Orthonormal vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \qquad Q^T Q = \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ 0 & & 1 \end{bmatrix}$$

If Q is square then $Q^T Q = I$ tell us $Q^T = Q^{-1}$

<u>examples.</u>

permutation $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{2}$$

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$Q = \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \cdot \frac{1}{3}$$

Q has orthonormal columns

Project onto its column space.

$$P = Q(Q^T Q)^{-1} Q^T = Q Q^T \Big\} = I \text{ if } Q \text{ is square.}$$

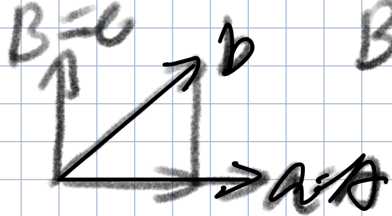
$(QQ^T)(QQ^T) = QQ^T$.

$A^TA\hat{x} = A^Tb \Rightarrow$ Now A is Q

$$Q^TQ\hat{x} = Q^Tb \qquad \boxed{\hat{x}_i = q_i^T b}$$

Gram-Schmidt.

Independent vectors a, b.



$B = b$

$\downarrow$

orthogonal vector A, B

$$B = b - \frac{A^Tb}{A^TA}A$$

$$A^TB = A^T\left(b - \frac{A^Tb}{A^TA}A\right) = 0.$$

$\downarrow$

orthonormal $q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$

Vector a, b, c

$q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$ $q_3 = \frac{C}{\|C\|}$

$A = a$

$$B = b - \frac{A^Tb}{A^TA}A$$

$$A^TB = A^T\left(b - \frac{A^Tb}{A^TA}A\right) = 0.$$

$C \perp A$

$C \perp B$

$$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$$

example.

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $B = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{b} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\underset{A}{\overset{\shortparallel}{a}}$ $A \perp B$

$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

$A = LU$ $A = QR$

$\begin{bmatrix} a_1 & a_2 \end{bmatrix} =$

$\begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} a_1^T q_1 & * \\ a_1^T q_2 & * \end{bmatrix}$