Ch. 2. Vector Spaces

§ Vector Spaces and subspaces.

Symmetric Matrix.

ex. $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$\Rightarrow A = A^T$

$$\underset{R^T}{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T} = \underset{R}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}}$$

$R^T R$ is always symmetric

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & \bullet & \bullet \\ 7 & \bullet & \bullet \end{bmatrix}$$

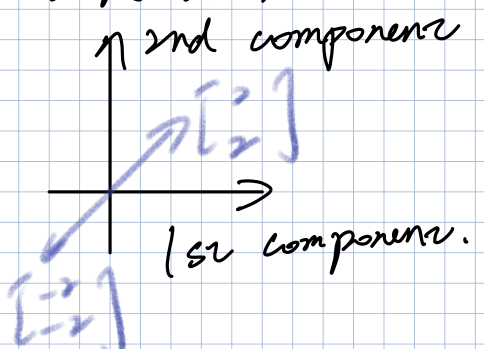
Why? We can take the transpose.

$(R^T R)^T = R^T R^{TT}$

$= R^T R$, proved.

Vector Space

ex. R^2 = all 2-dimensional real vectors.



$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \ldots,$

x-y plane

= All vectors with 3 components.

R^n = vectors with n column. (n components)

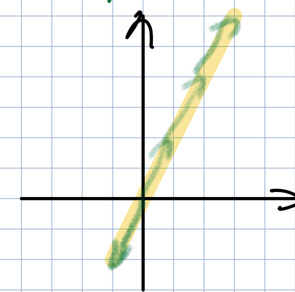
Q can we do those addition and still in the spaces?

Ans: Yes

ex. a vector space inside R^2

⇒ subspace of R^2

⇒ line in R^2 through zero vector



① all in R^2

② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

③ zero vector only "Z"

R^3: Plane / line / $Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

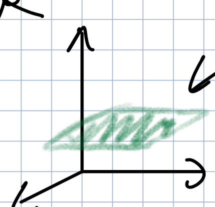
Q How they come out of Matrices.

Ans: all their comb. for a subspace call col space $C(A)$

Vector Space requirements $V+w$ and cV are in the space, comb. $w+d$ are in the space.

R^3

Plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is subspace of R^3



Q. give 2 subspaces: P and L.

$P \cup L$: all vectors in P or L or both. — (1)

$P \cap L$: all vectors in both P and L — (2)

Which is a subspace? Ans: (2)

Subspaces S and T intersection $S \cap T$ is a Subspace.

Column Space of A is a subspace of R^4

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$ - all linear comb. of cols.

$C(A)$

Q. Does $Ax = b$ have a solution for every b? Ans: No

4 equations, 3 unknowns

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

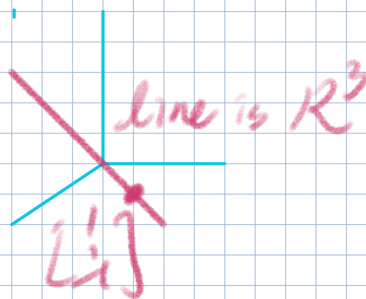
Q. Which vector B allow this system to be solve?

exactally when b is in $C(A)$ (in R^4)

Null space of A contains $c\begin{bmatrix} ? \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$ (in R^3)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Check that solutions to $Ax=0$ always give a subspaces.

If $Av=0$ and $Aw=0$.

then $A(v+w)=0$.

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$$

$\Rightarrow$ echlon $\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U.$

$\boxed{\text{rank of } A = \text{cnt of pivots} = 2}$

free col

2 pivots col

$$x=c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

$x_1+2x_2+2x_3+2x_4=0.$

$2x_3+4x_4=0$

$r=2$ cnt of variable.

$n-r=4-2$ free variable

R = reduced row echlon form
(zero above & below pivots)

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R = rref(A)$ Notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

in the pivot rows/col.

$x_1 + 2x_2 \quad - 2x_4 = 0$

$x_3 + 2x_4 = 0$

$Rx = 0$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ pivot cols $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ free cols

I $\qquad$ F

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow r \text{ pivot rows} \\ \leftarrow n-r \text{ free cols.} \end{matrix}$$

↑

r pivots cols

Null Space Matrix

$N = \begin{bmatrix} -F \\ I \end{bmatrix}$

$Rx = 0 \quad x_{pivot} = -F x_{free}$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

Redo Algorithm again.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

$r=2$ again U free $3-2=1$ free col.

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 2x_3 = 0.$$

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c\begin{bmatrix} -F \\ I \end{bmatrix}$$

N.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{array}{cc} I & F \\ \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.$$