Algo.

My strategy is to solve this recursively. We will find some particular stops along the way to be "storage" points and store gas there so that later we can come to pick them back to the car. Their locations are not determined at once, but for convenience purpose, we label their distances from the starting point to be ld_1, d_2, d_3, etc.

First, we load C gallons of gas at the depot where the distance from the depot $d = 0$, and drop all the gas tank's remaining gas of $(C - \frac{C^2}{B})$ gallons at the first storage point, $d = \frac{C^2}{B}$. Then, drive back to the depot and load another C gallons and drop another $(C - \frac{C^2}{B})$ gallons at $d = \frac{C^2}{B}$. Repeat until there is no gas at $d = 0$ and there will be $\frac{B}{C}(C - \frac{C^2}{B}) = (B - C)$ gallons of gas at $d = \frac{C^2}{B}$. Then, starting from $d = \frac{C^2}{B}$ with $(B - C)$ gallons of gas, we use the same equation to calculate the distance to the next storage point $d_2 - d_1 = \frac{C^2}{(B-C)}$. We repeat the process along the way until $(d_j - d_i)$ is less than or equal to the calculated interval, and then transport them all to our destination.

The above strategy will give optimal solution.

- **Greedy Choice**: An optimal solution contains the choice of transporting all the gas to the first storage point $d = \frac{C^2}{B}$.

  Pf.

  No matter how many gallons are in the gas tank, the car consumes same amount of gas. It is thus wise to minimize number of trips because the less trips it takes over a distance, the less gas consumed for transportation cost, and the more the remaining gas.

  Another observation is that, while hypothetically we can divide an amount of gas into infinity number of cans, as long as they are consumed on the way, all drops arrived at the destination must have been at some point passed each storage points by some "trip". The total gas consumed for transporting gas from $d = a$ to $d = b$ is

$$V_{gas-consumed} = \int_a^b f(x)\, dx,$$

  where x illustrates distance and $f(x)$ is the number of trips that go across $d = x$.

  According to our algorithm, there is a total of $(C - \frac{C^2}{B})$ gallons that have passed the first storage by $\frac{C}{B}$ trips. At $d = \frac{C^2}{B}$, $f(x) = \frac{B-C}{C}$, which is the hypothetical minimum because of the property that all drops arrived at the destination must have been at some point passed the first storage point. For $0 \leq x < \frac{C^2}{B}$, f(x) must be greater than $\frac{B-C}{C}$ because there are even more gallons that pass $d = x$. As the number of trips must be an integer, $n_{min} = \frac{B-C}{C} + 1 = \frac{B-C}{C} + \frac{C}{C} = \frac{B}{C}$, which is the number of trips it takes in our algorithm.

  To summarize, because our algorithm consumes the hypothetical minimum amount of pas it takes from transporting from $d = 0$ to $d = \frac{C^2}{B}$, the first choice by greedy algorithm is optimal.

- **Inductive Structure**: After we load all the gas from $d = 0$ to $d = \frac{C^2}{B}$, there is no external constraint on the subproblem of transporting the remaining of gas to the destination.

  Pf. After we load all of gas from $d = 0$ to $d = \frac{C^2}{B}$, there is no gas at $d = 0$ and there will be $\frac{B}{C}(C - \frac{C^2}{B}) = (B - C)$ gallons of gas at $d = \frac{C^2}{B}$. Starting from $d = \frac{C^2}{B}$ with $(B - C)$ gallons of gas, we use the same equation to calculate the distance to the next storage point $l_2 - l_1 = \frac{C^2}{(B-C)}$. Because $(B - C)$ is still a multiple of C, we have no other external constraints on how we transport next.

- **Optimal Substructure**: The solution to the subproblem of transporting gas from $d = 0$ to $d = d_i$ together with the choice of $(i + 1)$th storage point yields the optimal solution.

  Pf. According to the equation above, we derive

$$V_{gas-consumed} = \int_0^d {}_{i+1} f(x)\, dx = \int_0^i f(x)\, dx + \int_i^{i+1} f(x)\, dx,$$

the gas consumed for problem P is simply the sum of badness of fit for subproblem P' and that of the $(k+1)$th greedy choice. Applying the standard contradiction argument, we conclude that if the sum is minimum, then after adding the gas consumed for the last greedy choice the final solution will also be optimal.

Complexity Analysis

The overall time complexity of transporting takes O(n^2) while the overall time to compute locations is O(n), where $n = \frac{B}{C}$.

The total of these distance intervals sum up to D. As recursion goes, distance between consecutive storage points will decrease but is trivial to the complexity analysis in a way. The total number of trips each step of greedy choice is O(n) while each trip takes O(n) trips to drive back and forth.