A very short note on computing impulse response functions

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An impulse-response function describes the evolution of the variable of interest along a specified time horizon after a shock in a given moment. To make thinks easier and understand the intuition, let's focus first on the case of a univariate AR(1) process:

$$x_t = \phi x_{t-1} + u_t$$

where x_t is a scalar, $\phi < 1$ (what makes the process stationary) and u_t is a (scalar) random disturbance with mean 0.

The moving average representation

Since the above process is stationary, we can find the infinite moving average representation (define L as the lag operator such that $LX_t = X_{t-1}$ and $L^2 X_t = X_{t-2}$):

$$\begin{aligned}
x_t &= \phi x_{t-1} + u_t \\
(1-\phi L)x_t &= u_t \\
x_t &= (1-\phi L)^{-1} u_t \\
x_t &= \frac{1}{1-\phi L} u_t \\
x_t &= (\phi L)^0 u_t + (\phi L)^1 u_t + (\phi L)^2 u_t + (\phi L)^3 u_t + \dots \\
x_t &= u_t + \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \dots
\end{aligned}$$

Suppose now that x_t, instead of being a scalar is a column vector with dimensions $n \times 1$, i.e. we have now a VAR(1) instead of an AR(1). We are, however, interested in the evolution of x_t after a structural shock, rather than after an innovation in u_t. If we think of u_t as reduced-form innovations that are mixed combination of some structural shocks ε_t, we can assume the following relationship:

$$u_t = B\varepsilon_t$$

where B is a $n \times n$ matrix and ε_t is a column vector $(n \times 1)$ containing the n structural shock (the relationship between this B matrix and the A_0 matrix that we see in the problem sets is $B = A_0^{-1}$). We can then write the above moving average representation as:

$$\begin{aligned}
x_t &= B\varepsilon_t + \phi B\varepsilon_{t-1} + \phi^2 B\varepsilon_{t-2} + \phi^3 B\varepsilon_{t-3} + \dots \\
x_t &= C_0\varepsilon_t + C_1\varepsilon_{t-1} + C_2\varepsilon_{t-2} + C_3\varepsilon_{t-3} + \dots \qquad (1)
\end{aligned}$$

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Equation 1 is very important for our purposes since the coefficients of the moving average representation (which we define as $C_i = B\phi^i$) are the responses of variables contained in x to impulses in these structural shocks. Let's compute the response of x_t to a shock for different time periods.

What is the response of variables in x in period t to a shock in time t? [1](contemporaneous impact):

$$\frac{\partial x_t}{\partial \varepsilon_t} = B$$

What is the response of variables in x in period $t+1$ to a shock in time t? If we forward Equation 1 one period, we obtain:

$$\frac{\partial x_{t+1}}{\partial \varepsilon_t} = \phi B$$

What is the response of variables in x in period $t+j$ to a shock in time t? Following the same procedure:

$$\frac{\partial x_{t+j}}{\partial \varepsilon_t} = \phi^j B$$

An impulse-response function will be a plot of $\frac{\partial x_{t+j}}{\partial \varepsilon_t}$ for all $j = 0, \ldots, H$ (where H is the time horizon of our plot). This function will depict the response of variables x_{t+j} for all j after a shock at time t. Notice that all we need to plot this graph is an estimation of the autoregressive coefficients ϕ (obtained from the reduced-form regression) and B (obtained after imposing some identifying restrictions).

An equivalent representation

Since all we need is ϕ and B, an equivalent method to compute the impulse-response functions is the recursive simulation of the system:

$$x_t = \phi x_{t-1} + B\varepsilon_t$$

for all periods $t = 1, \ldots, H$, with $x_0 = 0$ (note that now we are being more specific about time notation: we start the analysis at time 1 rather than at time t). Check that you can obtain the same $\frac{\partial x_{t+j}}{\partial \varepsilon_t}$ for all j as we did above.

If we assume that the shock is a one-time impulse that happens in time $t = 1$ and has size 1 (i.e. $\varepsilon_j = 0$ for all $j > 1$ and $\varepsilon_1 = 1$), then the response of x to this one-time shock in periods 1, 2 and 3 is:

$$\begin{aligned}
x_1 &= \phi x_0 + B\varepsilon_1 = B\varepsilon_1 = B \\
x_2 &= \phi x_1 = \phi B \\
x_3 &= \phi x_2 = \phi(\phi x_1) = \phi^2 B \\
\ldots
\end{aligned}$$

[1]Note that we are considering a shock in ε_t, which is a vector, so we are actually considering as many shocks as elements in this vector. If we want to consider just a shock to one of the elements of ε_t, say, a shock to government spending ε_t^g, the contemporaneous impact will be given by the column of the matrix B that corresponds to the shock ε_t^g

How to compute the impulse-response in practice

In short, the variables in x reacts contemporaneously by amount B to the shocks (or just a column of B if we shock only one element of ε_t, and it then evolves according to the autoregressive coefficients.

We can easily compute this by first noticing what happens to variable x at the time of the shock, and then using a `for` loop to compute its evolution afterwards: [2]

```
response(t=1) = B
for t=2:H
    response(t) = phi * response(t-1)
end
```

Extensions

What if the process is an AR(P) rather than an AR(1)? In this case, the `for` loop should include all the P lags.

Readings

- James D. Hamilton: Time Series Analysis, *Princeton University Press* (1994). Chapter 11, Pages 318-320

[2] This is not Matlab code, but just a sketch of the algorithm that you could use to generate IR functions. Remember than when we are considering only one shock, the contemporaneous impact will be given by the relevant column of matrix B.