- **Algorithm**

  For the k-center problem, the algorithm first adds an arbitrary vertex to S. Then it iteratively adds vertices to S such that

  $$d(S,i) \geq \frac{1}{2} max_{j \in V} d(S,j)$$

  until S consists of k vertices.

- **The Algorithm has an approximation ratio of 4.**

  Suppose that, in an optimal solution, the maximum distance from any vertex to a center is r^*, and v_j^* the points selected in the jth cluster. Trivially, the approximation is always able to pick k vertices for S. When the algorithm terminates, there are two possible cases, each of which results in a lower bound of $4r^*$.

  - **Case 1: the solution selects exactly one point in every v_i^* for all i.**

    For every cluster v_i^*, suppose the optimal solution chooses vertex v^* as the cluster's center while the approximation chooses v'. By definition of r^*, any vertex in the cluster has a distance to v^* of at most r^*. Because of the metric property, any vertex in the cluster has a distance to v' of at most $2r^*$. Then, taking the maximum of all the distance values will result in a distance within $2r^*$, which is less than $4r^*$.

  - **Case 2: the solution selects two points in some v_i^*.**

    The two vertices v_a and v_b are at most $2r^*$ away from each other due to the metric property. Because of the way the algorithm adds vertices, if two points are in the same cluster, then

    $$\frac{1}{2} max_{j \in V} d(v_a, v_b) < 2r^*$$

    no matter whether v_a is selected before or after v_b. After multiplying both sides of the inequalities by 2, we can see that all the vertices are within $4r^*$.