The preimage in 2-D

David Coeurjolly and Isabelle Sivignon

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The recognition problem in Discrete Geometry

Discrete objects

- points
- set of points
- arcs
- curves

Analytical model

- Discrete straight segments
- Discrete hyperplanes
- Discrete circles and hyperspheres
- Discrete polygons, meshes, polynoms,. . .

Recognition step

Theoretical and algorithmic solutions to fill the gap between discrete objects and their analytical representations in a model

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General Definition

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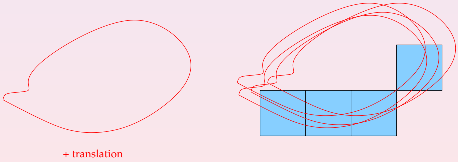
Let $\mathcal{O}$ be a family of Euclidean objects parametrized by a set of parameters $\{\alpha_i\}_{1..n}$
Let D_r be a digitization process on a grid with resolution r and S be a set of pixels

Definition

The preimage of S is the equivalence class of objects O_j in $\mathcal{O}$ such that $S \subset D_r(O_j)$



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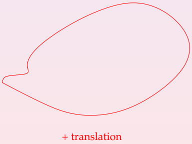

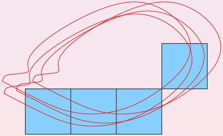

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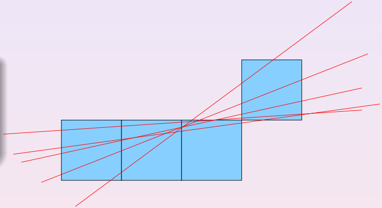
Preimage of straight lines

We need

- a parametrization of straight lines
- a digitization process

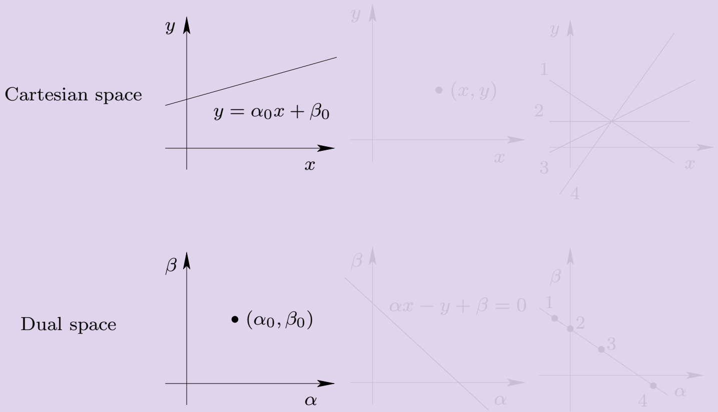


Parameter space

$\mathcal{O}$ = set of straight lines $y = \alpha x + \beta \Rightarrow$ 2-D parameter space = $\{(\alpha, \beta)\}$

Linear dual transform

$y = \alpha x + \beta \leftrightarrow (\alpha, \beta)$



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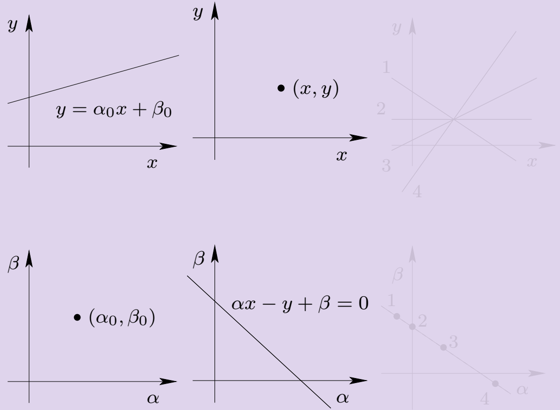
Cartesian space

Dual space



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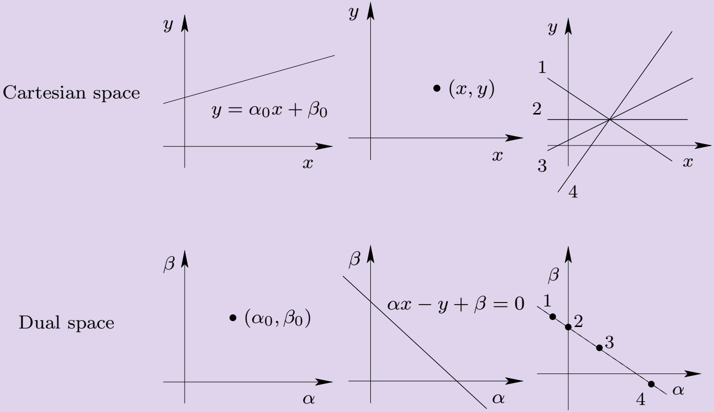
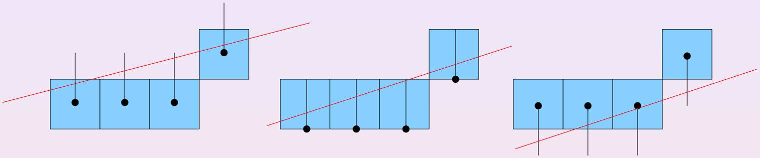


Digitization process associated to 8-connected DSS



From left to right

- Object boundary quantization : $y = \lfloor \alpha x + \beta \rfloor$ with $x \in \mathbb{Z}$
- Grid Intersect quantization : $y = [\alpha x + \beta]$ with $x \in \mathbb{Z}$
- Background boundary quantization : $y = \lceil \alpha x + \beta \rceil$ with $x \in \mathbb{Z}$

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A pixel = a strip in the dual space

Let $P(x_0, y_0)$ be a pixel and $l : y = \alpha x + \beta$ a straight line

$$P \in D_r(l) \Leftrightarrow y_0 = \lfloor \alpha x_0 + \beta \rfloor$$
$$0 \leq \alpha x_0 + \beta - y_0 < 1$$

Set of straight lines l such that $P \in D_r(l)$

(α, β) such that:

$$\mathcal{B}(x_0, y_0) = \begin{cases} 0 \leq \alpha x_0 + \beta - y_0 & (D) \\ \alpha x_0 + \beta - y_0 < 1 & (D') \end{cases}$$

$\Rightarrow$ strip in the dual space

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$\Rightarrow$ strip in the dual space

Preimage of a set of pixels

Definition ([DS84, McI85])

Let $S = \{(x_i, y_i)\}$ be a set of pixels, its preimage $\bar{S}$ is:

$$\bar{S} = \{(\alpha, \beta) \mid 0 \leq \alpha x_i + \beta - y_i < 1,\ \forall i\}$$

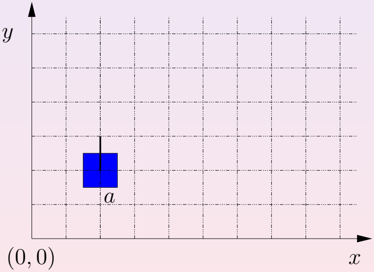
$$\bar{S} = \bigcap_i \mathcal{B}(x_i, y_i)$$

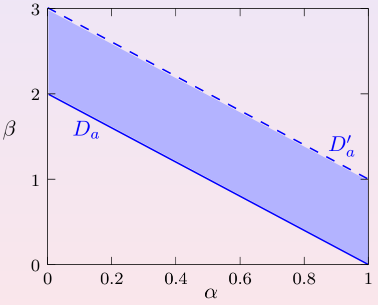
in other words

$(\alpha, \beta) \in \bar{S} \Leftrightarrow$ the straight line crosses the interval $[y, y+1[$ of each pixel

Strip intersection process

W.l.o.g we suppose straight lines in the first octant, hence $(\alpha, \beta) \in [0, 1] \times [0, 1[$





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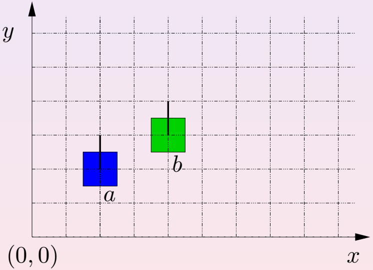

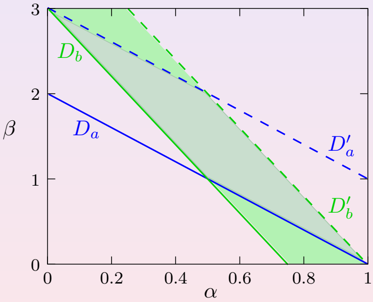

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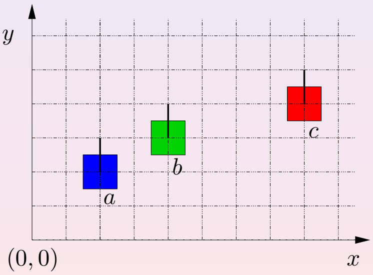

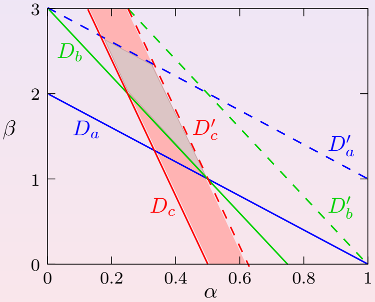

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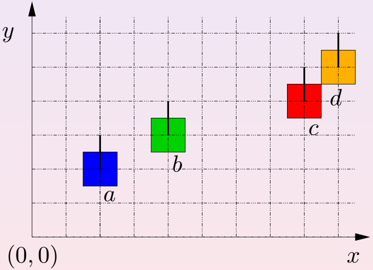

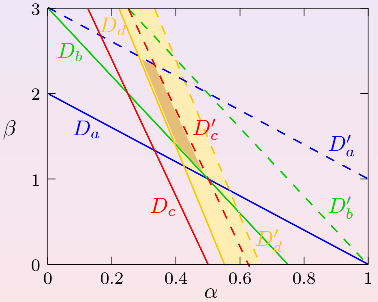

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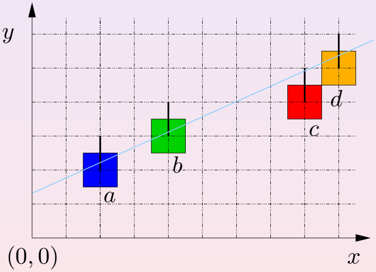

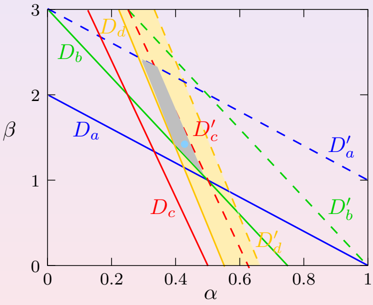

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Recognition

Definition

Given a set of pixels S, S is a piece of digital straight line iff the preimage of S is not empty

First computational analysis

- 1 pixel $\rightarrow$ 2 linear constraints
- n pixels $\rightarrow$ $2.n$ linear constraints

Intersection of $2.n$ linear constraints

- $O(n)$ to detect if $\bar{S}$ is empty or not [Meg84]
- $O(n \log n)$ to construct the vertices/faces of the preimage [PS85]

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Preimage Structure

The preimage is a convex polygon

If S is an 8-arc

- $\bar{S}$ has only 3 or 4 vertices
- $\bar{S}$ has a strong arithmetical structure

[DS84, McI85, LB93]

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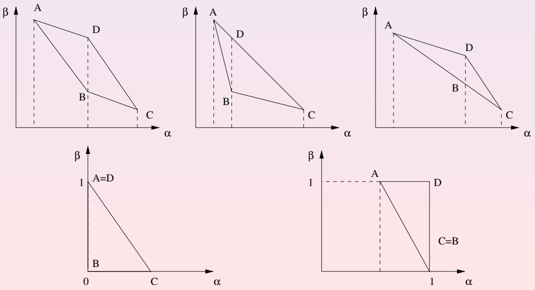

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Farey series

[HW75]

Farey series

The Farey series $\mathcal{F}_m$ of order m is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed m.

Ex:

$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

Properties

- if $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms in $\mathcal{F}_m$ (with $\frac{h}{k} < \frac{h'}{k'}$) , then $kh' - hk' = 1$
- if $\frac{h}{k}$, $\frac{h''}{k''}$ and $\frac{h'}{k'}$ are three successive terms in $\mathcal{F}_m$ (with $\frac{h}{k} < \frac{h''}{k''} < \frac{h'}{k'}$), then $\frac{h''}{k''} = \frac{h+h'}{k+k'}$. The fraction $\frac{h''}{k''}$ is called the mediant of $\frac{h}{k}$ and $\frac{h'}{k'}$.
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$$\mathcal{F}_6 = \left\{ \frac{0}{1}, \frac{\mathbf{1}}{\mathbf{6}}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{\mathbf{5}}{\mathbf{6}}, \frac{1}{1} \right\}$$

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Farey Fan [McI85]

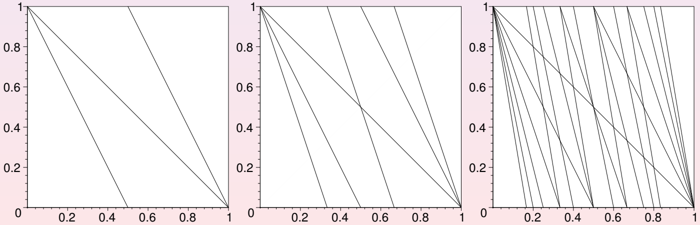
Definition (Rays)

$$R(x,y) = \{(\alpha,\beta) \mid \beta = -\alpha x + y\}$$

According to the strip decomposition, for each point (x,y), we have the pair of parallel rays $R(x,y)$ and $R(x,y+1)$ in the (α,β) dual space.

Farey Fans

If we focus on the domain $[0,1] \times [0,1[$ and segments in the first octant, rays $R(x,y)$ with $0 \leq y \leq x \leq n$ define a diagram called the Farey fan of order n



(Farey fans with order 2,3 and 6)

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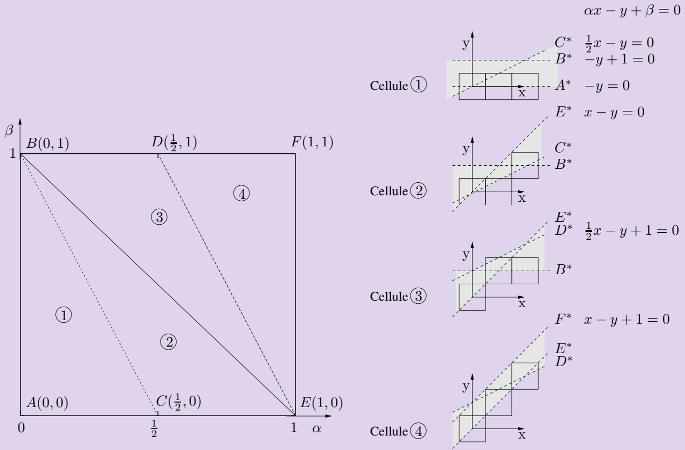
Preimage and DSS

The Farey fan of order n represents all the DSS of length $n+1$



Farey Fan (bis)

Farey fan = all possible strip intersections for all 8-arcs with $0 \leq y \leq x \leq n$

$R(x, y)$ and $R(x', y')$ with $x > x'$ intersect at

$$\left(\frac{y - y'}{x - x'}, \frac{xy' - x'y}{x - x'}\right).$$

If $0 \leq \alpha < 1$, the α-coordinate of this intersection can be written:

$$\alpha = \frac{p}{q} \quad 0 \leq p \leq q \quad 1 \leq q \leq max(x, n - x).$$

Furthermore,

$$\beta = y - \frac{px}{q}$$

Theorem [McI85]

- The coordinates (α, β) belong to the Farey series of order $max(x, n - x)$
- If we scan all rays $R(x', y')$, the $\alpha-$coordinates constitute the Farey series $\mathcal{F}_{max(x,n-x)}$
- Two adjacent vertices are adjacent fractions in $\mathcal{F}_{max(x,n-x)}$

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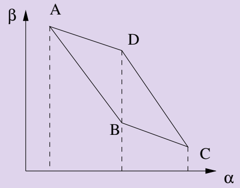
Farey fan and preimages

Theorem [McI85]

A facet of the Farey fan has at most 4 vertices and the two *middle vertices* have the same abscissa



Sketch of proof.

Recursive proof on n using the following properties:

- adjacent vertices are adjacent fractions in the Farey series
- when a new ray is considered and crosses the segment $[uw]$ of the Farey fan, the fraction of the intersection abscissa is the mediant of the abscissa of u and w

□

To sum up

The preimage of an 8-DSS is a facet of the Farey fan and when the strip of a new pixel modify it, the Farey structure controls the ways the intersections are.

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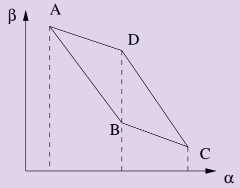

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Recursive proof on *n* using the following properties:

- adjacent vertices are adjacent fractions in the Farey series
- when a new ray is considered and crosses the segment [*uw*] of the Farey fan, the fraction of the intersection abscissa is the mediant of the abscissa of *u* and *w*

□

To sum up

The preimage of an 8-DSS is a facet of the Farey fan and when the strip of a new pixel modify it, the Farey structure controls the ways the intersections are.

Lindenbaum and Bruckstein algorithm [LB93]

UPDATE_PREIMAGE($p(x_{n+1}, y_{n+1})$)
$L_{low} \leftarrow \beta = -\alpha x_{n+1} + y_{n+1}$
$L_{high} \leftarrow \beta = -\alpha x_{n+1} + y_{n+1} + 1$
if $B \cap L_{low} \neq \emptyset$ **and** $\alpha_B \neq 0$ **then**
  $\alpha_A = \mathcal{F}^{+}_{n+1-x_{AD}}(\alpha_A)$; $\beta_A = -\alpha_A x_{AD} + y_{AD}$
  $x_{AB} = x_{n+1} = n+1$; $y_{AB} = y_{n+1}$
else
  if $D \cap L_{low} \neq \emptyset$ **then**
    if $\alpha_D = 1$ **then**
      return "$S \cup \{p\}$ **is not a DSS**"
    else
      $\alpha_A = \alpha_D$; $\beta_A = \beta_D$
      $\alpha_B = \alpha_D = \mathcal{F}^{-}_{n+1-x_{BC}}(\alpha_C)$
      $\beta_B = -\alpha_B x_{BC} + y_{BC}$; $\beta_D = -\alpha_D x_{CD} + y_{CD}$
      $x_{AB} = n+1$; $y_{AB} = y_{n+1}$
      $x_{AD} = x_{CD}$; $y_{AD} = y_{CD}$
    end if
  else
    if $B \cap L_{high} \neq \emptyset$ **then**
      if $\alpha_B = 0$ **then**
        return "$S \cup \{p\}$ **is not a DSS**"
      else
        $\alpha_C = \alpha_B$; $\beta_C = \beta_B$
        $\alpha_B = \alpha_D = \mathcal{F}^{+}_{n+1-x_{AD}}(\alpha_A)$
        $\beta_B = -\alpha_B x_{AB} + y_{AB}$; $\beta_D = -\alpha_D x_{AD} + y_{AD}$
        $x_{CD} = n+1$; $y_{CD} = y_{n+1} + 1$
        $x_{BC} = x_{AB}$; $y_{BC} = y_{AB}$
      end if
    else
      if $D \cap L_{high} \neq \emptyset$ **and** $\alpha_D \neq 1$ **then**
        $\alpha_C = \mathcal{F}^{-}_{n+1-x_{BC}}(\alpha_C)$; $\beta_C = -\alpha_C x_{BC} + y_{BC}$
        $x_{CD} = x_{n+1} = n+1$; $y_{CD} = y_{n+1} + 1$
      else
        if $B \in \mathcal{B}(p)$ **and** $D \in \mathcal{B}(p)$ **then**
          return "$S \cup \{p\}$ **is a DSS with the same preimage**"

Lindenbaum and Bruckstein algorithm [LB93]

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- $L_{low} \leftarrow \beta = -\alpha x_{n+1} + y_{n+1}$
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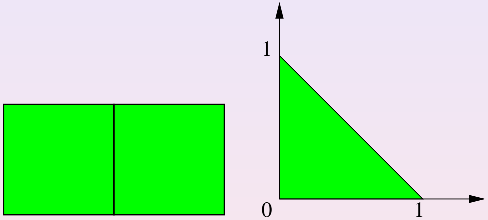
Example of the recognition algorithm



Step: Initialization
$\bar{S}$ coordinates: $\{(0,1),(0,0),(1,0),(0,1)\}$
$\bar{S}$ slopes: $\{(-,-),(0,0),(1,1),(-,-)\}$

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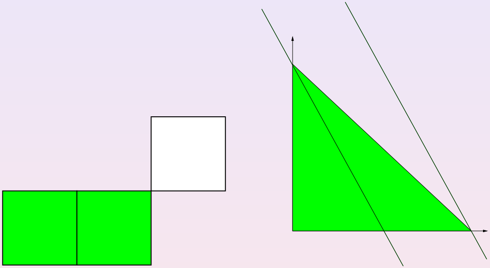
Example of the recognition algorithm

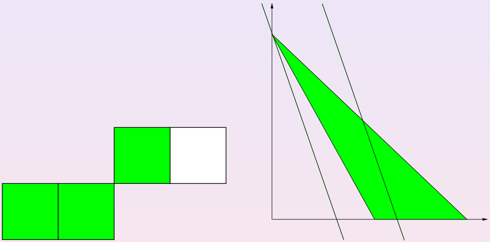


Step: $D \cap L_{low} \neq \emptyset$

$\bar{S}$ coordinates: $\{(0,1), (\frac{1}{2}, 0), (1,0), (\frac{1}{2}, \frac{1}{2})\}$

$\bar{S}$ slopes: $\{(2,1), (0,0), (1,1), (1,1)\}$

Example of the recognition algorithm



Step: $D \cap L_{high} \neq \emptyset$

$\bar{S}$ coordinates:$\{(0,1),(\frac{1}{2},0),(\frac{2}{3},0),(\frac{1}{2},\frac{1}{2})\}$

$\bar{S}$ slopes: $\{(2,1),(0,0),(3,1),(1,1)\}$

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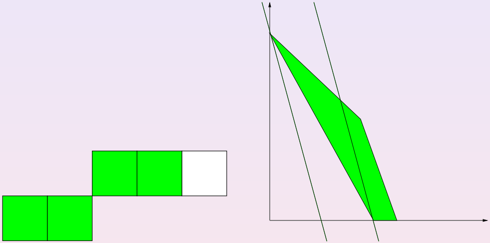
Example of the recognition algorithm



Step: $B \cap L_{high} \neq \emptyset$

$\bar{S}$ coordinates: $\{(0,1), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, 0), (\frac{1}{3}, \frac{2}{3})\}$

$\bar{S}$ slopes: $\{(2,1), (0,0), (4,1), (1,1)\}$

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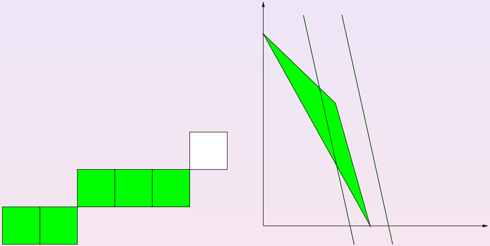
Example of the recognition algorithm

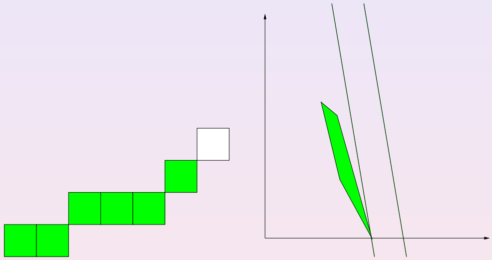


Step: $B \cap L_{low} \neq \emptyset$

$\bar{S}$ coordinates:$\{(\frac{1}{4}, \frac{3}{4}), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, 0), (\frac{1}{3}, \frac{2}{3})\}$

$\bar{S}$ slopes: $\{(5,2), (0,0), (4,1), (1,1)\}$

Example of the recognition algorithm



Step: $B \notin \mathcal{B}(p)$ or $D \notin \mathcal{B}(p)$
$\bar{S}$ coordinates: $\emptyset$
$\bar{S}$ slopes: $\emptyset$

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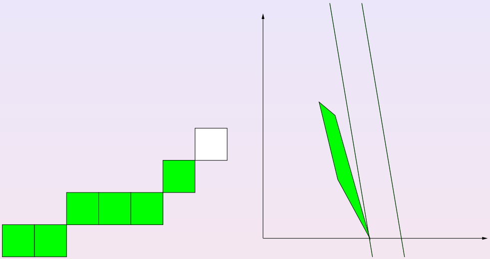
Example of the recognition algorithm



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To conclude on the recognition algorithm

- Optimal algorithm (preimage update in $O(1)$)
- Efficient preimage computation based on arithmetical structures

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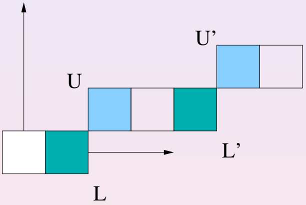
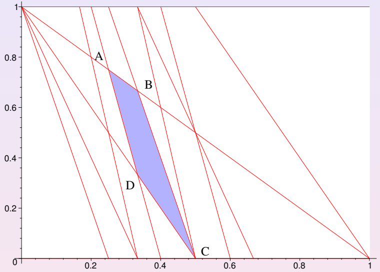
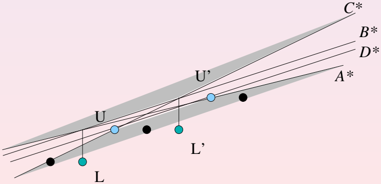
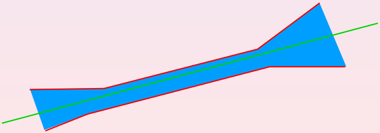
Preimage vertices





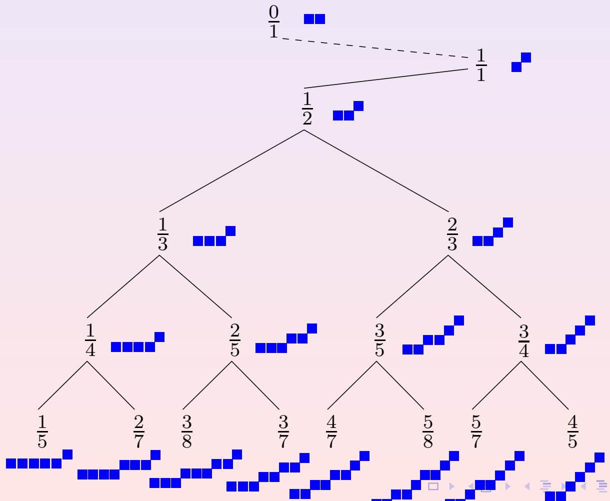




The Stern-Brocot Tree [HW75]

Binary search tree of $\mathcal{F}_n$

To represent the evolution of the recognition algorithm and equivalence classes of DSS



What is the structure of the preimage if S is not connected ?

Theorem

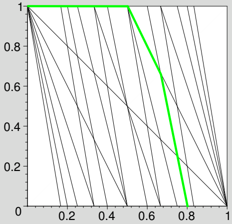
The preimage has $O(\log n)$ vertices.

Sketch of proof.

- The preimage is the union of adjacent Farey facets
- The maximum convex polyline in the Farey fan of order n has $O(\log n)$ vertices (= height of the Stern-Brocot tree)

$\square$

Example with $\mathcal{F}_6$



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Uncertain Geometry [Vee99, Vee00]

Idea

Use the preimage to define geometrical principles such that digital parallelism, concurrency,. . .

Parallelism between two discrete segments

Two discrete segments are digitally parallel if there exist at least one point in each preimages with the same $\alpha-$coordinate

Example: parallel line grouping [Vee00]

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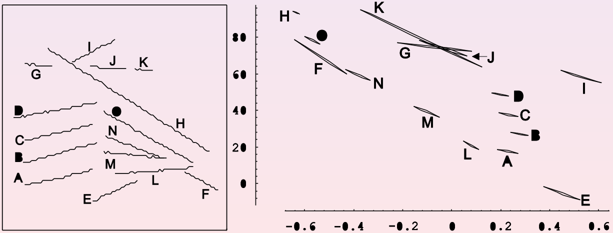

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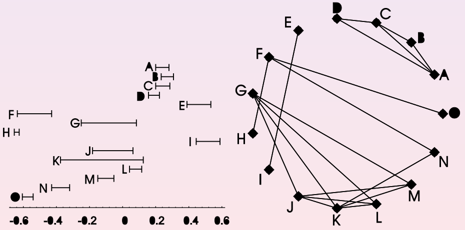

Parallel line groups = cliques

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Reversible reconstruction

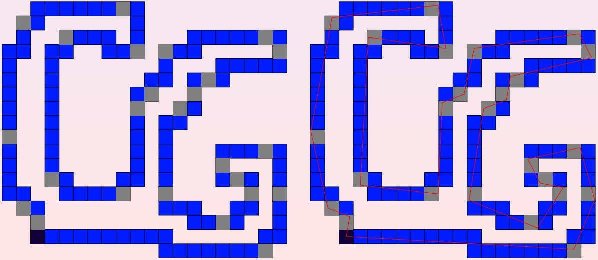
Statement

Construct from a discrete curve a polyline such that its digitization is exactly the original discrete curve

Idea [SBDA05]

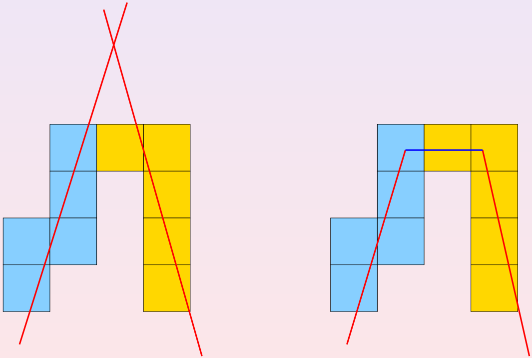
1. Construct a DSS segmentation (*i.e.* decomposition of the curve into maximal DSS)
2. Choose the edges of the polyline from the different preimages

(or a mix process between step 1 and 2)



The problem is not as trivial as it seems..

- The preimage must be filtered to ensure the reversibility of the polyline vertices
- artificial patches may be inserted



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Conclusion...

The preimage is the ultimate tool to

- recognize DSS
- define geometrical properties taking into account equivalence classes induced by the digitization
- reconstruct a reversible polygonal curve

DSS Recognition:

Brute-force algorithm

- $O(n \log n)$
- update $O(\log n)$
- memory $O(n)$

With the arthimetical properties

- $O(n)$
- update $O(1)$
- memory $O(1)$

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Conclusion...

The preimage is the ultimate tool to

- recognize DSS
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- reconstruct a reversible polygonal curve

DSS Recognition:

Brute-force algorithm

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