1. 计算下列极限.

(1) $\lim\limits_{x\to1}\frac{x^2+3x+5}{2x+1}=3$

(2) $\lim\limits_{x\to-1}\frac{x^3+1}{x+1}=3$

(3) $\lim\limits_{x\to2}\left(\frac{2}{x-2}-\frac{8}{x^2-4}\right)$

$=\lim\limits_{x\to2}\left(\frac{2(x+2)-8}{x^2-4}\right)$

$=\lim\limits_{x\to2}\frac{2x-4}{(x+2)(x-2)}=\frac{2}{x+2}=\frac{1}{2}$

(4) $\lim\limits_{h\to0}\frac{(x+h)^2-x^2}{h}$

$=\lim\limits_{h\to0}\frac{x^2+2xh+h^2-x^2}{h}$

$=\lim\limits_{h\to0}2x+h=2x$

(5) $\lim\limits_{x\to\infty}\frac{x^2-1}{x^4+2x+1}=0$

(6) $\lim\limits_{x\to\infty}\frac{x^2}{2x+1}=\infty$

(7) $\lim\limits_{x\to+\infty}\sqrt{x^2+x}-x=$

$=\lim\limits_{x\to+\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}=\frac{1}{2}$

(8) $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}\right)$

$=\lim\limits_{n\to\infty}1-\left(\frac{1}{2}\right)^n=1.$

(9) $\lim\limits_{n\to\infty}\left(\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots+\frac{1}{n(n+1)}\right)$

$=\lim\limits_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}\right)$

$=\lim\limits_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$

(10) $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right)$

$=\lim\limits_{n\to\infty}\frac{(n+1)n}{2n^2}$

$=\lim\limits_{n\to\infty}\frac{n^2+n}{2n^2}=\frac{1}{2}$

2. 计算下列极限

(1) $\lim\limits_{x\to0}\frac{1-\cos(2x)}{x\sin x}$

$=\lim\limits_{x\to0}\frac{2\sin^2x}{x\sin x}$

$=\lim\limits_{x\to0}\frac{2\sin x}{x}=2$

(2) $\lim\limits_{x\to1}\frac{\sin(x^2-1)}{\sin(x-1)}$ 令 $x-1=t$

$=\lim\limits_{t\to0}\frac{\frac{\sin t}{t}(t+2)}{\frac{\sin t}{t}}=2$

(3) $\lim\limits_{x\to0}\frac{\tan(x^2)}{x\sin x}$

$=\lim\limits_{x\to0}\frac{\frac{\sin x^2}{\cos x^2}}{x\sin x}$

$=\lim\limits_{x\to0}\frac{\sin x^2}{x\sin x}=\lim\limits_{x\to0}\frac{\sin x^2}{x}$

$=1.$

(4) $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^{\sqrt{x}}$

$=\lim\limits_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{-x}\right]^{-\frac{1}{x}\cdot\sqrt{x}}$

$=e^0=1$ #

(5) $\lim\limits_{x\to 1} x^{\frac{1}{\ln x}}$

$=\lim\limits_{x\to 1}\frac{1}{\ln x}\ln x^{e}$

$=\lim\limits_{x\to 1} e^{\frac{1}{\ln x}\ln x}$

$=e$

(6) $\lim\limits_{x\to 0}(1+x\sin x)^{\frac{1}{\ln(x^2+1)}}$

$=\lim\limits_{x\to 0}\frac{1}{\ln(x^2+1)}\ln(1+x\sin x)^{e}$

$=\lim\limits_{x\to 0}\frac{1}{\ln 1}\ln 1^{e}$

$=e\ln 1\cdot\frac{1}{\ln 1}=e$

(7) $\lim\limits_{x\to 0}\frac{\sqrt{1+x\tan x}-\cos x}{x^2}$

$=\lim\limits_{x\to 0}\frac{1+x\tan x-\cos^2 x}{x^2(\sqrt{1+x\tan x}+\cos x)}$

$=\lim\limits_{x\to 0}\frac{1}{\sqrt{1+x\tan x}+\cos x}\cdot\frac{\sin^2 x+x\tan x}{x^2}$

$=\frac{1}{2}\cdot 2=1$

(8) $\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\tan x}$

$=\lim\limits_{x\to 0}\frac{x(x\sin\frac{1}{x})}{\tan x}$

$=\lim\limits_{x\to 0}\frac{x}{\tan x}(x\sin\frac{1}{x})$

$=\lim\limits_{x\to 0} 1\cdot 0=0$

3. 设 $f(x)=\begin{cases}\sin x, & x\le 0\\ \ln(1+x), & x>0\end{cases}$，求极限 $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x}$

$\lim\limits_{x\to 0^+} f(x)=\frac{\ln(1+x)-\sin 0}{x}=1$

$\lim\limits_{x\to 0^-} f(x)=\frac{\sin x-\sin 0}{x}=1$ 　　 故 $\lim\limits_{x\to 0} f(x)=1$

4. 已知 $\lim\limits_{x\to\infty}\left[\frac{x^2}{1+x}-ax-b\right]=1$.

$\frac{x^2}{1+x}-ax=0$. 　 $a=1$

$\frac{x^2}{1+x}=\frac{(1+x)x-x}{1+x}=x-\frac{x}{1+x}$

$x-\frac{x}{1+x}-b=1$

5. 设 $f(x)=\begin{cases}\frac{\sqrt{1+x}-1}{x}, & x>0\\ 1/2, & x\le 0\end{cases}$. 求极限 $\lim_{x\to 0}\frac{f(x)-f(0)}{x}$

$\lim_{x\to 0}\frac{1/2-1/2}{0}=0.$

$\lim_{x\to 0^+}\frac{\frac{\sqrt{1+x}-1}{x}-1/2}{0}=$ 不存在, 故 $\lim_{x\to 0}\frac{f(x)-f(0)}{x}=1/2$

6. 设 $x_1>0$, $x_{n+1}=\frac{1}{2}(x_n+\frac{1}{x_n})$ $(n\in N)$ 証明 $\lim x_n$ exist. 并求值

$x_2=\frac{1}{2}(x_1+\frac{1}{x_1})$

$x_3=\frac{1}{2}(x_2+\frac{1}{x_2})$

7. 求极限 $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n^2}+\frac{\sqrt{n}}{(n+1)^2}+\cdots+\frac{\sqrt{n}}{(n+n)^2}\right)$

令原式 $=\{a\}$

$\Rightarrow \lim_{n\to\infty} n\cdot\frac{\sqrt{n}}{n^2}\le\{a\}\le\lim_{n\to\infty} n\cdot\frac{\sqrt{n}}{(2n)^2}$

$\Rightarrow 0\le\{a\}\le 0$, 由 squeeze theorem 得

$\{a\}=0$ #

8. 设 $a_1, a_2 \cdots a_m$ 为 m 个正数. 証明 $\lim_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}=\max\{a_1, a_2\cdots a_m\}$

令 $a=\max\{a_1, a_2, a_3\cdots a_m\}$

则 $a\le\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}\le m^{\frac{1}{n}}a$

由 squeeze theorem 得 $\lim_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}=\max\{a_1, a_2, a_3\cdots a_m\}$. proved #