# Euclidean Medial Axis computation

David Coeurjolly

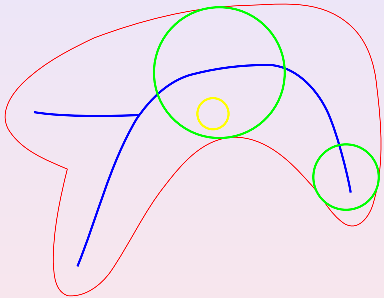
Laboratoire LIRIS
Université Claude Bernard Lyon 1
43 Bd du 11 Novembre 1918
69622 Villeurbanne CEDEX
France
david.coeurjolly@liris.cnrs.fr

2006

# Table of contents

# Table of contents

# Skeleton and medial axis in the continuous plane



## Possible definitions

- generalized symmetry axes
- *prairie fire model* and “self-intersections” of wave-fronts
- centers of maximal balls
- one dimensional topological equivalent
- . . .

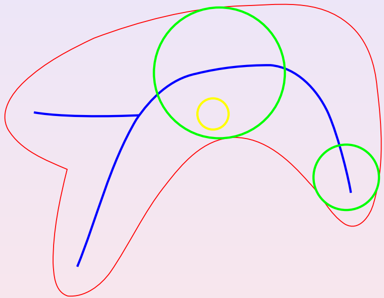
All theses definitions are equivalent in the continuous case

## In the discrete model

Skeleton : *minimal* topological equivalent

Medial axis : centers of maximal balls

# Skeleton and medial axis in the continuous plane



## Possible definitions

- generalized symmetry axes
- *prairie fire model* and "self-intersections" of wave-fronts
- centers of maximal balls
- one dimensional topological equivalent
- ...

All theses definitions are equivalent in the continuous case

## In the discrete model

Skeleton : *minimal* topological equivalent

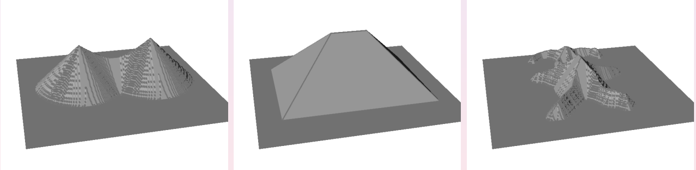
Medial axis : centers of maximal balls

# Medial axis and DT

## Link

DT(p) : maximal radius such that the disk centered at $p$ with radius $DT(p)$ is included in the shape

$\Rightarrow$ the medial axis can be viewed as the set of local maxima in the DT



# Maximal disks and Medial axis

## Definition (Maximal ball)

A maximal ball is a ball contained in the shape not exactly covered by another ball contained in the shape.

## Definition (Medial axis)

The medial axis (MA for short) of a shape is the set of maximal ball centers contained in the shape.

# Table of contents

David Coeurjolly

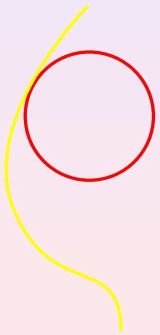


# MA extraction with Chamfer metrics

**Let $p$ be a point in the shape, how to decide if $p \in MA$ ?**

Look-up table $[DT(p), \vec{d}_i] \rightarrow [\textit{limit radius}]$

$$p \in MA \Leftrightarrow \forall i, DT(p + \vec{d}_i) < LUT_i(p)$$



David Coeurjolly

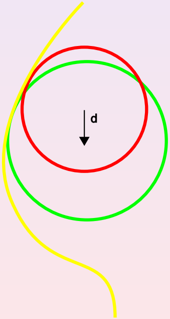


# MA extraction with Chamfer metrics

## Let $p$ be a point in the shape, how to decide if $p \in MA$ ?

Look-up table $[DT(p), \vec{d}_i] \rightarrow [\textit{limit radius}]$

$$p \in MA \Leftrightarrow \forall i, DT(p + \vec{d}_i) < LUT_i(p)$$

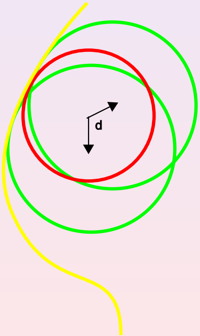


# MA extraction with Chamfer metrics

## Let $p$ be a point in the shape, how to decide if $p \in MA$ ?

Look-up table $[DT(p), \vec{d}_i] \rightarrow [limit\ radius]$

$$p \in MA \Leftrightarrow \forall i, DT(p + \vec{d}_i) < LUT_i(p)$$

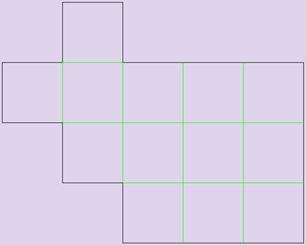


# Example

## Example ($d_{3,4}$)

$p \in MA \Leftrightarrow \forall i, DT(p + \vec{d_i}) < LUT_i(p)$ with $\{a, b\} = \{\rightarrow, \nearrow\}$

| $DT(p)$ | a | b |
|---|---|---|
| 3 | 4 | 5 |
| 4 | 7 | 8 |
| 6 | 8 | 9 |
| 7 | 10 | 11 |
| 8 | 11 | 12 |
| 9 | 12 | 13 |
| 10 | 13 | 14 |
| 11 | 14 | 15 |
| 12 | 15 | 16 |
| 13 | 16 | 17 |



David Coeurjolly



# Example

## Example ($d_{3,4}$)

$p \in MA \Leftrightarrow \forall i, DT(p + \vec{d}_i) < LUT_i(p)$ with $\{a, b\} = \{\rightarrow, \nearrow\}$

| $DT(p)$ | a | b |
|---|---|---|
| 3 | 4 | 5 |
| 4 | 7 | 8 |
| 6 | 8 | 9 |
| 7 | 10 | 11 |
| 8 | 11 | 12 |
| 9 | 12 | 13 |
| 10 | 13 | 14 |
| 11 | 14 | 15 |
| 12 | 15 | 16 |
| 13 | 16 | 17 |

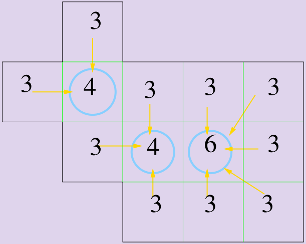
| | | | | |
|---|---|---|---|---|
| | 3 | | | |
| 3 | 4 | 3 | 3 | 3 |
| | 3 | 4 | 6 | 3 |
| | | 3 | 3 | 3 |

# Example

## Example ($d_{3,4}$)

$p \in MA \Leftrightarrow \forall i, DT(p + \vec{d}_i) < LUT_i(p)$ with $\{a, b\} = \{\rightarrow, \nearrow\}$

| $DT(p)$ | a | b |
|---|---|---|
| 3 | 4 | 5 |
| 4 | 7 | 8 |
| 6 | 8 | 9 |
| 7 | 10 | 11 |
| 8 | 11 | 12 |
| 9 | 12 | 13 |
| 10 | 13 | 14 |
| 11 | 14 | 15 |
| 12 | 15 | 16 |
| 13 | 16 | 17 |



# MA with Chamfer metrics - summary

:-)

- Local computations to detect maximal balls based on a LUT
- Can be generalized to higher dimensions

:-(

- Anisotropic representation
- The entire LUT must be computed
- A new LUT is required if you change the chamfer mask (weights or displacements)

# Table of contents

# Medial axis for the Euclidean metric

## LUT

We can also construct a Look-up table but the number of possible displacements is unbounded

## Separable techniques

Optimal algorithms to solve:

- the reversible Euclidean distance transform (REDT)
- the MA extraction problem (using Laguerre Diagram)

## The REDT

### Problem

Given a MA (*i.e.* a set of disks), how to reconstruct the shape ?

Let $(x_m, y_m, r_{x_m y_m})$ be the points of the MA. The object $P$ is given by:

$$P = \{(i,j) \mid \exists m,\ (i - x_m)^2 + (j - y_m)^2 < r_{x_m y_m}\}\,.$$

or

$$P = \{(i,j) \mid \max_{(x_m,y_m)\in MA} \{r_{x_m y_m} - (i - x_m)^2 - (j - y_m)^2\} > 0\}\,.$$

$\Rightarrow$ Same kind of maximization/minimization processes as in the SEDT

SEDT:

$$s(q) = \min_{p(x,y)\in P} \{(x - i)^2 + (y - j)^2\}$$

David Coeurjolly



# The REDT

## Problem

Given a MA (*i.e.* a set of disks), how to reconstruct the shape ?

Let $(x_m, y_m, r_{x_m y_m})$ be the points of the MA. The object $P$ is given by:

$$P = \{(i,j) \mid \exists m,\ (i-x_m)^2 + (j-y_m)^2 < r_{x_m y_m}\}\,.$$

or

$$P = \{(i,j) \mid \max_{(x_m,y_m)\in MA}\{r_{x_m y_m} - (i-x_m)^2 - (j-y_m)^2\} > 0\}\,.$$

$\Rightarrow$ Same kind of maximization/minimization processes as in the SEDT

SEDT:

$$s(q) = \min_{p(x,y)\in P}\{(x-i)^2 + (y-j)^2\}$$

# Separable process

$$P = \{(i,j) \mid \max_{(x_m,y_m)\in MA}\{r_{x_m y_m} - (i-x_m)^2 - (j-y_m)^2\} > 0\}\,.$$

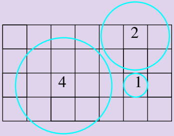
Let $F$ be the image where $f_{ij} = r_{ij}$ if $(i,j,r_{ij}) \in MA$ or 0 otherwise

1. Build from $F$ the map $G = \{g_{ij}\}$ such that:

$$g_{ij} = \max_x\{f_{xj} - (i-x)^2, 1 \le x \le n\}\,. \tag{1}$$

2. Build from $G$ the map $H$ such that:

$$h_{ij} = \max_y\{g_{iy} - (j-y)^2, 1 \le y \le n\}\,. \tag{2}$$




| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 2 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 3 | 4 | 3 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

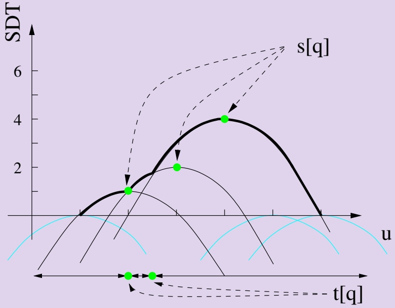
| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 2 | 1 |
| 0 | 2 | 3 | 2 | 0 | 1 | 0 |
| 0 | 3 | 4 | 3 | 0 | 1 | 0 |
| 0 | 2 | 3 | 2 | 0 | 0 | 0 |

David Coeurjolly



# Upper envelope computation

## $\{f_{xj} - (i - x)^2\}$: family of parabolas




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| F | 0 | 0 | 1 | 2 | 4 | 0 | 0 |
| G | 0 | 0 | 1 | 3 | 4 | 3 | 0 |
| REDT | 0 | 0 | 1 | 1 | 1 | 1 | 0 |

# REDT Computational Cost

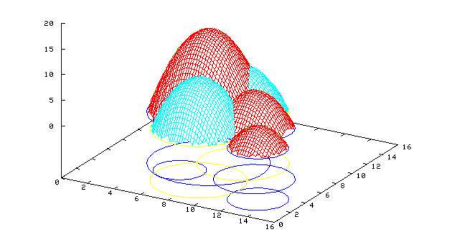
**Upper envelope computation in linear time**

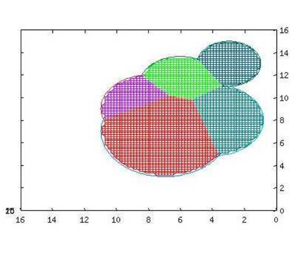
$\Rightarrow O(n^2)$ for a 2-D image
$\Rightarrow O(n^d)$ for a d-D image

# Geometrical interpretation

## Definitions

- 1 disk of the MA = 1 elliptic paraboloid $r_{x_m y_m} - (i - x_m)^2 - (j - y_m)^2$
- Compute the *max* function $\Leftrightarrow$ Compute the upper envelope of all the elliptic paraboloids




$\Rightarrow$ Laguerre Diagram

# Power diagrams / Laguerre diagrams

$$P = \{(i,j) \mid \max_{(x_m,y_m)\in MA} \{r_{x_m y_m} - (i-x_m)^2 - (j-y_m)^2\} > 0\}\,.$$

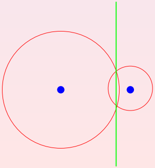
## Definitions

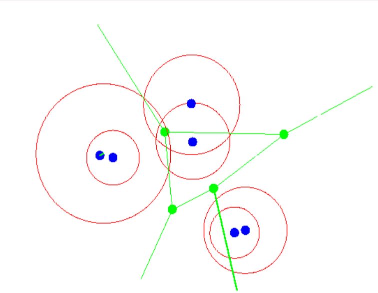
The power of a point with respect to a circle $(i,j,s_{ij})$ :

$$\sigma(x,y) = (x-i)^2 + (y-j)^2 - s_{ij}^2$$

Laguerre Cell : $L(\sigma_i) = \{x \in \mathbb{R}^2 \quad | \quad \sigma_i(x) \leq \sigma_j(x), 1 \leq j \leq n\}$

Laguerre Diagram : the set of non-empty cells with their faces



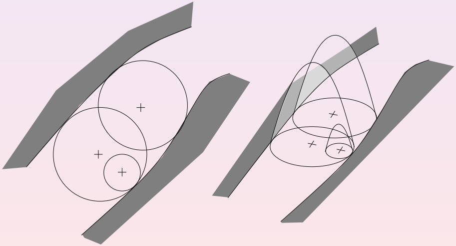


# Links between MA and Laguerre Diagram

## Properties

- Associate to the point $(i,j)$ the maximal elliptic paraboloid at $(i,j)$ = Find the Laguerre cell in which $(i,j)$ belongs to
- Equivalence between maximal disks and maximal elliptic paraboloids

$\Rightarrow$ Extract MA = find the sites of the non-empty cells in the Laguerre Diagram



(a)

# Links between MA, Laguerre Diagram and the REDT algorithm

## Observation

If we have two disks $B_1$ and $B_2$ contained in the shape such that $B_1 \subset B_2$, and if we mark the disks that belong to the upper envelope of elliptic paraboloids in the REDT algorithm, $B_1$ will not be marked

## Ideas of the algorithm

- Input: all disks contained in the shape, *i.e.* $\{(x, y, r_{xy})\}$ for all $(x, y) \in S$ and $r_{xy} = SDT(x, y)$
- Use the REDT algorithm and the upper envelope computation to *filter* the disks to obtain the MA

# Links between MA, Laguerre Diagram and the REDT algorithm

## Observation

If we have two disks $B_1$ and $B_2$ contained in the shape such that $B_1 \subset B_2$, and if we mark the disks that belong to the upper envelope of elliptic paraboloids in the REDT algorithm, $B_1$ will not be marked

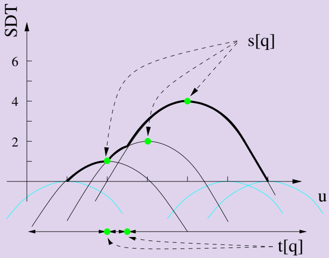
## Ideas of the algorithm

- Input: all disks contained in the shape, *i.e.* $\{(x, y, r_{xy})\}$ for all $(x, y) \in S$ and $r_{xy} = SDT(x, y)$
- Use the REDT algorithm and the upper envelope computation to *filter* the disks to obtain the MA

# REDT → Discrete Laguerre Labeling

## Sketch of the alg.

- For each point $(i,j)$ of the object, we associate to $(i,j)$ the label to the parabola in the the upper envelope that is maximal in $(i,j)$
- List of upper envelope parabolas are stored in $s[q]$
- Labeling performed dimension by dimension



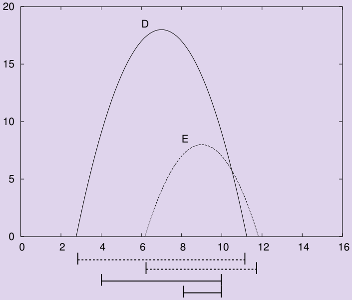

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| F | 0 | 0 | 1 | 2 | 4 | 0 | 0 |
| G | 0 | 0 | 1 | 3 | 4 | 3 | 0 |
| REDT | 0 | 0 | 1 | 1 | 1 | 1 | 0 |

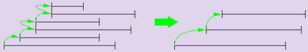
## Computational cost

⇒ $O(n^2)$ for a 2-D image
⇒ $O(n^d)$ for a d-D image

# Discrete Laguerre Diagram $\rightarrow$ MA centers

## Filtering of the Laguerre Diagram

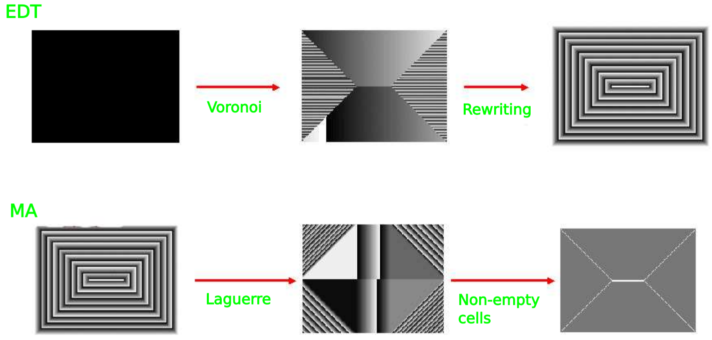


*D* and *E* belong to the upper envelope but the discrete disk *D* contains the discrete disk *E*

$\Rightarrow$ Linear in time filtering

# Summary



David Coeurjolly



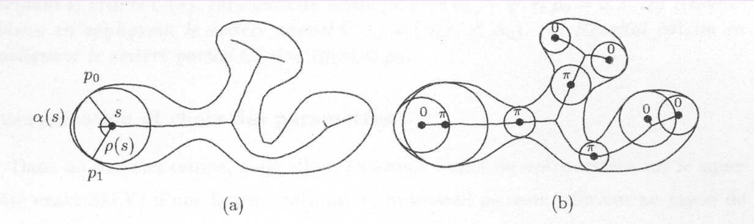
# MA filtering in computational geometry

## Observations

- small disks may not be essential for the shape geometry
- contact points between the disk and the object contour are important

## Filtering based on two parameters [D. Attali]

1. Radius of the disk
2. Contact angle *(bisector angle)*



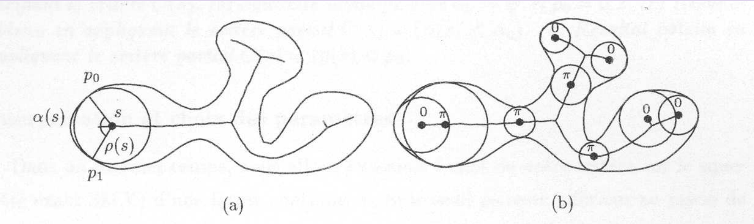
# MA filtering in computational geometry

## Observations

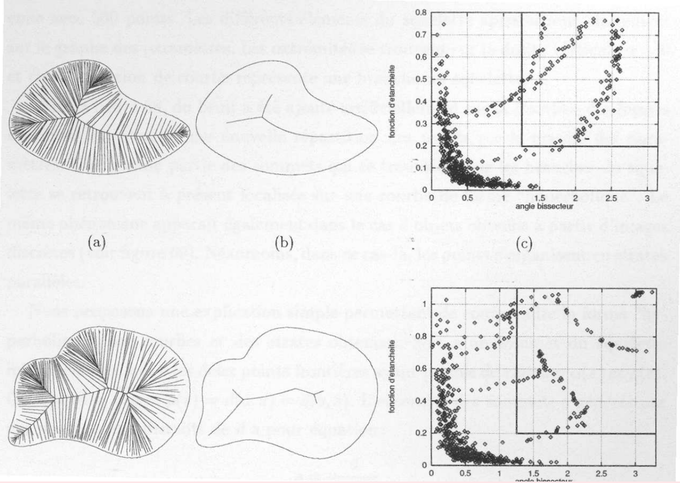
- small disks may not be essential for the shape geometry
- contact points between the disk and the object contour are important

## Filtering based on two parameters [D. Attali]

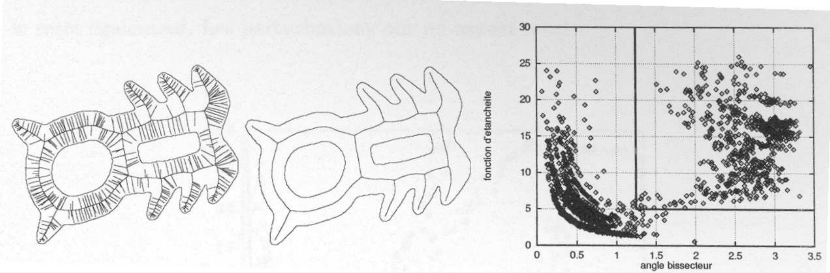
1. Radius of the disk
2. Contact angle *(bisector angle)*



# MA filtering results [D. Attali]



# MA filtering results [D. Attali]

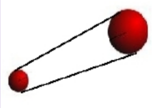


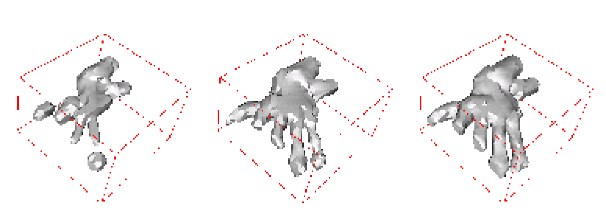
# Table of contents

# Shape coding and Transmission

## Idea

- Use MA balls to efficiently lossless encode binary shapes
- Use convex hull of 2 balls (*a.k.a. cone*)
- Use convex hull of n balls, . . .



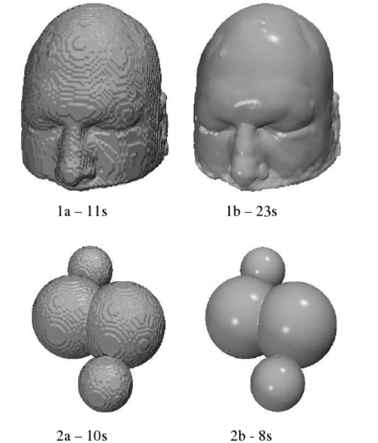


[F. Dupont]

David
Coeurjolly



# Visualization




[S. Prévost and L. Lucas]

Use MA balls to estimate the normal vectors and to smooth surfaces during ray-tracing

# Table of contents

# Conclusion

**We have:**

- optimal in time and separable algorithms to compute the REDT and the MA ;
- Algorithms based on the error free Euclidean metric ;
- deep links between Discrete Geometry and Computational Geometry.