$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$
$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$
$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

OK. $\boxed{b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}}$

$0 = b_3 - b_2 - b_1$

Augmented Matrix $= [A \;\; b]$

pivot cols

Solvability condition on b.

$Ax = b$ solvable when b is in $C(A)$

If a comb. of rows A gives zero rows.
then the same comb. of entries of b must give 0.

Algorithm

To find complete sol'n to $Ax = b$.

Step 1. x particular: set all free variables to zero. $x_2 = 0$, $x_4 = 0$

Solve $Ax=b$ pivot variables

solve $Ax=0$ pivots variables.

(+) $x_1 + 2x_3 = 1$, $2x_3 = 3$ $\quad x_1 = -2$, $x_3 = 3/2$ $\quad x_p = c\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

Step 2: X nullspace.

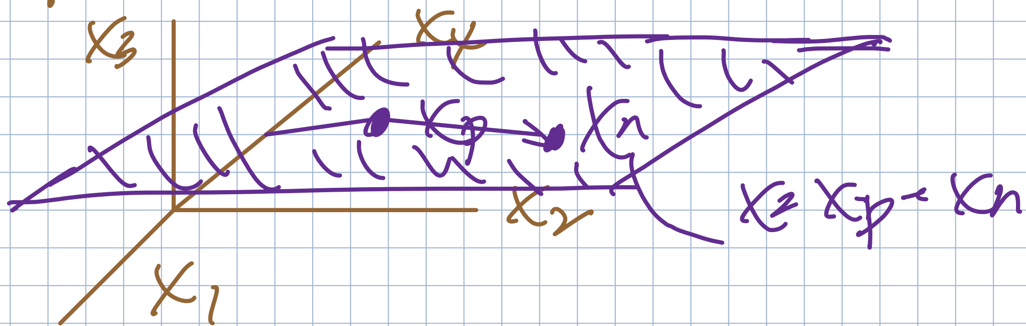
$$\begin{aligned} &Ax_p = b \\ +)\ &Ax_n = 0 \\ \hline &A(x_p + x_n) = b. \end{aligned}$$

$x = x_p + x_n$

$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

if $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$. can solve?

Plot all solutions x in R^4.



m by n matrix A of rank r. ($r \le m$, $r \le n$)

Full column rank means $r = n$:

No free variables. unique solution if it exists (0 or 1 solution)

$N(A) = \{\text{zero vector}\}$

Solution to $Ax = b$: $x = x_p$ $\quad A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ $\quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Full row rank means $\boxed{r=m}$

Can I solve $Ax=b$ for every b. Exist #

Left with $\underbrace{n-r}_{\substack{\| \\ n-m}}$ free variables

$$A=\begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R=\begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$$

$r=m=n$

$$A=\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$R=I$$

$r=m=n$	$r=n<m$	$r=m<n$	$r<m$, $r<n$
$R=I$	$R=\begin{bmatrix} I \\ 0 \end{bmatrix}$	$R=\begin{bmatrix} I & F \end{bmatrix}$	$R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 solution	0 or 1 solution	∞ solutions	0 or ∞ solutions