1. Suppose we have made a good guess on $M^*$ such that there is an assignment with makespan no greater than $M^*$. Then, we construct an integer solution as follows. $x_{i,j}$ is 1 if job $j$ is scheduled on machine $i$, and 0 otherwise. We will show that the integer solution satisfies the four LP constraints.

$$\sum_{j=1}^{n} p_{i,j} x_{i,j} \leq M^*, \forall i \in [m]$$

The first constraint means that the maximum load of jobs assigned to a machine cannot exceed $M$. This is true because we assume that no assignment on any machine exceeds $M^*$.

$$\sum_{i=1}^{m} x_{i,j} = 1, \forall j \in [n]$$

The second constraint specifies that each job is assigned to a machine. It is also true because by the way we construct the solution. For every job $j$, suppose it is assigned to machine $i$ in the solution, then only $x_{i,j} = 1$ among all $x_{k,j}$ where $k$ is any machine. The rest are 0s, which makes the sum to be exactly 1.

$$x_{i,j} = 0, \forall i, j : p_{i,j} > M^*$$

The third constraint says that a job cannot be assigned to a machine where its processing time exceeds $M$. If $x_{i,j}$ is not zero, then job $j$ is not on machine $i$ and thus $x_{i,j} = 1$. Then adding the duration $p_{i,j}$ will make the whole makespan exceed $M^*$, which leads to contradiction.

$$x_{i,j} \in [0, 1], \forall i, j$$

The fourth constraint is true because we only assign 1s and 0s in our construction.

Therefore, we prove that there exists such an integer LP solution.

2. The randomized rounding that assigns each job $j$ to machine $i$ with probability $x_{i,j}$ independently yields an O($logn + logm$)-approximation with a high probability of at least $1 - \frac{1}{n}$ where $n$ is the number of jobs.

Using the Chenoff bound, the probability of assigning each job is

$$Pr[\sum_{j=1}^{n} p_{i,j} x_{i,j} > \frac{M^*}{max(p_{i,j})} * \frac{\alpha * c * ln(k)}{ln(ln(k))}] \leq \frac{1}{k^c}.$$

Plug in $\alpha = 2$, $c = 1$, and set $\alpha * c * \frac{ln(k)}{ln(ln(k))} = log(n) + log(m)$, the above inequality simplifies to $k^2 > m * n$ and

$$Pr[\sum_{j=1}^{n} p_{i,j} x_{i,j} > \frac{M^*}{max(p_{i,j})} * \frac{\alpha * c * ln(k)}{ln(ln(k))}] < \frac{1}{mn}.$$

The probability that every job satisfies the condition is

$$Pr = (1 - \frac{1}{mn})^m \geq e^{-\frac{1}{n}} > 1 - \frac{1}{n},$$

which is high probability.

3. To prove that this LP has an integrality gap of $\Omega(m)$ where $m$ is the number of machines, we show a worst case that has one single job whose duration is one unit time for all $m$ machines. An optimal solution places the job on any of the machines, resulting in total time of 1. The fractional solution schedules $\frac{1}{m}$ of the job on every machine, which results in total time of $\frac{1}{m}$. The former is $m$ times than the later. Thus, the integrality gap is $\Omega(m)$.