Engagement Maximization

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Abstract

We consider the problem of a rational, Bayesian agent receiving signals over time for the purpose of taking an action. The agent chooses when to stop and take an action based on her current beliefs, and prefers (all else equal) to act sooner rather than later. However, the signals received by the agent are determined by a principal, whose objective is to maximize engagement (the total attention paid by the agent to the signals). We show that engagement maximization by the principal minimizes the agent's welfare; the agent does no better than if she gathered no information at all. Relative to a benchmark in which the agent chooses which signals to acquire, engagement maximization leads to excessive information acquisition and to more extreme beliefs. We show that an optimal strategy involves "suspensive signals" that lead the agent's belief to change while keeping it "less certain than the prior" and "decisive signals" that lead the agent's belief to jump to the stopping region.

Key Words: Information Acquisition, Recommendation Algorithms, Polarization, Rational Inattention

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1 Introduction

Free-to-use online platforms such as Facebook, Instagram, Youtube, and Pinterest are used by billions of people worldwide and earn substantial profits by displaying advertisements to their users. Their business models are powered by personalized recommendation algorithms that seek to maximize the "engagement" of each user by selectively displaying content (Lada et al. [2021], Sequoia [2018]). Large quantities of computing and other resources have been invested by these firms to develop algorithms that can predict billions of users' preferences in real time.

A key challenge for these algorithms is to manage users' incentives. Content is presented sequentially in "news feeds," "recommendations," or "timelines," and users can choose freely *what* to pay attention to and *when* to stop using the platform based on the entire history of content presented thus far. From the platform's perspective, providing the most useful content immediately is suboptimal, as a user might become satisfied and choose to stop using the platform, limiting the quantity of advertisements the platform can display to the user. On the other hand, if the user anticipates that the platform will never provide useful content, they will never begin to use the platform in the first place.

In this paper, we study the optimal design of sequential information presentation from a principal-agent perspective. The principal (the platform) provides the agent (the user) with information. The agent gets instrumental value from the information (i.e. the value of information is from facilitating decision making), and chooses when to stop engaging with the platform.[1] The rate at which the agent can process information is limited, and the agent experiences an opportunity cost of time spent on the platform net of any utility from using the platform. The principal's goal is to maximize the attention the agent allocates to the platform (which will be equivalent to maximizing the stopping time), and the principal accomplishes this by choosing the nature of the information provided to the agent. The key modeling assumptions we impose are that the principal knows perfectly the agent's preferences and that the principal can flexibly manipulate the entire information flow. These assumptions

[1] We show that it is without loss of generality to assume the agent does not process the information selectively; that is, the agents attends to the information the principal provides in equilibrium.

capture the state platforms aim to reach with unlimited data and computing power.

Our main result is the characterization of an optimal strategy for the principal. First, we completely characterize the optimal "overall" information structure (i.e. the signal-state joint distribution that is unconditional on time). It is characterized by the solution to an augmented static rational inattention (RI) problem: the information structure maximizes a linear combination of the instrumental value of information and the measure of information with endogenous weights. Second, we identify one optimal sequential information structure for the principal: the principal sends a "dilution" of the overall information structure (a compound Poisson process such that a signal arrives at a Poisson rate and upon arrival, the signal is distributed according to the overall information structure).

We show that any optimal strategy has the following features:

- **Welfare minimization**: Under the principal's optimally chosen information structure and the agent's optimal stopping policy, the agent engages substantially with the platform but is no better off than if she did not use the platform at all, and made her decision without acquiring any information. That is, engagement maximization by the principal minimizes the welfare of the agent. The agent's ability to stop using the platform at any time does not allow her to extract any surplus.

- **Belief polarization**: When the agent chooses to stop and act after engaging with the platform, the agent will hold extreme beliefs, relative to a benchmark model in which the agent can choose both what information to receive and when to stop and act.

- **Learning paths**: An optimal strategy contains (up to) two types of signals almost surely. The first type are *decisive* signals that are so informative that the agent makes a decision immediately after receiving the signal. The second type are *suspensive* signals that causes the agent's beliefs to move to a posterior that is "more uncertain than the prior".

After presenting our main results of welfare-minimization and belief polarization, we consider several extensions of our baseline model. First, we consider the possibility that the agent does not fully attend to the signals sent by the principal. We show that it is without loss of generality to assume the agent pays full attention to the signals provided. Second, we consider the case in which there is an increasing opportunity cost of time and/or decreasing marginal utility from using the platform. This case is less tractable; to make progress, we impose additional symmetry assumptions, and derive conditions under which our main results continue to hold. Third, we discuss the importance of the role of jumps in beliefs in our model. The optimal belief process in our baseline model will involve rare but informative signal realizations that cause beliefs to jump. This result speaks to the sensational nature of the information that the principal optimally provides the agent. To clarify this point, we extend our analysis to the case in which the size of jumps in beliefs is arbitrarily restricted. We characterize how the restriction on jump size restricts the set of implementable strategies, and show that under certain circumstances, if beliefs must follow a continuous process, the principal and agent are in fact fully aligned and it is instead the agent who receives the full surplus. Fourth, we consider a modification of the model in which the user has unlimited information processing capacity and the platform maximizes time spent (as opposed to attention). We show that certain of our results continue to apply to this modified model. Lastly, we apply our framework to a seemingly quite different setting, in which a teacher seeks to maximize the engagement of a student who cares only about passing a test.[2]

1.1 Example

We illustrate the main model and the logic of our main result in a simple example. An agent faces a choice between two actions, $A = \{l, r\}$. The payoffs from the actions are uncertain and depend on the state of the world $x \in X = \{L, R\}$. The agent assigns equal prior probability to both states. The agent gets utility one when the chosen action matches the state ($l|L$ or $r|R$) and utility negative one otherwise ($l|R$ or $r|L$). The agent is impatient and pays a constant cost of delay of two utils per unit of time. The agent's information processing capacity is bounded above — the mutual information of the state and the signal received in any period $t+\delta$, conditional on the signal history up to period t, is bounded by δ.

[2]We are grateful to Emir Kamenica for suggesting this alternative setting as an application of our model.

The agent-optimal strategy: The agent-optimal problem is exactly the problem studied by Hébert and Woodford [2021b]. Hébert and Woodford [2021b] shows that it reduces to a standard rational inattention problem, where the conditional distribution of the optimal action profile $p(a|x)$ solves:

$$\begin{aligned}(p(a|x)) \in \arg\max \sum_{a\in A, x\in X} \frac{1}{2} u(a|x) p(a|x) \\ -2 \sum_{a\in A, x\in X} \frac{1}{2} p(a|x) \ln\left(\frac{p(a|x)}{\sum_{x'\in X} \frac{1}{2} p(a|x')}\right).\end{aligned} \tag{1}$$

Per Matêjka et al. [2015], the optimal solution is logit:

$$p(l|L) = p(r|R) = \frac{e}{1+e};\ p(l|R) = p(r|L) = \frac{1}{1+e}.$$

The expected engagement (the total amount of information acquired, as measured by mutual information) is $I = \sum_{a\in A, x\in X} \frac{1}{2} p(a|x) \ln(\frac{p(a|x)}{\sum_{x'\in X} \frac{1}{2} p(a|x')}) \approx 0.11 < \ln(2)$; this quantity is also the expected time required for the agent reaches a decision. Per Hébert and Woodford [2021b] and Morris and Strack [2019], there are a continuum of admissible learning strategies the agent can take to implement this action profile. The agent's expected utility under any of these strategies is

$$V^B = \underbrace{\frac{e-1}{1+e}}_{\text{Utility Gain}} \underbrace{-\left(\frac{2e}{1+e}\ln\left(\frac{2e}{1+e}\right) + \frac{2}{1+e}\ln\left(\frac{2}{1+e}\right)\right)}_{\text{Cost of Delay}} \approx -1 - 2\ln\left(\frac{2}{1+e}\right) \approx 0.24.$$

Note that this is greater than zero, the expected utility the agent would achieve if she gathered no information at all.

A better (but not optimal) strategy for the principal: Instead of offering the agent her preferred signal structure, suppose the principal sends the agent a rare but completely revealing signal until time $T = 1$, and then switches to the agent-optimal strategy. The mutual information of a signal that either reveals x with probability $\lambda\delta$ or leaves beliefs unchanged is, given the uniform prior, $\lambda\delta\ln(2)$, so the principal can send this signal with rate $\lambda = \frac{1}{\ln(2)}$. The expected engagement under this strategy is

$$e^{-\lambda}I + (1-e^{-\lambda})\ln(2) > I.$$

The agent's continuation utility at time t is

$$V_t = \int_t^1 (1-2(s-t))\lambda e^{-\lambda(s-t)}ds + e^{-\lambda(1-t)}V^B,$$

which is strictly positive for all $t \geq 0$. We conclude that agent would still prefer this signal to her outside option at every point in time, and that this strategy generates a higher degree of expected engagement (and thus time spent observing advertisements) for the the principal.

This example illustrates the forces behind our main result. By assumption, the principal benefits from sending the agent signals that the agent will attend to, as it is assumed that this is what generates advertising revenue. The agent perceives delay as costly, and will only continue to attend to the principal's signals if she values the information being provided. The agent will in general be willing to receive "too much" information, relative to what she would choose for herself, provided that this still leaves her with some surplus. The principal can take advantage of this willingness by sending signals that always result in the agent either learning too much (learning x in the example above) or leaving beliefs almost unchanged (exactly constant in the example above). These signals generate larger expected stopping times, and thus more revenue for the principal.

Of course, this proposed strategy is far from ideal. It is not necessarily optimal the principal to send fully revealing signals, and in many cases it is never in the principal's interest to provide the agent with the agent-optimal signal. We will revisit this example after stating the main theorem and present the optimal solution.

1.2 Related Literature

Our paper contributes to several strands of literature on the dynamic provision of information. Viewing our principal as a media company, our model is related to work on models of media bias (see Gentzkow et al. [2015] for a survey). We share with Kleinberg et al. [2022] an interest in explaining why the users of internet platforms would engage heavily with those platforms while perceiving themselves as gaining little from doing so. We derive this outcome as a result of strategic behavior by rational agents with conflicting incentives; those authors emphasize the time-inconsistency of user preferences. We share with Acemoglu et al. [2021] an emphasis on explaining what kind of information is available on internet platforms; our analysis focuses on content selection algorithms, whereas their analysis focuses on information sharing between users.

Our model is equivalent to one where the principal's only goal is to ensure the agent continues to pay attention, as in Kawamura and Le Quement [2019]. However, our principal presents unbiased information, and hence is not engaged in cheap talk (Crawford and Sobel [1982]). Our principal is also not attempting to persuade the agent towards any particular action (as in Kamenica and Gentzkow [2011]) or improve his reputation by catering to the agent's prior beliefs (as in Gentzkow and Shapiro [2006]).

In these respects, our model is more akin to models of gradual information revelation over time, such as Ely et al. [2015] or Hörner and Skrzypacz [2016]. However, unlike these papers, the gradual nature of belief evolution in our model arises from the agent's information processing constraint, as opposed to a desire to maximize suspense or address problems of limited commitment. Closer to our work is Ely and Szydlowski [2020], who consider the problem of a principal providing information to an agent for the purpose of persuading the agent to continue her effort. The key difference between our model and theirs is that in our model, the information provided is about actions that do not directly affect the principal, whereas in theirs the information is about the cumulative effort necessary to induce a payment from the principal to the agent. As a result, their model can be viewed as a kind of Bayesian persuasion in which commitment to an information structure is essential; in contrast, the principal's ability to commit affects little in our analysis.

Formally, our approach is a principal-agent version of Hébert and Woodford [2021b]. Those authors consider a model in which a single decision maker chooses both what information to acquire and when to stop and act, whereas in our model the principal chooses the information and the agent chooses when to stop and act. We compare our model to a benchmark in which the agent chooses both the information and when to stop and act; this benchmark is characterized by results found in Hébert and Woodford [2021b]. We follow Hébert and Woodford [2021b] in assuming that the principal can choose any stochastic process for the agent's beliefs, subject only to the martingale requirement (which is imposed by Bayesian updating) and the upper bound on the agent's attention. We model this upper bound using a "uniformly posterior-separable" information cost, in the terminology of the rational inattention literature (Caplin et al. [2022]).[3]

The rest of the paper is organized as follows. We begin in section 2 by describing the basic environment of our model. Section 3 characterizes optimal policy in our baseline model. Section 4 discusses the key implications, welfare minimization and belief polarization. Section 5 discusses several extensions of our baseline model, and in Section 6 we conclude.

2 The Environment

2.1 The Agent's Problem

We study the problem of a rational, Bayesian agent receiving signals about an underlying state for the purpose of taking an action. We model information acquisition as a continuous time process, building on results in Hébert and Woodford [2021b] and Zhong [Forthcoming].

Let X be a finite set of possible states of nature. The state of nature is determined ex-ante, does not change over time, but is not known to the agent. Let $q_t \in \mathscr{P}(X)$ denote the DM's beliefs at time $t \in [0,\infty)$, where $\mathscr{P}(X)$ is the probability simplex defined on X. We will represent q_t as vector in $\mathbb{R}_+^{|X|}$ whose elements sum to one, each of which corresponds to the likelihood of a particular element of X, and use the notation $q_{t,x}$ to denote the likelihood under the agent's beliefs at time t over the true state being $x \in X$.

At each time t, the agent can either stop and choose an action from a finite set A, or continue to acquire information. Let τ denote the time at which the agent

[3]Examples of such information costs include mutual information, as applied in Sims [2010], as well as other proposed alternatives (Hébert and Woodford [2021a], Bloedel and Zhong [2020]).

stops and makes a decision, with $\tau = 0$ corresponding to making a decision without acquiring any information. The agent receives utility $u_{a,x}$ if she takes action a and the true state of the world is x, and pays a flow cost of delay per unit time, $\bar{\kappa} > 0$, until an action is taken. This flow cost captures the opportunity cost of the agent's time net of whatever utility is gained using the principal's platform. In 5.2, we provide an extension where there is an increasing time-dependent flow cost of delay $\kappa(t)$. .

Let $\hat{u}(q_\tau)$ be the payoff (not including the cost of delay) of taking an optimal action under beliefs q_τ:

$$\hat{u}(q_\tau) = \max_{a \in A} \sum_{x \in X} q_{\tau,x} u_{a,x}.$$

The agent's beliefs, q_t, will evolve as a martingale. This property follows from "Bayes-consistency." In a single-period model, Bayes-consistency requires that the expectation of the posterior beliefs be equal to the prior beliefs. The continuous-time analog of this requirement is that beliefs must be a martingale.

Formally, let Ω be the set of $\mathscr{P}(X)$-valued càdlàg functions, let $q : \Omega \times \mathbb{R}_+ \to \mathscr{P}(X)$ be the canonical stochastic process on this space, let $\{\mathscr{F}_t\}$ be the natural filtration associated with this canonical process, and let $\mathscr{F} = \lim_{t\to\infty} \mathscr{F}_t$. Let $\mathscr{T} \subset \Omega \to \mathbb{R}_+$ be the set of non-negative stopping times with respect to $\{\mathscr{F}_t\}$.

Given a probability measure P defined on $(\Omega, \mathscr{F})$, $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ defines a probability space. The agent's problem, given this probability space, is to choose a stopping time to maximize

$$V(P) = \sup_{\tau \in \mathscr{T}} \mathbb{E}^P[\hat{u}(q_\tau) - \bar{\kappa}\tau | \mathscr{F}_0].$$

2.2 The Principal's Problem

The principal chooses the information the agent receives so as to maximize engagement. In the background, we have in mind that the principal is showing ads (unrelated to the agent's current decision problem) or otherwise profiting from each additional increment of attention the agent pays to the information the principal provides.

We will assume the agent can process information only at a certain rate. We measure the agent's information processing using a continuous time version of what Caplin et al. [2022] call a "uniformly posterior-separable" cost function, as described in Hébert and Woodford [2021b]. Uniformly posterior-separable cost functions are defined in terms of a "generalized entropy function," $H : \mathscr{P}(X) \to \mathbb{R}_+$. We assume that H is strongly convex and twice continuously-differentiable.

We require the agent's belief process to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} I_t = \limsup_{h \downarrow 0} \frac{1}{h} \mathbb{E}^P[H(q_{t+h}) - H(q_{t^-}) | \mathscr{F}_{t^-}] \leq \chi, \tag{2}$$

where $\chi > 0$ is a finite constant. Here, I_t denotes the cumulative information acquired by the agent. This constraint can be understood as the continuous-time analog of requiring that $E^P[H(q_t) - H(q_{t-h}) | \mathscr{F}_{t-h}] \leq \chi \delta$ in a discrete time model with time interval δ.

The principal is assume to profit at a rate of $\rho > 0$ per unit of attention paid by the agent (as measured by the left-hand side of (2)). The principal's goal is to design the agent's belief process so as the maximize profits, taking into account the fact the agent will know the nature of the beliefs process and optimally choose when to stop paying attention.

Let $\bar{q}_0 \in \mathscr{P}(X)$ be the agent's prior. The principal chooses his policies from the set $\mathscr{A}(\bar{q}_0)$, which is the set of probability measures on $(\Omega, \mathscr{F})$ such that q is martingale belief processes with $q_0 = \bar{q}_0$, increasing adapted processes I with $I_0 = 0$, and non-negative stopping times τ, such that (2) is satisfied for all $t \in [0, \tau)$. Formally, the principal chooses the probability measure P, which is equivalent to choosing the law of of the belief process q. In this sense, the principal can choose any càdlàg martingale belief process with $q_0 = \bar{q}_0$, subject to the constraint imposed by the agent's information processing capacity.[4]

Definition 1. The principal's problem given initial belief $\bar{q}_0 \in \mathscr{P}(X)$ is to maximize

[4] An alternative interpretation is that the principal can send more information than the agent can process, in which case the agent chooses which information to attend to. Allowing the agent this choice cannot benefit the principal, and it is therefore without loss of generality to suppose the principal chooses a process that satisfies the agent's information processing constraint.

engagement,

$$J(\bar{q}_0) = \sup_{(P,I,\tau)\in\mathscr{A}(\bar{q}_0)} \rho\mathbb{E}^P[I_\tau|\mathscr{F}_0]$$

subject to the agent's stopping decision,

$$\tau \in \arg\max_{\tau'\in\mathscr{T}} \mathbb{E}^P[\hat{u}(q_{\tau'}) - \bar{\kappa}\tau'|\mathscr{F}_0].$$

Note that we have assumed that the agent is willing to follow the principal's recommended stopping rule whenever she is indifferent between that rule and another equally optimal stopping rule.

3 Optimal Policy

Throughout this section, we assume that it is prohibitively costly for the agent to perfectly learn the true state, to the point that the agent would prefer to learn nothing at all if confronted with only these two possibilities.

Assumption 1. *We assume that*

$$\hat{u}(\bar{q}_0) - \frac{\bar{\kappa}}{\chi}H(\bar{q}_0) > \sum_{x\in X} \bar{q}_{0,x}(\hat{u}(e_x) - \frac{\bar{\kappa}}{\chi}H(e_x)),$$

where $e_x \in \mathscr{P}(X)$ has full support on state $x \in X$.

This assumption will imply that perfectly learning the state with probability one cannot occur in our principal-agent problem, as the principal cannot simultaneously satisfy the agent's participation constraint and the restriction on the rate of information acquisition, (2). Note that this does not rule out the kind of signals considered in our example, because in our example the principal eventually switched to the agent-optimal policy (and as a result, the agent would not learn the state with probability one).

3.1 A Relaxed Problem

We start by defining a relaxed version of the principal's problem. Any probability measure and stopping rule the principal can implement will induce a probability measure over beliefs the agent will hold when she chooses to stop (i.e. a law for q_τ). We study a version of the principal's problem in this space. Let $\mathscr{P}(\mathscr{P}(X))$ denote the space of probability measures of beliefs on $\mathscr{P}(X)$, and, given any feasible policy chosen by the principal, let $\pi \in \mathscr{P}(\mathscr{P}(X))$ denote the probability measure of the stopped belief q_τ. The following lemma describes the date-zero participation constraint of the agent and an upper bound on the total engagement associated with the measure π.

Lemma 1. $\forall (P, I, \tau) \in \mathscr{A}(q_0)$ *satisfying the agent's optimal stopping in Definition 1, the following conditions are satisfied:*

1. $\mathbb{E}^\pi[\widehat{u}(q)] - \bar{\kappa}\mathbb{E}^P[\tau|\mathscr{F}_0] \geq \widehat{u}(\bar{q}_0)$*, and*
2. $\mathbb{E}^\pi[H(q) - H(\bar{q}_0)] \leq \chi\mathbb{E}^P[\tau|\mathscr{F}_0]$.

Proof. Condition (1): let $\tau' \equiv 0$, then the agent's utility from stopping is $\widehat{\mu}(q_0)$. The optimality of τ implies condition (i).

Condition (2): $\mathrm{d}I_t \leq \chi \mathrm{d}t \implies \mathbb{E}^P[I_\tau|\mathscr{F}_0] = \mathbb{E}^P\left[\int_0^\tau \mathrm{d}I_t|\mathscr{F}_0\right] \leq \mathbb{E}^P\left[\int_0^\tau \chi \mathrm{d}t|\mathscr{F}_0\right] = \chi\mathbb{E}^P[\tau|\mathscr{F}_0]$. □

Lemma 1 presents a necessary condition for any admissible policy for the principle in the optimization problem (Definition 1). The first condition states that the agent's optimal stopping utility is weakly greater than the utility from stopping immediately. The second condition states that the cumulative information acquired by the agent is weakly less than $\chi\mathbb{E}[\tau|\mathscr{F}_0]$—the maximal attention permitted by the information constraint (2).

Combining these two constraints,

$$\mathbb{E}^\pi[\widehat{u}(q) - \hat{u}(\bar{q}_0)] \geq \bar{\kappa}\mathbb{E}^P[\tau|\mathscr{F}_0] \geq \frac{\bar{\kappa}}{\chi}\mathbb{E}^\pi[H(q) - H(\bar{q}_0)].$$

Let us define the principal's relaxed optimization problem as incorporating only this combined constraint:

$$\bar{J}(q_0) = \sup_{\pi \in \mathscr{P}(\mathscr{P}(X)): \mathbb{E}^\pi[q]=q_0} \rho \mathbb{E}^\pi[H(q) - H(\bar{q}_0)] \tag{3}$$
$$s.t. \; \frac{\bar{\kappa}}{\chi} \mathbb{E}^\pi[H(q) - H(\bar{q}_0)] \leq \mathbb{E}^\pi[\widehat{\mu}(q) - \widehat{\mu}(\bar{q}_0)].$$

Because this combined constraint must hold in the original principal's problem, we must have $\bar{J}(\bar{q}_0) \geq J(\bar{q}_0)$.

Equation 3 is a convex optimization problem that satisfies the conditions in Theorem 4 of Zhong [2018], which immediately implies:

Proposition 1. *Equation 3 has a solution π^* with support size less than $2|X|$. Either $\bar{J}(q_0) = 0$ or $\bar{J}(q_0) > 0$ and there exists $\lambda > \frac{\rho\chi}{\bar{\kappa}}$ s.t. all π^* satisfy*

$$\pi^* \in \arg \max_{\pi \in \mathscr{P}(\mathscr{P}(X)): \mathbb{E}^\pi[q]=\bar{q}_0} \mathbb{E}^\pi \left[\hat{u}(q) - (\frac{\bar{\kappa}}{\chi} - \frac{\rho}{\lambda})(H(q) - H(\bar{q}_0)) \right].$$

This proposition demonstrates that π^* is the solution to a static rational inattention problem, with a general UPS cost function (of the sort studied by Caplin et al. [2022]). Those authors show that a necessary condition for π^* is that it *concavifies* the function $\widehat{\mu} + (\frac{\bar{\kappa}}{\chi} - \frac{\rho}{\lambda})H$.[5] Information acquisition is infeasible ($\bar{J}(q_0) = 0$) whenever the only π satisfying the constraint is the one with $\text{Supp}(\pi) = \{\bar{q}_0\}$. In this case, we will say that π^* is degenerate, and otherwise say that π^* is non-degenerate.

Let us take as given a solution π^* to this relaxed problem, and consider how it might be implemented in an incentive compatible way in the original principal's problem.

[5] Related results appear in earlier working papers by those authors and in the Bayesian persuasion literature.

3.2 Implementation

Take any non-degenerate $\pi \in \mathscr{P}(\mathscr{P}(X))$ s.t. $\mathbb{E}^{\pi}[q] = \bar{q}_0$, and define the stochastic process q_t as:

$$q_t = \bar{q}_0 + \mathbf{1}_{N_{\alpha}(t)\geq 1} \cdot (Q - \bar{q}_0), \tag{4}$$

where $Q \in \mathscr{P}(X)$ is a random variable distributed according to π and $N_{\alpha}(t)$ is an independent Poisson counting process with parameter α. Here, q_t is a compound Poisson process that jumps according to π at rate α. We call such process q_t an $\alpha-$*dilution* of π. [6]

Proposition 2. *For all non-degenerate $\pi \in \mathscr{P}(\mathscr{P}(X))$ that satisfy the constraint in* (3)*, let $\alpha = \frac{\chi}{\mathbb{E}^{\pi}[H(q)-H(\bar{q}_0)]}$ and let q_t be the α-dilution of π. Then q_t is feasible in the problem defined in 1 and implements utility level $\rho\mathbb{E}^{\pi}[H(q) - H(\bar{q}_0)]$.*

Proof. Let P be the law of the process defined in (5), and define the stopping time $\tau = \inf\{t \in \mathbb{R}_+ | q_t \neq q_0\}$. Evidently, conditional on continuation, $\lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}^P[H(q_t) - H(q_{(t-h)^-})|\mathscr{F}_{(t-h)^-}] = \lim_{h\downarrow 0} \frac{1}{h}(1 - e^{-\frac{\chi \cdot h}{\mathbb{E}^{\pi}[H(q)-H(q_0)]}})(\mathbb{E}^{\pi}[H(q) - H(\bar{q}_0)]) = \chi$. Thus, the information constraint is satisfied. Since $q_\tau \sim \pi$, the principle's utility is $\rho\mathbb{E}^{\pi}[H(q_t) - H(\bar{q}_0)]$. What remains to be verified is the agent's optimality condition.

Take any admissible stopping time τ',

$$\begin{aligned}
\mathbb{E}^P[\widehat{\mu}(q_{\tau'}) - \bar{\kappa}\tau'|\mathscr{F}_0] =& \text{Prob}(\tau' < \tau)\left(\widehat{\mu}(q_0) - \bar{\kappa}\mathbb{E}^P\left[\tau'|\tau' < \tau\right]\right) \\
&+ \text{Prob}(\tau' \geq \tau)\left(\mathbb{E}^P\left[\widehat{\mu}(q_\tau) - \bar{\kappa}\tau'|\tau' \geq \tau\right]\right) \\
\leq& \text{Prob}(\tau' < \tau)\left(\mathbb{E}^{\pi}[\widehat{\mu}(q) - \bar{\kappa}\mathbb{E}^P[\tau|\mathscr{F}_0] - \bar{\kappa}\mathbb{E}^P\left[\tau'|\tau' < \tau\right]\right) \\
&+ \text{Prob}(\tau' \geq \tau)\left(\mathbb{E}^{\pi}[\widehat{\mu}(q)] - \bar{\kappa}\mathbb{E}^P\left[\tau|\tau' \geq \tau\right]\right) \\
=& \text{Prob}(\tau' < \tau)\left(\mathbb{E}^{\pi}[\widehat{\mu}(q) - \bar{\kappa}\mathbb{E}^P\left[\tau|\tau' < \tau\right]\right) \\
&+ \text{Prob}(\tau' \geq \tau)\left(\mathbb{E}^{\pi}[\widehat{\mu}(q)] - \bar{\kappa}\mathbb{E}^P\left[\tau|\tau' \geq \tau\right]\right) \\
=& \mathbb{E}^P[\widehat{\mu}(q) - \bar{\kappa}\tau|\mathscr{F}_0].
\end{aligned}$$

[6] Pomatto et al. [2018] first introduce the notion of dilution. They define the dilution of an information structure π as "producing π with probability α and uninformative signal with probability $1-\alpha$." (the same notion appeared in Bloedel and Zhong [2020]) Our notion of α-dilution is essentially the repetition of a dilution in continuous time.

The first equality is from the definition of conditional expectations and the process defining q_t, (5). The first inequality is from the constraint $\mathbb{E}^{\pi}[\widehat{\mu}(q) - \widehat{\mu}(\bar{q}_0)] \geq \bar{\kappa}\mathbb{E}^{P}[\tau|\mathscr{F}_0]$ in the relaxed problem (3) and $E[\tau|\tau' \geq \tau] \leq E[\tau'|\tau' \geq \tau]$. The second equality is from the memorylessness property of τ,[7] and the last from the definition of conditional expectations. $\square$

Combining Lemma 1, Proposition 1, and Proposition 2, we obtain the main characterization of the optimal policy:

Theorem 1. *$\forall \bar{q}_0 \in \mathscr{P}(X)$, there exists $\pi^* \in \mathscr{P}(\mathscr{P}(X))$ with support size less than $2|X|$ solving Equation 3. If Supp$(\pi^*) = \{\bar{q}_0\}$, the agent will immediately stop and any feasible policy is optimal. Otherwise, let $\alpha^* = \frac{\chi}{\mathbb{E}^{\pi^*}[H(q) - H(\bar{q}_0)]}$, and let (P^*, τ^*) be the law and jumping time of the α^*-dilution of π^*. Then, $(P^*, I_t^* = \chi t, \tau^*)$ is an optimal solution to the principle's problem (Definition 1).*

There are generally many optimal policies in the principal's problem. First, there may be multiple π^* that solve the relaxed principal's problem (although uniqueness in static rational inattention problems is guaranteed under certain additional assumptions). Second, there are many stochastic processes q_t (equivalently, laws P) and stopping times τ that induce the same law for q_τ; provided that this law is equal to π^* and that incentive compatibility and the bounds on information acquisition are satisfied, all such policies are optimal. However, it is not the case that anything goes, as we show next.

3.3 Characterization of optimal policies

For the purpose of discussing optimal polices, we consider the case in which the optimal policy is to acquire some information ($\bar{J}(\bar{q}_0) > 0$ in the context of Proposition 1).

To begin, we define, given a measure $\pi \in \mathscr{P}(\mathscr{P}(X))$, the value of information acquisition in a hypothetical restricted static rational inattention problem:

$$V_R(q, \pi) = \max_{\pi' \in \mathscr{P}(\mathrm{Supp}(\pi)) : E^{\pi'}[q'] = q} \mathbb{E}^{\pi'}\left[\widehat{\mu}(q') - \hat{u}(q) - \frac{\bar{\kappa}}{\chi}(H(q') - H(q))\right].$$

[7]Specifically, $\mathbb{E}^{P}[\tau|\tau' < \tau] = \mathbb{E}^{P}[\tau - \tau'|\tau' < \tau] + \mathbb{E}^{P}[\tau'|\tau' < \tau] = \mathbb{E}^{P}[\tau|\mathscr{F}_0] + \mathbb{E}^{P}[\tau'|\tau' < \tau]$.

This problem maximizes the agent's expected utility subject to the usual constraint of Bayes consistency and the additional constraint that the posteriors lie in the support of π. We use the notation $V_R(q,\pi)$ to emphasize that the problem is restricted relative to the usual rational inattention problem. Note that the problem is feasible only for q that lie in in the convex hull of the support of π, which we denote by $\mathrm{Conv}(\mathrm{Supp}(\pi))$.

We interpret $V_R(q,\pi)$ as describing the value of information acquisition, in the sense that $V_R(q,\pi)$ is the difference between the utility the agent achieves with and without information acquisition.[8] We define the set of "suspensive" beliefs given π as the set for which the value of information acquisition is weakly positive.

Definition 2. Given $\pi \in \mathscr{P}(\mathscr{P}(X))$, the belief $q \in \mathscr{P}(X)$ is *decisive* if $q \in \mathrm{Supp}(\pi)$; the belief $q \in \mathscr{P}(X)$ is *suspensive* if q$\in$Conv(Supp(π)) $\setminus$ Supp(π) and satisfies $V_R(q,\pi) \geq 0$. Let $\Delta(\pi) \subseteq P(X)$ denote the set of all suspensive beliefs given π.

Given a measure π that describes the stopping beliefs, it is straightforward that beliefs in $\mathrm{Supp}(\pi)$ are "decisive" because they directly lead to stopping. At the suspensive beliefs in $\Delta(\pi)$, the agent benefits, relative to the prior consistent with π, from the signals that the principal is offering (which always result in some posterior in the support of π). Therefore, $\Delta(\pi)$ characterizes the beliefs at which the principle can possibly induce "suspense" before eventually leading the agent to stopping beliefs in $\mathrm{Supp}(\pi)$.

Suppose the optimal policy in (3), π^*, is unique. In this case, the stochastic process for beliefs q_t cannot leave the set of suspensive beliefs $\Delta(\pi^*)$ before $t = \tau$. If q_t left the convex hull of the support of π^*, then π^* would not describe the law of q_τ. If q_t was decisive or if $V_R(q_t,\pi^*) < 0$, the agent would choose to stop. In the case of multiple optimal policies, this argument applies for some optimal π^*, which leads to the following result.

Proposition 3. *Given $\bar{q}_0 \in \mathscr{P}(X)$, let Π^* be the set of solutions to* (3)*. Then, in all solutions to the principal's problem, for all $t \in [0,\tau^*)$, beliefs are suspensive given*

[8] The value of information acquisition is related to, but not the same as, the notion of uncertainty defined in Frankel and Kamenica [2019].

some $\pi^* \in \Pi^*$,

$$Supp(q_t) \subseteq \bigcup_{\pi^* \in \Pi^*} \Delta(\pi^*),$$

and at $t = \tau^*$, *beliefs are decisive given some* $\pi^* \in \Pi^*$,

$$Supp(q_\tau) \subseteq \bigcup_{\pi^* \in \Pi^*} Supp(\pi^*).$$

Proof. Let π^* be a solution to the problem defined in (3). We first show that for any $q' \in \text{Supp}(\pi^*)$, $\bar{J}(q') = 0$. Let π^{**} be the optimal policy associated with q'. The policy $\pi^{***}(q) = \pi^*(q)\mathbf{1}\{q \neq q'\} + \pi^*(q')\pi^{**}(q)$ is feasible in (3) given the prior $\bar{q}_0$ and would achieve strictly higher utility than π^* if $\bar{J}(q') > 0$. It follows in the principal-agent problem that if $q_t \in \text{Supp}(\pi^*)$, then $t = \tau^*$, because there is no way to induce the agent to continue.

By Theorem 1, there is some $\pi^* \in \Pi^*$ that is the law of q_{τ^*} under the optimal policies, and the second claim follows. By the martingale property of the belief process, $q_t = E^{P^*}[q_\tau | \mathscr{F}_t, t < \tau^*]$ for all $t \in [0, \tau^*)$, and thus q_t must lie in the convex hull of the support of π^*. The belief q_t cannot be decisive if $t < \tau^*$, by the argument above. By the definition of (3), $\frac{\bar{\kappa}}{\chi}\mathbb{E}^{\pi^*}[H(q) - H(\bar{q}_0)] \leq \mathbb{E}^{\pi^*}[\widehat{\mu}(q) - \widehat{\mu}(\bar{q}_0)]$, from which it follows immediately that $V_R(q_t, \pi^*) \geq 0$. $\square$

Solving (3) generically leads to the IC constraint binding, which suggests that the value of acquiring π^* is exactly zero at the prior for the agent. The set of suspensive beliefs $\Delta(\pi^*)$thus defines beliefs that are "more uncertain" than the prior. Therefore, the interpretation of Proposition 3 is that two types of signals appear in any optimal policy almost surely. The first type are *decisive* signals that are so informative that the agent makes a decision immediately after receiving the signal. The second type are *suspensive* signals that causes the agent's beliefs to move to a posterior that is "more uncertain than the prior."

3.4 The agent-optimal benchmark

We will compare our results to a benchmark in which the agent and not the principal chooses the probability space and martingale belief process (subject to (2)). This benchmark is a special case of the more general models described in Hébert and Woodford [2021b] and Zhong [Forthcoming].

Those authors show that the optimal policies of this benchmark dynamic model are also equivalent to the solution to a static rational inattention problem. That is, under the optimal policies given the initial belief $\bar{q}_0 \in \mathscr{P}(X)$,

$$\mathbb{E}^P[\hat{u}(q_\tau) - \bar{\kappa}\tau | \mathscr{F}_0] = V^B(\bar{q}_0)$$
$$= \max_{\pi \in \mathscr{P}(\mathscr{P}(X)): E^\pi[q] = \bar{q}_0} \mathbb{E}^\pi[\hat{u}(q) - \frac{\bar{\kappa}}{\chi}(H(q) - H(\bar{q}_0))]$$

This solution can be implemented, as above, by a compound Poisson process, but also by a diffusion (and many other processes).

This static rational inattention problem is essentially identical to the one that characterizes optimal policy in our principal-agent framework, except that the UPS information cost is scaled by $\frac{\bar{\kappa}}{\chi}$ in the benchmark problem and $\frac{\bar{\kappa}}{\chi} - \frac{e}{\lambda}$ in the principal agent case. The similarity in structure between these two problems will allow us to highlight the implications of engagement maximization.

Before proceeding, we should note that the principal is unable to induce any information acquisition whenever the only feasible π in the relaxed problem places full support on $\bar{q}_0$; this corresponds to the case in the benchmark model in which the agent strictly prefers to not gather information. In all other cases, the principal will induce information acquisition in the principal-agent model. Consequently, the "strict stopping" region of the benchmark model exactly coincides with the stopping region in the principal-agent model.

3.5 Example (continued)

We illustrate Theorem 1 by continuing the numerical example in Section 1.1. We maintain the same model primitives. Figure 1 illustrates the optimal q_τ for every prior belief: When $\bar{q}_0 \leq \underline{q}$ or $\bar{q}_0 \geq \bar{q}$, the agent stops immediately no matter what information he will receive. When $\bar{q}_0 \in (\underline{q}, \bar{q})$, the dashed lines illustrate the support of the agent-optimal policy. The lines are flat, meaning that the agent's stopping beliefs do not change with the prior belief, consistent with the prediction of the standard RI models.[9] The solid curves illustrate the support of the unique optimal π^*. They are closer to the boundaries zero and one, indicating that the posterior beliefs become more polarized under the principle's optimal policy.

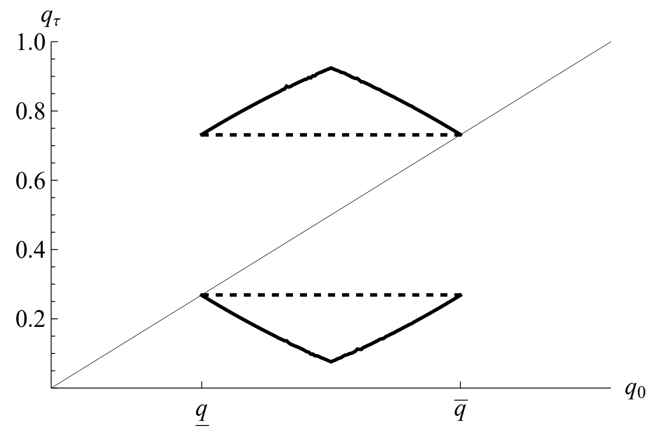


Figure 1: Supp(q_τ) as a correspondence of q_0

Figure 2 illustrates the engagement level for every prior belief. The dashed curves represent the agent-optimal policy and the solid curves represent the principle-optimal policy. The principle-optimal policy induces higher engagement level than the agent-optimal policy. However, the engagement level converges to zero when the prior belief goes to the boundary of the continuation region $(\underline{q}, \bar{q})$.

Now, focus on the $\bar{q}_0 = \frac{1}{2}$ case. As is illustrated in Figure 1, the unique optimal π^* involves two posterior beliefs $\{q^1, q^2\}$, where $q^1 < \underline{q}$ and $q^2 > \bar{q}$. Per Proposition 2, the dilution of π^* implements an optimal policy. In this case, the dilution of π^* is the unique optimal policy. It is apparent from the symmetry of the problem that the value of information acquisition, $V_R(q, \pi^*)$, is maximized at $q = \frac{1}{2}$. As a result, the set of suspensive beliefs, $\Delta(\pi^*)$, is a singleton, $\{\bar{q}_0\}$. By Proposition 3, beliefs will remain at $\bar{q}_0$ until they jump to either q^1 or q^2.

When $\bar{q}_0 < \frac{1}{2}$, the optimal policy is not unique. In this case, again by the sym-

[9] This property is known as the "locally invariant posteriors" (Caplin et al. [2022]).

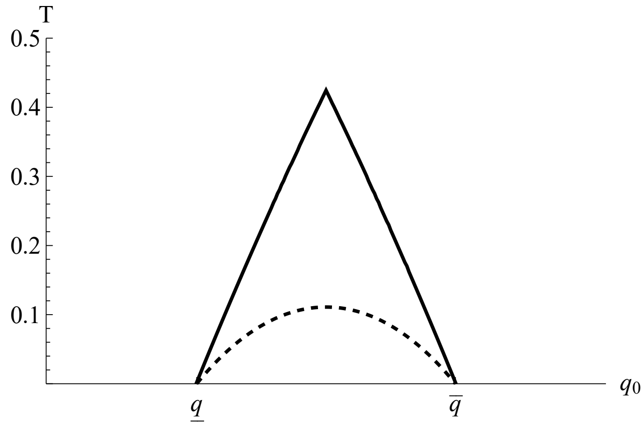


Figure 2: Engagement as a function of q_0

metry of the problem, it is apparent that the set of suspensive beliefs is $\Delta(\pi^*) = [q_0, 1-q_0]$. Beliefs will remain in this interval until eventually jumping to either q^1 or q^2. A symmetric argument applies when $\bar{q}_0 > \frac{1}{2}$. Figure 3 illustrates the sample paths of one optimal policy, in which beliefs jump between $\bar{q}_0$ and $1-\bar{q}_0$ before eventually jumping to q^1 or q^2.

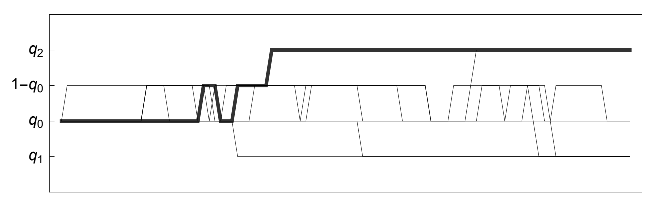


Figure 3: Sample paths of an optimal policy

4 Implications

In this section, we consider several implications of our main result.

Implication 1: Welfare Minimization. An immediate implication of Theorem 1 is that the agent's participation constraint binds ($\lambda > 0$). That is,

$$E[\hat{u}(q_{\tau^*}) - \bar{\kappa}\tau^* | \mathscr{F}_0^*] = \hat{u}(\bar{q}_0).$$

Strikingly, the agent is no better offer receiving information from an engagement-maximizing principal than if she could not receive any information at all. The principal extracts the full surplus generated by the ability to produce information and apply it in the agent's decision problem, despite the agent's ability to choose when to stop and act. The principal, in maximizing the engagement of the agent, minimizes the agent's welfare subject to a participation constraint.

A different way of viewing this result is that paying the principal for the information using attention as opposed to a monetary payment is inefficient. If the principal could commit to a signal structure in exchange for a payment from the agent (the transferable utility case), he would of course provide the optimal signal structure from the agent's perspective, as this would maximize revenue. Our model assumes that the principal is instead paid via attention, which distorts the principal's incentives for information provision.

In practice, there are likely good reasons why platforms such as Facebook and Twitter choose to make their services free to users. Providers that charge users in exchange for information (the New York Times, for example) generally have far fewer users than freely accessible, advertising-supported platforms. Our baseline model does not speak to this trade-off, but instead offers an explanation for why the information available on free-to-use platforms differs in nature from the information available from paid information providers.

Implication 2: Commitment is Unnecessary. Consider a modified version of our model in which the principal lacks commitment with respect to the stochastic process q_t and instead simply chooses signals for the agent to receive at each instant. It is immediate than an equilibrium of this game without commitment exists in which the principal's value function is $\bar{J}(q)$, the agent's value function is $\hat{u}(q)$, and the principal chooses at each moment the signals described in Proposition 2.

The intuition behind this result is that, under the optimal policy, the principal never promises the agent anything other than the minimum possible utility starting from any belief $q \in \mathscr{P}(X)$. That is, although it is in theory possible for a principal with commitment to induce the agent to continue by promising a larger-than-minimum level of utility in the future, such promises are not necessarily a feature of the optimal policy with commitment, and consequently commitment is unnecessary.

However, a lack of commitment can restrict the set of optimal policies. If beliefs change in a way that is strictly suspensive, $V_R(q_t, \pi^*) > 0$, then the agent's participation constraint becomes slack. The principal at this point would be tempted to re-optimize (inducing a different $\pi^{**} \neq \pi^*$). In our example, the set of suspensive beliefs was $[q_0, 1-q_0]$ for any $q_0 < \frac{1}{2}$. But it is apparent from Figure 2 that if $q_t \in (q_0, 1-q_0)$, the optimal posteriors given the initial belief q_t are no longer in the support of π^*. Commitment is thus necessary for the agent to tolerate moving to strictly suspensive beliefs. In the absence of commitment, the only optimal policies are the kinds of dilutions described in Proposition 2 and policies that mix such dilutions with jumps to other equally suspensive beliefs (such as the policies illustrated in Figure 3).

Implication 3: Extreme Beliefs. Let $Q^i(\bar{q}_0)$ be the union of the support of all optimal policies in the benchmark ($i = a$) and principal-agent ($i = p$) models. Let $\text{Conv}Q^i(\bar{q}_0)$ denote the convex hull of $Q^i(\bar{q}_0)$. The following proposition demonstrates that the beliefs the agent will hold when stopping after using the principal's platform are more extreme than the beliefs the agent would choose to acquire in the benchmark model. This result follows from the observation that in both models, stopping beliefs are characterized by the solution to a static rational inattention problem, with a lower information cost in the principal-agent case than in the benchmark case.

Proposition 4. $\forall q \in Q^p(\bar{q}_0), q$ *is not a relative interior point of* $\text{Conv}Q^i(\bar{q}_0)$.

Proof. We prove the contra-position of Position 4. Define

$$U^a(q) = \hat{u}(q) - \frac{\bar{\kappa}}{\chi} H(q).$$

Let $\hat{U}^a$ be the upper concave hull of U^a. $\forall q^*$ that is a relative interior point of $\text{Conv}Q^i(\bar{q}_0)$, by definition, $U^a(q^*) \leq \hat{U}^a(q^*)$.

Define

$$U^p(q) = \hat{u}(q) - \left(\frac{\bar{\kappa}}{\chi} - \frac{e}{\lambda}\right) H(q),$$

where λ is defined in Proposition 1. Then, by Proposition 1, $\forall q \in Q^p(\bar{q}_0)$, $\hat{U}^p(q) = U^p(q)$. Note that $U^p(q) = U^a(q) + \frac{e}{\lambda}H(q)$. Then, it is obvious that $\hat{U}^p$ is no higher than the upper concave hull of $\hat{U}^a + \frac{e}{\lambda}H$. Now we prove that $\hat{U}^p$ is exactly the upper concave hull of $\hat{U}^a + \frac{e}{\lambda}H$. If not, then there exists a finite support $\pi \in \mathscr{P}(\mathscr{P}(X))$ s.t.

$$\begin{aligned}
\sup_{\pi \in \mathscr{P}(\mathscr{P}(X))} U^p(q) <& \sum \pi_j (\hat{U}^a(q_j) + \frac{e}{\lambda} H(q_j)) \\
=& \sup \pi_j \left(E_{\pi^j}[U^a(q)] + \frac{e}{\lambda} H(q_j) \right) \\
\leq& \pi_j E_{\pi^j} \left[U^a(q) + \frac{e}{\lambda} H(q) \right] \\
=& E_{\sum \pi_j \pi^j} \left[U^p(q) \right].
\end{aligned}$$

The first inequality is by the assumption. The existence of π^j that satisfies the first equality is from the $\hat{U}^a$ being the upper concave hull of U^a. The second inequality is Jensen's inequality. However, we reach a contradiction. Therefore,

$$\begin{aligned}
U^p(q^*) =& U^a(q^*) + \frac{e}{\lambda} H(q^*) \\
\leq& \hat{U}(q^*) + \frac{e}{\lambda} H(q^*) \\
<& \hat{U}^p(q^*),
\end{aligned}$$

where the last strict inequality is from $\hat{U} + \frac{e}{\lambda}H$ being strictly convex in a open region containing q^*. As a result, $q^* \notin Q^p(\bar{q}_0)$. $\square$

This proposition is illustrated in Figure 2. For all priors $\bar{q}_0 \in (\underline{q}, \bar{q})$, which is the continuation region in the benchmark model defined by $V^B(\bar{q}_0) > \hat{u}(\bar{q}_0)$, $q^1 < \underline{q} < \bar{q} < q^2$, and consequently $Q^p(\bar{q}_0) = \{q^1, q^2\} \not\subset \hat{Q}^a(q_0) = [\underline{q}, \bar{q}]$.

Extreme beliefs are a natural consequence of the tradeoffs facing the principal. By forcing the agent to ultimately acquire more information than she would choose for herself (the extreme beliefs), the principal can simultaneously delay the agent's stopping decision while simultaneously providing information the agent is willing to attend to.

Implication 4: The Necessity of Jumps in Beliefs. We have shown in Proposition 3 that, at all times under an optimal policy, beliefs are either suspensive or decisive. This result immediately implies that the optimal policy cannot involve continuous sample paths for beliefs (we assume the initial prior lies in the continuation region). If sample paths were continuous, the belief process q_t would have to exit $\Delta(\pi^*)$, which would immediately induce the agent to stop. Consequently, beliefs must jump discontinuously from the set of suspensive beliefs (which lie in the continuation region of the benchmark model) to the set of decisive beliefs, which lie in the interior of the strict stopping region of the benchmark model (by Proposition 4).

Relatedly, our result that the agent's value function is equal to $\hat{u}(q)$ everywhere implies that a pure diffusion process is infeasible almost everywhere in the continuation region. Because the agent recognizes that the principal will leave her with no surplus, only information that causes her to change her beliefs about the currently optimal action is valuable. If the belief q_t is such that one action is strictly optimal given those beliefs (which will be true generically), the principal must offer at least the possibility of jumping to a different region in which another action is optimal; otherwise, the agent will perceive no benefit from the information provided. We interpret this result as suggesting that the principal provides that agent with news articles containing extreme or sensational claims, which should cause the agent to either move her beliefs a lot or not at all, as opposed to providing more nuanced or qualified information.

5 Extensions

In this section, we consider several extensions to the model analyzed above. We first discuss the possibility that the agent does not fully attend to the principal's signals, and show that this is never optimal for the agent. We next consider the case in which the cost of delay is increasing over time, representing either an increasing opportunity cost of time or diminishing utility from using the platform. We then discuss an alternative version of our model in which agents are not capacity constrained and the principal's objective is to maximize the time until a decision is made. Lastly, we provide an alternative interpretation of our model in the context of a teacher and student.

5.1 Incentive compatibility

In our main model, the agent solves a stopping problem given the principal's chosen information structure. In practice, the agent may choose to process the principal's chosen information structure in a selective way, which leads to an extra "incentive compatibility" requirement for the principal. We argue that the optimal policy is "incentive compatible" as is defined in Proposition 5.

Proposition 5. *Let π^* be defined as in Theorem 1. For all stopping times $\hat{\tau} \in \mathscr{T}$ and all $\hat{\pi} \in \mathscr{P}(\mathscr{P}(X))$ that are a mean preserving contraction of π^*,* $\mathbb{E}^{\hat{\pi}}\left[\hat{\mu}(q) - \bar{\kappa}\hat{\tau} \middle| \mathscr{F}_0\right] \leq \hat{\mu}(\bar{q}_0)$.

Proof.

$$\begin{aligned}
\mathbb{E}^{\hat{\pi}}\left[\hat{\mu}(q) - \bar{\kappa}\hat{\tau} \middle| \mathscr{F}_0\right] \leq & \mathbb{E}^{\pi^*}\left[\hat{\mu}(q_{\hat{\tau}}) - \bar{\kappa}\hat{\tau} \middle| \mathscr{F}_0\right] \\
\leq & \mathbb{E}^{\pi^*}[\hat{\mu}(q) - \bar{\kappa}\tau | \mathscr{F}_0] = \widehat{\mu}(\bar{q}_0)
\end{aligned}$$

The first inequality is from the convexity of $\widehat{\mu}$ and $\widehat{\pi}$ being a mean preserving contraction of $q_{\widehat{\tau}}$. The second inequality is the same as the main inequality of Proposition 2. The last equality is from the constraint in Equation 3 being binding for π^*. □

Any deviation of the agent leads to an alternative belief process $\widehat{q}_t$ and a corresponding stopping time $\widehat{\tau}$. Instead of modeling $\widehat{q}_t$ explicitly, observe that since $\widehat{q}_t$ is derived from a garbling of the signals that generate q_t, the law of $\widehat{q}_{\widehat{\tau}}$ (i.e. $\hat{\pi}$) is a mean preserving contraction of the law of $q_{\hat{\tau}}$. Proposition 5 therefore implies that the agent's expected utility from such a deviation is less than $\widehat{\mu}(\bar{q}_0)$—the utility of fully attending the principal's information.

The intuition of the result is simple: in the optimal policy, the principal is providing valuable information to the agent, to ensure that the agent does not stop using the platform. If the agent fails to attend to this information while continuing to use the platform, she incurs the cost of delay without fully benefiting from the information provided.

5.2 Increasing Costs of Delay

In this subsection, we consider the case in which the cost of delay is increasing over time. We denote the flow cost of delay at period t by $\kappa(t)$. This case is of interest because it is natural to assume that the utility gained from using a platform declines, and the opportunity cost of time rises, the more time is spent on the platform.

This case is not as tractable as the constant cost case. We will impose additional symmetry assumptions, not present in our main model, to allow us to prove our results. We should emphasize that we have no reason to believe, absent these symmetry assumptions, that our main results fail– but we have not been able to prove that they continue to hold, either. We will conjecture and verify that an equilibrium without commitment exists in which our characterization of optimal policy, suitably modified, continues to hold. Note that, that unlike the constant cost case, we will prove only the existence of an such an equilibrium in the absence of commitment, as opposed to proving the stronger claim that all equilibria share these properties.

Specifically, we will prove that there exists a $t_0 \in \mathbb{R}_+$ such that if q_{t_0} is uniform, then the optimal policy in the increasing cost case can be constructed from the optimal policies of the constant cost case.

Let $\pi^*(\kappa)$ be a solution to Proposition 1 under a constant cost of delay $\bar{\kappa} = \kappa$, given the uniform prior $\iota \in \mathscr{P}(X)$. The essence of our solution is that in each time t the principal offers a belief process that either jumps to an element of the support of $\pi^*(\kappa(t))$ or remains constant. Define the stochastic process $\hat{q}_s$ as:

$$\hat{q}_s = \iota + \mathbf{1}_{N_\alpha(s)\geq 1} \cdot (Q_s - \iota), \tag{5}$$

where $Q_s \in \mathscr{P}(X)$ is a random variable distributed according to $\pi^*(\kappa(s))$ and $N_\alpha(s)$ is an independent Poisson counting process with an arbitrary parameter $\alpha > 0$ and

$N_\alpha(s) = 0$ for $s \leq t_0$. Now define a time-changed version of this process,

$$q_t = \hat{q}_{s(t)},$$

with $s(t) = t$ for $t \in [0, t_0]$ and

$$\frac{ds}{dt} = \frac{\chi}{\alpha E^{\pi^*(\kappa(t))}[H(q') - H(\bar{q}_0)]}$$

for all $t > t_0$.

We will prove that if $q_{t_0} = \iota$, the law of this time-changed process from t_0 onward, P^*, along with $I^*_t = \chi(t - t_0)$ and the stopping rule $\tau^* = \inf\{t \in [t_0, \infty) : \hat{q}_t \neq \iota\}$, is part of a solution to

$$\sup_{(P,I,\tau) \in \mathscr{A}(\iota)} \rho \mathbb{E}^P [I_\tau | \mathscr{F}_{t_0}]$$

subject to

$$\tau \in \arg\max_{\tau' \in \mathscr{T}} \mathbb{E}^P [\hat{u}(q_{\tau'}) - \int_{t_0}^{\tau'} \kappa(t) dt | \mathscr{F}_{t_0}].$$

That is, the principal-optimal policy (P^*, I^*, τ^*) can be implemented via this kind of signal structure.

Additional assumptions. We introduce here a set of additional assumptions for the purpose of tractability. The first assumption states that the delay cost is non-decreasing, truncated at finite time, and high enough to guarantee interior solutions.

Assumption 2. *We assume that $\kappa(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, $\exists T$ s.t. $\kappa(t)$ is constant for $t \geq T$, and* $\text{Supp}(\pi^*(\kappa(0)))$ *is in the interior of $\mathscr{P}(X)$.*

The non-decreasing aspect of this assumption is natural, given that we would like to model an increasing opportunity cost of time or diminishing utility from using the platform. The existence of a T after which the cost is constant is a technical device; our results will hold with T arbitrarily large. The assumption that the optimal constant-cost policy is interior at time zero ensures that this remains true for all $t > 0$, and simplifies our analysis. We view these assumptions as relatively innocuous.

Our substantive assumption is rotational symmetry. Let R denote a matrix in $\mathbb{R}^{|X|^2}$. We say R is a *rotation* if $Re_x \in \mathscr{P}(X)$, R is full rank, stochastic, and satisfies $R^J = I$ for some $J \in \mathbb{N}^+$. A function $f : \mathscr{P}(X) \to \mathbb{R}$ is rotationally symmetric if $f(q) \equiv f(R(q))$ (for some rotation R).[10]

Assumption 3. *Both H and $\hat{\mu}$ are rotationally symmetric.*

This kind of symmetry holds in our example, as the utility function and Shannon's entropy are both symmetric about $q = \frac{1}{2}$.

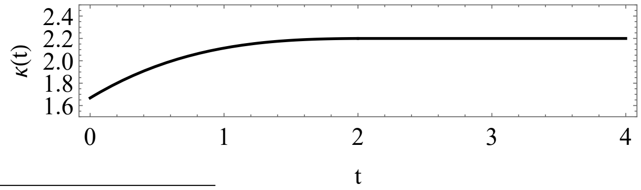
Theorem 2. *Under Assumptions 2 and 3, there exists a $t_0 < T$ s.t. if q_{t_0} is uniform, then (P^*, I^*, τ^*) is an optimal policy in the principal's problem from t_0 onward.*

Proof. See the appendix, section A.1. □

The role of the initial time t_0 is analogous to the role of Assumption 1 in our main analysis. At $t = t_0$, the agent's cost of delay is high enough to ensure that the incentive compatibility constraint is binding for the principal.

5.2.1 Example (continued)

We illustrate Theorem 2 by continuing the numerical example in Sections 1.1 and 3.5. We maintain the same model primitives except that $\kappa(t)$ no longer constant. The top panel of Figure 4 depicts $\kappa(t)$: $\kappa(t)$ strictly increasing from $[0,2]$ and stays constant afterwards. We choose $\kappa(t)$ take values close to 2 for the purpose of facilitating comparisons with our previous examples (which had fixed $\bar{\kappa} = 2$).



[10]For example, any permutation matrix is a rotation. Therefore, any exchangeable function is rotationally symmetric.

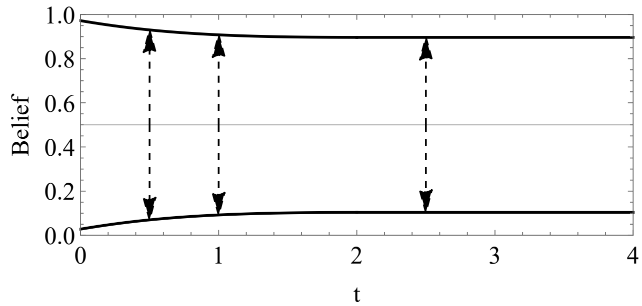


Figure 4: Increasing $\kappa(t)$ and the optimal policy.

The optimal policy is illustrated by the lower panel of Figure 4. The two black curves depict $\mathrm{Supp}(\pi^*(\kappa(t)))$. The optimal policy involves the same dilution structure as in Theorem 1, but with posterior beliefs varying over time. If one interpret how extreme the posterior belief is as the decision quality, then decision quality falls over time under increasing time cost.

5.3 Bounds on Belief Revision

In this subsection, we consider the case in which there is an exogenous bound on how much the agent can revise his posterior belief. Formally, for $d > 0$, let D^d denote the set of $\mathscr{P}(X)$-valued càdlàg functions whose jump size is bounded by d,

$$D^d = \left\{ f \in \Omega \middle| \forall t \in \mathbb{R}_+, \, |f(t) - f(t^-)| \leq d \right\}.$$

Let $\mathscr{A}^d(\bar{q}_0)$ denote the subset of $\mathscr{A}(\bar{q}_0)$ whose strategies are supported within D^d. Consider the optimization problem:

$$\begin{aligned} J^d(\bar{q}_0) = & \sup_{(P,I,\tau) \in \mathscr{A}^d(\bar{q}_0)} \mathbb{E}^P[\rho I_\tau | \mathscr{F}_0] \\ & \text{s.t.} \tau \in \arg\max_{\tau' \in \mathscr{T}} \mathbb{E}^P[\hat{u}(q_{\tau'}) - \bar{\kappa}\tau' | \mathscr{F}_0]. \end{aligned} \tag{6}$$

Equation 6 is the same as Definition 1 except that the set of admissible strategies is restricted to have no jumps of size greater than d.

We begin our analysis of this restricted problem by observing that the agent-optimal benchmark, V^B, is unchanged. This follows from results in Hébert and Woodford [2021b], who show that the agent-optimal policy can be implemented by a pure diffusion process. Let $E^A = \{q \in \mathscr{P}(X) | V^B(q) > \widehat{u}(q)\}$ be the continuation region of the agent-optimal benchmark.

Now define the set of beliefs that can be reached by a jump of size no greater than d from this continuation region:

$$\bar{Q}^d = \left\{ q \in \mathscr{P}(X) | \inf_{q' \in E^A} |q - q'| \leq d \right\}.$$

Note that the definition of $\bar{Q}^d$ is independent of the prior $\bar{q}_0$.

As we argued previously, the beliefs q_t must lie in the continuation region of the agent-optimal benchmark if $t < \tau$. Given that the maximum possible jump size is d, it follows that the stopping beliefs must lie in $\bar{Q}^d$.

Lemma 2. *If (P, I, τ) is admissible in Equation 6, then* $\text{Supp}(q_\tau) \subset \bar{Q}^d$.

Proof. See the appendix, section A.2. □

Trivially, if the bound d is sufficiently large, it does not restrict the principal, and the problem collapses to the problem considered in our main analysis.

Proposition 6. $\lim_{d \to \infty} J^d(\bar{q}_0) = J(\bar{q}_0)$.

Proof. For d larger than the diameter of $\mathscr{P}(X)$, $\mathscr{A}^d(\bar{q}_0) = \mathscr{A}(\bar{q}_0)$. □

Let us next consider the opposite case, in which the beliefs process must be continuous ($d = 0$). In this case, the only possible stopping beliefs are the ones on the boundary of the continuation region in the agent-optimal problem ($\bar{Q}^0$ is the closure of E^A). If in addition there are only two states ($|X| = 2$), the locally invariant posteriors property implies that the stopping beliefs will be identical to those the agent would choose in the agent-optimal problem; the only alternative the principal could choose would involve less information acquisition by the agent.

Proposition 7. *If $|X| = 2$, then $J^0(\bar{q}_0)$ is achieved by an agent-optimal policy.*

Proof. See the appendix, section A.3. □

When there are more than two states ($|X| > 2$), the principal's and agent's optimal policies need not coincide. Consider as an example the case of two actions ($|A| = 2$). The agent-optimal policy will in this case involve a diffusion on a line segment within the probability simplex (see Hébert and Woodford [2021b]). The principal cannot induce the agent to follow this line segment beyond its endpoints, but can send the agent signals that cause the agent's beliefs to move orthogonal to this line segment.

5.4 Optimal Policy without Capacity Constraints

In this subsection, we consider a modified version of our model with a constant cost of delay in which the agent has an unlimited capacity to acquire information, and the principal's goal is to maximize the time spent by the agent on the platform. The agent-optimal policy in this case is for the agent to learn the optimal action with certainty immediately, as this avoids entirely the cost of delay.

The principal in this case chooses his policies from the set $\bar{\mathscr{A}}(\bar{q}_0)$, which is the set of probability measures on $(\Omega, \mathscr{F})$ such that q is martingale belief processes with $q_0 = \bar{q}_0$ and non-negative stopping times τ. Note that this set does not impose the constraint on the rate of information acquisition, (2), that was imposed in our main analysis. The principal solves

$$J(\bar{q}_0) = \sup_{(P,\tau)\in\bar{\mathscr{A}}(\bar{q}_0)} \mathbb{E}^P[\tau|\mathscr{F}_0]$$

subject to the same constraint with respect to the agent's stopping decision,

$$\tau \in \arg\max_{\tau'\in\mathscr{T}} \mathbb{E}^P[\hat{u}(q_{\tau'}) - \int_0^{\tau'} \bar{\kappa} dt|\mathscr{F}_0].$$

The first part of Lemma 1 remains applicable: if π is the law of q_τ, we must have $\mathbb{E}^\pi[\widehat{u}(q) - \hat{u}(\bar{q}_0)] \geq \bar{\kappa}\mathbb{E}^P[\tau|\mathscr{F}_0]$. Now observe that the expected utility under π is bounded above by the utility of fully learning the state. Let $\pi^{\max} \in \mathscr{P}(X)$ be the unique probability measure that places full support on the extreme points of $\mathscr{P}(X)$ (i.e. the e_x basis vectors), with $\pi^{max}(e_x) = \bar{q}_{0,x}$.

By the convexity of $\hat{u}$, for all π such that $E^{\pi}[q] = \bar{q}_0$,

$$\mathbb{E}^{\pi^{max}}[\widehat{u}(q) - \hat{u}(\bar{q}_0)] \geq \mathbb{E}^{\pi}[\widehat{u}(q) - \hat{u}(\bar{q}_0)] \geq \bar{\kappa}\mathbb{E}^{P}[\tau|\mathscr{F}_0] \geq \bar{\kappa}J(\bar{q}_0).$$

It follows that $\bar{\kappa}^{-1}\mathbb{E}^{\pi^{max}}[\widehat{u}(q) - \hat{u}(\bar{q}_0)]$ is an upper bound on the utility achievable in this problem.

But now observe that the α-dilution of $\pi^{\max}$, as defined in (5), with intensity $\alpha = \bar{\kappa}(\mathbb{E}^{\pi^{max}}[\widehat{u}(q) - \hat{u}(\bar{q}_0)])^{-1}$, achieves this bound. Moreover, the policy is incentive-compatible: at each instant, the agent compares the utility benefit of the signal's arrival, $\alpha\mathbb{E}^{\pi^{max}}[\widehat{u}(q) - \hat{u}(\bar{q}_0)]$, against the cost of delay, $\bar{\kappa}$, and is willing to continue. It follows that this policy is optimal.

This policy is the optimal policy in a special case of our main model. Consider the case of our main model in which $\bar{\kappa} = \chi$ and $H(q) = \hat{u}(q)$. In this special case, $\pi^{\max}$ is an optimal policy in our relaxed problem (3), because the constraint in that problem satisfied for any policy, and the process described in Proposition 2 is exactly the process above. The intuition behind this equivalence is that in our main model, it is always optimal for the principal to exhaust the agent's information processing capacity. If exhausting this capacity necessarily involves satisfying the incentive compatibility constraint, then the principal is free to choose the process that simultaneously exhausts the agent's capacity and takes as long as possible to reach a decisive belief.

5.5 Engaging Test-Motivated Students

Our model was developed to describe the process of engagement maximization by an internet platform. However, the problem of engagement maximization appears in many contexts. In this subsection, we provide an alternative interpretation of our engagement maximization model in the context of a teacher (the principal) and a student (the agent). We first discuss this interpretation in the context of our general framework, and then provide a concrete example.

Let R be a set of possible responses to a test. Let the states of nature be $X = T \times Q$, where T denotes the true state (a finite set) and Q is the set of possible questions (functions $T \to R$ describing the correct response conditional on the true state). We will think of the test question realization $\iota \in Q$ as unknown to both the student and teacher (e.g. the test is a standardized test outside the control of the teacher).[11]

The action A is a function $Q \to R$ that describes the student's responses given the question. The student's utility given action $a \in A$ in state $x = (\omega, \iota) \in T \times Q$ is

$$u_{a,x} = v(\sum_{i=1}^{N} \mathbf{1}\{a_i(\iota) = \iota_i(\omega)\}),$$

where $a_i(\iota)$ denotes the student's response to question i and $\iota_i(\omega)$ is the correct response to question i given the state ω, and $v(\cdot)$ is an increasing function. That is, the student's utility is an increasing function of the number of questions answered correctly.

The teacher commits to a teaching strategy without regards to the realization of ω, which determines the sequence of signals the student will receive.[12] The teacher's objective is to maximize the student's learning. Let $q_\omega = \sum_{\iota \in Q} q_{(\omega,\iota)}$ denote the total probability of the true state $\omega \in T$ under $q \in \mathscr{P}(X)$. We will assume initially that the teacher's goal is to maximize the expectation of the log of this probability.

The expected utility for the teacher when the teaching strategy is designed (prior to the state being revealed, so that the expectation is taken over stopping beliefs and ω) is

$$E^P[\ln(q_{\tau,\omega})] = E^\pi[\sum_{\omega \in T} q_{\tau,\omega} \ln(q_{\tau,\omega})]$$

where π is the law of the posterior stopping beliefs q_τ under P. The rate at which the student can learn the truth is governed by the mutual information between her prior and posterior over the truth, and it is impossible for the student to acquire any

[11] A natural extension of this framework would be to consider the case in which the teacher designs and can reveal information about the test questions, in addition to revealing the truth. Exploring this case is beyond the scope of the present paper.

[12] Committing to a teaching strategy before $\omega \in T$ is revealed is analogous, in this context, to the assumption of commitment in Bayesian persuasion models. It avoids the need to consider strategic interactions between teachers with different "types" (knowledge of the truth).

information about the test questions. Define

$$H(q) = \begin{cases} \sum_{\omega \in T} q_{\tau,\omega} \ln(q_{\tau,\omega}) & \text{proj}(q, \mathscr{P}(Q)) = \text{proj}(\bar{q}_0, \mathscr{P}(Q)) \\ \infty & \text{otherwise.} \end{cases}$$

This function is infinite when the projections of q and $\bar{q}_0$ onto $\mathscr{P}(Q)$ are not equal, ensuring that the student will not acquire information about the test questions until she takes the test, and is otherwise equal to the negative of the Shannon's entropy of the projection of $q \in \mathscr{P}(X)$ onto $\mathscr{P}(T)$. Under these assumptions, the information processing constraint is exactly that (2) in our main model, and the teacher's objective is to maximize $E^P[H(q_\tau)]$. Note that we have constructed this example so that the constraint and principal's objective are governed by the same H function, in keeping with our main analysis.[13] If we assume that the student has a constant opportunity cost of time spent on the course equal to $\bar{\kappa}$, then this teacher-student model is equivalent to our principal-agent model.

The inherent conflict in this model is that the student is concerned only about the test-relevant portions of the truth, whereas the teacher would like the student to learn both the test-relevant and test-irrelevant aspects of the truth. The teacher must balance covering all aspects of the subject with the need to keep students engaged by providing test-relevant information. Our main results demonstrate that the teacher will drive the student to her reservation value (so that she is indifferent between studying and not) and cause the student to learn more (arrive at more extreme beliefs) than the student would choose for herself.

More subtly, our results show that the information the teacher provides is equivalent to the information the student would choose for herself if she had a lower opportunity cost of time (by Proposition 1 and Theorem 1). One implication of this result, which we highlight in the next example, is that the teacher will never induce the student to acquire test-irrelevant information (i.e. the teacher will "teach to the test"). The teacher optimally prefers to induce the student to acquire more test-relevant information than the student would choose to acquire on her own as

[13] This equivalence is a consequence of assuming the log-probability objective and is well-known in the proper scoring rule literature.

opposed to inducing the student to acquire some test-irrelevant information. We illustrate this in the following example. We also show that this conclusion arises as a consequence of the alignment between the teacher's objective and the student's information processing constraint (i.e. that they use the same H function).

5.5.1 Example (continued)

We adapt the example studied in Sections 1.1 and 3.5 to study engaging test-motivated students. Suppose the true state of the world T consists of both a test-relevant and test-irrelevant dimension, $T = T_1 \times T_2 = \{L,R\} \times \{0,1\}$. The student and teacher know that the student will be asked a single question, $Q = \{Q_0\}$, whose responses are $R = \{l,r\}$, with $Q_0(L0\} = Q_0(L1) = l$ and $Q_0(R0) = Q_0(R1) = r$. That is, the $\{L,R\}$ component of the true state is relevant, and the $\{0,1\}$ component is irrelevant. The student (agent) again faces a binary choice $A = \{l,r\}$. The student's utility is one if she answers the question correctly and negative one otherwise, and therefore is identical (ignoring the test-irrelevant dimension) to that in Sections 1.1 and 3.5. The prior belief is uniform. We assume the agent's information processing capacity is defined by the mutual information of state $i$and the signal, and continue to use the parameters $\bar{\kappa} = 2, \chi = 1$ and $\rho = 1$.

Engagement maximization. We first study the benchmark case where the teacher maximizes the engagement of the student. This setting is a special case of our main model. By Proposition 1, the teacher's optimal strategy maximizes

$$\mathbb{E}^{\pi}[\hat{u}(q) - (2-\lambda)(H(q) - H(\bar{q}_0))]$$

for some $\lambda < 2$. The student-optimal benchmark maximizes

$$\mathbb{E}^{\pi}[\hat{u}(q) - 2(H(q) - H(\bar{q}_0))].$$

Both optimization problem can be written as:

$$\sup_{P(a|t_1,t_2)} \sum_{a,t_1,t_2} \frac{1}{4} u(a,t_1) P(a|t_1,t_2) - C \cdot (\sum_a \sum_{t_1,t_2} \frac{1}{4} P(a|t_1,t_2) \log(P(a|t_1,t_2))$$

$$- \sum_a (\sum_{t_1,t_2} \frac{1}{4} P(a|t_1,t_2)) \log(\sum_{t_1,t_2} \frac{1}{4} P(a|t_1,t_2))),$$

for some positive constant C. Note that replacing $P(a|t_1,t_2=0)$ and $P(a|t_1,t_2=1)$ with $\frac{P(a|t_1,0)+P(a|t_1,1)}{2}$ (denoted by $P(a|t_1)$) does not change the positive term while strictly reduces the negative term if $P(a|t_1,0)$ and $P(a|t_2,1)$ are not identical. Evidently, w.l.o.g., the optimization problem reduces to

$$\sup_{P(a|t_1)} \sum_{a,t_1} \frac{1}{2} u(a,t_1) P(a|t_1) - C \cdot (\sum_{a,t_1} \frac{1}{2} P(a|t_1) \log(P(a|t_1))$$

$$- \sum_{a,t_1} \frac{1}{2} P(a|t_1) \log(\sum_{t_1} \frac{1}{2} P(a|t_1))),$$

which is equivalent to Equation 1.[14] The analysis above implies that when there is a test-irrelevant state (which is costly to learn about), no information about the state will be acquired either in the student-optimal benchmark or in the principal-agent problem. This is interesting because the student learning about state t_2 enters the teacher's payoff function (the measure of engagement), but the teacher optimally provides only test-related information to the student.

Figure 5 illustrates the analysis above. The tetrahedron depicts the probability simplex that lies in $\mathbb{R}^3$. Each vertex of the tetrahedron is a degenerate belief (labeled by the state that occurs with probability one). The red dot is the uniform prior. The hyperplane represents the cutoff beliefs where the student is indifferent between actions l and r. To the right of the hyperplane, $q_R > q_L$ and r is optimal. Vice versa for the left of the hyperplane. The two blue dots are the student-optimal posterior beliefs. The two green dots are the solution to the principal-agent problem. Evidently, all the dots are one the same straight line, which illustrates that the teacher only

[14] This equivalence is special to mutual information and is the implication of a more general "compression invariance" property introduced by Caplin et al. [2022], Bloedel and Zhong [2020].

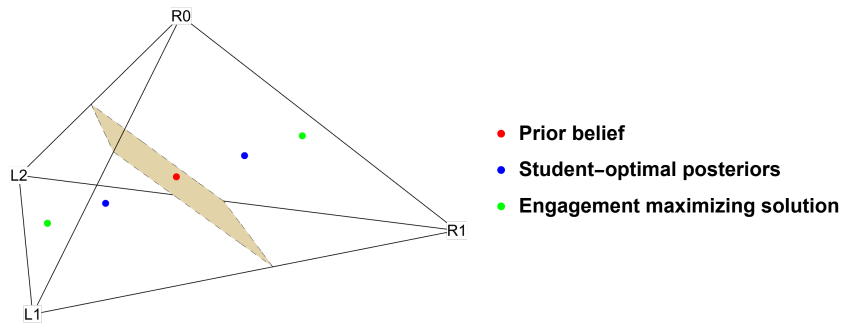


Figure 5: Engagement maximization

releases test-related information to the student, but at a higher precision comparing to what the student prefers.

Knowledge maximization. Now, we turn to the case where the teacher only cares about how much the student knows about the state $t_2 \in T_2$ (the test-irrelevant dimension). We assume that the teacher's payoff is $2\mathbb{E}^{\pi}[|q_{t_2} - 0.5|]$ if the measure of stopping belief is π, where q_{t_2} denotes the probability of $t_2 = 1$ under the student's stopping belief. Note that this payoff function is equivalent to the instrumental value of information in a hypothetical binary decision problem where the utility of matching (mismatching) the state y is 1 (-1). Proposition 1 can be extended to show that the teacher's optimal policy maximizes the following auxiliary problem (for some $\lambda > 0$):

$$\sup_{\pi} \mathbb{E}^{\pi}[2|q_{t_2} - 0.5| + \lambda \hat{u}(q) - 2\lambda(H(q) - H(\bar{q}_0))],$$

which is equivalent to a hypothetical static RI problem where the teacher has four possible actions, whose utilities are

$u(a,t)$	$L0$	$R0$	$L1$	$R1$
a_1	$1+\lambda$	$1-\lambda$	$-1+\lambda$	$-1-\lambda$
a_2	$1-\lambda$	$1+\lambda$	$-1-\lambda$	$-1+\lambda$
a_3	$-1+\lambda$	$-1-\lambda$	$1+\lambda$	$1-\lambda$
a_4	$-1-\lambda$	$-1+\lambda$	$1-\lambda$	$1+\lambda$

.

Per Matêjka et al. [2015], the solution is logit, given by

$P(a\|t)$	$L0$	$R0$	$L1$	$R1$
a_1	$\frac{e^{1+\lambda}}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e^{\lambda}}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{1}{e^{1+\lambda}+e+e^{\lambda}+1}$
a_2	$\frac{e}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e^{1+\lambda}}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{1}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e^{\lambda}}{e^{1+\lambda}+e+e^{\lambda}+1}$
a_3	$\frac{e^{\lambda}}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{1}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e^{1+\lambda}}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e}{e^{1+\lambda}+e+e^{\lambda}+1}$
a_4	$\frac{1}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e}{e^{1+\lambda}+e+e^{\lambda}+1}$	$\frac{e^{1+\lambda}}{e^{1+\lambda}+e+e^{\lambda}+1}$

,

and the default rule is uniform $P(a) \equiv \frac{1}{4}$. Note that the conditional distribution,

$P(a\|t)$	$L0\&L1$	$R0\&R1$
$a_1\&a_3$	$\frac{e}{e+1}$	$\frac{1}{e+1}$
$a_2\&a_4$	$\frac{1}{e+1}$	$\frac{e}{e+1}$

,

is identical to the student-optimal solution (and independent to λ). The unknown parameter λ can be pinned down by setting the student's IC condition binding. The analysis above suggests that when the teacher only cares about the test-irrelevant knowledge, she will give the student exactly his preferred test-relevant information and extra knowledge such that the student gets barely enough welfare to be willing to participate.

Figure 6 illustrates the analysis above. Except the green dots, the figure is exactly the same as Figure 5. The green dots are the optimal posteriors of the principal-agent problem. For any pair of green dots on the same side of the hyperplane, they equals the blue dot on expectation. Their distance to the hyperplane is also the same as the blue dot, illustrating that the teacher provides the student his preferred test-relevant information together with extra test-irrelevant knowledge.

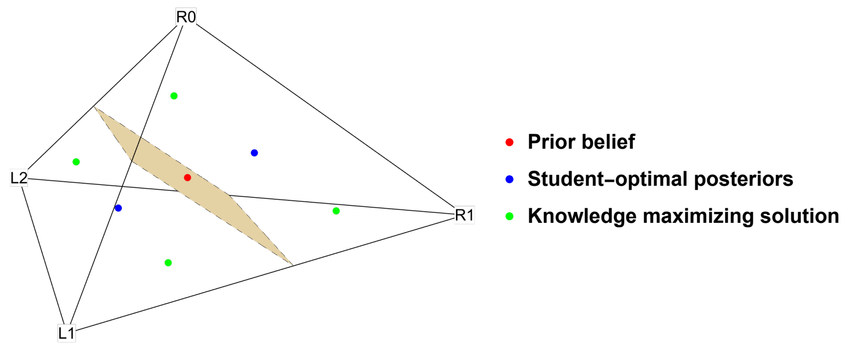


Figure 6: Knowledge maximization

6 Conclusion

We have considered the problem of a principal who provides information to an agent so as to maximize the attention the agent pays to the principal's information (engagement). The agent values this information for instrumental purposes, is rational and Bayesian, and faces a constraint on the rate at which she can process information. Our main result is that, by maximizing engagement, the principal leaves the agent no better off than if she could not receive any information at all. Moreover, the agent will end up holding extreme beliefs (relative to a benchmark in which the agent could choose the information for herself). Our results highlight the pitfalls of presenting users with information for the purposes of maximizing engagement, a standard practice on popular internet platforms.

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A Additional Proofs

A.1 Proof of Proposition 2

Proof. By Assumption 3, it is w.l.o.g. to consider strategies whose marginal distribution of (q_τ, τ), which we denote $F(q,t) \in \Delta(\Delta(X) \times \mathbb{R}_+)$, is rotationally symmetric, because any implementable process q_t^* can be rotated by R and remains implementable. Therefore, $\mathbb{E}^{F(q,t)}[q|t \geq t'] = \mathbb{E}^{F(q,t)}\left[\sum_{j=0}^{J-1} \frac{1}{J} R^j q \middle| t \geq t'\right] = q_0$ for all $t' \geq t_0$. Now, consider the following relaxed problem for the principal:

$$\begin{aligned}
&\sup_{F \in \Delta(\Delta(X) \times \mathbb{R}^+)} \rho \int_{\mathscr{P}(X)} \int_0^\infty H(q) F(dq, dt) \qquad (7)\\
&\text{s.t.} \int_{\mathscr{P}(X)} \int_{s \leq t} (H(q) - H(q_0)) F(dq, ds) \leq \chi \int_{s \leq t} (1 - F(s)) ds;\\
&\int_{\mathscr{P}(X)} \int_{s > t} (\widehat{\mu}(q) - \widehat{\mu}(q_0)) F(dq, ds) \geq \int_{s > t} K_t(s) dF(s),
\end{aligned}$$

where $K_t(s) = \int_t^s \kappa(\tau) d\tau$ and $F(s) = \int_{\mathscr{P}(X)} \int_{\tau \leq s} F(dq, ds)$. Note that the first inequality in equation 7 is a necessary condition for the information constraint and the second inequality in equation 7 is a necessary condition for the agent's IC.

Let F^* be the joint distribution of $(q_{\tau^*}^*, \tau^*)$ under the policies (P^*, I^*, τ^*). We prove that F^* solves equation 7, and hence that the policies (P^*, I^*, τ^*) achieve weakly higher utility than any other policies.

We first write down the Lagrangian: let $\lambda(t)$ and $\gamma(t)$ be the multipliers of the first and second constraints, respectively,

$$
\begin{aligned}
\mathscr{L} =& \rho \int_{\mathscr{P}(X)} \int_0^\infty H(q) F(dq,dt) + \chi \int_{\mathscr{P}(X)} \int_0^\infty \left(\int_0^t \Lambda(s) ds \right) F(dq,dt) \\
& - \int_{\mathscr{P}(X)} \int_0^\infty \Lambda(t) H(q) F(dq,dt) \\
& + \int_{\mathscr{P}(X)} \int_0^\infty \Gamma(t) \widehat{\mu}(q) F(dq,dt) - \int_0^\infty \left(\int_0^t \gamma(s) K_s(t) \right) F(dt) \\
& + \int_{\mathscr{P}(X)} \int_0^\infty \xi(q,t) F(dq,dt) + \nu (1 - \int_{\mathscr{P}(X)} \int_0^\infty F(dq,dt)),
\end{aligned}
$$

where $\Lambda(t) = \int_t^\infty \lambda(s) ds$ and $\Gamma(t) = \int_{t_0}^t \gamma(s) ds$. Note that we normalize $H(q_0) = \widehat{\mu}(q_0) = 0$ to simplify notations. It is sufficient to show that F^* together with a set of multipliers jointly maximize the Lagrangian. Since the Lagrangian is linear in F, it is sufficient to verify the FOC: $\forall (\mu, t)$ in the support of F^*:

$$
\rho H(q) + \chi \int_{t_0}^t \Lambda(s) ds - \Lambda(t) H(q) + \Gamma(t) \widehat{\mu}(q) - \int_{t_0}^t \gamma(s) K_s(t) ds = \nu \tag{8}
$$

Note that the characterization applies to the constant κ case as well: there exist Λ_t and Γ_t (both are constant fixing t) s.t.

$$
\begin{cases}
\nabla H(q)(\rho - \Lambda(t)) + \Gamma(t) \nabla \widehat{\mu}(q) = 0 \ \forall q \text{ in the support of } \pi^*(\kappa(t)) \\
\chi \Lambda_t - \Gamma_t \kappa(t) = 0
\end{cases}
$$

The first equality is from concavification. The second equality is from the FOC. Now define $\Lambda(t)$ and $\Gamma(t)$:

$$
\begin{cases}
\frac{\rho - \Lambda(t)}{\Gamma(t)} = \frac{\rho - \Lambda_t}{\Gamma_t} \\
\chi \Lambda(t) - \Gamma(t) \kappa(t) + \gamma(t) E_{\pi^*(\kappa(t))}[\widehat{\mu}(q)] + \lambda(t) E_{\pi^*(\kappa(t))}[H(q)] = 0
\end{cases}
$$

Replace $\Lambda(t)$ with $\rho + \left(\frac{\kappa(t)}{\chi} - \frac{\rho}{\Gamma_t}\right)\Gamma(t)$, we get an ODE for $\Gamma(t)$:

$$\rho\chi - \frac{\rho\chi}{\Gamma_t}\Gamma(t) - E^{\pi^*(t)}[H(q)]\left(\frac{\kappa'(t)}{\chi} + \frac{\rho\Gamma_t'}{\Gamma_t^2}\right)\Gamma(t) + \\ \left(E^{\pi^*(\kappa(t))}\left[\widehat{\mu}(q) - \left(\frac{\kappa(t)}{\chi} - \frac{\rho}{\Gamma_t}\right)H(q)\right]\right)\gamma_t = 0$$

Note that the IC implies $\left(E^{\pi^*(\kappa(t))}\left[\widehat{\mu}(q) - \frac{\kappa(t)}{\chi}H(q)\right]\right) = 0$. Reorganizing terms, we get:

$$\gamma(t) = \frac{\Gamma_t}{\rho E^{\pi^*(\kappa(t))}[H(q)]}\left(\rho\chi\left(\frac{\Gamma(t)}{\Gamma_t} - 1\right) + E^{\pi^*(t)}[H(q)]\left(\frac{\kappa'(t)}{\chi} + \frac{\rho\Gamma_t'}{\Gamma_t^2}\right)\Gamma(t)\right).$$

The ODE with initial conditions $\gamma(T) = 0$, $\Gamma(T) = \Gamma_T$ satisfies the Lindelof condition and has a unique solution. Note that when $\Gamma(t) = \Gamma_t$, $\gamma(t) = \frac{\kappa'(t)}{\rho\chi}\Gamma_t^2 + \Gamma_t' > 0$, so $\Gamma(t)$ stays below Γ_t. When $\Gamma(t) = 0$, $\gamma(t) < 0$, so $\Gamma(t)$ stays above 0. Therefore, there exists $\underline{t} < T$ s.t. on $[\underline{t}, T]$, $\Gamma(t) \geq 0$ and weakly increases.

By the construction of $\underline{t}$, for all $t_0 \geq \underline{t}$, there exist multipliers $\lambda(t)$ and $\gamma(t)$ on $[t_0, T]$ s.t. the Lagrangian $\mathscr{L}$ is maximized jointly by $(\lambda(t), \gamma(t))$ and F^*. $\square$

A.2 Proof of Lemma 2

Proof. Define:

$$\tau' = \tau \wedge q_t \text{ first leaves } E^A.$$

By definition $\tau' \leq \tau$. Since $\mathrm{Supp}(P) \subset D^d$ and $\mathrm{Supp}(\mathrm{q}_{\tau'-}) \subset E^A$, $\mathrm{Supp}(q_{\tau'}) \subset \bar{Q}^d$. Now, we prove by contradiction that $\tau' = \tau$. Suppose $\tau' < \tau$ for a positive measure of events, on which

$$\begin{aligned}\mathbb{E}^P\left[\widehat{u}(q_\tau) - (\tau - \tau')\cdot\bar{\kappa}\middle|\tau' < \tau\right] =& \mathbb{E}^P\left[\mathbb{E}\left[\widehat{u}(q_\tau) - (\tau - \tau')\cdot\bar{\kappa}\middle|\tau'\right]\middle|\tau' < \tau\right] \\ <& \mathbb{E}^P\left[\widehat{u}(q_{\tau'})\middle|\tau' < \tau\right].\end{aligned}$$

The inequality is from the fact that $\tau' < \tau \implies q_{\tau'} \notin E^A$. Therefore, τ' strictly improves upon τ and hence τ is not incentive compatible. The contradiction implies that $\text{Supp}(q_\tau) \subset \bar{Q}^d$. $\square$

A.3 Proof of Proposition 7

Proof. As is discussed in Section 3, the agent-optimal policy can be solved by concavifying $\widehat{u}(q) - \frac{\bar{\kappa}}{\chi}H(q)$. Therefore, there exists a linear function $L(q)$ that is weakly higher than $\widehat{u}(q) - \frac{\bar{\kappa}}{\chi}H(q)$ and tangents it at two beliefs $q^1 < \bar{q}_0 < q^2$. WLOG, let q^1 and q^2 be the smaller and largest such beliefs, respectively. Since $\widehat{u} - \frac{\bar{\kappa}}{\chi}H$ is piece-wise strictly concave, the interval $[q^1, q^2]$ is bounded away from the rest of E^A.

Proposition 2 implies that any admissible principal's strategy has $\text{Supp}(q_\tau) \subset \bar{Q}^0$, which is the closure of E^A. Moreover, any continuous path that starts from $\bar{q}_0$ and ends outside of $[q^1, q^2]$ leaves E^A; hence, it is not admissible. Therefore, $\text{Supp}(q_\tau) \subset [q^1, q^2]$.

Therefore, $J^0(\bar{q}^0)$ is bounded above by the following relaxed problem:

$$\sup_{\pi \in \mathscr{P}(\mathscr{P}([q^1,q^2]))} \mathbb{E}^\pi[\rho(H(q) - H(\bar{q}_0))].$$

Since H is strictly convex, the relaxed problem is solved by π^* with support $\{q^1, q^2\}$ (such π^* is unique). By Hébert and Woodford [2021b], there exists a Gaussian process that implements π^* and satisfies the information constraint. Note that this Gaussian process also implements the agent maximal continuation payoff (the upper concave hull of $\widehat{u}(q) - \frac{\bar{\kappa}}{\chi}(H(q) - H(q_t))$) for every interim belief; hence, it is incentive compatible. $\square$