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The Signal-to-Noise Ratio of the Nuclear Magnetic Resonance Experiment

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A fresh approach to the calculation of signal-to-noise ratio, using the Principle of Reciprocity, is formulated. The method is shown, for a solenoidal receiving coil, to give the same results as the traditional method of calculation, but its advantage lies in its ability to predict the ratio for other coil configurations. Particular attention is paid to the poor performance of a saddle-shaped (or Helmholtz) coil. Some of the practical problems involved are also discussed, including the error of matching the probe to the input impedance of the preamplifier.

INTRODUCTION

An important reason for the development of high-resolution NMR spectrometers employing superconducting, rather than iron, magnets has been the belief that the signal-to-noise ratio (S:N) available is proportional to the Larmor frequency to the three-halves power (*1*, *2*), and thus that an increase of, for example, three times in frequency from 90 to 270 MHz would bring as a return an improvement of a factor of 5.2 in S:N. It is always difficult accurately to compare the performances of different instruments as a variety of factors (to be discussed further) come into play; nevertheless, it is generally felt by those with experience of superconducting systems that the improvement obtained is disappointing—for example, about 2 for the frequencies quoted. As it is acknowledged that the electronics needed at very high and ultrahigh frequencies are "difficult," the accusing finger has tended to point in the engineers' direction. However, recent developments in the design of low-noise amplifiers (*3*) and the general improvement in frequency-changing techniques, due mainly to the increased use of hot carrier diodes, have caused us to consider afresh a fundamental dictum of NMR., the $\frac{3}{2}$ power law, and to examine anew the derivation of the formula, *which was derived prior to the advent of superconducting systems.*

PRIMARY AND SECONDARY CONSIDERATIONS

The usual formula for the signal-to-noise ratio available after a 90° pulse is given by (*1*, *2*)

$$\Psi_{\text{rms}} = K\eta M_0 (\mu_0 Q \omega_0 V_c / 4FkT_c \Delta f)^{1/2}, \qquad [1]$$

where K is a numerical factor ($\sim$1) dependent on the receiving coil geometry; η is the "filling factor," i.e., a measure of the fraction of the coil volume occupied by the sample; M_0 is the nuclear magnetization which is proportional to the field strength B_0; μ_0 is the permeability of free space; Q is the quality factor of the coil; ω_0 is the Larmor angular frequency; V_c is the volume of the coil; F is the noise figure of the preamplifier; k is Boltzmann's constant; T_c is the probe (as opposed to sample) temperature; and Δf is the bandwidth (in Hertz) of the receiver.

We may consider that the primary factors involved in any analysis of S:N are those contained within the equation. Secondary factors, such as whether or not quadrature detection is used, the availability of Fourier transform techniques, sweep rate or pulsing rate, the use of decoupling or Overhauser effect, though of great importance, are not of such a fundamental nature and will therefore not be considered further.

For all its usefulness, Eq. [1] is not a "fundamental" equation. It contains four unknowns, K, η, Q and F, only two of which (Q and F) are easily measurable. The definition of filling factor $\eta = V_s/2V_c$ (V_s is the sample volume) may well be satisfactory for a solenoidal coil, but its validity for other coil configurations must be questioned. Further, the equation contains little information as to the dependency of S:N on various physical parameters; for example, if we quadruple the coil volume while keeping η constant, do we obtain only a doubling of S:N, or does the change in the coil dimensions alter K and Q also? Table 1 shows how complex the use of Eq. [1] may be. The interactions between the various factors are manifold and mostly strong. This is partly a consequence of the method of calculation. The reader is referred to Ref. (*1*, *2*) for further details, but it is perhaps worth quoting Abragam who states that "the above calculation gives only an order of magnitude."

TABLE 1

INTERACTION OF TERMS IN THE TRADITIONAL EQUATION FOR SIGNAL-TO-NOISE RATIO AS GIVEN BY EQ. [1]

	Temperature T	Preamplifier noise figure F	Coil Q	Filling factor and geometry $K\eta$	Volume V_c
Preamplifier noise figure F	Strong				
Coil Q	Strong				
Filling factor and geometry $K\eta$			Weak		
Volume V_c			Strong	Strong for small samples	
Field strength B_0		Strong	Strong		Strong

THE PRINCIPLE OF RECIPROCITY

Clearly it would be of use to formulate a different method of calculation which gives a direct insight into the various factors involved and which removes some of the interactions inherent in Eq. [1]. This may be done to a reasonable extent by invoking the principle of reciprocity. Consider the induction field $\mathbf{B}_1$ produced by a coil C carrying unit current (See Fig. 1). Obviously, the field at point A is much stronger than at point B. Intuitively, one would expect therefore that if a magnetic dipole $\mathbf{m}$ were placed at point A and set rotating about the z axis, the alternating signal it induced in the coil would be much greater than that induced by the same dipole placed at point B. This is indeed the case, and it may easily be shown that the induced emf is given by

$$\xi = -(\partial/\partial t)\{\mathbf{B}_1 \cdot \mathbf{m}\}, \tag{2}$$

where $\mathbf{B}_1$ is the field produced by the unit current at $\mathbf{m}$. It follows that for a sample of volume V_s, which has been recently subjected to a 90° pulse, we need only know the

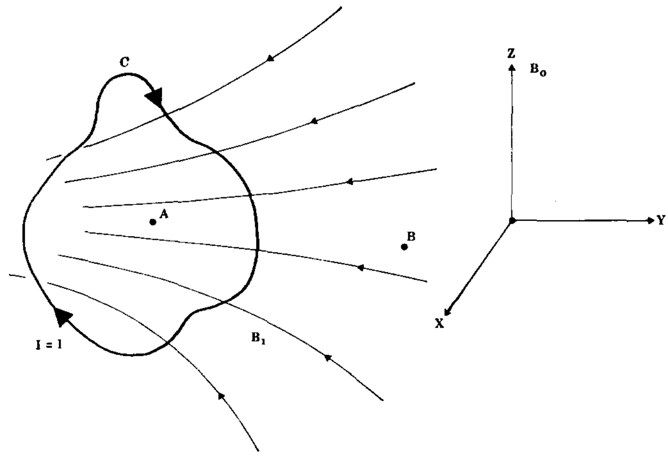

FIG. 1. The induction field $\mathbf{B}_1$ produced by a coil C carrying unit current.

value of $\mathbf{B}_1$ at all points in the sample to be able to calculate the emf induced in the coil. Thus, if $\mathbf{M}_0$ lies in the xy plane,

$$\xi = -\int_{\text{sample}} (\partial/\partial t)\{\mathbf{B}_1 \cdot \mathbf{M}_0\}\, dV_s. \tag{3}$$

The calculation of $\mathbf{B}_1$ is feasible for most shapes of coil; of course, if $\mathbf{B}_1$ may be considered to be reasonably homogeneous over the sample volume, the calculation is considerably simplified as the integration of Eq. [3] becomes trivial, giving

$$\xi = K\omega_0 (B_1)_{xy} M_0 V_s \cos \omega_0 t, \tag{4}$$

where K is an "inhomogeneity factor" which may if necessary be calculated, $(B_1)_{xy}$ is the component of $\mathbf{B}_1$ perpendicular to the main field B_0, and phase has been neglected. The magnetization M_0 is given by

$$M_0 = N\gamma^2 \hbar^2 I(I+1) B_0 / 3kT_s, \tag{5}$$

where N is the number of spins at resonance per unit volume, γ is the magnetogyric ratio, and T_s is the sample temperature. As $\omega_0 = -\gamma B_0$ it follows from Eqs. [4] and [5] that the EMF induced in the coil is proportional to the square of the Larmor frequency.

THE NOISE

Having laid the basis of the calculation for the emf induced by the nuclear magnetization in the receiving coils, we now consider the noise. In a correctly designed system, this should originate solely from the resistance of the coil. As the dimensions of the coil are inevitably much less than the wavelength of the radiation involved, the radiation resistance is negligible, and it should therefore be possible to predict the thermal noise present purely on the basis of the equation

$$V = (4kT_c \Delta f R)^{1/2}. \quad [6]$$

Here, T_c is the temperature of the coil and R its resistance. Unfortunately, the calculation of R may not be performed accurately, and it is solely this factor which leads to uncertainty in the theoretical prediction of signal-to-noise ratio. At the frequencies of most interest in NMR spectroscopy (say >5 MHz) the "skin effect" associated with the magnetic field generated by a current ensures that that current flows only in regions of the conductor where there is also a magnetic flux. Thus, for a long, straight cylindrical conductor, the current flows in a skin on the surface. This situation is easily amenable to calculation and yields the result that

$$R = (l/p)\,(\mu\mu_0\,\omega_0\,\rho(T_c)/2)^{1/2}, \quad [7]$$

where l is the length of the conductor; p is its circumference, μ is the permeability of the wire; and $\rho(T_c)$ is the resistivity of the conductor, which is of course a function of temperature. However, the situation is considerably more complicated if the conductor is not cylindrical, and worse still, if there are many conductors in close proximity, as is, in effect, the case with a coil, the magnetic field created by the current of one conductor influences the distribution of current in another. This "proximity effect" (*4*), which is also manifest when conductors such as the silvering on a dewar, or even the sample itself, are close to the coil, normally tends to reduce the surface area over which current is flowing, and thus the resistance is increased from that calculated from Eq. [7] by a factor ζ of about 3. Attempts have been made to calculate ζ, but only for a single-layer solenoid may any confidence be placed in the results (*5*).

THE SIGNAL-TO-NOISE RATIO

By combining Eqs. [4] to [7] we may arrive at an equation for the signal-to-noise ratio,

$$\Psi_{\text{rms}} = \frac{K(B_1)_{xy}\,V_s\,N\,\gamma\,\hbar^2\,I(I+1)}{7.12\,kT_s}\cdot\left(\frac{p}{FkT_c l\zeta\Delta f}\right)^{1/2}\cdot\frac{\omega_0^{\,7/4}}{[\mu\mu_0\,\rho(T_c)]^{1/4}}. \quad [8]$$

Note that the proximity factor ζ and the noise figure F of the preamplifier have been included. At first sight, this equation appears unmanageable, but this is in fact not the case. First, the unknowns η and Q of Eq. [1] are absent; they have been replaced by a single function ζ, which, from experience, is reasonably well known, and second, the number of factors in Eq. [8] which are variable in a given experimental situation is small. These factors are:

(a)	$K(B_1)_{xy}$	the effective field over the sample volume produced by unit current flowing in the receiving coil.
(b)	p	the perimeter of the conductor.
(c)	l	the length of the conductor.

The above factors are dependent only on coil geometry.

(d)	T_c	the temperature of the coil.
(e)	$\rho(T_c)$	the material from which the coil is made.
(f)	F	the quality of the preamplifier.

It is of interest to note that the frequency dependence is to the power of 7/4. This is not a new conclusion; it has also been postulated by Soutif and Gabillard (*6*) and it must be stressed that, *for a solenoid*, in no way is Eq. [8] in contradiction with Eq. [1]. Figure 2 shows an experimental plot of Q versus frequency for a set of solenoids all wound in the same manner with the same overall dimensions. This plot clearly recon-

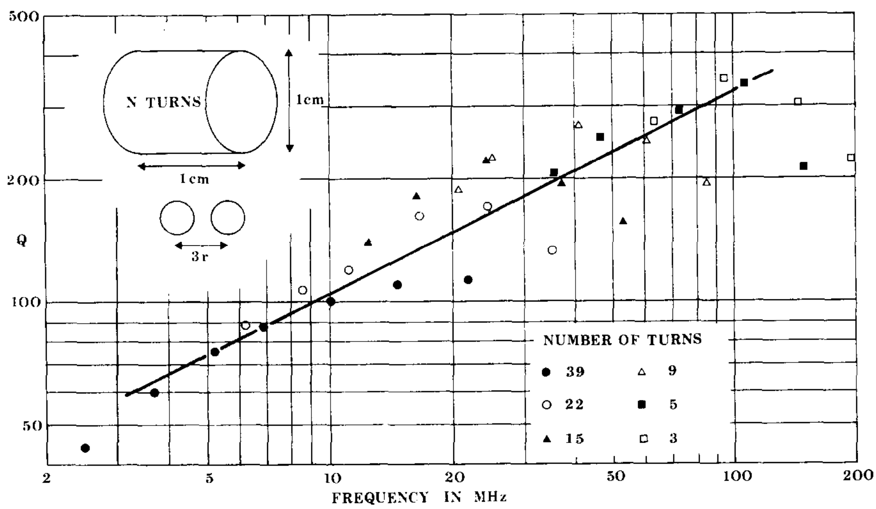

FIG. 2. The Q of a set of solenoids. The dimensions of each solenoid are the same, but the number of turns varies with frequency. The winding geometry is shown in the inset, and gives the optimum Q (*4*). For many turns, and well below the self-resonant frequency, $Q \propto f^{1/2}$.

ciles the two equations and the derivation of Eq. [1] from Eq. [8] is shown in Appendix 1. The major advantage of the calculation from first principles which results in Eq. [8] is that it is applicable to any coil geometry, for it is found that ζ changes but little with a change of configuration provided the separation of the windings is small in comparison with the overall dimensions of the coil. Of particular interest are saddle-shaped (or Helmholtz) coils used mainly with superconducting instruments and solenoidal coils used predominantly with conventional machines. Let us therefore compare the performance of these two configurations.

SADDLE-SHAPED AND SOLENOIDAL COILS

The factors of interest are (a) to (c) above, and so, assuming the same volume of sample is used in each case, the essential part of Eq. [8] is

$$\Psi \propto V_s (B_1)_{xy}/R^{1/2} \qquad \text{or} \qquad \Psi \propto V_s (B_1)_{xy} (p/l)^{1/2}. \qquad [9]$$

Our first task therefore is to calculate $(B_1)_{xy}$ for the two coils. While this is trivial for a many-turn solenoid, it is not for the saddle-shaped coils, and the essence of the calculation is indicated in Appendix 2. Let a be the radius of the coils, and $2g$ the lengths, as shown in Fig. 3.

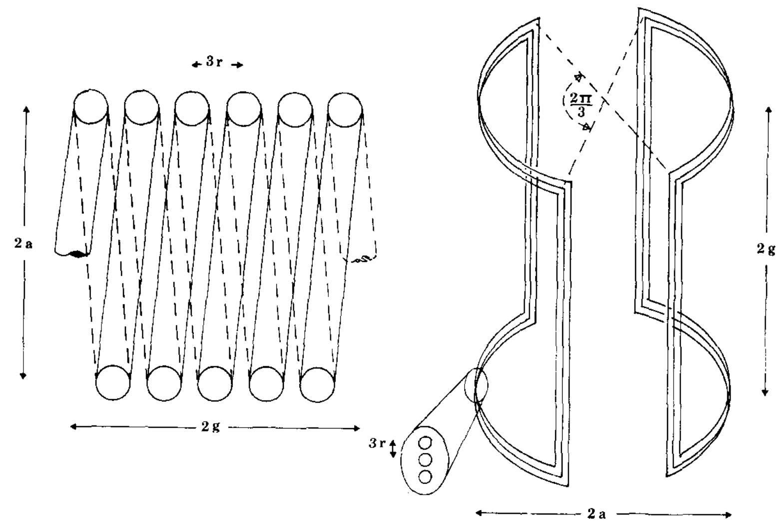

FIG. 3. The two winding geometries considered. The angular width of the saddle-shaped coils is 120°, as this value gives the best homogeneity, and the width of the windings is approximately $g/5$.

Saddle-shaped	Solenoid
$(B_1)_{xy} = \frac{n\sqrt{3}\,\mu_0}{\pi}\left\{\frac{ag}{[a^2+g^2]^{3/2}} + \frac{g}{a[a^2+g^2]^{1/2}}\right\},$	$(B_1)_{xy} = \frac{\mu_0 n}{2}\frac{1}{[a^2+g^2]^{1/2}},$
$n \geqslant 1.$	$n \gg 1.$

Typically, $a \simeq g$, and hence

$(B_1)_{xy} = 0.585\, n\mu_0/a.$ $\qquad\qquad$ $(B_1)_{xy} = 0.354\, n\mu_0/a.$

We must now calculate p and l. Let us assume that wire is used in both cases. We then have that

$l \simeq 8na\{(g/a) + (\pi/3)\}.$ $\qquad\qquad$ $l \simeq 2\pi an.$

When $a \simeq g$,

$l \simeq 16.4\, na.$ $\qquad\qquad$ $l \simeq 6.3\, na.$

The length of wire per unit turn is thus greater for the saddle-shaped coil by a factor of about 2.6, a significant fact which will be considered later. The radius r of the wire used is dependent upon the manner in which the coils are wound, but assuming that a planar structure is used, the n turns must fit in a length $2g$ for the solenoid and in a width $g/5$ approximately for the saddle-shaped coils. To optimize the performance of each coil with respect to proximity effect, the distance between the centers of each turn should be roughly $3r$ (*4*), and so, assuming this value, we have that for $a \simeq g$,

$$3r(n-1) \simeq a/5, \qquad\qquad 3r(n-1) = 2a,$$

$$\therefore \quad p \simeq 2\pi a/15(n-1). \qquad\qquad \therefore \quad p = 4\pi a/3(n-1).$$

Thus, for $n \gg 1$,

$$p \simeq 0.42a/n, \qquad\qquad p \simeq 4.2\,a/n,$$

$$\therefore \quad \psi \propto 0.094\,\mu_0\,V_s/a. \qquad\qquad \therefore \quad \psi \propto 0.29\,\mu_0\,V_s/a. \qquad [10]$$

Thus the performance of the solenoid would appear to be approximately three times better than that of the saddle-shaped coils. It is difficult to explain this result on the basis of the traditional formula as represented by Eq. [1]. This is not surprising, as there is a critical assumption in the derivation (shown in Appendix 1) involving the energy stored by current flowing in the coil. Briefly, this assumption is that half the energy is stored, via the field $\mathbf{B}_1$, within the confines of the coil and that the field is homogeneous within those confines. While this is broadly true for a solenoid, due to the continuity and closed form of the winding, it is certainly not true for the saddle-shaped coils, where the open structure dictates that much of the magnetic energy is stored in flux which lies close to the wires and which does not pass through the sample. A further deficiency of Eq. [1] concerns the length of a 90° pulse when the probe is of the single-coil variety. One would be led to believe that the length was dependent only on $Q^{-1/2}$. Typically a saddle-shaped coil has a lower Q than a solenoid, and so the 90° pulse length should be a little longer (up to, say, 60%) but Eq. [10] show the inadequacy of this statement. This may be seen from the following argument.

90° PULSE LENGTHS

When the probe is matched to the transmitter (a point to be considered later), the power supplied W is dissipated entirely in the resistance R of the coil. The current flowing through the coil is thus given by

$$I = (W/R)^{1/2}.$$

Now the $\mathbf{B}_1$ field employed in the calculations to date has been derived for unit current. It follows that when the transmitter is on, the irradiating field $(B_1^*)_{xy}$ is given by

$$(B_1^*)_{xy} = I(B_1)_{xy} = (B_1)_{xy}(W/R)^{1/2}.$$

Thus, from Eq. [9],

$$(B_1^*)_{xy} \propto \mathrm{W}^{1/2}\,\psi/V_s. \qquad [11]$$

The 90° pulse length is thus a direct measure of the S:N obtainable from a single coil system and if the preceding calculations are correct, one would expect the length to be about three times longer for the saddle-shaped coils. To check the results obtained, experiments were performed at two frequencies: 20 MHz, where the condition $n \gg 1$ holds, and 129 MHz, where it does not. Table 2 shows the data collected, which are in good agreement with the theory. Finally, it is of interest to note that the results of Eq. [10] are independent of the number of turns on the coil. This is equivalent to saying that

TABLE 2

EXPERIMENTAL RESULTS[a] SHOWING THE SUPERIOR PERFORMANCE OF A SOLENOIDAL RECEIVING COIL AS COMPARED TO A SADDLE-SHAPED COIL WHEN USING THE SAME SAMPLE: A SPHERE 7.5 MM IN DIAMETER

Frequency	Coil type	Number of turns	Q	Height (mm)	Radius (mm)	90° Pulse[b] (μsec)	Signal[c] (volts)
129 MHz, ^{31}P	Saddle-shaped	2	210	7	4	27	1.0
	Solenoid	4	300	6	4	9	2.6
			Ratio 1.4			Ratio 3	Ratio 2.6
20 MHz, ^{1}H	Saddle-shaped	6	80	10	5	46	0.78
	Solenoid	18	208	10	5	18	2.20
			Ratio 2.5			Ratio 2.6	Ratio 2.8

[a] All values obtained are accurate to better than 10%.
[b] No comparison is intended between the performances at the two frequencies as the two transmitters are of different powers.
[c] The signals are measured relative to a constant noise background.

the Q of a coil is mainly dependent on the overall dimensions of that coil rather than the number of turns within those dimensions. That this is broadly true may be seen from Fig. 2.

DEPENDENCIES

From the various equations derived, it is possible to determine how the signal-to-noise ratio varies as parameters are altered. With regard to coil geometry, the latter is usually determined by the homogeneity of the main magnetic field B_0. Thus, for example, a superconducting magnet with a field of 11 tesla might have a homogeneity of one part in 10^9 over 1 ml, whereas with a 4 tesla magnet the equivalent volume might be 25 ml. For simplicity, we shall assume that coil length and diameter are approximately equal, though it should be stressed that this ratio is not necessarily optimal for the best utilization of the main field homogeneity nor, in the case of a saddle-shaped coil, does it produce the best B_1 homogeneity (7). Thus, from Eq. [10] we can see that if *all* linear dimensions are scaled in the same manner, the S:N varies as a^2 or as $V^{2/3}$. Assuming a frequency dependence of $\omega_0{}^{7/4}$, it is thus easy to show that the low-field magnet gives 25% *better* signal to noise than its high-field counterpart. It has of course been assumed that 25 ml of sample is available. In most biochemical applications of NMR, such extravagence would be unthinkable and so, for higher sensitivity, a higher field is still called for.

The major variable in Eq. [8] not yet considered is temperature. While the sample temperature T_s may well be fixed by the chemistry of the system under observation, there is no such limitation on the temperature T_c of the probe, and if the latter is cooled, significant improvements may be obtained. From Eq. [10], *for a constant sample volume*, the signal-to-noise ratio varies only as a^{-1}, and so the insertion of a dewar vessel between the coil and the sample may well only decrease ψ by 30%. On the other hand, at liquid nitrogen temperature (77 K), T_c is a quarter of room temperature and thus, if the preamplifier has an excellent noise figure, the noise is reduced by a factor of 2 on cooling. However it must not be overlooked that the conductivity of the conductor is temperature dependent, and for copper, $\rho(T_c)$ is one-tenth of its room-temperature value at 77 K. The net gain in signal-to-noise ratio obtainable by cooling the probe is therefore about 2.5 (Note, however, that the 90° pulse length should decrease by only about 25%.) Of course, if the sample may also be cooled, further improvement is possible.

LIMITATIONS OF THE FORMULA

So far, little indication has been given of the limitations of Eq. [8]. These are predominantly twofold. First, the calculation breaks down below about 5 MHz, where, for an average size of sample, the radius of the wire used becomes comparable with the skin depth. Second, the calculation breaks down when the distributed capacitance between the turns on the coil is sufficiently large at the frequency of interest to change the phase of the emf induced in one part of the coil relative to another. In its extreme manifestation, this effect causes self-resonance, a condition whereby the coil resonates without an external tuning capacitor. This should be avoided, and the onset of self-resonance may be seen in each of the curves of Fig. 2, where, with increasing frequency, the Q of each coil begins to drop away from the $f^{1/2}$ line. To avoid this effect, the number of turns on the coil must be decreased with increasing frequency. There is a limit of course to this procedure; for the saddle-shaped coil, the limit is a single turn. In this case, the formula still holds, provided attention is paid to the fact that the leads from the coil and the links between the two sections contribute appreciable resistance. A further modification may be to connect the two halves in parallel rather than series. This situation may also be accounted for and allows satisfactory use up to about 300 MHz. Also, it is usual for the conductor to be foil rather than wire with such a configuration.

With a solenoid, the calculation breaks down when the condition $n \gg 1$ is not obeyed and this may be seen in Fig. 2 to occur for frequencies in excess of 150 MHz. However, a single-turn solenoid fabricated from foil is a special case for which provision can be made, and whereas a single-turn saddle-shaped coil is useful up to about 200 MHz, a single-turn solenoid is useful to 600 MHz. The reason for this behavior lies in the fact, mentioned earlier, that a saddle-shaped coil requires 2.6 times the length of conductor per unit turn of a solenoid, and thus has greater inductance and self-capacitance per unit turn. This is ironic when it is remembered that it is only at the high frequencies made available by superconducting systems that the saddle-shaped configuration is required. It is the authors' opinion that the disappointing signal-to-noise ratio experienced with superconducting systems is a direct consequence of the use of saddle-shaped coils, and the construction of a spectrometer to work at a frequency of 470 MHz being undertaken in this laboratory presents a considerable challenge, as the only satisfactory coil configuration so far found is solenoidal.

A further limitation is the amount of space available for the construction of the coil. If there is appreciable coupling, magnetic or electrostatic, between the coil and, say, a shield or a strongly conducting sample (for example, a saline solution), then in general, the S:N ratio will be degraded. The mechanism by which the degradation takes place is dependent upon geometry and frequency; the field generated by unit current may be lessened by induction, proximity effect may change the coil resistance, and there may be resistive losses in the coupled element. The simplest way of monitoring these effects is to measure the Q of the coil in free air and then to observe the change when the coil is in place in the probe. It should preferably be less than 10%, and a good rule of thumb for obtaining this value is that no conductor should be closer to the coil than the largest dimension of the latter. The effects of coupling are particularly noticeable at low temperatures. It is quite easy to obtain Q's of over 600 at 77 K in free liquid nitrogen, but another matter in the confines of the probe. Of particular importance is coupling to conductors at room temperature. The component of resistance introduced into Eq. [6] by this coupling carries with it a temperature four times greater than that of liquid nitrogen and its noise contribution is thus disproportionately large. It might be added that the construction of a low-temperature probe with a room-temperature sample is no easy matter, and that to date, we do not have a reliable system.

THE PREAMPLIFIER AND TRANSMITTER

The only factor not so far considered is the noise figure of the preamplifier. For the frequency range 50 to 500 MHz the best semiconductor now available is probably a gallium arsenide field effect transistor. The authors have described elsewhere the design and construction of a preamplifier with a noise figure of 0.3 db at 129 MHz (*3*), and it

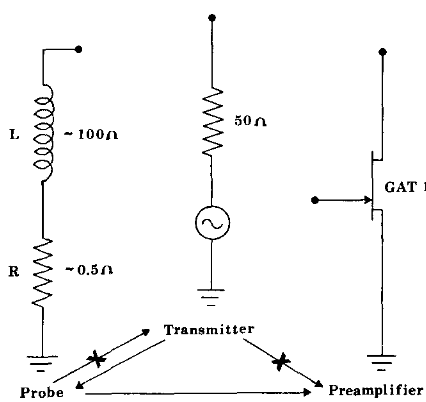

FIG. 4. The three elements probe, transmitter, and transistor must be interfaced in such a way that the signal-to-noise ratio is not degraded and the transistor is not damaged.

remains therefore to consider in what manner the probe coil and the preamplifier can be interfaced in order to obtain the best noise performance. If a single-coil probe is used, there is also the problem of interfacing to the transmitter, while protecting the preamplifier from the pulses and conserving the noise performance. As this is the most difficult situation likely to be encountered, it is to this that we turn our attention. The problem is illustrated in Fig. 4. Considering first the interface between probe coil and transmitter, it is obvious that at the frequency of interest, the impedance of the coil $Z \sim 0.5 + j\ 100$ ohms $[j = (-1)^{1/2}]$ must be transformed in such a manner as to power match the transmitter. As is well known, an essentially lossless transformation may be effected with the aid of the circuit of Fig. 5, provided high-Q capacitors are used. However, a word of warning is required here. It is desirable to keep the leads from the coil

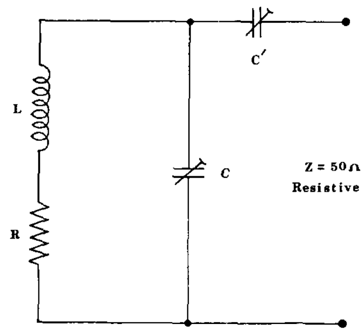

FIG. 5. Impedance transformation using a tuned circuit. The value of C is slightly less than that required to tune to resonance and the matching capacitance C' is given by $C' \simeq (C/50Q\ \omega_0)^{1/2}$.

to the capacitors as short as possible to minimize the resistance R. In this situation, the capacitor C is physically close to the coil and unfortunately, many high-Q variable capacitors are ferromagnetic. The main field homogeneity is thus disturbed.

Turning now to the interface between the coil and the F.E.T. one must ask under what conditions the transistor gives its best performance. Robinson (*8*) has considered this problem and has shown that the optimum noise figure is obtained when the input capacitance is almost tuned out and the signal source has a source impedance which is resistive and given by

$$R_{SO} \simeq 1.6 f_T / f g_m, \qquad [12]$$

f_T is the figure of merit for the F.E.T. and is given by

$$f_T = g_m / 2\pi C_j, \qquad [13]$$

where g_m is the transconductance of the device and C_j is the junction capacitance. Alternatively,

$$R_{SO} \simeq 1.6 / 2\pi f C_j. \qquad [14]$$

With a junction capacitance of 2 pF and a frequency of $f = 129$ MHz, $R_{SO} \simeq 980\ \Omega$. A practical value obtained with the amplifier of Ref. (*3*) gave $R_{SO} = 800\ \Omega$. On the other hand, the input impedance of the F.E.T. with its junction capacitance tuned out is resistive, and given by

$$R_{in} \simeq g_m / (2\pi f C_j)^2. \qquad [15]$$

At 129 MHz, and taking the values $C_j = 2$ pF, $g_m = 15$ mA/V, we have $R_{in} \simeq 5.7$ kΩ. Obviously, the source and the F.E.T. are grossly mismatched powerwise when they are noise matched, and it follows that one should never tune a probe by looking for the maximum signal from the receiver. The signal may well be a maximum; the signal-to-noise ratio most certainly will not be.

The probe has been matched to 50 Ω resistance in order to power match to the transmitter. It follows, to make a noise match to the F.E.T., that 50 Ω must be transformed to R_{SO} within the preamplifier. This, and the tuning out of the gate capacitance C_j, is easily accomplished using the transformation properties of tuned circuits and the reader is referred to Ref. (*3*) for further details. In general, the greater the ratio f_T/f, the better the noise performance. However, in the pursuit of excellence one must beware, particularly at low frequencies, of making the optimum source impedance higher than is practical. Above a value of about 2 kΩ, losses in the transforming device must also be taken into account. Hence to obtain very low noise figures, it may be necessary to cool the preamplifier.

Finally, the preamplifier must be protected from the potentially destructive transmitter pulses, and if a class A transmitter is used, there must be a gate of some sort which prevents the injection of noise from transmitter to receiver. Nor must the protection or the gate degrade the noise performance of the system. As one progresses to higher frequencies, crossed diodes (*9*) become increasingly ineffective due to their junction capacitance—typically 4 pF. Not only does such a large value allow noise to pass from the transmitter; it also ensures that any diodes used to protect the F.E.T. form, above say 50 MHz, a major part of the tuning capacitance. This may, depending on the diode, be disastrous, for after passing heavy current, many diodes exhibit a change of capacitance which lasts many milliseconds. This change can ruin the noise performance of a tuned amplifier and even cause it to oscillate. A far more elegant way to protect the preamplifier is to use PIN diodes (*10*) but unfortunately, PIN diodes have a low resistance to radio frequencies when they are passing a heavy direct current (say, 2 Ω at 40 mA). A PIN diode in the line from the probe to the preamplifier therefore introduces shot noise and can degrade the noise figure of the receiver by as much as 4 db. Fortunately, it is possible to construct a PIN diode circuit which not only protects the preamplifier (60 db isolation) from the transmitter, but which also has all the diodes "off" when the spectrometer is receiving signal. Details may be found in Ref. (*11*).

CONCLUSION

The authors have attempted to provide a direct physical picture of the factors governing the signal-to-noise ratio in an NMR experiment. It has been shown that the signal received from a sample by a set of coils is directly proportional to the magnetic field that would be created at the sample if unit current were passed through the coils, while the noise present in the coils has been shown to be purely a function of the coil resistance. This simple argument allows a direct comparison of the efficiency of different coil configurations to be made. For example, while unit currents flowing through saddle-shaped and solenoidal coils create similar B_1 fields, and the coils thus receive similar signals from the sample, the resistance of a saddle-shaped coil is considerably larger than that of a solenoid and so the signal-to-noise ratio is much less. The correct manner of interfacing between the probe, the transmitter and the preamplifier has been discussed, and attention has been drawn to the importance of noise matching the probe to the amplifying device used, and the distinction between noise and power matching. Finally, the problems of protecting the receiver from the transmitter pulses and noise have been considered, and the use of PIN diodes advocated as a solution.

APPENDIX 1: THE EQUIVALENCE OF THE PRESENT AND THE TRADITIONAL FORMULATIONS

From Eqs. [4] and [6], the signal-to-noise ratio may be written as

$$\Psi_{\mathrm{rms}} = K\omega_0 (B_1)_{xy} M_0 V_s/(8kT_c R\Delta f)^{1/2}. \qquad [16]$$

To convert to the traditional formula of Eq. [1], we must find a relationship between the energy stored in the B_1 field, which is a measure of the coil inductance, and the value of $(B_1)_{xy}$ at the sample. If over the sample volume, the B_1 field is predominantly homogeneous and in the xy plane, we may say that $(B_1)_{xy} \simeq B_1$ for the sample. The energy stored in the sample volume is given by

$$E = \frac{1}{2\mu_0} \int_{\text{sample}} B_1^2 \, dV \simeq (B_1)_{xy}^2 (V_s/2\mu_0). \qquad [17]$$

The inductance of the coil is given by

$$L = (1/\mu_0) \int_{\text{all space}} B_1^2 \, dV. \qquad [18]$$

If, following Hill and Richards (*2*), we define the filling factor as

$$\eta = \int_{\text{sample}} B_1^2 \, dV \Big/ \int_{\text{all space}} B_1^2 \, dV, \qquad [19]$$

then from Eqs. [17] to [19],

$$(B_1)_{xy} \simeq (\mu_0 \eta L/V_s)^{1/2}$$

or $\qquad [20]$

$$K(B_1)_{xy} = K(\mu_0 \eta L/V_s)^{1/2},$$

where mean and root mean square inhomogeneity factors have been introduced, $K \simeq K \simeq 1$.

Substituting in Eq. [16] and adding the noise figure of the preamplifier we thus obtain

$$\Psi_{\mathrm{rms}} = KM_0 \left[\left(\frac{\omega_0 L}{R} \right) \frac{\eta \mu_0 V_s}{8kT_c F\Delta f} \right]^{1/2}. \qquad [21]$$

For the special case of a solenoid, it may be shown that

$$\int_{\text{coil volume } V_c} B_1^2 \, dV \simeq \tfrac{1}{2} \int_{\text{all space}} B_1^2 \, dV.$$

Thus if the field within the solenoid is homogeneous

$$\eta \simeq V_s/2V_c. \qquad [22]$$

Substitution in Eq. [21] gives Eq. [1],

$$\Psi_{\mathrm{rms}} = K\eta M_0 (\mu_0 Q\omega_0 V_c/4FkT_c \Delta f)^{1/2},$$

which is valid only for a solenoid.

APPENDIX 2: THE FIELD AT THE CENTER OF A SADDLE-SHAPED COIL

The vector magnetic potential **A** at point P due to an element of arc **ds** is given by

$$d\mathbf{A} = (\mu\mu_0 I/4\pi)\,(d\mathbf{s}/v), \qquad [23]$$

where $v = |\mathbf{p} - \mathbf{a}|$ is the distance of P from **ds** (see Fig. 6). For the special case of P at

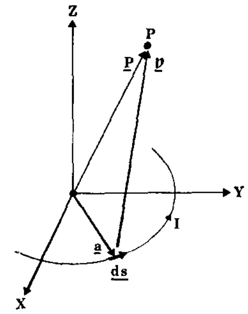

FIG. 6. The coordinate system.

the center of the coil of Fig. 3, we have that

$$v = (a^2 + g^2)^{1/2}$$

and further, the contributions to $(B_1)_{xy}$ from all four arcs add. Thus we have that

$$(\mathbf{B}_1)_{\text{arcs}} = \text{curl}\left\{\frac{\mu\mu_0 I}{\pi}\int_{-\pi/3}^{+\pi/3} \frac{(-a\sin\phi)\,\mathbf{i} + (a\cos\phi)\mathbf{j}}{(a^2+g^2)^{1/2}}\,d\phi\right\}$$

where **i** and **j** are unit vectors in the x and y directions.

$$\therefore \quad (\mathbf{B}_1)_{\text{arcs}} = \text{curl}\left\{\frac{\sqrt{3}\,\mu\mu_0\, Ia}{\pi(a^2+g^2)^{1/2}}\mathbf{j}\right\};$$

$$\therefore \quad (\mathbf{B}_1)_{\text{arcs}} = -\frac{\sqrt{3}\,\mu\mu_0 I}{\pi}\frac{ag}{[a^2+g^2)^{3/2}}\mathbf{i}. \qquad [24]$$

The potential due to one of the four verticals of the coil is given by

$$\mathbf{A} = \int_{-g}^{+g} \frac{\mu\mu_0 I}{4\pi(a^2+z^2)^{1/2}}\,dz$$

and the field produced by them, which is parallel to that produced by the arcs, is given by

$$(\mathbf{B}_1)_{\text{verticals}} = \frac{\sqrt{3}\,\mu\mu_0 I}{2\pi}\frac{\partial}{\partial a}\left\{\int_{-g}^{+g}\frac{dz}{(a^2+z^2)^{1/2}}\right\}\mathbf{i}$$

$$= \frac{\sqrt{3}\,\mu\mu_0 I}{\pi}\frac{\partial}{\partial a}\left\{\sinh^{-1}\left(\frac{g}{a}\right)\right\}\mathbf{i}; \qquad [25]$$

$$\therefore \quad (\mathbf{B}_1)_{\text{verticals}} = -\frac{\sqrt{3}\,\mu\mu_0 I}{\pi}\frac{g}{a(a^2+g^2)^{1/2}}\mathbf{i}.$$

Hence, from Eqs. [24] and [25], the field at the center of a saddle-shaped coil which is passing unit current is given by

$$(B_1)_{xy} = \frac{\sqrt{3}\,\mu\mu_0}{\pi}\left\{\frac{ag}{(a^2+g^2)^{3/2}} + \frac{g}{a(a^2+g^2)^{1/2}}\right\}. \qquad [26]$$

By using the type of analysis, briefly indicated above, for a point P which is off-center, the total magnetic field can be analyzed in a series of spherical harmonics, from which it may be shown that the optimum homogeneity is obtained when the angular width of the coil is 120°, as shown in Fig. 3, and the length is twice the diameter.

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