Ch. 2. Vector Spaces

§ Vector Spaces and subspaces.

Symmetric Matrix.

ex. $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$\Rightarrow A = A^T$

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

R^T $\qquad$ R

$R^T R$ is always symmetric

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & \bullet & \bullet \\ 7 & \bullet & \bullet \end{bmatrix}$$

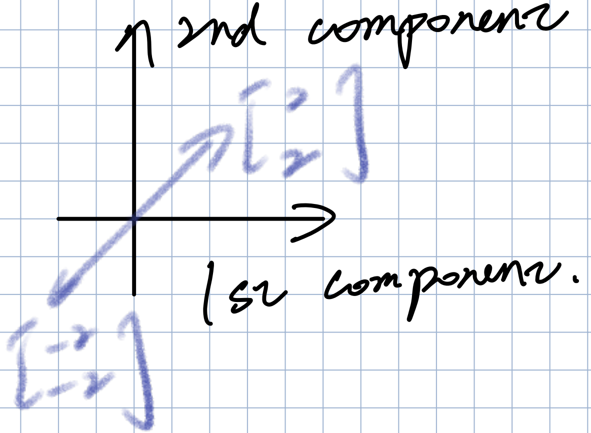
Why? we can take the transpose.

$(R^T R)^T = R^T R^{TT}$

$= R^T R$, proved.

Vector Space:

ex. R^2 = all 2-dimensional real vectors.



$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \ldots$

x-y plane.

= all vectors with 3 components.

R^n = vectors with n column. (n components)

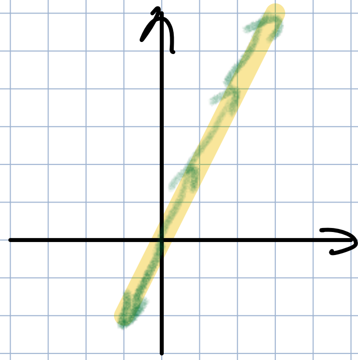
Q can we do those addition and still in the spaces?

Ans: Yes

ex. a vector space inside R^2

⇒ subspace of R^2

∮ line in R^2 through zero vector



↘ ① all in R^2

② any line through $\begin{bmatrix}0\\0\end{bmatrix}$

③ zero vector only "Z"

R^3: Plane / line / $Z = \begin{bmatrix}0\\0\\0\end{bmatrix}$

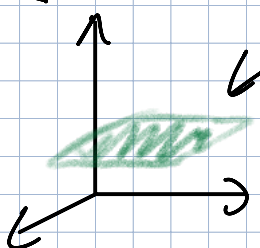
Q How they come out of Matrices.

Ans: all their comb. for a subspace

call col space $C(A)$

Vector Space requirements $V+W$ and cV are in the space, comb. $V+W$ are in the space.

R^3



Plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is subspace of R^3

Q. give 2 subspaces: P and L.

$P \cup L$: all vectors in P or L or both. — (1)

$P \cap L$: all vectors in both P and L — (2)

Which is a subspace? Ans: (2)

Subspaces S and T intersection $S \cap T$ is a subspace.

Column space of A is a subspace of R^4

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$ - all linear comb. of cols.

$C(A)$

Q. Does $Ax = b$ have a solution for every b? Ans: No

4 equations, 3 unknowns

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

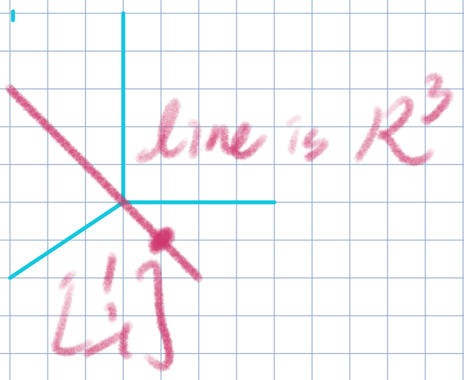
Q. Which vector B allow this system to be solve?

exactly when b is in $C(A)$ (in R^4)

Null space of A — contains $c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$ (in R^3)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Check that solutions to $Ax=0$ always give a subspaces.

If $Av=0$ and $Aw=0$.

then $A(v+w)=0$.

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$$

$\rightarrow$ echlon $\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U.$

$$\boxed{\begin{array}{l}\text{rank of } A\\ =\text{CNT of pivots}\\ =2\end{array}}$$

free col

2 pivots col

$$x=c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

$x_1+2x_2+2x_3+2x_4=0.$

$2x_3+4x_4=0$

$r=2$ CNT of variable.

$n-r=4-2$ free variable

R = reduced row echlon form
(zero above & below pivots)

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R = \text{rref}(A)$ Notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

in the pivot rows/col.

$$x_1 + 2x_2 \quad - 2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

$Rx = 0$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ pivot cols $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ free cols

I F

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

← r pivot rows

← n−r free cols.

r pivots cols

Null Space Matrix

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$Rx = 0$ $x_{\text{pivot}} - F^{x} \text{free}$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x \text{ pivot} \\ x \text{ free} \end{bmatrix} = 0$$

Redo Algorithm again.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

r=2 again

U

free 3-2=1 free col.

$x_1 + 2x_2 + 3x_3 = 0$

$2x_2 + 2x_3 = 0.$

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c\begin{bmatrix} -7 \\ \\ 2 \end{bmatrix}$$

N.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} = R.$$

I F

$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$
$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$
$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

pivot col.

Augmented Matrix

$= [A \quad b]$

solvability condition on b.

$Ax = b$ solvable when b is in $C(A)$

if a comb. of rows A gives zero rows.

then the same comb. of entries of B must give 0

Algorithm

Q: To find complete sol'n to $Ax = b$

Step1. X particular:

set all free variable to zero

Solve $Ax = b$ pivots variables.

$x_1 + 2x_3 = 1 \qquad x_1 = -2 \qquad x_p = c\begin{bmatrix} 2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

$2x_3 = 3 \qquad x_3 = 3/2$

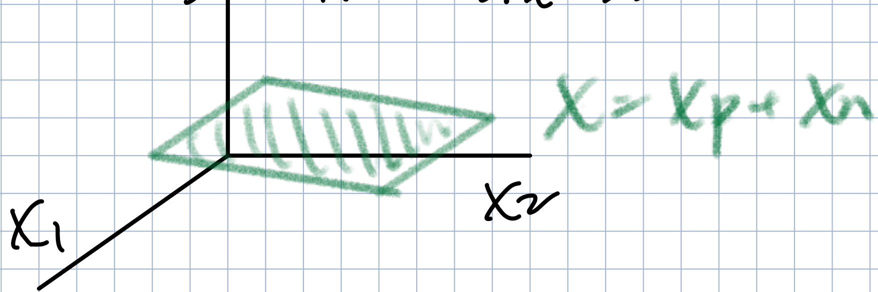
Step2 X nullspace.

$$\begin{array}{l} Ax_p = b \\ +)\ Ax_n = 0 \\ \hline Ax_p + Ax_n = b \ \# . \end{array}$$

$$X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Q if $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$ can solve?

X_3 Plot all solutions X in R^4



m by n matrix A of rank r ($r \le m$, $r \le n$)

Full column rank means $r = n$. (0 or 1 solutions)

No free variables unique solution if it exists.

$N(A) = \{\text{zero vectors}\}$

Solution to $Ax=b$: $x = x_p$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Full row rank means $\boxed{r=m}$.

Q can I solve $Ax=b$ for every b.

Yes. Left with $\underbrace{n-r}_{=n-m}$ free variables.

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & ? & ? \\ 0 & 1 & ? & ? \end{bmatrix}$$

$r=m=n$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$R = I$

$r=m=n$	$r=n<m$	$r=m<n$	$r<m$ $r<n$
$R=I$	$R=\begin{bmatrix} I \\ 0 \end{bmatrix}$	$R=[I\ F]$	$R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 sol.	0 or 1 sol	∞ sol	0 or ∞ sol.