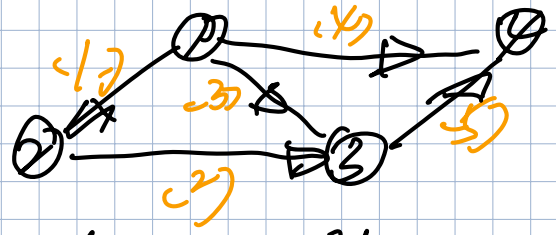
Graph and Network.
Incidence Matrices.
Kirchhoff's Laws.

Example Graph.

Graph: Nodes, Edges.



n = 4 nodes
m = 5 edges

Incidence Matrix.

$$A = \begin{array}{c} \text{node} \\ \\ \\ \\ \\ \text{edge} \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \quad \text{loop.} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$A^T y = 0$ $\quad \dim N(A^T) = m - r = 5 - 3 = 2.$

$n \times m$
4×5

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{A^T C A x = f}$$

$Ax=0$.

$\Delta x = \begin{bmatrix} x_2-x_1 \\ x_3-x_2 \\ x_4-x_3 \\ x_4-x_1 \\ x_4-x_5 \end{bmatrix}$

$x = x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

potentials at nodes.

$E \nabla Ax$.

x_2-x_1, etc.

potential difference.

$y = Ce$

$x = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Ohm's Law on edges

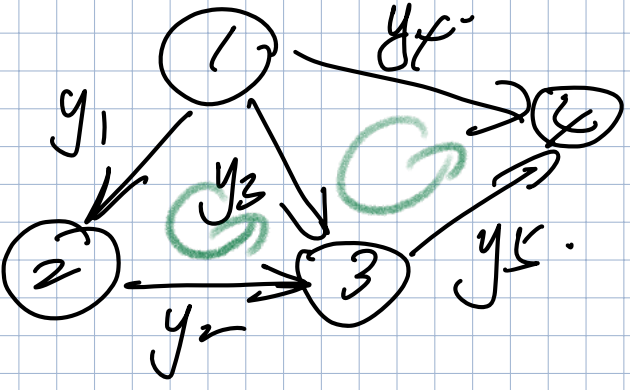
currents y_1, y_2, y_3, y_4, y_5

$\downarrow A^T$

$A^T y = 0$ kirchhoff's C2.

$\dim N(A) = 1$.

RANK $= 3$



$y_1 - y_3 - y_4 = 0$.

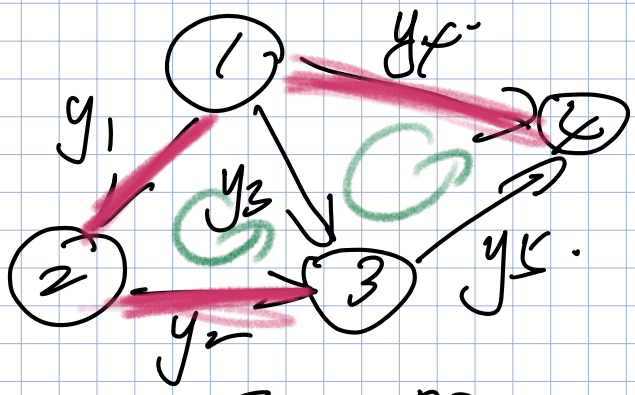
$y_1 - y_2 = 0$

$y_2 + y_3 - y_5 = 0$

$y_4 + y_5 = 0$.

Basis for $N(A^T)$

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$



TREE: no loop.

$\dim \mathcal{O}(\mathcal{G}) = m - r$

loops = # edges − (# nodes − 1)

(rank = n − 1)

nodes − # edges + # loops = 1.

Euler's formula.

5 − 7 + 3 = 1.