1. **Subgame perfection warmup**

   Below is the payoff matrix for this sequential game. BJ takes an action first of either c or w. LJ takes an action given BJ chooses c (the first entry shown), and an action given BJ chooses w (the second entry).

BJ | LJ	c, c	c, w	w, c	w, w
c	5, 3	4, 4*	5*, 3	4*, 4*
w	9*, 1*	9*, 1*	0, 0	0, 0

   The pure-strategy Nash equilibria are [w, (c, w)], [w, (c, c)] and [c, (w, w)]. This is because none of the two players obtains more by switching to the other action under any of the two circumstances. To find them, we mark the best responses with stars for each column and row. If both players are playing their best responses, then it is a Nash equilibrium.

   Among the equilibria, Only [w, (c, w)] is subgame perfect according to the backward induction. LJ will choose c if BJ chooses w, and choose w if BJ chooses c. BJ knows it and thus will choose w to get 9 instead of 4.

2. **A very unfair final**

   (a) To obtain a subgame perfect outcome of this game, we have a backward induction starting from the last student. In the subgame when there only remains two students, Student 14 proposes to own 100 herself, which will pass regardless of whether Student 15 agrees. This becomes a part of common knowledge, so in the subgame of three students, Student 13 will propose $(99, 0, 1)$ so that Student 15 will vote for her to get 1 more point than in Student 14's proposal. The induction goes on as shown in the chart below. The last row shows the subgame perfect outcome.

Round | Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
14														100	0
13													99	0	1
12												99	0	1	0
11											98	0	1	0	1
10										98	0	1	0	1	0
9									97	0	1	0	1	0	1
8								97	0	1	0	1	0	1	0
7							96	0	1	0	1	0	1	0	1
6						96	0	1	0	1	0	1	0	1	0
5					95	0	1	0	1	0	1	0	1	0	1
4				95	0	1	0	1	0	1	0	1	0	1	0
3			94	0	1	0	1	0	1	0	1	0	1	0	1
2		94	0	1	0	1	0	1	0	1	0	1	0	1	0
1	93	0	1	0	1	0	1	0	1	0	1	0	1	0	1

   (b) If ties are now broken, we use a similar way to reason, except that the students will propose different plans to distribute points. We assume the indifferent rejection rule that Student i will reject Student $k+1$'s proposal if it is indifferent than Student k's. When the subgame has two students, Student 15 will reject Student 14's proposal even if Student 15 offers 100 to her, due to our assumption. Student 13 knows that and will offer 0 to Student 14. Student 14 has to vote for Student 13 because getting a zero is better than to fail. The induction goes on as shown in the chart below. The last row shows a subgame perfect outcome.

Round | Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
15															100
14														0/F	100
13													100	0	0
12												98	0	1	1
11											97	0	1	0	2
10										96	0	1	2	1	0
9									96	0	1	0	0	2	1
8								95	0	1	0	1	1	0	2
7							95	0	1	0	1	0	2	1	0
6						94	0	1	0	1	0	1	0	2	1
5					94	0	1	0	1	0	1	0	1	0	2
4				93	0	1	0	1	0	1	0	1	2	1	0
3			93	0	1	0	1	0	1	0	1	0	0	2	1
2		92	0	1	0	1	0	1	0	1	0	1	1	0	2
1	92	0	1	0	1	0	1	0	1	0	1	0	2	1	0

Note that when a student needs to win one more votes among several other students of equivalent status, he or she randomly assigns one more point to one of them. Technically, there is no unique subgame perfect outcome.

3. **Lighten my load**

It is a weighted potential game if we define the potential function Φ as the sum of squares of machine loads.

Without loss of generality, suppose that there are k machines and that player i switches job i from machine 1 to 2. Let m_1 denote the loading time of all jobs scheduled on machine 1 and m_2 the loading time on machine 2. Then we have the following expressions for the potential of the game:

$$\Phi(a_1, a_{-i}) = (m_1 + t_i)^2 + m_2^2 + \sum_{i=3}^{k} m_i^2 \tag{1}$$

$$\Phi(a_2, a_{-i}) = m_1^2 + (m_2 + t_i)^2 + \sum_{i=3}^{k} m_i^2. \tag{2}$$

Then the potential difference between job i on machine 1 and 2 is

$$\begin{aligned} \Phi(a_2, a_{-i}) - \Phi(a_1, a_{-i}) &= m_1^2 + (m_2 + t_i)^2 - (m_1 + t_i)^2 - m_2^2 \\ &= m_1^2 + m_2^2 + 2m_2 t_i + t_i^2 - m_1^2 - 2m_1 t_i - t_i^2 - m_2^2 \\ &= -2t_i(m_1 - m_2). \end{aligned} \tag{3}$$

We can then assign each job a weight that only depends on i,

$$w_i = -2t_i. \tag{4}$$

Because the utility of player i is the negative of the total load of the machine job i is scheduled to,

$$u(a_2, a_{-i}) - u(a_1, a_{-i}) = m_1 - m_2 \tag{5}$$

Equation (3) is then simplified to

$$\Phi(a_2, a_{-i}) - \Phi(a_1, a_{-i}) = w_i(u_2 - u_1) \tag{6}$$

which accords with the definition of a weighted potential game.

4. **Iterated elimination**

   Yes, there are pure strategy Nash equilibria. They are (H, L), (T, L), (R, T), and (R, E). By definition, none of the players benefits from switching to some other strategy given the other player's strategy at these equilibria. For example, at (R, E), column player will not switch from R to L to lose 1; row player will not either because 1 is the most to get when the other chooses R.

   Yes, there are weakly dominated strategies. If I followed the iterated elimination of weakly dominated strategies, however, different orders of iteration will lead to different results. For example, we can eliminate the row player's pure strategy of choosing E because it is weakly dominated by choosing T. We are then left with a $2 * 2$ grid. After that, we can eliminate the column player's strategy of choosing R, which is weakly dominated by L. Then we have (H, L) and (T, L). Nevertheless, we can eliminate choosing H and then L for the same reasons. In this case, we obtain different results (T, R) and (E, R).

   The phenomenon of reaching different results after the iteration of weakly dominated strategies is in fact comprehensible. Weakly dominated strategies can make part of a Nash equilibrium. Thus, it is possible to eliminate one or more Nash equilibria during iterations. (https://www.umass.edu/preferen/Game %20Theory%20Evolving/GTE%20Public/GTE%20Eliminating%20Dominated%20Strategies.pdf)

5. **Alternating ultimatums** The following strategies are a subgame perfect equilibrium: at any time, Alice offers $a^* = 1/(1+\gamma)$ and accepts any $b \leq b^*$, while Bob offers $b^* = 1/(1+\gamma)$ and accepts any $a \leq a^*$. To prove that it is a subgame perfect equilibrium, we use backward induction at a smaller scale and then use mathematical induction to derive our final conclusion.

   Suppose the game ends after t turns. At the last turn, or Turn t, Bob gets to offer and will propose to have it all; Alice can do nothing but accept. Therefore, at Turn $t-1$, Alice tries to avoid Bob's rejection. Because Bob's rejection leads to a multiplication factor of γ, by ensuring Bob the same amount as that he proposes at the last turn, Alice gives him γ to win Bob's approval. Making this action, Alice gains $1-\gamma$ of the total available amount at Turn $t-1$. Bob also wants that indifference point where Alice accepts his proposal to avoid delay penalty, so he gives her $\gamma(1-\gamma)$. The same reasoning goes on until the first turn.

Turn	Alice	Bob
t	0	1
t - 1	1 - γ	γ
t - 2	$\gamma(1-\gamma)$	1 - $\gamma(1-\gamma)$
t - 3	1 - $\gamma(1-\gamma(1-\gamma))$	$\gamma(1-\gamma(1-\gamma))$
...	...	...
1	1 - $\gamma + \gamma^2 - \gamma^3 + ...\gamma^{t-1}$	$\gamma - \gamma^2 + \gamma^3 + ...\gamma^{t-1}$

   According to the backward induction, the subgame perfect for the alternating game with t turns is the last row of the chart above. The formula holds true for any positive integer t. Notice that the last terms of the two geometric series change signs based on whether t is even or odd. The two series, however, converge when t approaches infinity anyways, so we do not have to worry about that. Thus, at the first turn, Alice earns

$$1 - \gamma + \gamma^2 - \gamma^3 + ... = \frac{1}{1+\gamma}$$

while Bob earns

$$\gamma - \gamma^2 + \gamma^3 + ... = \frac{\gamma}{1+\gamma}.$$

So, if they start at Turn 1 and follow the strategies that have been analyzed in the backward induction, they will come to a subgame perfect equilibrium.

(https://www.youtube.com/watch?v=d4umvT3_iF0)

- **Extra Credit** For any $x \in [0, 1]$, there is some Nash equilibrium of the game where the reward of 1 is divided as x going to Alice and $1 - x$ going to Bob (γ is set arbitrarily).

  It is a Nash equilibrium when Alice always proposes $a^* = x$ and only accepts $b \leq b^*$, and Bob always proposes $b^* = 1 - x$ and only accepts offers $a \leq a^*$. Given that Alice offers $a^* = x$ and accepts any $b \leq 1 - x$, any deviation for Bob does not yield a better outcome. If Bob does not accept Alices proposal, then the next time he is offered the same fraction, he gains less due to the multiplication factor γ. If Bob offers less than how much Alice wants, the game goes on, the ultimate outcome of which is that they will both get nothing due to disagreement. Bob also has no incentive to offer more or accept less than in his current strategy since they are strictly dominated. Similarly, Alice has no incentive to deviate given Bobs strategy. Because both of the players do not regret given their opponent's choice, it is a Nash equilibrium.

6. **On selling and rebuying** Find a Bayes-Nash equilibrium for this game, where each bid b_i is an increasing linear function of the received signal s_i. Suppose a Bayes-Nash equilibrium is for everyone to bid

$$b_i = k * s^i + b$$

where k and b are constants. We calculate the revenue expectation of signal s_i winning the book and look for maximum values.

The probability of s_i winning is when everyone else receives a lower signal,

$$Pr(winning|s_i) = (\frac{s_i}{M})^{n-1}$$

where the constant n is the total number of players.

Because s_i is the largest among all signals and that the distribution is uniformed, they uniformly spread among the real-value interval [0, s_i]. This means that the expectation of another signal other than s_i is $\frac{s_i}{2}$. The buying price is b_i while the selling price is the average of all signal values. Thus, the The expected utility to buy the book is their price difference

$$u_i = \frac{1}{n}(s_i + \frac{s_i}{2} * (n-1)) - b_i.$$

The revenue expectation is the multiplication of the probability of s_i winning and its expected utility,

$$f(b_i) = (\frac{s_i}{M})^{n-1} * [\frac{1}{n}(s_i + \frac{s_i}{2}(n-1)) - b_i].$$

When the revenue expectation is maximized, its derivative $f'(s_i, b_i) = 0$. Thus, we have

$$f'(b_i) = \frac{d}{ds_i}[(\frac{s_i}{M})^{n-1} * (\frac{1}{n}(s_i + \frac{s_i}{2}(n-1)) - b_i)].$$

According to Mathematica's output,

$$b_i = \frac{1+n}{2n}s_i + Cs^{1-n}$$

where C is an arbitrary constant that specifies an initial value for the deferential equation. When we set $C = 0$, b_i fits our linear equation model,

$$b_i = \frac{1+n}{2n}s_i.$$

It is part of Nash equilibrium because, for each player, the best response is to assume winning and follow the calculation above. If other people in the auction remain the same strategies, maximizing the revenue expectation using the linear function is the best response.

(Ryan Wan, Charlie Wu)

7. **Alice and Silent Bob**

In normal form, here is the payoff matrix for their actions. If Alice is feisty ($p = \lambda$):

Alice | Bob	Argue	Silent
Beer	(1, 0)	(3, 1)
Wine	(0, 0)	(2, 1)

If Alice is Sedate ($p = 1 - \lambda$):

Alice | Bob	Argue	Silent
Beer	(0, 1)	(2, 0)
Wine	(1, 1)	(3, 0)

There are two pure strategy Bayes-Nash equilibria. In order to obtain Bayes-Nash equilibria, we look for best responses for both players. Below is a table of their expectations for playing different strategies. For Alice's column, the first entry represents what she orders if she is feisty while the second entry sedate; for Bob's row, the first entry represents his reaction if Alice orders beer while the second if Alice orders wine. Best responses depend on the value of λ. Knowing $\lambda \geq 0.5$, we mark the best responses with stars.

Alice | Bob	Argue, Argue	Silent, Argue	Argue, Silent	Silent, Silent
Beer, Beer	$\lambda, 1-\lambda$	$(2+\lambda)^*, \lambda^*$	$\lambda, 1-\lambda$	$2+\lambda, \lambda^*$
Beer, Wine	$1^*, 1-\lambda$	$1+2\lambda, 1^*$	$3 - 2\lambda, 0$	$3, \lambda$
Wine, Beer	$0, 1 - \lambda$	$2 - 2\lambda, 0$	$2\lambda, 1^*$	$2, \lambda$
Wine, Wine	$1 - \lambda, 1-\lambda$	$1 - \lambda, 1-\lambda$	$(3 - \lambda)^*, \lambda^*$	$3 - \lambda, \lambda^*$

As we see, the two pure strategy Bayes-Nash equilibria are where both players have their best responses: [(Beer, Beer), (Silent, Argue)] and [(Wine, Wine), (Argue, Silent)]. In either case, Alice orders the same regardless of her type.

Both pure strategy Bayes-Nash equilibria are also Perfect Bayesian Equilibria. At a perfect Bayesian equilibrium, players are sequentially rational and their strategies consist with some beliefs. In this signaling game, Alice sends a signal indicating her type to Bob via her drink choice. It is rational for

Alice to the same drink, either beer or wine, regardless of her type. By doing this, Bob does not know her type and has to guess. Suppose Alice orders beer. It is rational for Bob to either argue or remain silent, each of which is a best response for some possible beliefs of Alice's type and her strategy. And so is the case when Alice orders wine.

(http://isites.harvard.edu/fs/docs/icb.topic818969.files/beer%20quiche%20fall%202010.pdf,

http://web.mit.edu/14.12/www/02F_lecture1518.pdf)