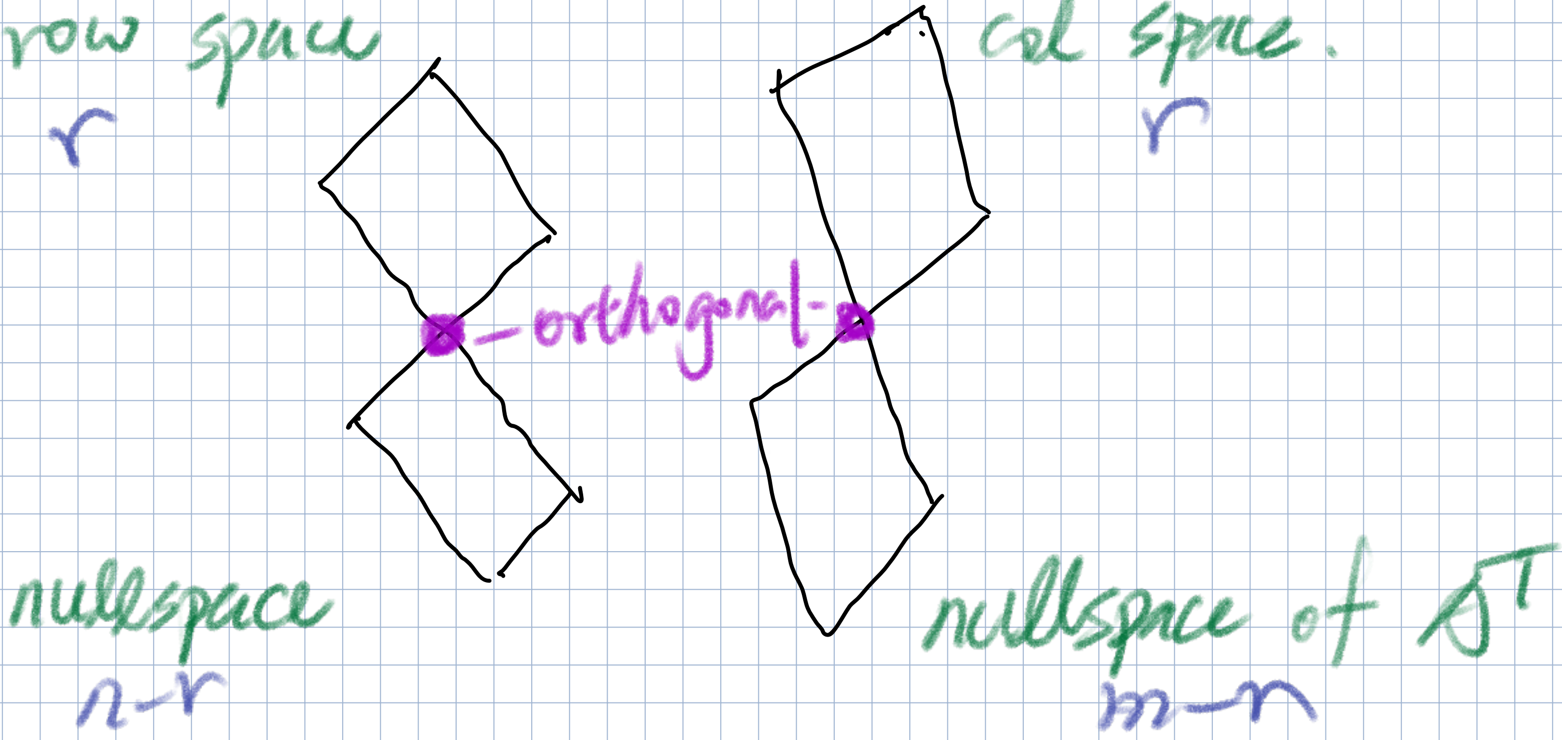
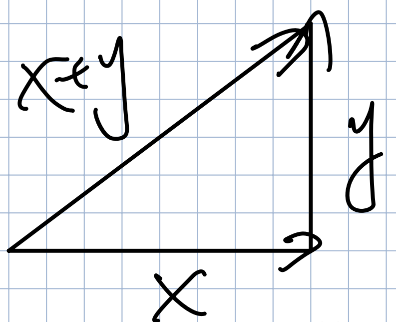


orthogonal vectors.



pythagoras

$x^T y = 0$

$x^T x$ is positive

$$\boxed{\|x+y\|^2 = \|x\|^2 + \|y\|^2}$$

$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\quad x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$\|x\|^2 = 14$ $\quad \|y\|^2 = 5$ $\quad \|x+y\|^2 = 19$

$$\cancel{x^T x} + \cancel{y^T y} = (x+y)^T(x+y) = \cancel{x^T x} + \cancel{y^T y} + x^T y + x^T y$$

subspace S is orthogonal to subspace T.

means: every vector in S is orthogonal to every vector in T.

rowspace is orthogonal to nullspace.

why?

$Ax = 0$.

$$\begin{bmatrix} \text{rows 1 of } A \\ \text{rows 2 of } A \\ \vdots \\ \text{rows m of } A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} c_1(\text{row}1)^T = 0 \\ c_2(\text{row}2)^T = 0 \end{matrix}\right\}$$

$\Rightarrow (c_1\text{row}1 + c_2\text{row}2)^T x = 0$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$n = 3$

$r = 1$

$\dim N(A) = 2$

Nullspace and rowspace are orthogonal complements in R^n.

Nullspace contains all vector $\perp$ row space

Solve $Ax = b$ when there is no solution.

A^TA symmetric

$n \times m \quad m \times n$

$n \times n$

$(A^TA)^T = A^TA^{TT}$

$A^TAx = A^Tb \quad (m = 3 > n = 2)$

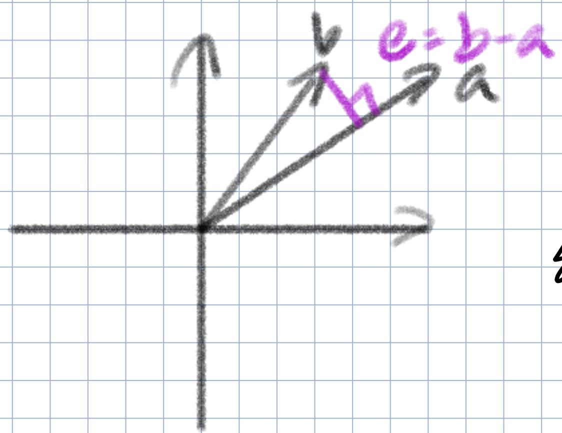
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

$N(A^TA) = N(A)$

rank of A^TA = rank of A

A^TA is invertible exactly if A has indep. cols.



$a^T(b - xa) = 0.$

$xa^Ta = a^Tb$

$$\begin{cases} x = \dfrac{a^Tb}{a^Ta} \\ p = ax \end{cases} \Rightarrow p = a\dfrac{a^Tb}{a^Ta}$$

$\Rightarrow$ projection $p = Pb.$

MATRIX

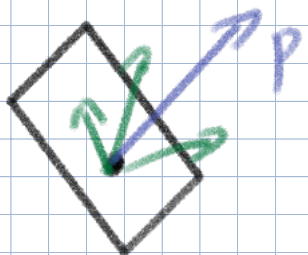
$P = a\dfrac{a^Tb}{a^Ta}$ $C(P)$ = line through a rank$(P) = 1$.

$P^T = P$ $P^2 = P.$

$\boxed{\text{Why Project?}}$

Because $Ax = b$ may have no solution,

Solve $Ax = p$ instead project of b onto column space.



e=b-p prependicular to the plane

plane ⊥ $a_1 \cdot a_2$ = col space ⊥ $A = [a_1\ a_2]$

$p = x_1 a_1 + x_2 a_2$ $\quad p = A\hat{x}$ $\quad$ Find $\hat{x}$

Key: $b - A\hat{x}$ is prep. to plane.

$$a_1^T(b - A\hat{x}) = 0$$
$$a_2^T(b - A\hat{x}) = 0$$

$$\Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \underbrace{\boxed{(b - A\hat{x})}}_{=e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

e in $N(A^T) \Rightarrow e \perp C(A)$

☆ $A^TA\hat{x} = A^Tb \qquad \hat{x} = (A^TA)^{-1}A^Tb$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

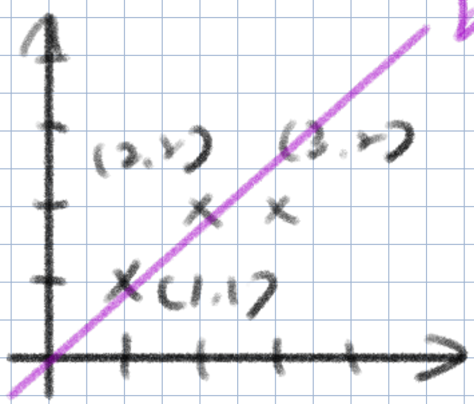
project: $A\hat{x} = A(A^TA)^{-1}A^Tb$

1D $\quad \dfrac{aa^T}{a^Ta}$

nD

MATRIX $P = \boxed{A(A^TA)^{-1}A^T}$

Application: least squares fitting by a line



$b = C + Dt.$

$$\begin{cases} C + D = 1 \\ C + 2D = 2 \\ C + 3D = 2 \end{cases}$$

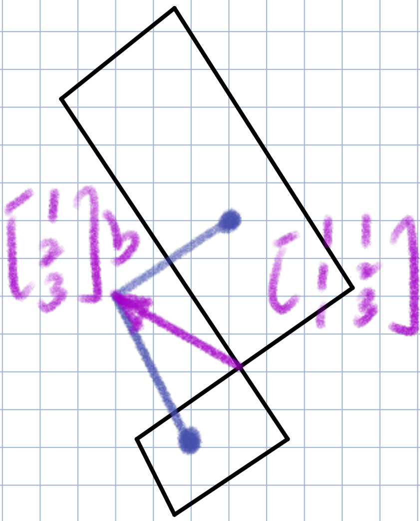
$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

Project. matrix

$$P = A(A^TA)^{-1}A^T$$

if b in column space $Pb = b$.

if b prep. column space $Pb = 0$.

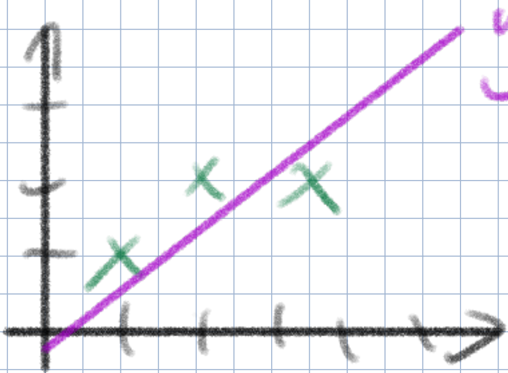


col space.

$p+e=b$

$=(I-P)b$

$N(A^T)$ proj. onto $\perp$ space.



How to pick the best straight line.

$Ax=b$

$$\begin{bmatrix}1 & 1\\ 1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

Minimize $\|Ax-b\|^2=\|e\|^2$

$$=e_1^2+e_2^2+e_3^2$$

$$=(C+D-1)^2+(C+2D-2)^2+(C+3D-2)^2$$

Find $\hat{x}=\begin{bmatrix}\hat{C}\\ \hat{D}\end{bmatrix}$, p

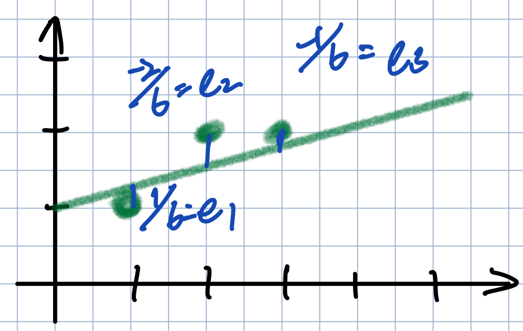
$A^TA\hat{x}=A^Tb$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\end{bmatrix}\left[\begin{array}{cc|c}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 3 & 2\end{array}\right]=\left[\begin{array}{cc|c}3 & 6 & 5\\ 6 & 14 & 11\end{array}\right]$$

$3C+6D=5$
$3C+14D=11$
$\overline{\quad 8D=4 \quad}$

$D=\frac{1}{2}$ $\quad C=\frac{2}{3}$

$y=\frac{2}{3}+\frac{1}{2}t$



$e_1=-\frac{1}{6}$
$e_2=\frac{2}{6}$
$e_3=-\frac{1}{6}$

$b = p + e$

$$\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}7/6\\10/6\\13/6\end{bmatrix}\begin{bmatrix}-1/6\\2/6\\-1/6\end{bmatrix}$$

$$\begin{cases}A^TA\hat{x}=A^Tb\\ p=A\hat{x}\end{cases}$$

If A has indepent column the A^TA is invertible

Suppose $A^TAX=0 \Rightarrow$ To prove X must be 0.

$\boxed{\text{IDEA}}$ $X^TA^TAX=0=(AX)^TAX \Rightarrow AX=0.$

$\Rightarrow X=0$ $\quad$ A has indepent cols

☆ Col definitely indepent if they are perpendicular unit vector

orthonormal vectors.

Orthonormal Vectors.

$$q_i^Tq_j\begin{cases}0 \,;\ \text{if } i\neq j\\ 1 \,;\ \text{if } i=j\end{cases}$$

$$Q=\begin{bmatrix}q_1 \cdots q_n\end{bmatrix} \qquad Q^TQ=\begin{bmatrix}q_1^T\\ q_n\end{bmatrix}\begin{bmatrix}q_1 \cdots q_n\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

if Q is square then $Q^TQ=I$ tell us $Q^T=Q^{-1}$

example.

permutation $Q=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=I$

$Q=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$Q=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}$

$Q=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\frac{1}{2}$

$Q=\begin{bmatrix}1&-2\\2&-1\\2&2\end{bmatrix}\cdot\frac{1}{3}$

Q has orthonormal cols.
Project onto its cols. space

$P=Q(Q^TQ)^{-1}Q^T=QQ^T \Rightarrow I$ if Q is square.

$(QQ^T)(QQ^T)=QQ^T$

$A^TA\hat{x}=A^Tb \Rightarrow$ now A is Q

$Q^TQ\hat{x}=Q^Tb$ $\boxed{\hat{x}_i=q_i^T b}$

Gram-Schmidt Orthogonalization.

independent vectors a, b

↓

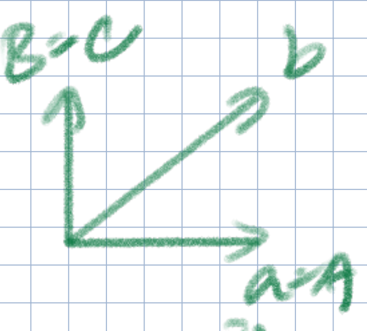
orthogonal vectors A, B

↓

orthonormal $q_1=\frac{A}{\|A\|}$ $q_2=\frac{B}{\|B\|}$

vector a, b, c

$q_1=\frac{A}{\|A\|}$ $q_2=\frac{B}{\|B\|}$ $q_3=\frac{C}{\|C\|}$



B=b

$B=b-\frac{A^Tb}{A^TA}A$

$A^TB=A^T\left(b-\frac{A^TA}{A^TA}A\right)$

$=0.$

$A = a$

$B = b - \frac{A^T b}{A^T A} A = 0.$ $\quad C = C - \frac{A^T C}{A^T A} A - \frac{B^T C}{B^T B} B$

example

$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $B = \underset{b}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} - \underset{A}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \underset{\perp}{=} \underset{B}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

$Q = [q_1 \ q_2] = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

$A = LU \quad A = QR$

$[a_1 \ a_2] = [q_1 \ q_2] \begin{bmatrix} a_1^T q_1 & * \\ a_2^T q_2 & * \end{bmatrix}$