- **Greedy is not optimal.**

  Pf. A counterexample when greedy strategy does not generate an optimal solution is as follows.

  knapsack 1: $B_1 = 5$

  knapsack 2: $B_2 = 5$

  Item 1: $s_1 = 3$, $p_1 = 1$

  Item 2: $s_2 = 3$, $p_2 = 1$

  Item 3: $s_3 = 2$, $p_3 = 100$

  Item 4: $s_4 = 2$, $p_4 = 100$

  An optimal solution is to pack item 1 and 3 into knapsack 1, and item 2 and item 4 into knapsack 2, which gives a total value of $p_1 + p_2 + p_3 + p_4 = 202$.

  Greedy solution, however, will pick item 3 and item 4 for knapsack 1. Then item 3 and 4 cannot fit into knapsack 2 at the same time and the total value will be $p_1 + p_3 + p_4 = 201$, which is smaller than 202 calculated above and therefore not optimal.

- **Greedy gives O(1)-approximation for identical knapsacks.**

  Pf.

  Suppose x_i is the total value of items covered by ith knapsack, y_i the total elements covered by the time ith knapsack finishes packing, and $z_i = OPT - y_i$ where OPT is the maximum value of packed items at the end. Let k denote the total number of knapsacks.

  Because capacity of the knapsacks are identical and every individual solution of a knapsack is solved optionally at its greedy step, we have

$$x_1 \geq x_2 \geq ... \geq x_i.$$

  Therefore we know that the ith knapsack covers at least a fraction of the remaining items that are eventually covered, written as $x_{i+1} \geq \frac{z_i}{k}$.

  Then we use mathematical induction to show that $z_{i+1} \leq (1 - \frac{1}{k})^{i+1} * OPT$. The base case is when $i = 1$ and is trivially true. Suppose $z_i \leq (1 - \frac{1}{k})^i * OPT$ is true, then

$$z_{i+1} \leq z_i - x_{i+1} \leq z_i * (1 - \frac{1}{k}) \leq (1 - \frac{1}{k})^{i+1} * OPT.$$

  After we plug in $i = k - 1$, it becomes

$$z_k \leq (1 - \frac{1 - \epsilon}{k})^k * OPT = (1 - \frac{1}{e}) * OPT.$$

  Because $(1 - \frac{1}{e})$ is constant, greedy gives a O(1)-approximation.

- **Greedy gives O(1)-approximation for knapsacks ordered in decreasing size.**

  Pf.

  Let g_i be the total value of chosen items in ith knapsack by greedy algorithm, OPT be that in an optimal solution. We also denote d_i to be the total value of "discarded" items, or items chosen by an optimal solution but not in g_i, for ith knapsack.

  We will first show that $g_i \geq d_i$ for all knapsacks. Suppose $g_i < d_i$, then if we replace the current items in ith knapsack with the "discarded" items, the new total value will be greater than the original. This contradicts the fact that greedy algorithm picks the combination of items that generates a greatest total value at that specific step. Adding up values for all i, it must be the case that

$$g_{TOT} \geq d_{TOT}.$$

The value of all items chosen in greedy and in optimal solution is $OPT - d_{TOT}$. In addition, greedy may choose other items that are not in optimal solution, which means

$$g_{TOT} \geq OPT - d_{TOT}.$$

Adding up the above two inequalities, we get $2 * g_{TOT} \geq OPT$, or $g_{TOT} \geq \frac{OPT}{2}$. Thus, greedy algorithm gives a 2-approximation whose ratio is constant.

(Discussed with Diqiu Zhou)