$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow[\text{echelon}]{} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

2 pivots column

free columns

rank of A.
= # of pivots = 2.

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0$

$r = 2$ # of variable

$n - r = 4 - 2$ free variable.

R = reduced row echelon form.

zero above & below pivots.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$=1$

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A)$$

notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

in the pivot rows/cols

$$x_1 + 2x_2 \qquad -2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

$Rx = 0$

I $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ pivot cols

F $\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$ free cols

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ ← r pivot rows

r pivots cols

n-r free cols.

nullspace matrix (columns = special sols.) ✓

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$Rx = 0$ $\boxed{x_{pivot} = -F x_{free}}$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0.$$

Do the algorithm again.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

$x_1 + 2x_2 + 3x_3 = 0$

$2x_2 + 2x_3 = 0$

$r = 2$ again! u free $3 - 2 = 1$ free cols.

$$x = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c \begin{bmatrix} -F \\ I \end{bmatrix}$$

N.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} \begin{array}{cc} \quad I \quad & F \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = R.$$