Tutorial on Digital Topology, Geometry and Application

2^{nd} talk: Minimal path-costs

Computation and applications

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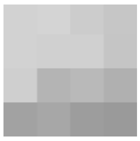
Outline

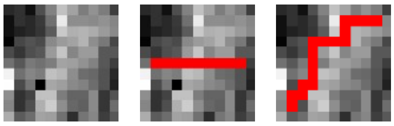
- What is a path cost?
- Methods
  - Geodesic distance
  - Fuzzy connectedness
  - Fuzzy distance
  - Minimum barrier distance
- Algorithms for efficient computation
- Application - segmentation

Digital images, paths

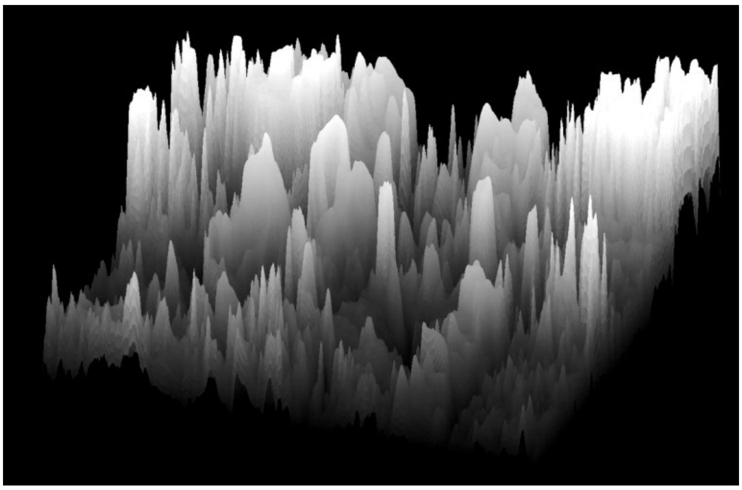


210	212	204	198
210	208	208	197
207	180	175	176
162	167	157	154

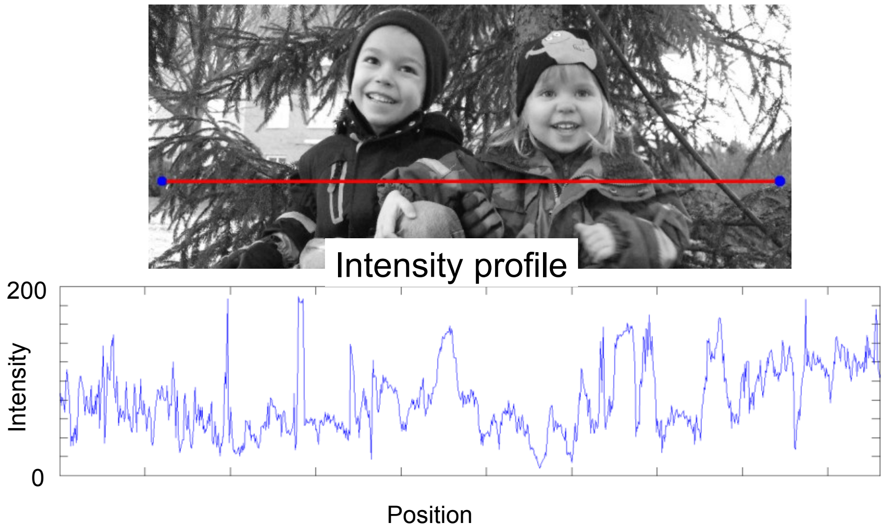
4-connected paths



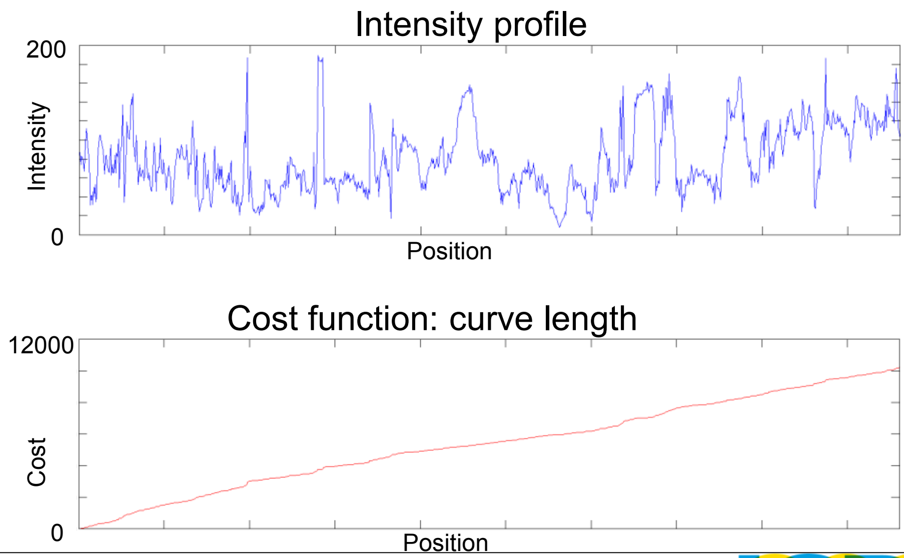
Topographic representation



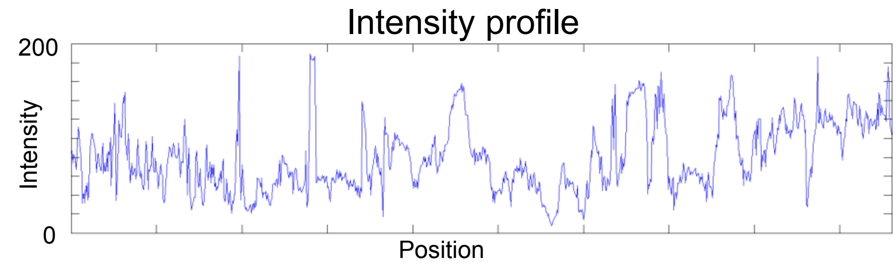
Path cost

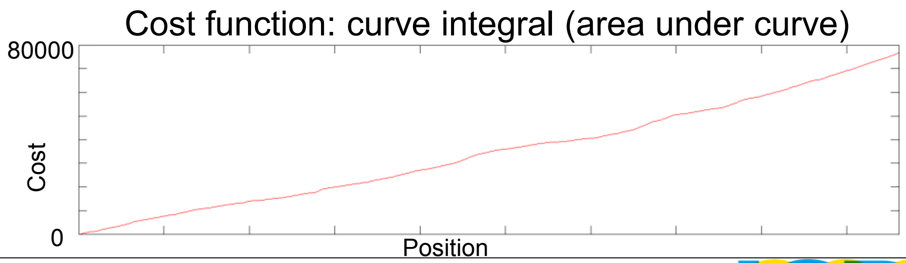


Path cost



Path cost





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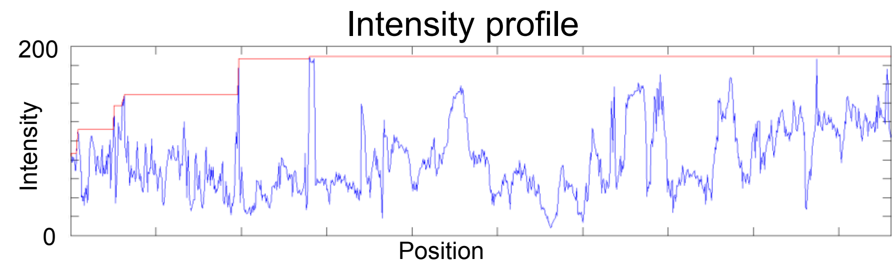
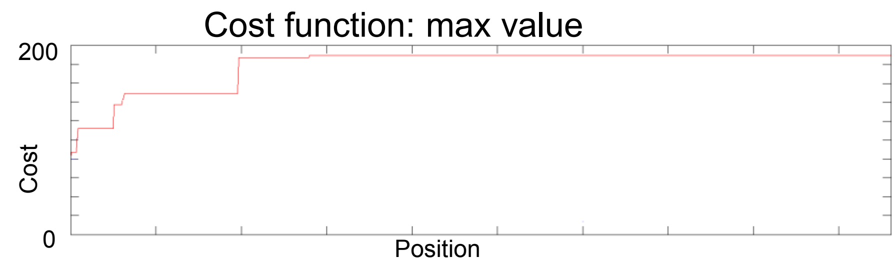
Path cost





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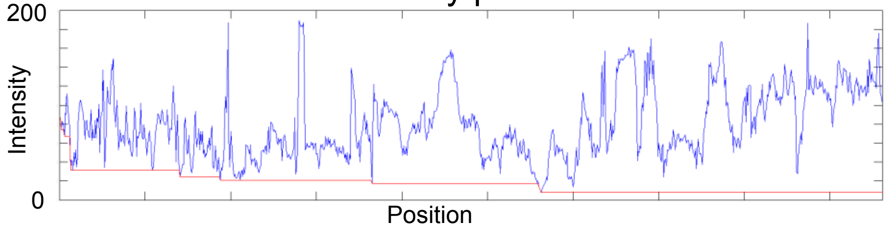
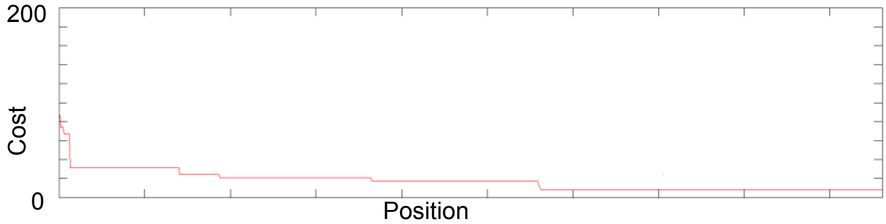
Path cost

Intensity profile



Cost function: min value



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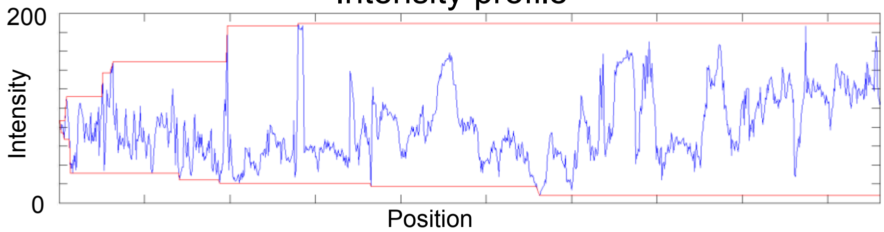
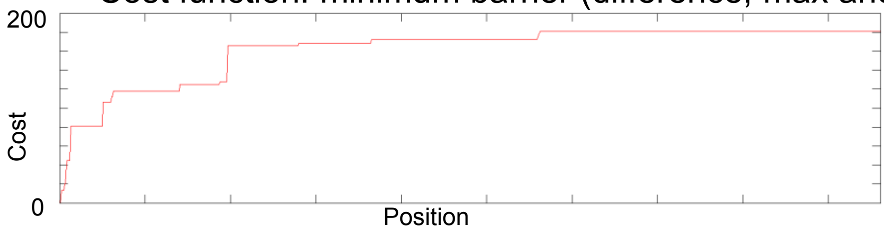
Path cost

Intensity profile



Cost function: minimum barrier (difference, max and min)



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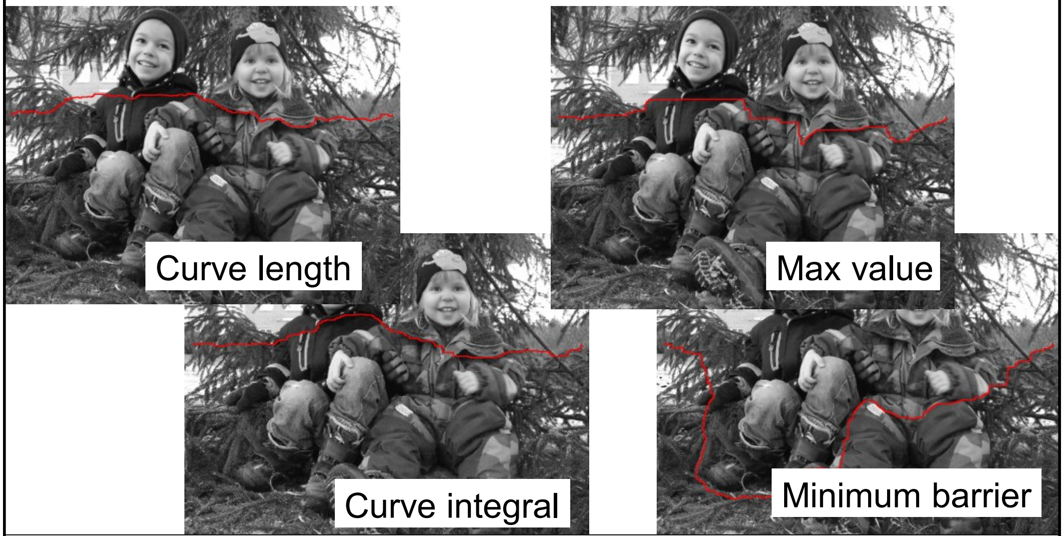
Optimal/minimal cost paths



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Methods

- Curve length - Geodesic distance
- Max/min value - Fuzzy connectedness
- Curve integral - Fuzzy distance
- Minimum barrier distance

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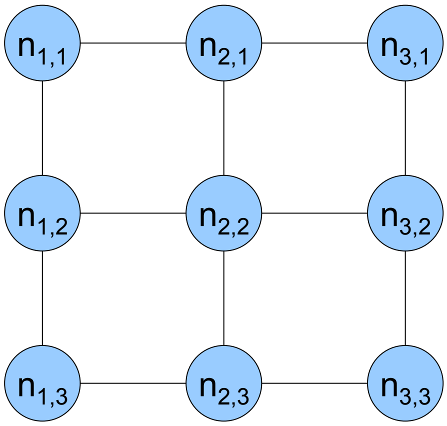
Methods

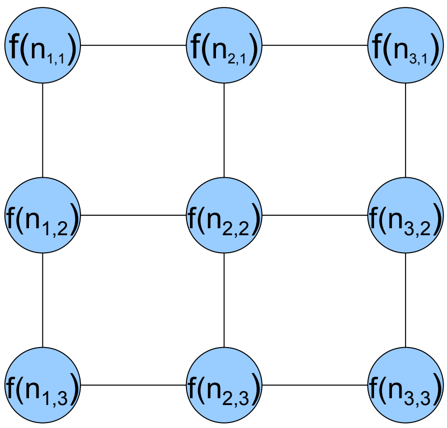
$n_{1,1}$	$n_{2,1}$	$n_{3,1}$
$n_{1,2}$	$n_{2,2}$	$n_{3,2}$
$n_{1,3}$	$n_{2,3}$	$n_{3,3}$

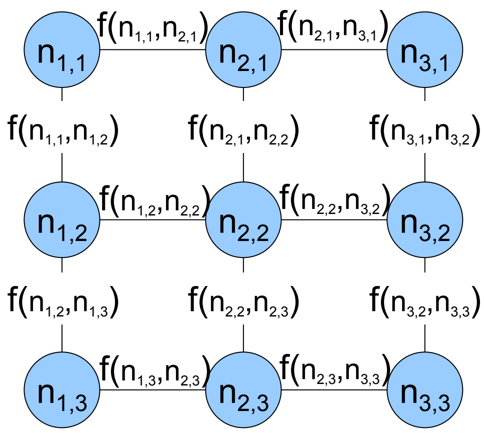


Image, intensity values

Graph representation

Methods

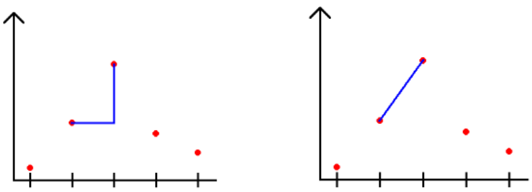




Node weighted graph

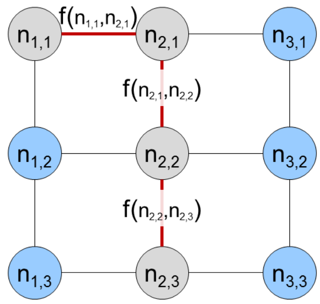
Edge weighted graph

Curve length - Geodesic distance



- Geodesic distance – Curve length on topographic image representation
- Cost: Sum of edge weights
- Simple $f(m,n) = |m-n| + 1$
  (or $f(m,n) = \sqrt{(m-n)^2 + 1}$)
- Example path cost:

$f(n_{1,1}, n_{2,1}) + f(n_{2,1}, n_{2,2}) + f(n_{2,2}, n_{2,3}) =$
$3 + |n_{1,1} - n_{2,1}| + |n_{2,1} - n_{2,2}| + |n_{2,2} - n_{2,3}|$



Geodesic distance

Extensions of the simple cost function:

- Distance Transform on Curved Spaces (DTOCS)
  - Approximation of local cost
- Weighted DTOCS (WDTOCS)
  - Approximation of local distances

P.J. Toivanen “New geodesic distance transforms for gray-scale images”. Pattern Recognition Letters 17(5) (1996) 437–450
L. Ikonen "Distance transforms on gray-level surfaces." *Acta Universitatis Lappeenrantaensis* (2006).

Fuzzy Connectedness

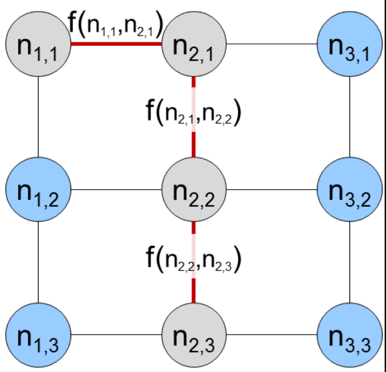
Fuzzy set, fuzzy membership functions:
A fuzzy set S in a fuzzy space X (here $X \subset \mathbb{Z}^2$) is a set of ordered pairs $S=\{(x,\mu_S(x))|x \in X\}$, where the membership function $\mu_S:X\rightarrow[0,1]$ represents the grade of membership of x in S.

Max/min value - Fuzzy Connectedness

- f defined by fuzzy affinity
- Cost: minimum of edge weights
- Simple $f(m,n) = 1 - |m - n|$
- Example path cost:

$$\min\left(f(n_{1,1},n_{2,1}), f(n_{2,1},n_{2,2}), f(n_{2,2},n_{2,3})\right) = 1 - \max(|n_{1,1} - n_{2,1}|, |n_{2,1} - n_{2,2}|, |n_{2,2} - n_{2,3}|)$$

- Gives fuzzy membership values



Fuzzy Connectedness

Extension of the simple cost: support for prior information:

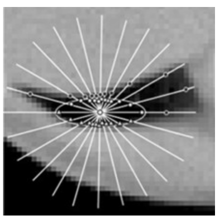
- Homogeneity-based component: The degree of local hanging-togetherness due to the similarity in intensity
- Object-feature based component: The degree of local hanging-togetherness with respect to some given feature

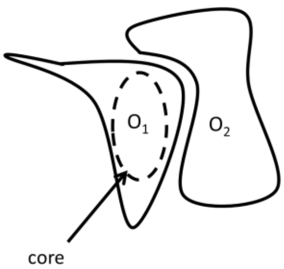
A. Rosenfeld, “The fuzzy geometry of image subsets”, Pattern Recognition Letters, 2(5), 1984
J. K. Udupa and S. Samarasekera, "Fuzzy connectedness and object definition: theory, algorithms, and applications in image segmentation," *Graphical Models and Image Processing,* vol 58, 1996.
J. K. Udupa and P. K. Saha, "Fuzzy connectedness and image segmentation," *Proceedings of the IEEE,* vol. 9, 2003.

Fuzzy Connectedness

Extensions/variants

- Local scale
- Vectorial fuzzy connectedness
- Relative fuzzy connectedness
- Iterative relative fuzzy connectedness
- …



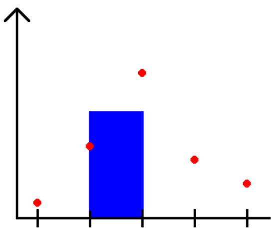


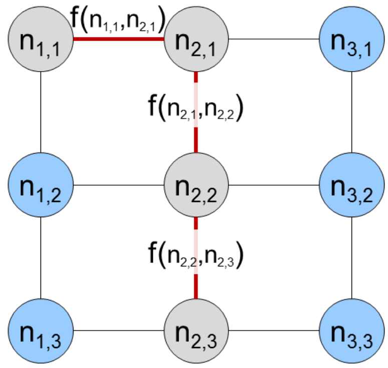
Curve integral

- Cost: Sum of edge weights
- Simple $f(m,n) = \frac{|m+n|}{2}$
- Example path cost:

$f(n_{1,1}, n_{2,1}) + f(n_{2,1}, n_{2,2}) + f(n_{2,2}, n_{2,3}) =$

$\frac{n_{1,1} + 2n_{2,1} + 2n_{2,2} + n_{2,3}}{2}$





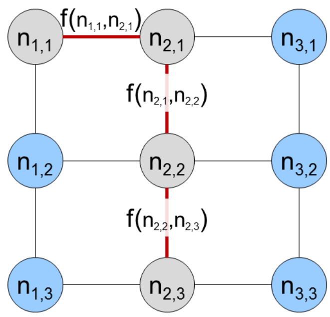
Curve integral

- Stability to noise and blur
- Generalization of path costs (chessboard/city block distances) for binary images
- Fuzzy distance - Curve integral typically applied to, for example, (inverted) fuzzy membership values or edge magnitude images

G. Levi, U. Montanari, “A grey-weighted skeleton”, Information and Control, Volume 17(1), 1970
D. Rutovitz “Data structures for operations on digital images” Pictorial Pattern Recognition, Washington, Thompson 1968
Saha, P., Wehrli, F.W., Gomberg, B.R.: “Fuzzy distance transform: Theory, algorithms, and applications” Computer Vision and Image Understanding 86(3) (2002)

Minimum Barrier Distance

- f vector valued function
- Cost: minimum interval
- $f(m, n) = (\min(m, n), \max(m, n))$



- Example path cost:

$$|\min(f_1(n_{1,1}, n_{2,1}), f_1(n_{2,1}, n_{2,2}), f_1(n_{2,2}, n_{2,3})) - \max(f_2(n_{1,1}, n_{2,1}), f_2(n_{2,1}, n_{2,2}), f_2(n_{2,2}, n_{2,3}))| = \max(n_{1,1}, n_{2,1}, n_{2,2}, n_{2,3}) - \min(n_{1,1}, n_{2,1}, n_{2,2}, n_{2,3})$$

Minimum barrier distance

- Efficient algorithms
- Metricity
- Convergence properties
- Vectorial minimum barrier distance

R. Strand, K.C. Ciesielski, F. Malmberg, P.K. Saha. "The Minimum Barrier Distance" Computer Vision and Image Understanding 117 (4), 2013
K.C. Ciesielski, R. Strand, F. Malmberg, P.K. Saha. "Efficient algorithms for finding the exact minimum barrier distance" Computer Vision and Image Understanding vol 123, 2014

Minimum barrier distance property

Let Π be the set of all paths between two points p and q in $\mathbb{Z}^n$ and $I: \mathbb{Z}^n \to \mathbb{R}$ (the *image*).
The minimum barrier distance between the points is

$$\min_{\pi \in \Pi} \left(\max_t I(\pi(t)) - \min_t I(\pi(t)) \right)$$

Let Π' be the set of all paths between points p and q in $\mathbb{R}^n$ and $I: \mathbb{R}^n \to \mathbb{R}$, then

$$\inf_{\pi \in \Pi'} \left(\max_t I(\pi(t)) - \min_t I(\pi(t)) \right) = \inf_{\pi \in \Pi'} \max_t I(\pi(t)) - \sup_{\pi \in \Pi'} \min_t I(\pi(t))$$

for bounded, continuous I!

Properties

- Geodesic distance
  - -Gradient based (noise, blur sensitive)
  - -A parameter gives spatial and intensity distance trade-off
- Fuzzy distance
  - -Favors high/low intensity values
    - ⇒ suited for gradient magnitude image (noise sensitive)
  - -Developed for fuzzy membership values
- Fuzzy connectedness
  - -Gradient based (noise, blur sensitive)
  - -No spatial distance term
  - -Support for apriori information
- Minimum barrier distance
  - -Noise and blur stability
  - -No spatial distance term
  - -Interval weighted graph

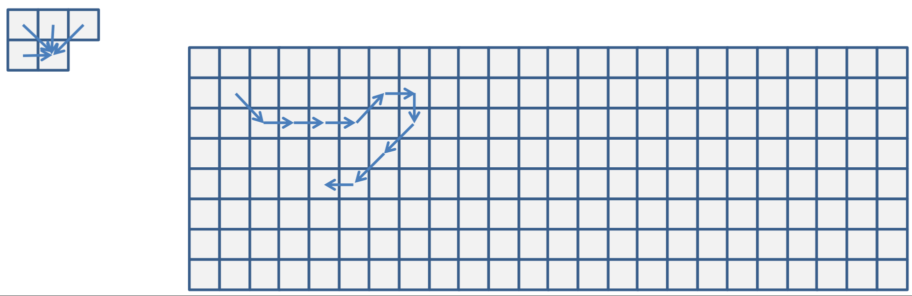
Algorithms

- Efficient algorithms needed in order to
  -process large amount of information and
  -minimize execution time

Algorithms

Repeated raster scan until convergence.

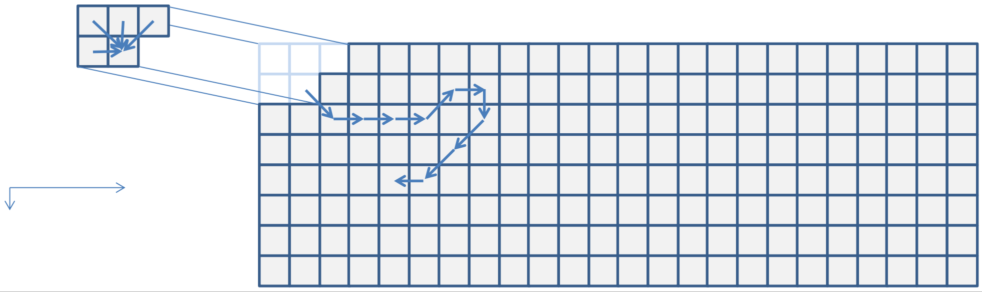
Drawback: Highly image content dependent
Slow convergence in some cases



Algorithms

Repeated raster scan until convergence.

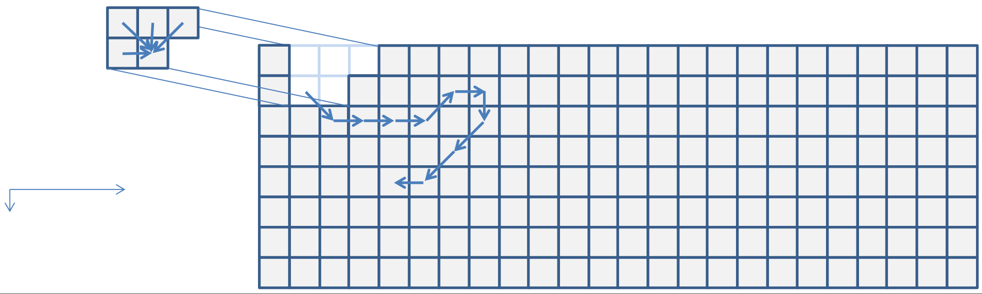
Scan 1, forward pass



Algorithms

Repeated raster scan until convergence.

Scan 1, forward pass



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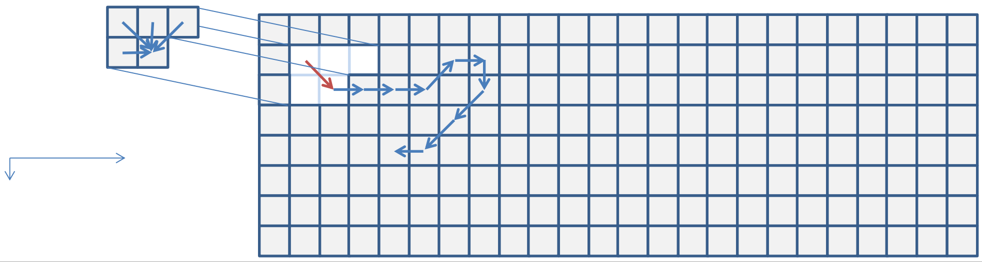
Algorithms

Repeated raster scan until convergence.

Scan 1, forward pass



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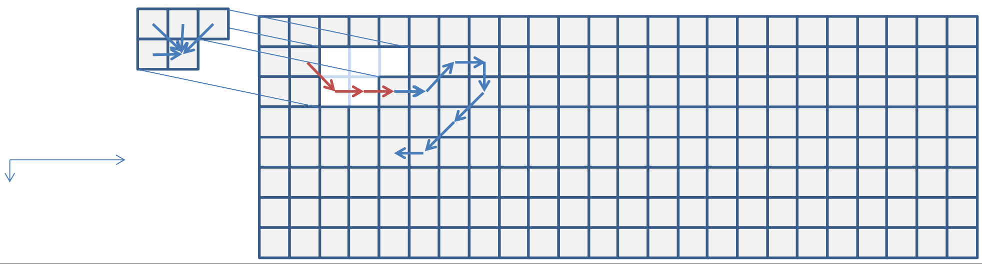
Algorithms

Repeated raster scan until convergence.

Scan 1, forward pass



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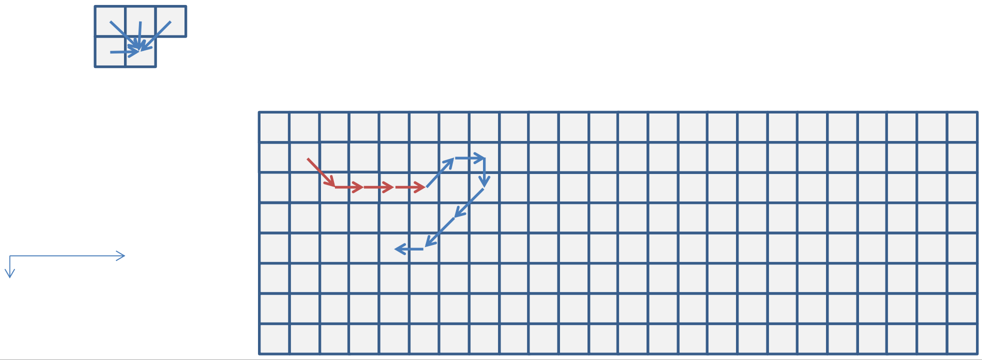
Algorithms

Repeated raster scan until convergence.

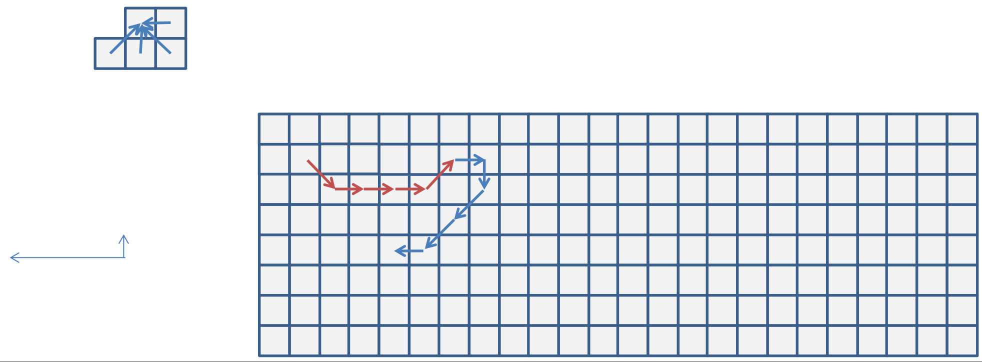
After scan 1, forward pass



Algorithms

Repeated raster scan until convergence.

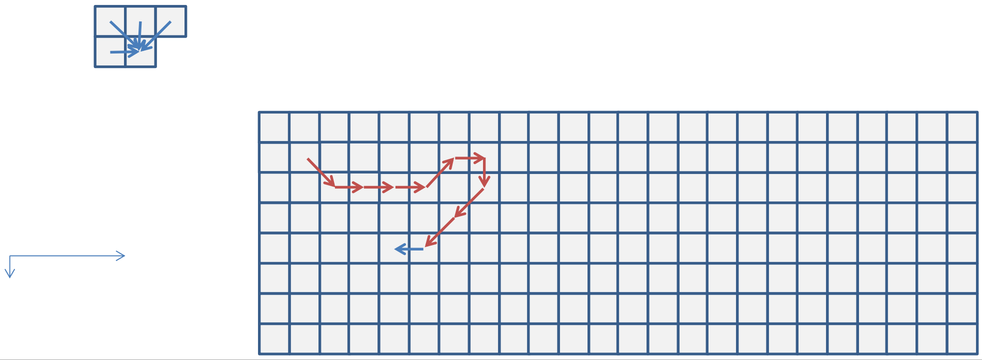
After scan 1, backward pass



Algorithms

Repeated raster scan until convergence.

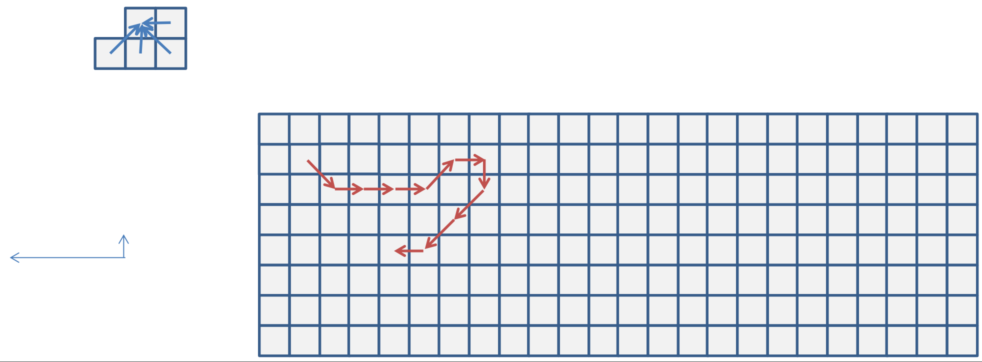
After scan 2, forward pass



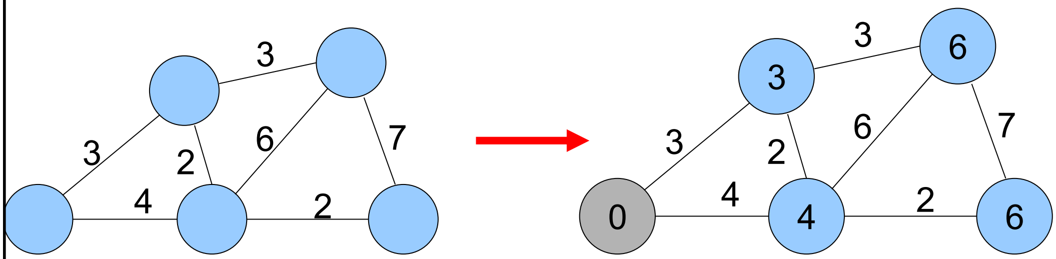
Algorithms

Repeated raster scan until convergence.

After scan 2, backward pass



Algorithms
Dijkstra's algorithm / wavefront propagation



Sufficient for *monotonic incremental* path cost functions

Algorithms
Dijkstra's algorithm / wavefront propagation

A path cost function is *monotonic incremental* if

- $f(\pi \cdot \langle s,t \rangle) = f(\pi) \circ (s,t)$, where $\circ$ satisfies
  - $x' \geq x \Rightarrow x' \circ (s,t) \geq x \circ (s,t)$
  - $x \circ (s,t) \geq x$

In other words:

- Adding the same element to two paths preserves the order of the cost of the paths.
- The path cost function is non-decreasing (adding an element to a path gives higher or equal cost).

A. Frieze, "Minimum Paths in Directed Graphs," Operational Research Quarterly 28(2), 1977.
A. X. Falcão, J. Stolfi, R. Lotufo, "The Image Foresting Transform: Theory, Algorithms, and Applications", TPAMI 26(1), 2004

Algorithms

Dijkstra’s algorithm / wavefront propagation

Monotonic incremental:

- Curve length
- Curve integral (if non-negative)
- Max/min

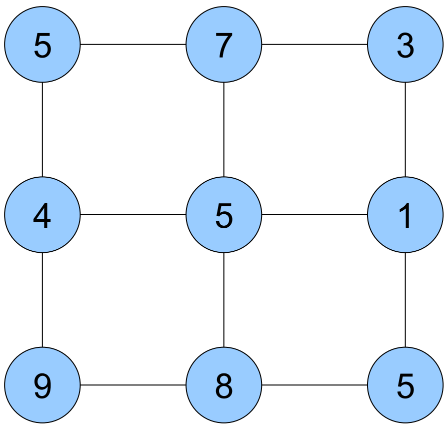
Not monotonic incremental:

- Minimum barrier distance

Minimum barrier

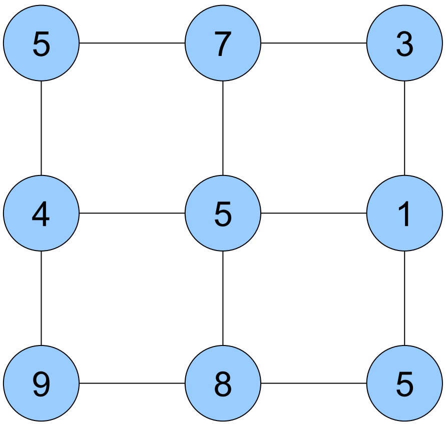
5	7	3
4	5	1
9	8	5

Image, intensity values

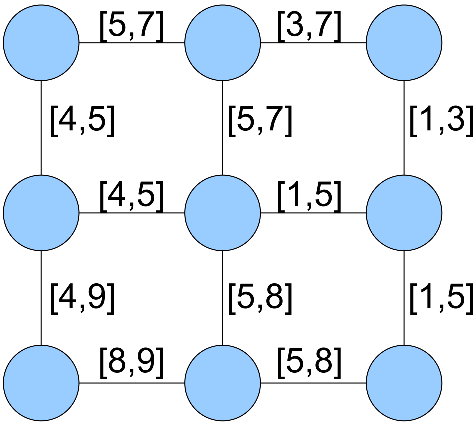


Graph representation

Minimum barrier



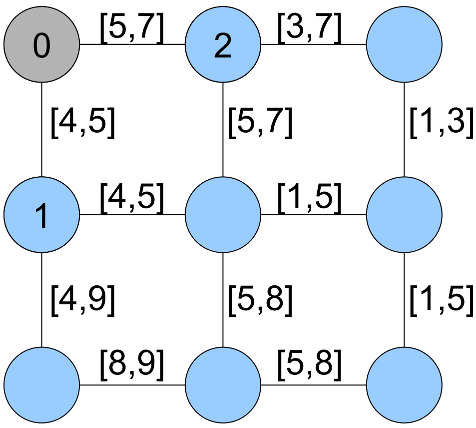
Graph representation



Interval weighted graph

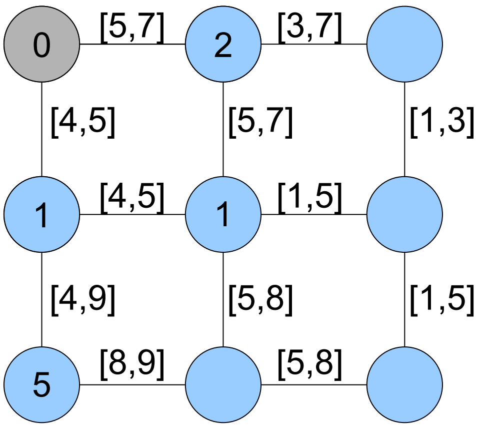
Minimum barrier Dijkstra approximation

Interval weighted Dijkstra

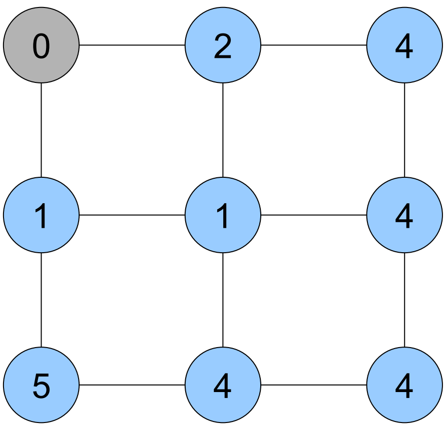


Minimum barrier Dijkstra approximation

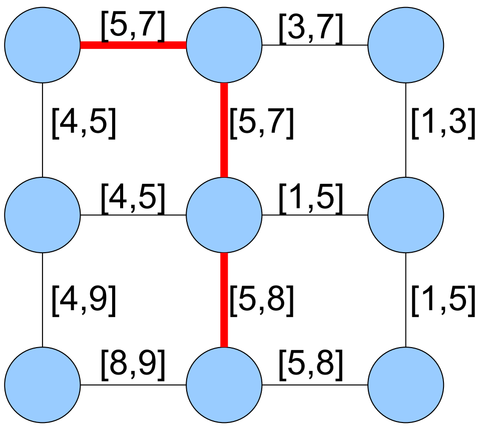
Interval weighted Dijkstra



Minimum barrier Dijkstra approximation



Result from interval weighted Dijkstra



The cost of this path is 3!

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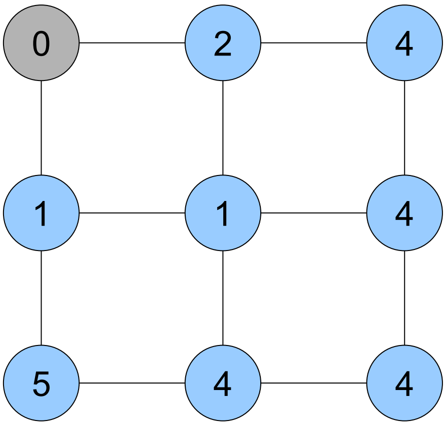
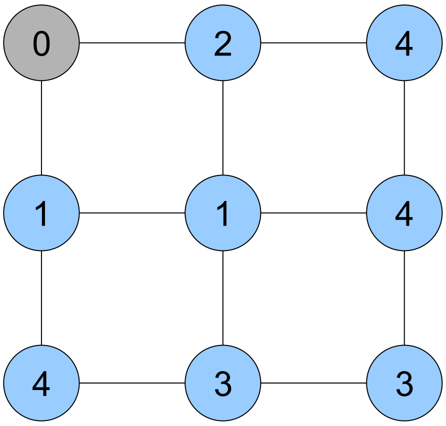
Minimum barrier Dijkstra approximation



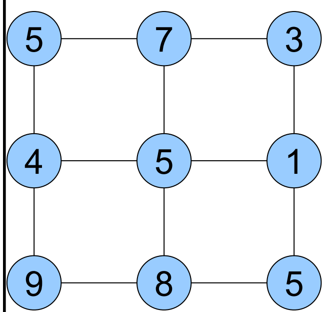
Approximation



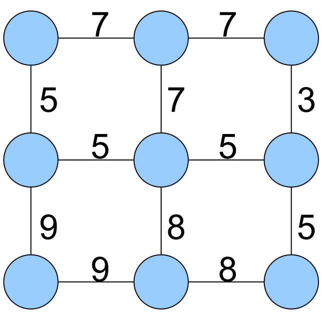
Correct

Another approximation: min max - max min

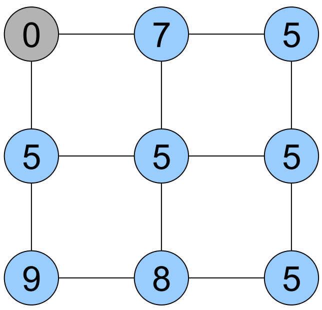
i) minimize max



Graph representation



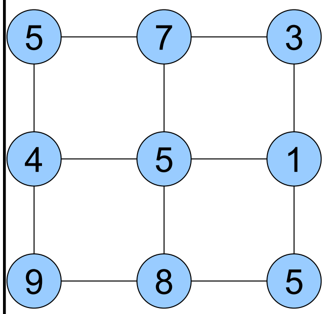
Weighted graph



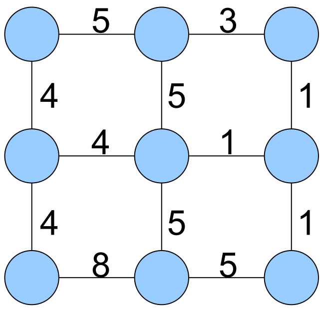
Result

Another approximation: min max - max min

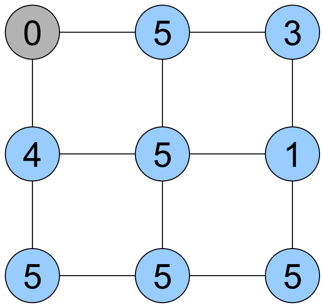
ii) maximize min



Graph representation

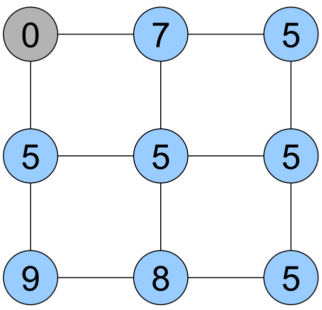


Weighted graph

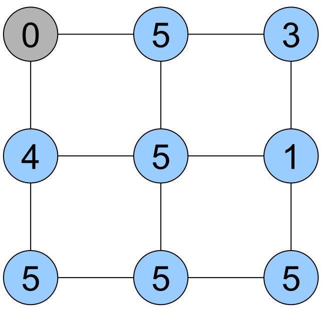


Result

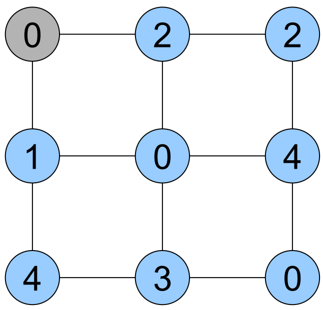
Another approximation: min max - max min



Min max

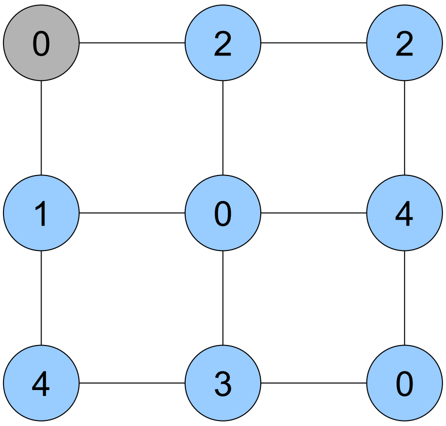


Max min

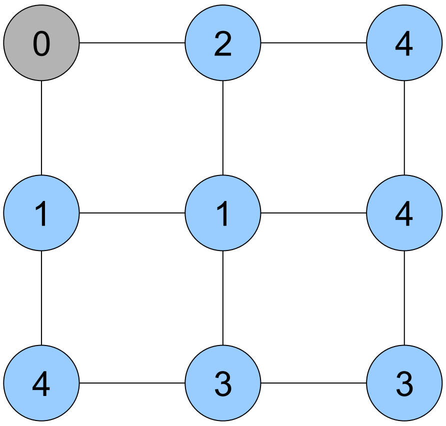


Difference

Another approximation: min max - max min



Approximation



Correct

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Minimum path cost/Distance transform

Seed point

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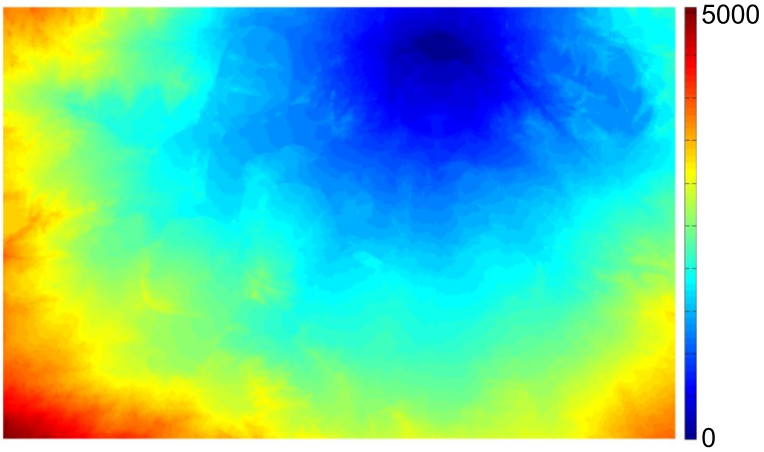
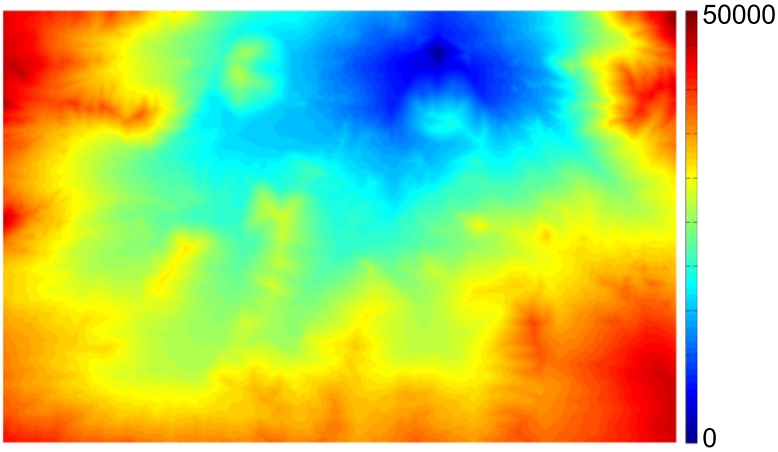
Minimum path cost transform

1) Geodesic distance, $f(m,n) = \sqrt{(m-n)^2 + 1}$



Minimum path cost transform

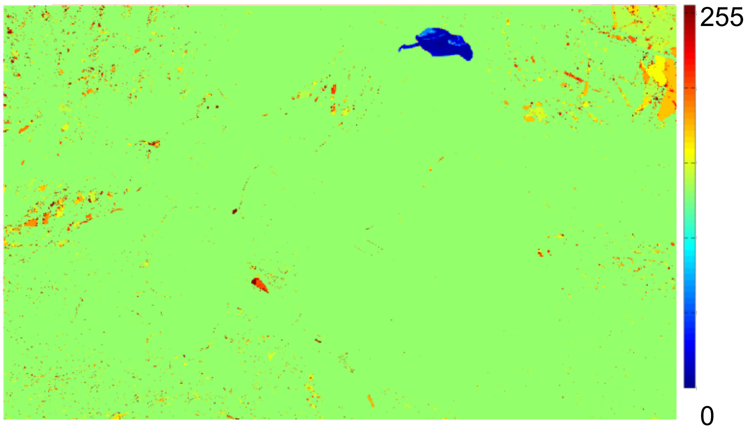
2) Curve integral/Fuzzy distance, $f(m,n) = \frac{|m+n|}{2}$



Minimum path cost transform

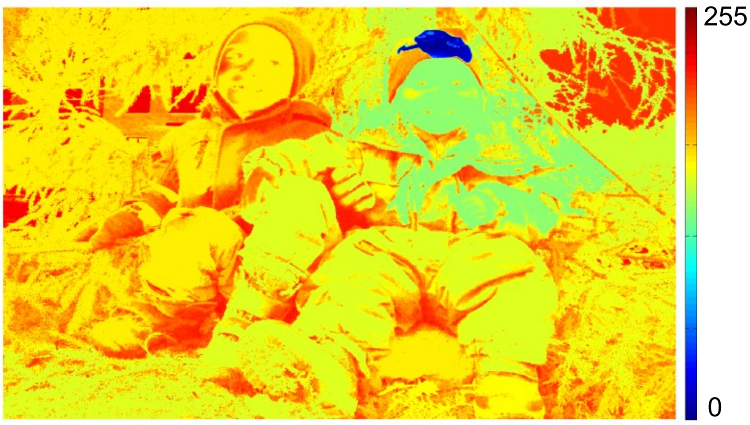
3) Gradient magnitude ('inverted fuzzy connectedness')

$f(m,n) = |m - n|$



Minimum path cost transform

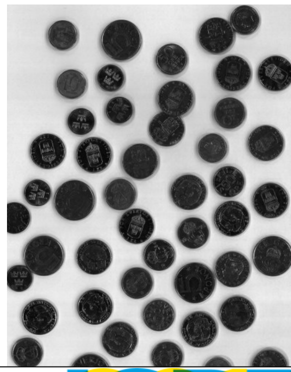
4) Minimum barrier, $f(m,n) = (\min(m,n), \max(m,n))$



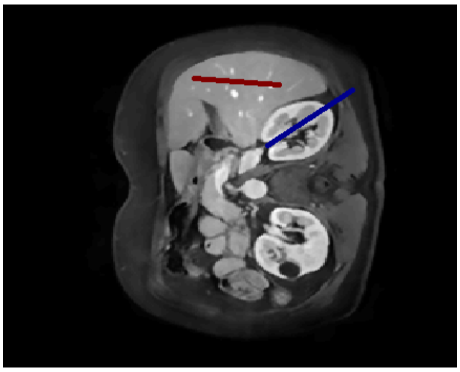
Interactive (semi-automatic) image processing

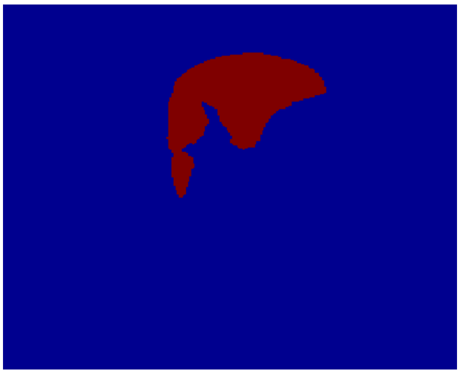
Human
Complex patterns
Interpret based on experience

Computer
quantitative analysis
objective
fast



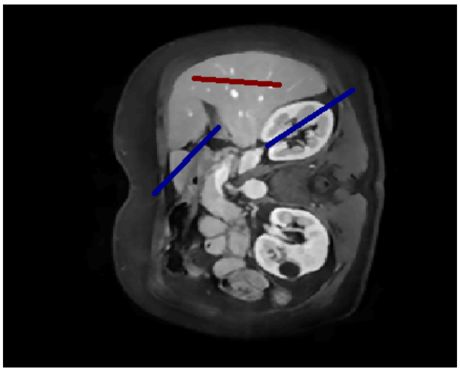
Segmentation

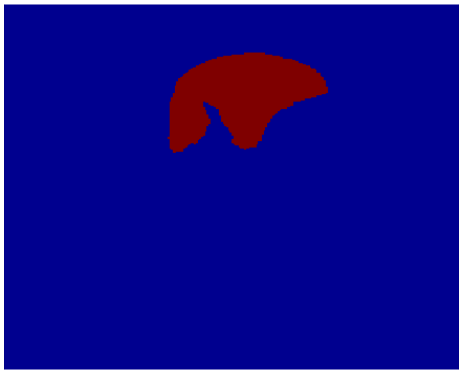




MR image
Interactive segmentation: The user adds background and object seed points. Labels are propagated together with cost values.

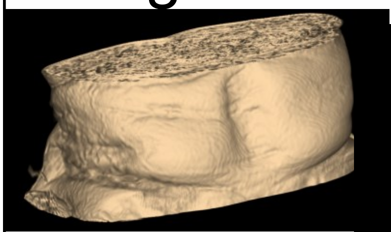
Segmentation

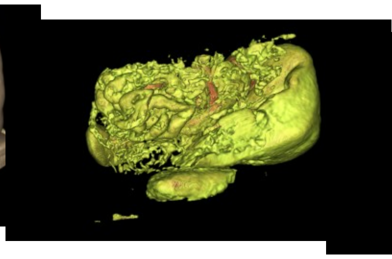


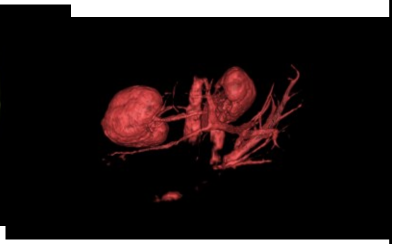


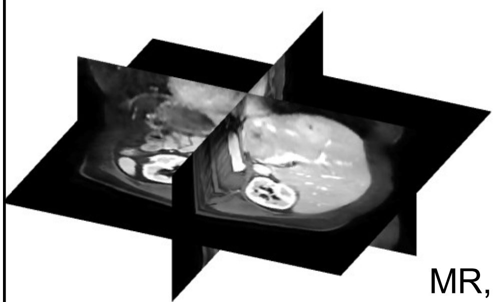
MR image
Interactive segmentation: The user adds seed points and the segmentation is updated locally.

Segmentation of volume images



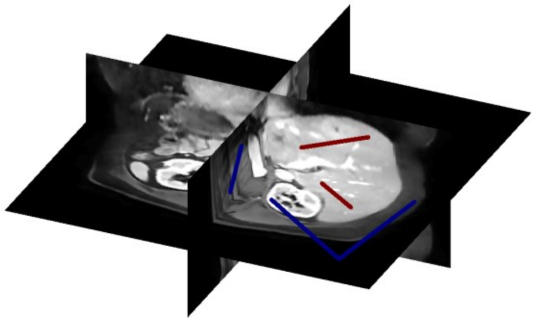




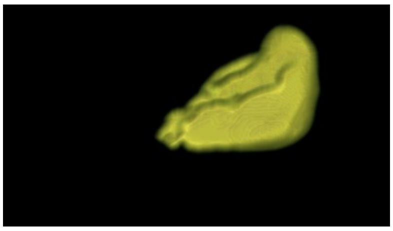


MR, volume image

Segmentation of volume images



Seed points



Segmentation result

Summary

Minimum path cost methods

- Geodesic distance
- Fuzzy connectedness.
- Fuzzy distance
- Minimum barrier distance

Efficient computation by wave-front propagation

Some applications in segmentation