

1. We can show an instance with an integrality gap of $\Omega(D)$. Suppose that there is only one pump with $s_i = d^2$ and $w_i = d^2$, and $D = d$. The fractional solution gives d while the optimal costs d^2 , which leads to a gap of at least d , or D .
2. This IP is a valid relaxation because, any feasible (integral) solution for the original problem satisfies all IP constraints.

Suppose there is a feasible integral solution whose corresponding indicator variables are X . Trivially, it satisfies the constraint

$$\sum_{i \in [n]} s_i x_i \geq D.$$

Then we have a look at the new constraint,

$$\sum_{i \in [n]} \min[s_i, D] x_i \geq D.$$

For every pump i , if $x_i = 0$, then the change does not affect the value of the left hand side; if $x_i = 1$, then there are two cases, either $s_i < D$ or $s_i \geq D$. When $s_i < D$, the value of the left hand side does not change. When $s_i \geq D$, choosing pump i alone will satisfy the constraint as $D \geq D$. In all the cases analyzed above, the constraint is satisfied.

The gap is still $\Omega(D)$. Suppose that there are two pumps, $s_1 = d - 1$, $w_1 = d$, $s_2 = d^2$, $w_2 = d^2$, and $D = d$. The fractional solution gives $d + \frac{1}{d^2}$ while the optimal cost is $d + d^2$, which leads to a gap of at least d , or D .