Abstract

0.1 Introduction

4d theories with $\mathcal{N} \geqslant 1$ supersymmeries on manifolds of the form $S^1 \times L(q,p)$ where $L(r,p) = S^3/\mathbb{Z}_p$ is a Lens space have been stuided in [?, ?, ?, ?, ?, ?]. It is also possible to study 3d theories with $\mathcal{N} \geqslant 2$ on said Lens spaces.

0.2 The Lens spaces $L(q_1, q_2, q_3; p)$

Let p, q_1, q_2, q_3 be integers such that $gcd(q_i, p) = 1$ for all $i = 1, 2, 3$. The Lens space $L(q_1, q_2, q_3; p) \subset \mathbb{C}^3$ is the quotient $S^5/\mathbb{Z}_p$ generated by the free $U(1)^3 \supset \mathbb{Z}_p \circlearrowright \mathbb{C}^3$ action

$$\mathbb{Z}_p : (z_1, z_2, z_3) \mapsto (\gamma^{q_1} z_1, \gamma^{q_2} z_2, \gamma^{q_3} z_3) \tag{1}$$

where $\gamma = e^{2\pi i/p}$ and $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. When there is no ambiguity, in order to save on notation, we will often write $L_p \equiv L(q_1, q_2, q_3; p)$. The Lens space satisfies:

$$\pi_i\left(L_p\right) = \begin{cases} \mathbb{Z}_p & i = 1 \\ \pi_i\left(S^5\right) & i \geqslant 2\,. \end{cases}, \quad H_i\left(L_p\right) = \begin{cases} \mathbb{Z} & i = 0, 5 \\ \mathbb{Z}_p & i = 1, 3 \\ 0 & i \neq 0, 1, 3, 5 \end{cases}. \tag{2}$$

Notice that the above depend only on p and are independent of the q_i. They are of physical interest because $\pi_1(L_p) = \mathbb{Z}_p$ is non-trivial and therefore the Lens space index is sensitive to the line operator spectrum of the theory.

0.3 The $S^1 \times L(q_1, q_2, q_3; p)$ partition function of 6d $\mathcal{N} = (2, 0)$ SCFTs

0.3.1 Lens space index

In this paper we will be interested in theories with $\mathcal{N} = (2, 0)$ supersymmetry in six dimensions. These theories (at the level of the local operator spectrum) are in one to one correspondence with the finite subgroups of $SU(2)$ [?, ?]. It is therefore common to label them of type $\mathfrak{g} = ADE$. Compactification of the theory of type $\mathfrak{g}$ on a circle with radius $\beta \to 0$ gives rise to the 5d $\mathcal{N} = 2$ SYM theory with gauge algebra $\mathfrak{g}$. It is conjectured that (atleast at the level of BPS states) that the 6d $\mathcal{N} = (2, 0)$ theory of type $\mathfrak{g}$ on $S^1_\beta \times S^5$ is equivalent to the 5d $\mathcal{N} = 2$ theory with gauge algebra $\mathfrak{g}$ and gauge coupling $g^2_{\text{YM}} = 2\pi\beta$. It is therefore believed that the S^5 partition function for the 5d theory is equal to the superconformal index of the $\mathcal{N} = (2, 0)$ theory. The parameters ω_i, $\mu/2$ are identified with the squashing parameters of the S^5 and the hypermultiplet mass [?] respectively.

The $\mathcal{N} = (2,0)$ superconformal algebra is $\mathfrak{osp}(8|4)$. Representations of $\mathfrak{osp}(8|4)$ are labelled by the Cartans of the maximal bosonic subalgebra $\mathfrak{so}(2,6)\oplus\mathfrak{usp}(4)$. The Cartans are $(E, h_1, h_2, h_3, R_1, R_2)$, where E corresponds to dilatations, (h_1, h_2, h_3) to 2-plane rotations in $\mathbb{R}^6$ and R_1, R_2 to $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$ Cartans. The theory has 16 Poincaré supercharges $\mathcal{Q}^{R_1R_2}_{h_1h_2h_3}$ with $-8h_1h_2h_3 = 2E = 1$ and 16 conformal supercharges with $-8h_1h_2h_3 = 2E = -1$.

We will define the superconformal index with respect to the supercharge $\mathcal{Q} := \mathcal{Q}^{++}_{---}$ and its conjugate $\mathcal{Q}^\dagger = \mathcal{Q}^{--}_{+++}$ which have $\frac{R_1+R_2}{2} = E = -h_i = \frac{1}{2}$ and $\frac{R_1+R_2}{2} = E = -h_i = -\frac{1}{2}$ respectively. Under the action (1) the supercharges transform as

$$\mathcal{Q}^{R_1R_2}_{h_1h_2h_3} \mapsto \gamma^{q_1h_1+q_2h_2+q_3h_3}\mathcal{Q}^{R_1R_2}_{h_1h_2h_3} \tag{3}$$

therefore, in order to preserve $\mathcal{Q}, \mathcal{Q}^\dagger$, we have to specialise to

$$q_1 + q_2 + q_3 = 0 \bmod 2p\,, \tag{4}$$

for simplicity in this paper we will drop the modulo condition and simply write $q_1+q_2+q_3 = 0$, this also means that we have to take $gcd(q_1+q_2, p) = 1$. In that case the action (1) preserves $\mathcal{Q}^{R_1R_2}_{+++}$ and their conjugates. Note that one could, in order to preserve more supersymmetries, also decide to turn on non-trivial background R-symmetry currents, however we will not pursue this here.

The Lens space index is then defined to be [?, ?, ?, ?, ?, ?]

$$\begin{aligned}\mathcal{I}_p\left(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}\right) &= \mathrm{Tr}_{L_p}(-1)^F e^{-\beta\left(\delta+h_1+h_2+h_3+\frac{3R_1+3R_2}{2}\right)-\beta(a_1h_1+a_2h_2+a_3h_3)-\beta\mu\frac{R_2-R_1}{2}}\\ &= \mathrm{Tr}_{L_p}(-1)^F e^{-\beta\delta}\mathfrak{q}_1^{h_1+\frac{R_1+R_2}{2}}\mathfrak{q}_2^{h_2+\frac{R_1+R_2}{2}}\mathfrak{q}_3^{h_3+\frac{R_1+R_2}{2}}\mathfrak{p}^{R_2-R_1}\end{aligned} \tag{5}$$

with $a_1 + a_2 + a_3 = 0$, or equivalently $\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 = e^{-3\beta}$. The trace is taken over the Hilbert space on $L_p = L(q_1, q_2, q_3; p)$ in the radial quantisation. We also defined

$$\mathfrak{q}_i := e^{-\beta(a_i+1)} = e^{-\beta\omega_i}\,, \quad \mathfrak{p} := e^{-\beta\mu/2}\,. \tag{6}$$

Here the $a_i = \omega_i - 1$ are related to $U(1)^3 \circlearrowright \mathbb{C}^3$ corresponding to

$$(z_1, z_2, z_3) \mapsto (e^{ia_1}z_1, e^{ia_2}z_2, e^{ia_3}z_3)\,. \tag{7}$$

The Lens space index (5) receives contribution only from states satisfying

$$\delta := 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} = E - h_1 - h_2 - h_3 - 2R_1 - 2R_2 = 0\,. \tag{8}$$

Since (5) receives contribution only from states with $\delta = 0$ it is; considered as a formal power series in the $\mathfrak{q}_i$ & $\mathfrak{p}$, independent of β.

0.3.2 Casimir energy

Recall that the $p = 1$ superconforma index (5) is expected to be equal to the $S^1 \times L_1$ partition function $\mathcal{Z}_{p=1}$ up to an overall Casimir energy factor [?, ?, ?, ?]. We expect that such a structure also holds for $p > 1$, namely,

$$\mathcal{Z}_p(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}) = e^{-\beta E_p(\mathfrak{g})} \mathcal{I}_p(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}) \,. \tag{9}$$

For $p = 1$ it is conjectured that the Casimir energy $E_{p=1}(\mathfrak{g})$ is equal to an equivariant integral of the anomaly polynomial of the $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g}$ [?]

$$E_{p=1}(\mathfrak{g}) = -\int A_8(\mathfrak{g}) \,. \tag{10}$$

For $\mathfrak{g} = \mathfrak{u}(1)$ (we use this notation to denote the free theory on a single M5-brane) and $\mathfrak{g} = ADE$ the anomaly polynomial is respetively given by [?, ?, ?]

$$A_8(\mathfrak{u}(1)) = \frac{1}{48}\left[p_2(NM) - p_2(TM) + \frac{1}{4}(p_1(NM) - p_1(TM))^2\right] , \tag{11}$$

$$A_8(\mathfrak{g}) = r_{\mathfrak{g}} A_8(\mathfrak{u}(1)) + d_{\mathfrak{g}} h^\vee_{\mathfrak{g}} \frac{p_2(NM)}{24} \,. \tag{12}$$

Here $r_{\mathfrak{g}}$, $d_{\mathfrak{g}}$, $h^\vee_{\mathfrak{g}}$ denote the rank, dimension and dual coexter number of $\mathfrak{g}$; for $\mathfrak{g} = A_{N-1}$ these are $N-1$, N^2-1 and N respectively. NM and TM denote the normal and tangent bundles to the six-manifold $M = S^1 \times S^5$ and $p_j(V)$ denotes the j^{th} Pontryagin class of the bundle V. The integral (10) may be promoted to an equivariant integral with respect to the $U(1)^4 \circlearrowright S^1 \times L_{p=1}$ action [?]

$$E_{p=1}(\mathfrak{u}(1)) = -\frac{1}{16\beta} - \frac{1}{48\omega_1\omega_2\omega_3}\left[\frac{1}{8}\prod_{l_1,l_2\in\{1,-1\}}(\omega_1 + l_1\omega_2 + l_2\omega_3) + 2\mu^4 - \mu^2\sum_{i=1}^{3}\omega_i^2\right] , \tag{13}$$

$$E_{p=1}(\mathfrak{g}) = r_{\mathfrak{g}} E_{p=1}(\mathfrak{u}(1)) - d_{\mathfrak{g}} h^\vee_{\mathfrak{g}} \frac{\left(\frac{9}{4} - \beta^2\mu^2\right)^2}{24\beta} , \tag{14}$$

we also still have the condition $\sum_{i=1}^3 \omega_i = 3$. Assuming that $|a_i| < 1$ we can therefore write

$$e^{-\beta E_{p=1}(\mathfrak{g})} = e^{-\beta \frac{-2r_{\mathfrak{g}} + d_{\mathfrak{g}} h^\vee_{\mathfrak{g}}\left(\frac{9}{4} - \mu^2\right)^2}{24}} e^{-\beta r_{\mathfrak{g}}} \tag{15}$$

0.3.3 The Lens space index of the free tensor multiplet

Let us begin with the most simple example - the free tensor multiplet. In this case we can actually letter counting so it should serve as a good example. The Lens space index is given by projecting onto $\mathbb{Z}_p$ invariant quantities. We can write

$$\mathcal{I}_p(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}) = \text{PE}\left[\frac{1}{|\mathbb{Z}_p|}\sum_{\gamma\in\mathbb{Z}_p} i_p(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}; \gamma)\right] . \tag{16}$$

Letters	E	h_1	h_2	h_3	R_1	R_2	$i_p(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p};\gamma)$
ϕ	2	0	0	0	1	0	$\mathfrak{p}^{-1}\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}$
ϕ	2	0	0	0	0	1	$\mathfrak{p}\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}$
λ^{++}_{++-}	5/2	1/2	1/2	$-1/2$	1/2	1/2	$-\gamma^{\frac{q_1+q_2-q_3}{2}}\mathfrak{q}_1\mathfrak{q}_2$
λ^{++}_{+-+}	5/2	1/2	$-1/2$	1/2	1/2	1/2	$-\gamma^{\frac{q_1-q_2+q_3}{2}}\mathfrak{q}_1\mathfrak{q}_3$
λ^{++}_{-++}	5/2	$-1/2$	1/2	1/2	1/2	1/2	$-\gamma^{\frac{-q_1+q_2+q_3}{2}}\mathfrak{q}_2\mathfrak{q}_3$
$\partial\lambda=0$	7/2	1/2	1/2	1/2	1/2	1/2	$\gamma^{\frac{q_1+q_2+q_3}{2}}\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$
$\partial_{z_{i=1,2,3}}$	1	$1,0,0$	$0,1,0$	$0,0,1$	0	0	$\gamma^{q_1}\mathfrak{q}_1,\ \gamma^{q_2}\mathfrak{q}_2,\ \gamma^{q_3}\mathfrak{q}_3$

Table 1: *The abelian tensor multiplet has a scalar in the fundamental of* $\mathfrak{so}(5)$*, 16 fermions* $\lambda^{R_1R_2}_{h_1h_2h_3}$ *with* $8h_1h_2h_3=-1$ *and a self-dual 3-form* $H=\star H$*.*

i_p is the single letter index and is computed by enumerating the single letters, weighted by the orbifold action (1). These are listed in Table 1. The sum over $\mathbb{Z}_p$ projects onto $\mathbb{Z}_p$-invariant quantities. After enforcing the condition (4) the Lens space index is given by

$$i_p(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p};\gamma)=\frac{\left(\mathfrak{p}+\mathfrak{p}^{-1}\right)\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}-\gamma^{q_1+q_2}\mathfrak{q}_1\mathfrak{q}_2-\gamma^{-q_2}\mathfrak{q}_1\mathfrak{q}_3-\gamma^{-q_1}\mathfrak{q}_2\mathfrak{q}_3+\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}{\left(1-\gamma^{q_1}\mathfrak{q}_1\right)\left(1-\gamma^{q_2}\mathfrak{q}_2\right)\left(1-\gamma^{-q_1-q_2}\mathfrak{q}_3\right)}\,. \tag{17}$$

In order to evaluate the sum (16) we have to evaluate sums of the form

$$S(b)=\frac{1}{p}\sum_{\gamma\in\mathbb{Z}_p}\sum_{n_1,n_2,n_3=0}^{\infty}\gamma^{q_1(n_1-n_3)+q_2(n_2-n_3)+b}\mathfrak{q}_1^{n_1}\mathfrak{q}_2^{n_2}\mathfrak{q}_3^{n_3}\,, \tag{18}$$

with, $b\in\mathbb{Z}$. The sum can be reduced to

$$S(b)=\frac{\sum_{(n_1,n_2,n_3)\in s_b(\eta)}\mathfrak{q}_1^{n_1}\mathfrak{q}_2^{n_2}\mathfrak{q}_3^{n_3}}{(1-\mathfrak{q}_1^p)(1-\mathfrak{q}_2^p)(1-\mathfrak{q}_3^p)}\,, \tag{19}$$

where

$$s_b(\eta)=\{(n_1,n_2,n_3)|\eta+b=0\bmod p\,,\ 0\leqslant n_1,n_2,n_3\leqslant p-1\}\subset\mathbb{N}^3\,, \tag{20}$$

and $\eta:=q_1n_1+q_2n_2+q_3n_3$. Unfortunately, we were unable to further reduce the set s_b for general values of the q_i, p. The proof goes as follows: For $p=1$ the sum (18) is simply the generating function for triplets of natural numbers $(n_1,n_2,n_3)\in\mathbb{N}^3$. On the other hand for $p>1$ (18) is the generating function for triplets $(n_1,n_2,n_3)\in\mathbb{N}^3$ with the orbifold constraint $\eta+b=0\bmod p$. This space is simply given by

$$S=\mathrm{span}_{p\mathbb{N}}\{(n_1,n_2,n_3)|(n_1,n_2,n_3)\in s_b(\eta)\}\,. \tag{21}$$

The generating function for S is given precisely by (19). One can also easily verify (19) via expansion of (18) in `Mathematica` for given any given values of the q_i, p

and b. Then

$$\begin{aligned} \frac{1}{p}\sum_{\gamma\in\mathbb{Z}_p} i_p(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p};\gamma) =& (\mathfrak{p}+\mathfrak{p}^{-1})\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}S(0) - \mathfrak{q}_1\mathfrak{q}_2 S(q_1+q_2) \\ & - \mathfrak{q}_1\mathfrak{q}_3 S(q_2) - \mathfrak{q}_2\mathfrak{q}_3 S(q_1) + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 S(0)\,. \end{aligned} \tag{22}$$

For example, for $p = 3$ and $2q_1 = 2q_2 = -q_3 = 2$ we have

$$\begin{aligned} &\frac{1}{3}\sum_{\gamma\in\mathbb{Z}_3} i_3(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p};\gamma) = \frac{1}{(1-\mathfrak{q}_1^3)(1-\mathfrak{q}_2^3)(1-\mathfrak{q}_3^3)}\Big\{ \left(\mathfrak{p}+\mathfrak{p}^{-1}+\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}\right)\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3} \\ &\times\left(1+\mathfrak{q}_2\mathfrak{q}_3(\mathfrak{q}_2+\mathfrak{q}_3)+\mathfrak{q}_1(\mathfrak{q}_2^2+\mathfrak{q}_2\mathfrak{q}_3+\mathfrak{q}_3^2)+\mathfrak{q}_1^2(\mathfrak{q}_2+\mathfrak{q}_3+\mathfrak{q}_2^2\mathfrak{q}_3^2)\right) \\ &-(\mathfrak{q}_2+\mathfrak{q}_1(1+\mathfrak{q}_1\mathfrak{q}_2^2)+\mathfrak{q}_3+\mathfrak{q}_1\mathfrak{q}_2(\mathfrak{q}_1+\mathfrak{q}_2)\mathfrak{q}_3+(\mathfrak{q}_1^2+\mathfrak{q}_1\mathfrak{q}_2+\mathfrak{q}_2^2)\mathfrak{q}_3^2)(\mathfrak{q}_2\mathfrak{q}_3+\mathfrak{q}_1(\mathfrak{q}_2+\mathfrak{q}_3))\Big\} \end{aligned} \tag{23}$$

$\beta \to 0$ limit Let us define $Q = e^{-\beta}$. We would like to take the $\beta \to 0$ ($Q \to 1$) limit. Following [**?**, **?**] we can write the (16) as

$$\begin{aligned} \mathcal{I}_p\left(Q^{\omega_1},Q^{\omega_2},Q^{\omega_3},Q^{\mu}\right) =& \prod_{r_1,r_2,r_3=0}^{\infty}\left\{ \frac{\prod_{(n_1,n_2,n_3)\in s_{q_1+q_2}(\eta)}\left[\omega_1+\omega_2+\sum_{i=1}^3\omega_i(n_i+pr_i)\right]_Q}{\prod_{(n_1,n_2,n_3)\in s_0(\eta)}\left[\mu+\sum_{i=1}^3\omega_i(n_i+pr_i+\frac{1}{2})\right]_Q}\times\right. \\ & \left.\frac{\prod_{(n_1,n_2,n_3)\in s_{q_2}(\eta)}\left[\omega_1+\omega_3+\sum_{i=1}^3\omega_i(n_i+pr_i)\right]_Q \prod_{(n_1,n_2,n_3)\in s_{q_1}(\eta)}\left[\omega_2+\omega_3+\sum_{i=1}^3\omega_i(n_i+pr_i)\right]_Q}{\prod_{(n_1,n_2,n_3)\in s_0(\eta)}\left[-\mu+\sum_{i=1}^3\omega_i(n_i+pr_i+\frac{1}{2})\right]_Q \prod_{(n_1,n_2,n_3)\in s_0(\eta)}\left[\sum_{i=1}^3\omega_i(n_i+pr_i+1)\right]_Q}\right\}, \end{aligned} \tag{24}$$

where $[n]_Q = (1-Q^n)/(1-Q)$ is the Q-number. The Q number satisfies the property that $\lim_Q [n]_Q = n$. Therefore

$$\begin{aligned} \lim_{\beta\to 0}\mathcal{I}_p\left(Q^{\omega_1},Q^{\omega_2},Q^{\omega_3},Q^{\mu}\right) =& \frac{\prod_{(n_1,n_2,n_3)\in s_0(\eta)}\Gamma_3\left(\mu+\sum_{i=1}^3\omega_i(n_i+\frac{1}{2})|p\vec{\omega}\right)}{\prod_{(n_1,n_2,n_3)\in s_{q_1+q_2}(\eta)}\Gamma_3\left(\omega_1+\omega_2+\sum_{i=1}^3\omega_i n_i|p\vec{\omega}\right)} \\ &\times\frac{\prod_{(n_1,n_2,n_3)\in s_0(\eta)}\Gamma_3\left(-\mu+\sum_{i=1}^3\omega_i(n_i+\frac{1}{2})|p\vec{\omega}\right)\prod_{(n_1,n_2,n_3)\in s_0(\eta)}\Gamma_3\left(\sum_{i=1}^3\omega_i(n_i+1)|p\vec{\omega}\right)}{\prod_{(n_1,n_2,n_3)\in s_{q_2}(\eta)}\Gamma_3\left(\omega_1+\omega_3+\sum_{i=1}^3\omega_i n_i|p\vec{\omega}\right)\prod_{(n_1,n_2,n_3)\in s_{q_1}(\eta)}\Gamma_3\left(\omega_2+\omega_3+\sum_{i=1}^3\omega_i n_i|p\vec{\omega}\right)}, \end{aligned} \tag{25}$$

where $\Gamma_3(z|\vec{\omega})$ is the Barnes triple Gamma function, defined in (97) and $p\vec{\omega} = (p\omega_1, p\omega_2, p\omega_3)$.

0.3.4 Unrefined limits

Chiral algebra limit

The chiral algebra limit is defined by

$$\mu \to \frac{1}{2}(\omega_1 + \omega_2 - \omega_3) = \omega_1 + \omega_2 - \frac{3}{2} = \frac{1}{2} + a_1 + a_2 \,. \tag{26}$$

This is equivalent to $\mathfrak{p}^2 \to \sqrt{\frac{\mathfrak{q}_1\mathfrak{q}_2}{\mathfrak{q}_3}}$. Note that this distinguishes a particular fugacity, however, by permuting $q_i, a_i \leftrightarrow q_j, a_j$ appropriately one may obtain the other limits. The Lens space index is then

$$\begin{aligned}\mathcal{I}_p\left(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \sqrt{\frac{\mathfrak{q}_1\mathfrak{q}_2}{\mathfrak{q}_3}}\right) &= \mathrm{Tr}(-1)^F e^{-\beta(E-R_1)-\beta\left(a_1\left(h_1-h_3+\frac{R_2-R_1}{2}\right)+a_2\left(h_2-h_3+\frac{R_2-R_1}{2}\right)\right)} \\ &= \mathrm{Tr}(-1)^F e^{-\beta(E-R_1)-\beta\left(a_1\left(h_1-h_3+\frac{R_2-R_1}{2}\right)+a_2\left(h_2-h_3+\frac{R_2-R_1}{2}\right)\right)}\end{aligned} \tag{27}$$

We can also consider the further specialisation to

$$\mu = \frac{1}{2}\,, \quad a_1 = a_2 = a_3 = 0\,, \tag{28}$$

or, equivalently, $\mathfrak{q}_i = \mathfrak{p}^4 = e^{-\beta}$. This preserves 8 supercharges $Q^{\pm\pm}_{---}$, $Q^{\pm\pm}_{+++}$ preserved by the Lens space for $p \geqslant 2$. The $\frac{1}{2}$-BPS Lens space index is given by

$$\mathcal{I}_p^{(\frac{1}{2})}\left(e^{-\beta}\right) := \mathcal{I}_p\left(e^{-\beta}, e^{-\beta}, e^{-\beta}, e^{-\frac{\beta}{2}}\right) = \mathrm{Tr}(-1)^F e^{-\beta(E-R_1)} = \mathrm{Tr}(-1)^F \mathfrak{q}_3^{2E-2R_1}\,, \tag{29}$$

it commutes with four supercharges $Q^{+\pm}_{---}$, $Q^{-\mp}_{+++}$. For example, for $p = 3$, $q_1 = q_2 = 1$ we have

$$\mathcal{I}_p^{(\frac{1}{2})} = \mathrm{PE}\left[\frac{\mathfrak{q}_3 + \mathfrak{q}_3^2 - 8\mathfrak{q}_3^3 + 7\mathfrak{q}_3^4 + 7\mathfrak{q}_3^5 - 11\mathfrak{q}_3^6 + \mathfrak{q}_3^7 + \mathfrak{q}_3^8 + \mathfrak{q}_3^9}{(1-\mathfrak{q}_3)^3}\right] \tag{30}$$

$\frac{1}{2}$-BPS limit

The $\frac{1}{2}$-BPS limit of the Lens space index is given by taking

$$\mathfrak{q}_i \to 0,\ \mathfrak{p} \to 0 \text{ with } \mathsf{x} = \frac{\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}}{\mathfrak{p}} \text{ held fixed} \tag{31}$$

or equivalently, $\beta \to \infty, \omega_i \to 1$ with $-3\beta + \beta\mu = 2\log\mathsf{x}$ fixed. For the $\mathfrak{u}(1)$ theory, in this limit the single letter index is actually independent of p and becomes

$$i_p \to \mathsf{x}\,. \tag{32}$$

0.3.5 S^5 partition function review

Let us review the case $p = 1$. The S^5 partition function of $\mathcal{N} = 1^*$ with gauge algebra $\mathfrak{g} = \mathfrak{u}(N)$ can be expressed as [?, ?, ?, ?, ?, ?]

$$Z_{S^5} = \int [d\alpha] e^{\frac{2\pi^2(\vec{\alpha},\vec{\alpha})}{\beta\omega_1\omega_2\omega_3}} \prod_{i=1}^{3} Z_{\text{nek}}^{(i)} \left(\vec{\alpha}, \mu + \frac{3\omega_i}{2}, \omega_{i+1}, \omega_{i+2}, \frac{2\pi}{\omega_i}, 2\pi\beta \right) , \tag{33}$$

where the index i is taken modulo 3 and $(\cdot,\cdot)$ denotes the standard metric on $\mathfrak{t}^*$ where $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra. The domain of integration is $i\mathbb{R}^N$ and the integration measure is given by

$$[d\alpha] = \frac{i^N}{N!} \prod_{I=1}^{N} d\alpha_I \, . \tag{34}$$

The partition function is expressed as three copies of the K-theoretic Nekrasov partition function $Z_{\text{nek}}^{(i)}(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r, g_{\text{YM}}^2)$ on $S_r^1 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ where the radius is r. The Nekrasov partition function includes both the perturbative and instanton contributions

$$Z_{\text{nek}}^{(i)}(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r, g_{\text{YM}}^2) = Z_{\text{pert}}^{(i)}(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r) Z_{\text{inst}}^{(i)}(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r, g_{\text{YM}}^2). \tag{35}$$

Each of the three factors corresponds to the fact that the Localisation locus $\mathcal{L} = \coprod_{i=1}^{3} S^1_{2\pi/\omega_i}$ on S^5 factorises into three fixed circles of the $U(1)^3$ action (7).

	r	ϵ_1	ϵ_2	m
$Z_{\text{nek}}^{(1)}$	$\frac{2\pi}{\omega_1}$	ω_2	ω_3	$\mu + \frac{3}{2}\omega_1$
$Z_{\text{nek}}^{(2)}$	$\frac{2\pi}{\omega_2}$	ω_3	ω_1	$\mu + \frac{3}{2}\omega_2$
$Z_{\text{nek}}^{(3)}$	$\frac{2\pi}{\omega_3}$	ω_1	ω_2	$\mu + \frac{3}{2}\omega_3$

Table 2: Arguments for the Nekrasov partition functions associated to the three localisation circles

Classical contribution For the following section, it will be convenient to expresses the $SL(3,\mathbb{Z})$ form explicitly. This can be achieved by writing the classical piece as [?]

$$e^{\frac{2\pi^2(\vec{\alpha},\vec{\alpha})}{\beta\omega_1\omega_2\omega_3}} = \prod_{I>J} e^{\frac{-2\pi i}{3!}\left[B_{33}\left(\alpha_I - \alpha_J - \frac{i\pi}{2N\beta} + \frac{3}{2}\right) - B_{33}\left(-\frac{i\pi}{2N\beta} + \frac{3}{2}\right)\right]} = \prod_{i=1}^{3} Z_{\text{cl}}^{(i)}(\vec{\alpha}, \omega_i, \omega_{i+1}, \omega_{i+2}) \, , \tag{36}$$

$$Z_{\text{cl}}^{(i)}(\vec{\alpha}, \omega_i, \omega_{i+1}, \omega_{i+2}) = \prod_{I>J} \frac{\Gamma\left(e^{-\alpha_I + \alpha_J + \frac{i\pi}{2\beta N} - \frac{3}{2}}; e^{-\omega_{i+1}}, e^{-\omega_{i+2}}\right)}{\Gamma\left(e^{\frac{i\pi}{2\beta N} - \frac{3}{2}}; e^{-\omega_{i+1}}, e^{-\omega_{i+2}}\right)} \, . \tag{37}$$

Here, $\Gamma(z;p,q)$ denotes the Elliptic gamma function, defined as

$$\Gamma(z;p,q) := \prod_{n,m=0}^{\infty} \frac{1-\frac{pq}{z}p^n q^m}{1-zp^n q^m} = \mathrm{PE}\left[\frac{z-pq/z}{(1-p)(1-q)}\right] . \tag{38}$$

The benefit of this is that the partition function (33) can now be manifestly expressed in the $SL(3,\mathbb{Z})$ covariant way, with

$$Z_{S^5} = \int [d\alpha] \prod_{i=1}^{3} \widetilde{Z}^{(i)}_{\text{nek}}\left(\vec{\alpha}, \mu + \frac{3\omega_i}{2}, \omega_{i+1}, \omega_{i+2}, \frac{2\pi}{\omega_i}, 2\pi\beta\right) , \quad \widetilde{Z}^{(i)}_{\text{nek}} := Z^{(i)}_{\text{cl}} Z^{(i)}_{\text{nek}} . \tag{39}$$

Perturbative contribution We collect all of the perturbative factors $Z_{\text{pert}} = \prod_{i=1}^{3} Z^{(i)}_{\text{pert}}$ The perturbative piece factorise into a contribution from the vector multiplet and adjoint hypermultiplet

$$Z_{\text{pert}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3, \mu\right) = Z_{\text{vec}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3\right) Z_{\text{hyp}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3, \mu\right) . \tag{40}$$

They are given by [**?**, **?**, **?**]

$$Z_{\text{vec}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3, \mu\right) = \prod_{I,J=1}^{N} S_3'\left(\alpha_I - \alpha_J | \vec{\omega}\right) , \tag{41}$$

$$Z_{\text{hyp}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3, \mu\right) = \prod_{I,J=1}^{N} \frac{1}{S_3\left(\alpha_I - \alpha_J + \mu + \frac{3}{2}|\vec{\omega}\right)} , \tag{42}$$

where $S_3(z|\vec{\omega})$ is the triple sine function, defined in (98). The prime indicates that when $I = J$ the $n_1 = n_2 = n_3 = 0$ term in the infinite product product of the triple sine function should be removed.

Instanton contribution The instanton contribution may be computed from equivariant integration over the moduli space $\mathcal{M}_{k,N}$ of k $U(N)$ instantons. $\mathcal{M}_{k,N}$ carries a torus action $T := T^3_{\epsilon_1,\epsilon_2,m} \times T(U(N)) \circlearrowright \mathcal{M}_{k,N}$ where $T(G)$ denotes a maximal torus of G. ϵ_1, ϵ_2, m and $\vec{\alpha} \in \mathfrak{t}$. The instanton moduli space $\mathcal{M}_{k,N}$ may be described as a algebraic variety using the ADHM construction [**?**]. Let V, W be vector spaces of dimension $\dim_{\mathbb{C}} V = k$ and $\dim_{\mathbb{C}} W = N$. Let us introduce linear maps

$$B^{(l)} : V \to V , \quad P : W \to V , \quad Q : V \to W . \tag{43}$$

for $l = 1, 2, 3, 4$. The ADHM equations are

$$\mathcal{E}^{(1)}_{\mathbb{C}} := [B^{(1)}, B^{(2)}] + [B^{(3)\dagger}, B^{(4)\dagger}] + PQ , \quad \mathcal{E}^{(2)}_{\mathbb{C}} := [B^{(1)}, B^{(3)}] - [B^{(2)\dagger}, B^{(4)\dagger}] \tag{44}$$

$$\mathcal{E}^{(3)}_{\mathbb{C}} := [B^{(1)}, B^{(4)}] + [B^{(2)\dagger}, B^{(3)\dagger}] , \quad \mathcal{E}_{\mathbb{R}} := \sum_{l=1}^{4} [B^{(l)}, B^{(l)\dagger}] + PP^\dagger - Q^\dagger Q . \tag{45}$$

The moduli space is given by

$$\mathcal{M}_{k,N} := \left\{ B^{(l)} \,,\, P \,,\, Q \,\middle|\, \mathcal{E}_{\mathbb{C}}^{(i)} = \mathcal{E}_{\mathbb{R}} = 0 \right\} / U(k) \,, \tag{46}$$

where the $g \in U(k)$ acts by

$$\left(B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}, P, Q \right) \mapsto \left(gB^{(1)}g^{-1}, gB^{(2)}g^{-1}, gB^{(3)}g^{-1}, gB^{(4)}g^{-1}, gP, Qg^{-1} \right) . \tag{47}$$

The torus action $T^3_{\epsilon_1,\epsilon_2,m}$ acts on the ADHM data by

$$\left(B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}, P, Q \right) \mapsto (e^{\epsilon_1} B^{(1)}, e^{\epsilon_2} B^{(2)}, e^{m-\epsilon_+} B^{(3)}, e^{-m-\epsilon_+} B^{(4)}, P, e^{2\epsilon_+} Q) \,. \tag{48}$$

The fixed points of the torus action are labelled by N-tuples of Young diagrams $\vec{\mu}$ such that $|\vec{\mu}| = k$. See Appendix 0.A.1 for various identities. Choosing bases

$$W = \operatorname{span}_{\mathbb{C}} \left\{ w_I | I = 1, 2, \ldots, N \right\} \,, \quad V = \operatorname{span}_{\mathbb{C}} \left\{ v_I^{(i,j)} \middle| I = 1, 2, \ldots, N, s = (i,j) \in \mu_I \right\} . \tag{49}$$

The torus action T acts by

$$w_I \mapsto e^{\alpha_I} w_I \,, \quad v_I^{(i,j)} \mapsto e^{(1-i)\epsilon_1 + (1-\epsilon_2)} v_I^{(i,j)} \,. \tag{50}$$

Then

$$B^{(1)} v_I^{(i,j)} = v_I^{(i+1,j)} \,, \quad B^{(2)} v_I^{(i,j)} = v_I^{(i,j+1)} \,, \quad P w_I = v_I^{(1,1)} \,, \quad Q = B^{(3)} = B^{(4)} = 0 \,. \tag{51}$$

The character of the tangent space $T\mathcal{M}_{k,N}$ at the fixed point labelled by $\vec{\mu}$ is then

$$\chi_{\vec{\mu}} \left(T\mathcal{M}_{k,N} \right) := \chi_{\vec{\mu}}^{\text{Vec}} + \chi_{\vec{\mu}}^{\text{Hyp}} \,, \tag{52}$$

where,

$$\chi_{\vec{\mu}}^{\text{Vec}} = \left(W^* V + e^{2\epsilon_+} V^* W - (1 - e^{\epsilon_1})(1 - e^{\epsilon_2}) V^* V \right) \sum_{t \in \mathbb{Z}} e^{\frac{2\pi t}{r}} \,, \quad \chi_{\vec{\mu}}^{\text{Hyp}} = e^{m-\epsilon_+} \chi_{\vec{\mu}}^{\text{Vec}} \,. \tag{53}$$

Here we abused notation an identified the vector spaces with their characters:

$$V = \sum_{I=1}^{N} \sum_{(i,j) \in \mu_I} e^{\alpha_I + (1-i)\epsilon_1 + (1-j)\epsilon_2} \,, \quad W = \sum_{I=1}^{N} e^{\alpha_I} \,, \tag{54}$$

where the conjugation flips the sign of the exponents. We also added the dressing by momentum factors along the S^1. Using the identity (93) it can be shown that

$$\chi_{\vec{\mu}}^{\text{Vec}} = \sum_{t \in \mathbb{Z}} e^{\frac{2\pi t}{r}} \sum_{I,J=1}^{N} \sum_{s \in \mu_J} \left(e^{E_{IJ}(s)} + e^{2\epsilon_+ - E_{IJ}(s)} \right) \,, \tag{55}$$

where,

$$E_{IJ}(s) := \alpha_I - \alpha_J - (\mu^{\mathrm{T}}_{J;j} - i)\epsilon_1 + (\mu_{I;i} - j + 1)\epsilon_2\,. \tag{56}$$

The contribution of the fixed point $\vec{\mu}$ to the instanton partition is obtained from the character by

$$\chi_{\vec{\mu}}\left(T\mathcal{M}_{k,N}\right) := \sum_i n_i e^{w_i} \quad \rightarrow \quad z_{\vec{\mu}} = \prod_i w_i^{-n_i} \tag{57}$$

Finally, after applying the infinite product (94) for the sine function, the instanton partition function is given by a weighted sum over all possible N-tuples $\vec{\mu}$

$$Z_{\text{inst}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r, g^2_{\text{YM}}\right) = \sum_{\vec{\mu}} q^{|\vec{\mu}|} z_{\vec{\mu}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r\right)\,, \tag{58}$$

$$z_{\vec{\mu}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r\right) = \prod_{I,J=1}^{N}\prod_{s\in\mu_J} \frac{\sin\frac{r(E_{IJ}(s)+m-\epsilon_+)}{2}\sin\frac{r(E_{IJ}(s)-m-\epsilon_+)}{2}}{\sin\frac{rE_{IJ}(s)}{2}\sin\frac{r(E_{IJ}(s)-2\epsilon_+)}{2}}\,, \tag{59}$$

where $q := e^{-4\pi^2 r/g^2_{\text{YM}}} = e^{-2\pi r/\beta}$ and $2\epsilon_\pm = \epsilon_1 \pm \epsilon_2$. Note that, $(m, \epsilon_1, \epsilon_2))$ are periodic is $\frac{2\pi}{r}$ shifts.

Unrefined limits

We would like to discuss the limit that will allow us to compute the unrefined limits of the 6d index. Let us first examine the perturbative piece. The answer, for $\mu = \frac{1}{2}(\omega_1 + \omega_2 - \omega_3)$ was given in [] and is given by

$$Z_{\text{pert}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3, \omega_1 + \omega_2 - \frac{3}{2}\right) = \prod_{I>J} 2\sinh\frac{\alpha_I - \alpha_J}{\omega_1} 2\sinh\frac{\alpha_I - \alpha_J}{\omega_2}\,. \tag{60}$$

Now for the instanton pieces. In this limit, using the periodicity of the variables, Table **??** becomes We can see that, from Table 3, in the 5d description the chiral

	r	ϵ_1	ϵ_2	m
$Z^{(1)}_{\text{nek}}$	$\frac{2\pi}{\omega_1}$	ω_2	ω_3	$2\omega_1 + \frac{\omega_2-\omega_3}{2} \sim \frac{\omega_2-\omega_3}{2} = \epsilon_-$
$Z^{(2)}_{\text{nek}}$	$\frac{2\pi}{\omega_2}$	ω_3	ω_1	$2\omega_2 + \frac{\omega_1-\omega_3}{2} \sim \frac{\omega_1-\omega_3}{2} = -\epsilon_-$
$Z^{(3)}_{\text{nek}}$	$\frac{2\pi}{\omega_3}$	ω_1	ω_2	$\omega_3 + \frac{\omega_1+\omega_2}{2} \sim \frac{\omega_1+\omega_2}{2} = \epsilon_+$

Table 3: Table to test captions and labels

algebra limits correspond to studying the Nekrasov partition function in the cases $m = \pm\epsilon_-$ and $\epsilon_+ = m$. It is known that, in these limits

$$z_\mu(\vec{\alpha}, m = \pm\epsilon_-, \epsilon_1, \epsilon_2, r, g^2_{\text{YM}}) \equiv 0\,, \quad z_\mu(\vec{\alpha}, m = \pm\epsilon_+, \epsilon_1, \epsilon_2, r, g^2_{\text{YM}}) \equiv 1\,. \tag{61}$$

So, in the former cases the partition function gets non-zero contributions only from the zero instanton sector. In the latter case, the instanton partition function is simply counting coloured young diagrams with a single fugacity q for the number of boxes:

$$Z_{\text{inst}}\left(\vec{\alpha}, m = \epsilon_+, \epsilon_1, \epsilon_2, r, g_{\text{YM}}^2\right) = \sum_{\vec{\mu}} q^{|\vec{\mu}|} = q^{-N/24}(q;q)^{-N} = \eta(q)^{-N} \,. \tag{62}$$

So, in total, in this limit (33) becomes

$$Z_{S^5} = \frac{1}{\eta(e^{-\frac{4\pi^2}{\beta\omega_3}})^N} \int [d\alpha] e^{\frac{2\pi^2(\vec{\alpha},\vec{\alpha})}{\beta\omega_1\omega_2\omega_3}} \prod_{I>J} 2\sin\frac{\alpha_I - \alpha_J}{\omega_1} 2\sin\frac{\alpha_I - \alpha_J}{\omega_2} \,. \tag{63}$$

This integral was computed in [] and in total reads

$$Z_{S^5} = e^{-\beta E_{p=1}(\mathfrak{u}(N))} \prod_{I=1}^{N} \frac{1}{(\mathfrak{q}_3^I; \mathfrak{q}_3)} \,. \tag{64}$$

0.3.6 The $L(q_1, q_2, q_3; p)$ partition function

Let us now discuss how to obtain the partition function for the 5d $\mathcal{N} = 1^*$ theory on the Lens space. We *assume* that the $\mathbb{Z}_p$ projections do not change the Localisation structure. In particular we assume that the Localisation locus is still given by $\mathcal{L} = \coprod_{l=1}^3 S^1_{2\pi/\omega_l}$. Under that assumption the partition function takes the form of three copies of an 'orbifolded' Nekrasov partition function $\widetilde{Z}^{\text{orb}}_{\text{nek}} = Z^{\text{orb}}_{\text{cl}} Z^{\text{orb}}_{\text{nek}}$. Let us focus on the Nekrasov partition function at the l^{th} localisation circle $S^1_{2\pi/\omega_l}$. Recall that, before the orbifolding, it is given by $Z_{\text{nek}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r, 2\pi\beta\right)$ with

$$m = \mu + 3\omega_l/2\,, \quad \epsilon_1 = \omega_{l+1}\,, \quad \epsilon_2 = \omega_{l+2}\,, \quad r = 2\pi/\omega_l\,, \tag{65}$$

where, as before, we take $l = 1, 2, 3$ modulo 3. We have to project onto states left invariant by the orbifold action (1). Note that the third transformation in (66) implies that the instanton fugacity transforms. For example $\mathbb{Z}_p : q = e^{\frac{-2\pi r}{\beta}} \mapsto e^{\frac{-2\pi r p}{\beta}(p + iq_l r)^{-1}}$.

According to the Douglas and Moore prescription [?] we should also turn on $\pi_1(L_p) = \mathbb{Z}_p$ valued holonomies breaking $U(N) \to \prod_{A=1}^p U(N_A)$ such that $\sum_{A=1}^p N_A = N$. Since the $U(N)$ was integrated we then have to sum over all possible holonomies in the partition function computation. To each $U(N_A)$ we assign equivariant parameters $\alpha_{A,I}$ with $I = 1, 2, \ldots, N_A$ such that $\vec{\alpha} = (\alpha_{1,1}, \ldots, \alpha_{1,N_1}, \ldots, \alpha_{p,1}, \ldots, \alpha_{p,N_p})$. With the identifications (65) the orbifold action may be traded for an action on the equivariant parameters

$$\alpha_{A,I} \mapsto \alpha_{A,I} - \frac{2\pi i A}{p}\,, \quad m \mapsto m + \frac{6\pi i q_l}{2p}\,, \quad \frac{2\pi}{r} \mapsto \frac{2\pi}{r} + \frac{2\pi i q_l}{p} \tag{66}$$

$$\epsilon_1 \mapsto \epsilon_1 + \frac{2\pi i q_{l+1}}{p}\,, \quad \epsilon_2 \mapsto \epsilon_2 + \frac{2\pi i q_{l+2}}{p}\,. \tag{67}$$

The Lens space partition function then takes the form

$$Z_{L(q_1,q_2,q_3;p)} = \int [d\alpha]' \prod_{i=1}^{3} \widetilde{Z}^{\text{orb}}_{\text{nek}} \left(\vec{\alpha}, \mu + 3\omega_i/2, \omega_{i+1}, \omega_{i+2}, 2\pi/\omega_i, 2\pi\beta\right) \,, \tag{68}$$

where the measure is given by

$$[d\alpha]' = \sum_{\{N_1,N_2,\ldots,N_p \mid \sum_{A=1}^{p} N_A = N\}} \frac{i^N}{N_1! N_2! \ldots N_p!} \prod_{A=1}^{p} \prod_{I=1}^{N_A} d\alpha_{A,I} \,. \tag{69}$$

The very first sum is summing over all holonomies.

0.3.7 Perturbative contribution

Let us detail how to implement the orbifold projections at the level of the perturbative part (40) of the partition function. Before the orbifold the perturbative piece is built out of triple sine functions which contain infinite products of the form

$$\prod_{n_1,n_2,n_3=0}^{\infty} (z + \vec{n} \cdot \vec{\omega}) \,. \tag{70}$$

Under the $\mathbb{Z}_p$ transformations (66), (67) let us say that $z \mapsto z + \frac{2\pi i b}{p}$. The fixed points of the transformation acting on the above product are labelled by the integers (n_1, n_2, n_3) subject to the condition $\eta + b = 0 \mod p$ with $\eta = q_1 n_1 + q_2 n_2 + q_3 n_3$. Unsurprisingly, the solution is similar to that of (19). The result of keeping only the fixed points of the transformation yields the infinite product

$$\prod_{r_1,r_2,r_3=0}^{\infty} \prod_{(n_1,n_2,n_3)\in s_b(\eta)} (z + \vec{\omega} \cdot (\vec{n} + p\vec{r})) \tag{71}$$

where $s_b(\eta) \subset \mathbb{N}^3$ is the same set that we defined in (20). Therefore the orbifold of the perturbative factors is

$$Z^{\text{orb}}_{\text{vec}} = \prod_{A,B=1}^{p} \prod_{I=1}^{N_A} \prod_{J=1}^{N_B} \prod_{r_1,r_2,r_3=0}^{\infty} \prod_{(n_1,n_2,n_3)\in s_{B-A}(\eta)} \Big\{ (\alpha_{A,I} - \alpha_{B,J} + \vec{\omega} \cdot (\vec{n} + p\vec{r})) \times (\alpha_{A,I} - \alpha_{B,J} + \vec{\omega} \cdot (\vec{n} + p\vec{r} + 1)) \Big\} \,, \tag{72}$$

$$Z^{\text{orb}}_{\text{hyp}} = \prod_{A,B=1}^{p} \prod_{I=1}^{N_A} \prod_{J=1}^{N_B} \prod_{r_1,r_2,r_3=0}^{\infty} \prod_{(n_1,n_2,n_3)\in s_{B-A}(\eta)} \left\{ \left(\alpha_{A,I} - \alpha_{B,J} + \mu + \frac{3}{2} + \vec{\omega} \cdot (\vec{n} + p\vec{r})\right)^{-1} \times \left(\alpha_{A,I} - \alpha_{B,J} - \mu + \frac{3}{2} + \vec{\omega} \cdot (\vec{n} + p\vec{r} + 1)\right)^{-1} \right\} . \tag{73}$$

As before $Z^{\text{orb}}_{\text{vec}}$ should be understood such that if a term with $n_1 = n_2 = n_3 = r_1 = r_2 = r_3 = 0$ appears it should be removed by hand.

0.3.8 The ramified instanton partition function

Let us focus on the instanton part of the Nekrasov partition function at the l^{th} localisation circle $S^1_{2\pi/\omega_l}$. Now we will describe how to implement the $\mathbb{Z}_p$ action. We decompose the vector spaces W, V (for a fixed momentum mode around the $S^1_{2\pi/\omega_l}$) with respect to their $\mathbb{Z}_p$ grading

$$W = \bigoplus_{A=1}^{p} W_A\,, \quad V = \bigoplus_{A=1}^{p} V_A\,, \tag{74}$$

of dimension $\dim_{\mathbb{C}} W_A = N_A$ and $\dim_{\mathbb{C}} V_A = k_A$. Moreover, we also take the index $A = 1, \ldots, p$ modulo p. Under the $\mathbb{Z}_p$ action the ADHM data transforms as

$$B^{(1)} \mapsto \gamma^{q_{l+1}} B^{(1)}\,, \quad B^{(2)} \mapsto \gamma^{q_{l+2}} B^{(2)}\,, \quad B^{(3)} \mapsto \gamma^{2q_l} B^{(3)}\,, \quad B^{(4)} \mapsto \gamma^{-q_l} B^{(4)}\,, \tag{75}$$

$$P \mapsto P\,, \quad Q \mapsto \gamma^{q_{l+1}+q_{l+2}} Q\,. \tag{76}$$

Note we used the condition $\sum_{i=1}^{3} q_i = 0$ of equation (4) to simplify the action on $B^{(3)}$, $B^{(4)}$. In order to have a non-trivial result, following [?], we also quotient by a $\mathbb{Z}_p \hookrightarrow U(k)$ corresponding to (47) with $g = \operatorname{diag}\left(\gamma \mathbb{I}_{k_1}, \gamma^2 \mathbb{I}_{k_2} \ldots,\right) \in U(k)$. This breaks $U(k) \to \prod_{A=1}^{p} U(k_A)$ with $k = \sum_{A=1}^{p} k_A$. The surviving components are

$$\begin{aligned} &B^{(1)}_A \in \operatorname{Hom}\left(V_A, V_{A+q_{l+1}}\right)\,, \quad B^{(2)}_A \in \operatorname{Hom}\left(V_A, V_{A+q_{l+2}}\right)\,, \quad B^{(3)}_A \in \operatorname{Hom}\left(V_A, V_{A+2q_l}\right)\,, \\ &B^{(4)}_A \in \operatorname{Hom}\left(V_A, V_{A-q_l}\right)\,, \quad P_A \in \operatorname{Hom}\left(V_A, V_A\right)\,, \quad Q_A \in \operatorname{Hom}\left(V_A, V_{A+q_{l+1}+q_{l+2}}\right)\,. \end{aligned} \tag{77}$$

The ADHM equations $\mathcal{E}^{(n=1,2,3,4)}_{\mathbb{C},A} = \mathcal{E}_{\mathbb{R},A} = 0$ are given by performing the projections to (44) and (45). The ramified instanton moduli space is then given by

$$\mathcal{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}} = \left\{ B^{(n)}_A\,, P_A\,, Q_A \middle| \mathcal{E}^{(n)}_{\mathbb{C},A} = \mathcal{E}_{\mathbb{R},A} = 0 \right\}\,. \tag{78}$$

The fixed points after the $\mathbb{Z}_p$ quotient are still labelled by N-tuples of Young diagrams $\vec{\mu}$ which we now label by $\vec{\mu} = \{\mu_{A,I}\}$. We choose bases $W_A = \operatorname{span}_{\mathbb{C}} \{w_{A,I} | I = 1, \ldots, N_A\}$ and $V_{A+q_{l+1}i+q_{l+2}j} = \operatorname{span}_{\mathbb{C}} \left\{ v^{(i,j)}_{A+q_{l+1}i+q_{l+2}j,I} \middle| I = 1, \ldots, N_{A+q_{l+1}i+q_{l+2}j}, (i,j) \in \mu_{A,I} \right\}$. The torus action acts by

$$w_{A,I} \mapsto e^{\alpha_{A,I}} w_{A,I}\,, \quad v^{(i,j)}_{A+q_{l+1}i+q_{l+2}j,I} \mapsto e^{(1-i)\epsilon_1 + (1-\epsilon_2)} v^{(i,j)}_{A+q_{l+1}i+q_{l+2}j,I}\,. \tag{79}$$

The fixed point configuration is given by the orbifold projection of (51), namely

$$B^{(1)}_A v^{(i,j)}_{A,I} = v^{(i+1,j)}_{A+q_{l+1},I}\,, \quad B^{(2)}_A v^{(i,j)}_{A,I} = v^{(i,j+1)}_{A+q_{l+2},I}\,, \quad P_A w_{A,I} = v^{(1,1)}_{A,I}\,, \quad Q_A = B^{(3)}_A = B^{(4)}_A = 0\,. \tag{80}$$

$$
\begin{tikzcd}
 & v_1^{(4,1)} & & \\
 & v_3^{(3,1)} \arrow[u, "B_3^{(1)}"] \arrow[r, "B_3^{(2)}"] & v_1^{(3,2)} & \\
 & v_2^{(2,1)} \arrow[u, "B_2^{(1)}"] \arrow[r, "B_2^{(2)}"] & v_3^{(2,2)} \arrow[u, "B_3^{(1)}"] \arrow[r, "B_3^{(2)}"] & v_1^{(2,3)} \\
 & v_1^{(1,1)} \arrow[u, "B_1^{(1)}"] \arrow[r, "B_1^{(2)}"] & v_2^{(1,2)} \arrow[u, "B_2^{(1)}"] \arrow[r, "B_2^{(2)}"] & v_3^{(1,3)} \arrow[u, "B_3^{(1)}"] \\
w_1 \arrow[ur, "P_1"] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v_3^{(2,1)} \arrow[r, "B_3^{(2)}"] & v_1^{(2,2)} \\
 & v_2^{(1,1)} \arrow[u, "B_2^{(1)}"] \arrow[r, "B_2^{(2)}"] & v_3^{(1,2)} \arrow[u, "B_3^{(1)}"] \\
w_2 \arrow[ur, "P_2"] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & v_2^{(3,1)} & \\
 & v_1^{(2,1)} \arrow[u, "B_1^{(1)}"] \arrow[r, "B_1^{(2)}"] & v_2^{(2,2)} \\
 & v_3^{(1,1)} \arrow[u, "B_3^{(1)}"] \arrow[r, "B_3^{(2)}"] & v_1^{(1,2)} \arrow[u, "B_1^{(1)}"] \\
w_3 \arrow[ur, "P_3"] & &
\end{tikzcd}
$$

Figure 1: *Example with* $p = 3$, $q_l = q_{l+1} = 1$, $q_{l+2} = -2$, $N_1 = N_2 = N_3 = 1$, $\mu_{1,1} = \{4,3,2\}$, $\mu_{2,1} = \{2,2\}$ *and* $\mu_{3,1} = \{3,2\}$. *Because* $N_A = 1$ *we drop the* I *indices, for example* $v_{A,1}^{(i,j)} = v_A^{(i,j)}$. $k_1 = 7$, $k_2 = 5$ *and* $k_3 = 6$*; in agreement with* (81).

The dimension of V_B is then given by

$$k_B = k_B(\vec{\mu}) = \dim_{\mathbb{C}} V_B = \sum_{A=1}^{p} \sum_{I=1}^{N_A} \sum_{(i,j)\in y_{A,B}^{(I)}} 1\,, \tag{81}$$

where $y_{A,B}^{(I)}$ is given by

$$y_{A,B}^{(I)} = \{(i,j) | (i,j) \in \mu_{A,I}, A + q_{l+1} i + q_{l+2} j = B \bmod p\}\ . \tag{82}$$

For $q_{l+1} = 0$ and $q_{l+2} = 1$ equation (81) reduces to (2.37) of [**?**]. We demonstrate an explicit example in Figure 1. Finally the character of $T\mathcal{M}_{\{k_A\},\{N_A\}}^{\mathbb{Z}_p}$ at the fixed point $\vec{\mu}$ is given by the $\mathbb{Z}_p$ invariant part of (52), namely

$$\chi_{\vec{\mu}}\left(T\mathcal{M}_{\{k_A\},\{N_A\}}^{\mathbb{Z}_p}\right) := \chi_{\vec{\mu},\mathbb{Z}_p}^{\text{Vec}} + \chi_{\vec{\mu},\mathbb{Z}_p}^{\text{Hyp}}\,, \tag{83}$$

with

$$\chi^{\text{Vec}}_{\vec{\mu},\mathbb{Z}_p} = \sum_{t\in\mathbb{Z}} e^{\frac{2\pi pt}{r}} \sum_{A=1}^{p} e^{\frac{2\pi A}{r}} \chi_A \,, \quad \chi^{\text{Hyp}}_{\vec{\mu},\mathbb{Z}_p} = \sum_{t\in\mathbb{Z}} e^{\frac{2\pi pt}{r}} \sum_{A=1}^{p} e^{\frac{2\pi A}{r}} e^{m-\epsilon_+} \chi_{A+2q_l} \,, \tag{84}$$

$$\chi_A := \sum_{B=1}^{p} \Big[W^*_{B+q_l A} V_B + e^{2\epsilon_+} V^*_{B+q_l A+q_{l+1}+q_{l+2}} W_B - V^*_{B+q_l A} V_B \\ + e^{\epsilon_1} V^*_{B+q_l A+q_{l+1}} V_B + e^{\epsilon_2} V^*_{B+q_l A+q_{l+2}} V_B - e^{2\epsilon_+} V^*_{B+q_l A+q_{l+1}+q_{l+2}} V_B \Big] \,, \tag{85}$$

where we used (4). As before conjugation reverses the signs of the exponents. We also abused the notation and identified the vector spaces and their characters

$$V_A = \sum_{C,D=1}^{p} \sum_{I=1}^{N_{q_{l+1}C+q_{l+2}D-A}} \sum_{(pi-C+1,pj-D+1)\in y^{(I)}_{q_{l+1}C+q_{l+2}D-A,A}} e^{\alpha_{q_{l+1}C+q_{l+2}D-A,I}+(C-pi)\epsilon_1+(D-pj)\epsilon_2} \,, \tag{86}$$

$$W_A = \sum_{I=1}^{N_{p-A+1}} e^{\alpha_{p-A+1,I}} \,, \tag{87}$$

under the orbifold $\mathbb{Z}_p : V_A, W_A \mapsto \gamma^A V_A, \gamma^A W_A$. At this point it is very important to stress that we understand A, B, C, D to be taken modulo p when and only when they are considered as indices used to label quantities for example $\alpha_{A,I} = \alpha_{A+p,I}$. According to the conversion rule (57) we can compute

$$\chi_{\vec{\mu}} \left(T\mathcal{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}} \right) \to z^{\mathbb{Z}_p}_{\vec{\mu}} \left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r \right) \,. \tag{88}$$

Appendices

0.A Identities

In this section we collect various definitions and identities used within this paper

0.A.1 Young diagrams

We use Greek letters μ, λ, ν to denote partitions of natural numbers. We denote the empty partition by $\varnothing$. A non-empty partition is a set of integers λ

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \lambda_l \geqslant \dots \geqslant \lambda_{\ell(\lambda)} > 0\,, \tag{89}$$

with $\ell(\lambda)$ the number of parts of λ. This definition is also extended to include $\lambda_{l>\ell(\lambda)} \equiv 0$. λ^{T} is the transpose. We denote

$$|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_l\,, \quad ||\lambda||^2 := \sum_{l=1}^{\ell(\lambda)} \lambda_l^2 = \sum_{(i,j)\in\lambda^{\mathrm{T}}} \lambda_l\,. \tag{90}$$

We give a box s in the Young diagram coordinates $s = (i,j)$ such that

$$\lambda = \{(i,j)|i = 1,\dots,\ell(\lambda); j = 1,\dots,\lambda_i\}\ . \tag{91}$$

We will also be interested in N-tuples of Young diagrams

$$\vec{\mu} = \{\mu_I | I = 1,2,\dots,N\}\ . \tag{92}$$

We write $\lambda_{I;i}$ to denote the number of boxes in the i^{th} column of the diagram λ_I. We will also make use the identity [**?**, **?**]

$$\begin{aligned}&\sum_{(i,j)\in\mu} e^{i\epsilon_1+j\epsilon_2} + \sum_{(i',j')\in\mu'} e^{(1-i')\epsilon_1+(1-j')\epsilon_2} - (1-e^{\epsilon_1})(1-e^{\epsilon_2}) \sum_{(i,j)\in\mu}\sum_{(i',j')\in\mu'} e^{(i-j')\epsilon_1+(j-j')\epsilon_2} \\ &= \sum_{(i,j)\in\mu} e^{-(\mu'^{\mathrm{T}}_j-i)\epsilon_1+(\mu_i-j+1)\epsilon_2} + e^{2\epsilon_+} \sum_{(i',j')\in\mu'} e^{(\mu^{\mathrm{T}}_{j'}-i')\epsilon_1-(\mu'_{i'}-j'+1)\epsilon_2}\,.\end{aligned} \tag{93}$$

0.A.2 Special functions

The Euler infinite product representation for the sine function is

$$\sin(x) = x \prod_{t=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 t^2}\right) . \tag{94}$$

The multiple zeta function is

$$\zeta_r(z, s|\vec{\omega}) = \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} (\vec{n} \cdot \vec{\omega} + z)^{-s} , \tag{95}$$

for $z \in \mathbb{C}$ and $\Re s > r$. Multiple gamma functions are defined as

$$\Gamma_r(z|\vec{\omega}) = e^{\frac{\partial}{\partial s} \zeta_r(z,s|\vec{\omega})|_{s=0}} . \tag{96}$$

A regularised infinite product can be defined

$$\Gamma_r(z|\vec{\omega}) \sim \prod_{n_1, n_2, \ldots, n_r = 0}^{\infty} (\vec{n} \cdot \vec{\omega} + z)^{-1} \tag{97}$$

and the multiple sine function is defined as

$$S_r(z|\vec{\omega}) = \Gamma_r(z|\vec{\omega})^{-1} \Gamma_r(|\vec{\omega}| - z|\vec{\omega})^{(-1)^r} \sim \prod_{n_1, n_2, \ldots, n_r = 0}^{\infty} (\vec{n} \cdot \vec{\omega} + |\vec{\omega}| - z) \, (\vec{n} \cdot \vec{\omega} + z)^{(-1)^{r+1}} , \tag{98}$$

with $|\vec{\omega}| = \omega_1 + \cdots + \omega_r$. It has the symmetry property

$$S_r(z|\vec{\omega}) = S_r(|\vec{\omega}| - z|\vec{\omega})^{(-1)^{r+1}} . \tag{99}$$

0.B Ramified instanton partition function

$$\chi_{\vec{\mu}} \left(T\mathcal{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}} \right) := \chi^{\mathrm{Vec}}_{\vec{\mu},\mathbb{Z}_p} + \chi^{\mathrm{Hyp}}_{\vec{\mu},\mathbb{Z}_p} , \tag{100}$$

with

$$\chi^{\mathrm{Vec}}_{\vec{\mu},\mathbb{Z}_p} = \sum_{t \in \mathbb{Z}} e^{\frac{2\pi p t}{r}} \sum_{A=1}^{p} e^{\frac{2\pi A}{r}} \chi_A , \quad \chi^{\mathrm{Hyp}}_{\vec{\mu},\mathbb{Z}_p} = \sum_{t \in \mathbb{Z}} e^{\frac{2\pi p t}{r}} \sum_{A=1}^{p} e^{\frac{2\pi A}{r}} e^{m - \epsilon_+} \chi_{A + 2q_l} , \tag{101}$$

$$\begin{aligned} \chi_A := \sum_{B=1}^{p} \Big[& W^*_{B+q_l A} V_B + e^{2\epsilon_+} V^*_{B+q_l A + q_{l+1} + q_{l+2}} W_B - V^*_{B+q_l A} V_B \\ & + e^{\epsilon_1} V^*_{B+q_l A + q_{l+1}} V_B + e^{\epsilon_2} V^*_{B+q_l A + q_{l+2}} V_B - e^{2\epsilon_+} V^*_{B+q_l A + q_{l+1} + q_{l+2}} V_B \Big] , \end{aligned} \tag{102}$$

$$V_A = \sum_{C,D=1}^{p} \sum_{I=1}^{N_{q_{l+1}C+q_{l+2}D-A}} \sum_{(pi-C+1,pj-D+1)\in y^{(I)}_{q_{l+1}C+q_{l+2}D-A,A}} e^{\alpha_{q_{l+1}C+q_{l+2}D-A,I}+(C-pi)\epsilon_1+(D-pj)\epsilon_2}\,, \tag{103}$$

$$W_A = \sum_{I=1}^{N_{p-A+1}} e^{\alpha_{p-A+1,I}}\,, \tag{104}$$

$$z_1^{\text{vec}} = \prod_{A,B,C,D=1}^{p} \prod_{I=1}^{N_{p-B-q_lA+1}} \prod_{J=1}^{N_\sigma} \prod_{s\in y^{(J)}_{\sigma,B}} \times \sin\frac{r}{2p}\left[\alpha_{\sigma-B,J}-\alpha_{p-B-q_lA+1,I}+(C-pi)\epsilon_1+(D-pj)\epsilon_2+\frac{2\pi}{r}A\right]^{-1} \tag{105}$$

$$z_2^{\text{vec}} = \prod_{A,B,C,D=1}^{p} \prod_{I=1}^{N_{p-B+1}} \prod_{J=1}^{N_{\sigma-q_lA-q_{l+1}-q_{l+2}}} \prod_{s\in y^{(I)}_{\sigma-q_lA-q_{l+1}-q_{l+2},B+q_lA+q_{l+1}+q_{l+2}}} \times \sin\frac{r}{2p}\left[\alpha_{p-B+1,J}-\alpha_{\sigma-q_lA-q_{l+1}-q_{l+2},I}+(C-pi-1)\epsilon_1+(D-pj-1)\epsilon_2+\frac{2\pi}{r}A\right]^{-1} \tag{106}$$

$$z_3^{\text{vec}} = \prod_{A,B,C,D,C',D'=1}^{p} \prod_{I=1}^{N_{\sigma-q_lA}} \prod_{J=1}^{N_\sigma} \prod_{s\in y^{(I)}_{\sigma-q_lA,B+q_lA}} \prod_{s'\in y^{(I)}_{\sigma,B}} \times \sin\frac{r}{2p}\left[\alpha_{\sigma,J}-\alpha_{\sigma-q_lA,I}+(C'-C-pi'+pi)\epsilon_1+(D'-D-pj'+pj)\epsilon_2+\frac{2\pi}{r}A\right] \tag{107}$$

we defined $\sigma = q_{l+1}C+q_{l+2}D-B$, $\sigma' = q_{l+1}C'+q_{l+2}D'-B$, $s = (C-pi+1, pj-D+1)$ and $s' = (C'-pi'+1, pj'-D'+1)$