,I} \middle| I = 1, \ldots, N_{A+q_1 i+q_2 j}, (i,j) \in \mu_{A,I} \right\} . \tag{10}$$

The torus action acts by

$$w_{A,I} \mapsto e^{\alpha_{A,I}} w_{A,I} , \quad v^{(i,j)}_{A+q_1 i+q_2 j,I} \mapsto e^{(1-i)\epsilon_1 + (1-\epsilon_2)} v^{(i,j)}_{A+q_1 i+q_2 j,I} . \tag{11}$$

The fixed point configuration is given by the orbifold projection of (**??**), namely

$$B^{(1)}_A v^{(i,j)}_{A,I} = v^{(i+1,j)}_{A+q_1,I} , \quad B^{(2)}_A v^{(i,j)}_{A,I} = v^{(i,j+1)}_{A+q_2,I} , \quad P_A w_{A,I} = v^{(1,1)}_{A,I} , \tag{12}$$

$$Q_A = B^{(3)}_A = B^{(4)}_A = 0 . \tag{13}$$

The dimension of V_B is then given by

$$k_B = k_B(\vec{\mu}) = \dim_{\mathbb{C}} V_B = \sum_{A=1}^{p} \sum_{I=1}^{N_A} \sum_{(i,j)\in y^{(I)}_{A,B}} 1 , \tag{14}$$

where $y^{(I)}_{A,B}$ is given by

$$y^{(I)}_{A,B} = \{(i,j) | (i,j) \in \mu_{A,I}, A + q_1 i + q_2 j = B \bmod p\} . \tag{15}$$

For $q_1 = 0$ and $q_2 = 1$ equation (11) reduces to (2.37) of [**?**]. We demonstrate an explicit example with $p = 3$, $q_1 = 1$, $q_2 = -2$, $N_1 = N_2 = N_3 = 1$, $\mu_{1,1} = \{4, 3, 2\}$, $\mu_{2,1} = \{2, 2\}$ and $\mu_{3,1} = \{3, 2\}$ in Figure 1. Because $N_A = 1$ we drop the I indices, for example $v^{(i,j)}_{A,1} = v^{(i,j)}_A$. $k_1 = 7$, $k_2 = 5$ and $k_3 = 6$; in agreement with (11). Finally the character of $T\mathbf{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}}$ at the fixed point $\vec{\mu}$ is given by the $\mathbb{Z}_p$ invariant part of (**??**), namely

$$\chi_{\vec{\mu}} \left(T\mathbf{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}} \right) := \chi^{\mathrm{Vec}}_{\vec{\mu},\mathbb{Z}_p} + \chi^{\mathrm{Hyp}}_{\vec{\mu},\mathbb{Z}_p} , \tag{16}$$

with

$$\begin{aligned} \chi^{\mathrm{Vec}}_{\vec{\mu},\mathbb{Z}_p} = \sum_{t\in\mathbb{Z}} e^{\frac{2\pi t}{r}} \sum_{B=1}^{p} \Big[& W^*_B V_B + e^{2\epsilon_+} V^*_{B+q_1+q_2} W_B - V^*_B V_B \\ & + e^{\epsilon_1} V^*_{B+q_1} V_B + e^{\epsilon_2} V^*_{B+q_2} V_B - e^{2\epsilon_+} V^*_{B+q_1+q_2} V_B \Big] , \end{aligned} \tag{17}$$

$$\begin{aligned} \chi^{\mathrm{Hyp}}_{\vec{\mu},\mathbb{Z}_p} = - \sum_{t\in\mathbb{Z}} e^{\frac{2\pi t}{r}} e^{m-\epsilon_+} \sum_{B=1}^{p} \Big[& W^*_{B-q_1-q_2} V_B + e^{2\epsilon_+} V^*_B W_B - V^*_{B-q_1-q_2} V_B \\ & + e^{\epsilon_1} V^*_{B-q_2} V_B + e^{\epsilon_2} V^*_{B-q_1} V_B - e^{2\epsilon_+} V^*_B V_B \Big] . \end{aligned} \tag{18}$$

$$\begin{tikzcd}
 & v_1^{(4,1)} & & \\
 & v_3^{(3,1)} \arrow[u, "B_3^{(1)}"] \arrow[r, "B_3^{(2)}"] & v_1^{(3,2)} & \\
 & v_2^{(2,1)} \arrow[u, "B_2^{(1)}"] \arrow[r, "B_2^{(2)}"] & v_3^{(2,2)} \arrow[u, "B_3^{(1)}"] \arrow[r, "B_3^{(2)}"] & v_1^{(2,3)} \\
 & v_1^{(1,1)} \arrow[u, "B_1^{(1)}"] \arrow[r, "B_1^{(2)}"] & v_2^{(1,2)} \arrow[u, "B_2^{(1)}"] \arrow[r, "B_2^{(2)}"] & v_3^{(1,3)} \arrow[u, "B_3^{(1)}"] \\
w_1 \arrow[ur, "P_1"] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v_3^{(2,1)} \arrow[r, "B_3^{(2)}"] & v_1^{(2,2)} \\
 & v_2^{(1,1)} \arrow[u, "B_2^{(1)}"] \arrow[r, "B_2^{(2)}"] & v_3^{(1,2)} \arrow[u, "B_3^{(1)}"] \\
w_2 \arrow[ur, "P_2"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & v_2^{(3,1)} & \\
 & v_1^{(2,1)} \arrow[u, "B_1^{(1)}"] \arrow[r, "B_1^{(2)}"] & v_2^{(2,2)} \\
 & v_3^{(1,1)} \arrow[u, "B_3^{(1)}"] \arrow[r, "B_3^{(2)}"] & v_1^{(1,2)} \arrow[u, "B_1^{(1)}"] \\
w_3 \arrow[ur, "P_3"] & &
\end{tikzcd}$$

Figure 1: *Example of the fixed point structure for* $p = 3$, $q_1 = 1$, $q_2 = -2$, $N_1 = N_2 = N_3 = 1$.

As before conjugation reverses the signs of the exponents. We also abused the notation and identified the vector spaces and their characters

$$V_A = \sum_{C,D=1}^{p} \sum_{I=1}^{N_{q_1C+q_2D-A}} \sum_{(pi-C+1,pj-D+1)\in y^{(I)}_{q_1C+q_2D-A,A}} e^{\alpha_{q_1C+q_2D-A,I}+(C-pi)\epsilon_1+(D-pj)\epsilon_2}\,, \tag{19}$$

$$W_A = \sum_{I=1}^{N_{p-A+1}} e^{\alpha_{p-A+1,I}}\,, \tag{20}$$

under the orbifold $\mathbb{Z}_p : V_A, W_A \mapsto \gamma^A V_A, \gamma^A W_A$. At this point it is very important to stress that we understand A, B, C, D to be taken modulo p when and only when they are considered as indices used to label quantities for example $\alpha_{A,I} = \alpha_{A+p,I}$. These quantities are significantly more complicated than those of the case $q_1 = 0$, $q_2 = 1$ of (**??**).

According to the conversion rule (**??**) we can, in principle, compute the partition function

$$\chi_{\vec{\mu}}\left(T\mathbf{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}}\right) \to z^{\mathbb{Z}_p}_{\vec{\mu}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r\right)\,. \tag{21}$$

The instanton partition function then reads

$$Z^{\text{orb}}_{\text{inst}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r; \mathbf{q}_A\right) = \sum_{\vec{\mu}} \left(\prod_{B=1}^{p} q_B^{k_B(\vec{\mu})}\right) z^{\mathbb{Z}_p}_{\vec{\mu}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r\right)\,. \tag{22}$$

In the limit $m = -\epsilon_+$ the supersymmetry of the 5d theory enhances $\mathcal{N} = 1^* \to \mathcal{N} = 2$. Correspondingly the instanton partition drastically simplifies and one can see that

$$\chi^{\text{Vec}}_{\vec{\mu},\mathbb{Z}_p}\Big|_{m=-\epsilon_+} = -\left(\chi^{\text{Hyp}}_{\vec{\mu},\mathbb{Z}_p}\right)^*\Big|_{m=-\epsilon_+}\,. \tag{23}$$

From (**??**) we conclude that

$$z^{\mathbb{Z}_p}_{\vec{\mu}}\left(\vec{\alpha}, -\epsilon_+, \epsilon_1, \epsilon_2, r\right) \equiv 1\,. \tag{24}$$

So,

$$Z^{\text{orb}}_{\text{inst}}\left(\vec{\alpha}, -\epsilon_+, \epsilon_1, \epsilon_2, r; \mathbf{q}_A\right) = \sum_{\vec{\mu}} \left(\prod_{B=1}^{p} q_B^{k_B(\vec{\mu})}\right)\,. \tag{25}$$