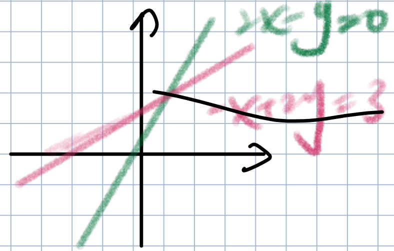
Ch.1 MATRIX AND GAUSSIAN ELIMINATION

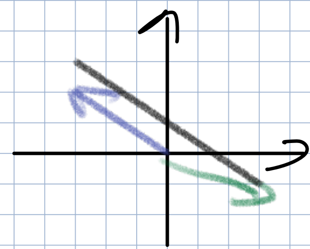
§ The geometry of linear equations.

(一) Row Picture



$$\begin{bmatrix} 2x-y=0 \\ -x+2y=3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(二) Column Picture

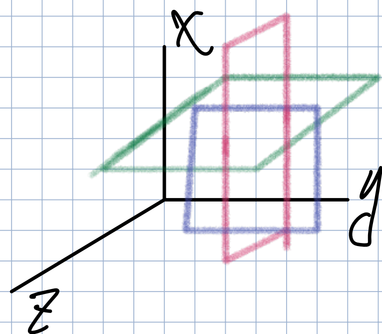


$$\begin{cases} 2x-y=3 \\ -x+2y=3 \end{cases} \Rightarrow x\begin{bmatrix} 2 \\ -1 \end{bmatrix}+y\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

linear combination of col

Row Picture 假設圖

$$\begin{cases} 2x-y=0 \\ -x+2y-z=-1 \\ -3y+4z=4 \end{cases}$$



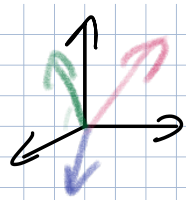
$$\Rightarrow A=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad B=\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Column Picture

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}+y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}+z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

col1 col2 col3

linear comb. $\Longrightarrow$



$(x, y, z) = (0, 0, 1)$

Q.

Can we solve $Ax=b$ for every b

$\equiv$ Do the linear combs. of the columns in 3D place.

Ans: For the Matrix A it is.

P.S: It's a non-singular matrix $\Rightarrow$ An invertible Matrix

$Ax=b$ (AX is a comb. of col A

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

消元法. 回代法

2要的有一矩阵.

pivot

$$\begin{matrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix} \xrightarrow{(2,1)} \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{matrix} \xrightarrow{(3,2)} \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{matrix}$$

$$\boxed{\begin{matrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix}}\ \boxed{\begin{matrix} 2 \\ 12 \\ 2 \end{matrix}} \rightarrow \begin{matrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{matrix}$$

$$\Rightarrow \begin{aligned} x+2y+z&=2 \\ 2y-2z&=6 \\ 5z&=-10 \end{aligned} \Rightarrow (x, y, z) = (2, 1, -2)$$

Matrix · col = col

$$\begin{bmatrix} col1 & col2 & col3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3\,col1 + 4\,col2 + 5\,col3$$

Row · Matrix = Row

$$\begin{bmatrix} row1 \\ row2 \\ row3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 7 \end{bmatrix} = \begin{matrix} 1 \times row1 \\ 2 \times row2 \\ 7 \times row3 \end{matrix}$$

§ Elimination Matrices.

Step 1. subtract 3 × row1 from row2

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

E_{21} = elementary or elimination

Step 2. subtract 2 × row2 from row3.

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$E_{32}(E_{21}A) = U \Rightarrow (E_{32}E_{21})A = U$

置换 Permutation

Exchange row 1 and 2 $L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Exchang col 1 and col 2 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} R = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & e \end{bmatrix}$$

可逆 Inversable

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

§ Matrix operations and inverse.

Matrix comb. → 5 ways

case1: $\begin{bmatrix} \text{row3} \end{bmatrix}\begin{bmatrix} \text{col4} \end{bmatrix} = \begin{bmatrix} c_{34} \bullet \end{bmatrix}$

A B $C=AB$

$m \times n$ $n \times p$ $m \times p$

$c_{34} = (\text{row3 of } A)(\text{col4 of } B)$

$= a_{31}b_{14} + a_{32}b_{24} + \cdots = \sum_{k=1}^{n} a_{3k}a_{k4}.$

Case 2.

$$\underset{A}{\begin{bmatrix} & & \end{bmatrix}} \underset{B}{\begin{bmatrix} \vert\ \text{col} & & \end{bmatrix}} = \begin{bmatrix} \vert\ A(\text{col}) & & \end{bmatrix}$$

Col & c are comb. of col of c.

$$\underset{A}{\begin{bmatrix} \text{---} \\ \\ \end{bmatrix}} \underset{B}{\begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \end{bmatrix}} = \underset{C=AB}{\begin{bmatrix} A\,\text{col} \end{bmatrix}}$$

rows of c are comb. of rows of B

Case 3.

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$m \times 1 \qquad 1 \times p \quad \to \quad m \times p$

case 4.

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

AB = Sum of [col of A] × [rows of B]

case 5 (Block)

$A_1B_1 + A_2B_3$

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} \square & \\ \hline & \end{array}\right]$$

Inverse (square matrix)

$A^{-1}A = AA^{-1} = I$

⇒ if A^{-1} exist

, matrix is (invertible / no singular)

singular case: no inverse.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Q

Does exist a vector $x \neq 0$ with $Ax = 0$.

$$\Rightarrow \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

⇒ $A \times$ col j of A^{-1} = col j of I

§ LU and LDU factorization

Gaussian Elimination.

let A.B inverse-able

$(AB)(B^{-1}A^{-1})=I, \quad (B^{-1}A^{-1})(AB)=I$

Transport

$AA^{-1}=I \quad \boxed{(A^{-1})^T}A^T=I$

This is $(A^{-1})^T$

inverse of A^{-1}

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

E_{21} $\quad A \quad U$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$A \quad L \quad U.$

In 2x2 Matrix, E_{21} same as L

But 3x3 matrix having big Difference.

$E_{32}E_{31}E_{21}A = U$ (no row exchange)

$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU$.

Typical Case:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & 5 & 0 \end{bmatrix} = E$$

E_{32} $\quad$ E_{21}

left of A

[$EA = U$. $A = LU$]

Reverse Direction.

Inverse (Reverse Order)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L$$

$A = LU$ if no row exchange.

multipliers go directly into L.

Q How expensive is elimination?

(= How many operation we do on $n \times n$ matrix)

※ Multiple + Subtract = an operation.

let $n=100$

$\begin{bmatrix} & \\ & \end{bmatrix}_{100\times100} \Rightarrow \begin{bmatrix} & \\ & \end{bmatrix}_{99\times99} \Rightarrow \cdots \Rightarrow \begin{bmatrix} & \\ & \end{bmatrix}_{1\times1}$ and go on

we can get about $\sum n^2 \approx n^3$

§ Transposes and Permutation

$3\times3 = p's$ $\quad P^{-1} = P^T$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

24 $4\times4 \Rightarrow 24$ $p's$

$n\times n \to n!$ $p's$

§ Permutation

P: execute row Exchanges. becomes $PA=LU$

$\Rightarrow$ Any inverseble case.

P = Indentity matrix with reordered rows.

$= n! = n(n-1)(n-2)\cdots\times2\times1$

counts reorders

$\Rightarrow$ all $n\times n$ permutations.

$$\begin{cases} P^TP = I \\ P^{-1} = P^T \end{cases}$$

Transposed.

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

3×2 $\quad\quad$ 2×3

$\Rightarrow$ Transposed $(A^T)_{ij} = (A)_{ji}$