① det $I = 1$
② Sign reverse with row exchange.
③ det is linear in each row separately

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = ad - cb.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Big Formula.

$$\det A = \sum \pm\; a_{1\alpha}a_{2\beta}a_{3r}\cdots a_{n\omega}$$

$n!$ terms

$(\alpha, \beta, r, \cdots, \omega) = $ Perm of $(1.2.3\cdots n)$

Example.

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$(4,3,2,1) \to +1$

$(3,2,1,4) \to -1$

Cofactors 3×3

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$$
$$+ a_{12}(-a_{21}a_{33} + a_{23}a_{31})$$
$$+ a_{13}$$
$$\vdots$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ - & - & 0 \\ - & - & 0 \end{vmatrix}$$

Cofactor of $a_{ij} = C_{ij}$

$$\pm \det \begin{pmatrix} \text{n-1 matrix with} \\ (\text{row } i, \text{ col } j) \text{ erased} \end{pmatrix}$$

$i+j$ even $\quad i+j$ odd.

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

Cofactor formula (along row 1)

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots a_{1n}C_{1n}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c)$$