

• **Claim 1: The general's algorithm indeed works.**

Suppose we are given a BLOCKING instance x of an undirected graph G that might contain k "garrison-vertexes" and none or some "empty vertexes."

The algorithm will separate all the vertexes into at least k subgraphs G_i , each of which contains exactly one garrison. It is obvious that, starting from any garrison, a soldier can only travel to the vertexes in the same subgraph and thus cannot travel to another garrison. Therefore all garrisons are isolated from one another.

• **Claim 2: The algorithm has an approximation ratio of 2.**

An optimal solution partitions the graph G into k subgraphs to ensure that none of the garrisons can connect to each other.

We use E_{ij} to denote the set of all edges that connect subgraphs G_i to G_j in an optimal solution. That is, every edge in the set has one vertex in G_i and one in G_j , and is counted only once in one of the subsets. For convenience purpose we set $E_{ii} = 0$. Then we can represent the optimal solution as

$$C^*(x) = |E_{12}| + |E_{13}| + |E_{23}| + |E_{14}| + \dots$$

We use E_A to denote the set of all edges starting from G_A in an optimal solution. That is, every edge in the set has one vertex in G_A and one in some other subgraph. By definition, we have

$$|E_i| = |E_{i1}| + |E_{i2}| + |E_{i3}| + \dots + |E_{ik}|$$

For every pair of the subgraphs G_i and G_j , every cut edge between them will be counted in both E_{ij} and E_{ji} . In addition, for any $i \neq j$ we have $|E_{ij}| = |E_{ji}|$ because they contain the same edges. Therefore,

$$|E_1| + |E_2| + |E_3| + \dots + |E_k| = 2(|E_{12}| + |E_{13}| + |E_{23}| + |E_{14}| + \dots) = 2C^*(x).$$

By the general's algorithm, we are ensured that B_i is the optimal for every single garrison,

$$|B_i| \leq |E_i|.$$

The general's solution counts the union of B_i for every garrison i where overlapping edges are counted only once,

$$C(x) = |B_1 \cup B_2 \cup B_3 \cup \dots \cup B_{k-1}| \leq |B_1| + |B_2| + \dots + |B_{k-1}|.$$

Taking into account what has been analyzed above,

$$C(x) \leq |B_1| + |B_2| + \dots + |B_{k-1}| \leq |E_1| + |E_2| + \dots + |E_{k-1}| = 2C^*(x) - |E_k| < 2C^*(x).$$

Since $2C^*(x)$ is the upper bound of $C(x)$,

$$1 \leq \frac{C(x)}{C^*(x)} < 2.$$

Therefore, the algorithm has an approximation ratio of 2.