COUPLED OSCILLATORS

OBJECTIVE:

The objective of this experiment is to allow you to examine the motion of two coupled, harmonic oscillators. This experiment will introduce you to the concepts of beats and modes of oscillations through the detailed study of two mechanical oscillators.

DISCUSSION:

This experiment will allow you to study a system of mechanically coupled oscillators that exhibit phenomena that may be beyond your intuition and may surprise you. The behavior of coupled oscillators can be seen in many applications such as the design of stereo equipment, analysis of electronic circuits, properties of lasers, the nature of the force that binds molecules, and the properties of elementary particles. The study of the mechanical coupled oscillators is easy to observe and to measure directly because the frequencies can be made so small. You can gain some insight then about the behavior of more complex systems from this simple, easily observed example.

METHOD:

In summary you will

a. measure or record the physical characteristics of two identical pendula. (Location of center of mass, locations of coupling spring and the spring constant.)

b. observe the phenomena of coupled oscillation and beats when the pendula are coupled by a spring.

c. measure the beat period, i.e., the total time required to transfer the motion from one pendulum to the other and back again.

d. measure the frequencies of two normal modes of vibration; Mode I with the pendula oscillating in the same direction, Mode II in which the pendula oscillate in opposite directions.

e. examine the relationship between the beat frequency and frequencies of the two normal modes as a function of location of the coupling spring.

f. calculate the moments of inertia from the periods of each individual, uncoupled pendulum; then calculate the predicted beat frequencies and compare these theoretical results to your measurements of actual beat frequencies.

APPARATUS:

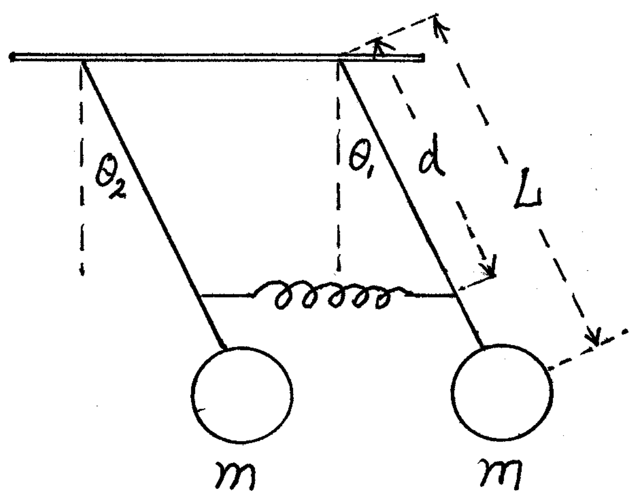
Two physical pendula constructed of identical materials and dimensions, each with pins for fastening a weak spring at varying distances from the pivot points.
Spring, meter stick, Hooke's law apparatus, weights, computer, motion sensor and DataStudio.

THEORY:

The detailed theory of this experiment has been written and made part of this experiment as an appendix. These derivation and equations are not found in your textbook. What follows is a summary that should allow you to perform the experiment and make most of the required calculations.

Mechanically Coupled Oscillators

This experiment uses a coupled pendulum. It consists of two identical pendula coupled by a weak spring each connected to the pendula a distance d below the pivot point.



In addition to the torque produced by gravity, each pendula also experiences a torque from the spring. This spring torque is a function of the spring constant, k, the displacement of the spring and the distance, d, from the pivot point. Because of this extra torque, the motion of the coupled system is more complicated and more interesting than the simple pendulum.

Normal Mode

The description of motion is simplified if we describe the motion in terms of two so-called Normal Modes of oscillation. We shall call these Mode I and Mode II for our pendula in this experiment. The general motion will be expressed as a sum of these two normal modes.

Mode I

In Mode I, the two pendula move together. This is $\theta_1 = \theta_2$ at all times. In this case the effect of the spring is zero since it does not stretch or compress. It follows then that the angular frequency of Mode I is equal to the angular frequency of the uncoupled pendula, i.e. $\omega_I = \omega_0$.

Mode II

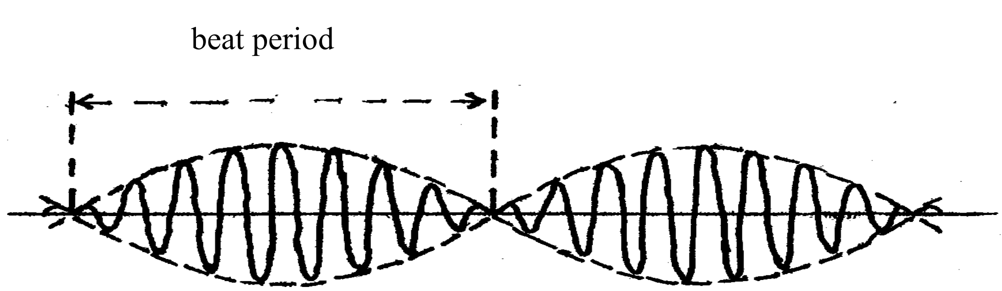
In mode II, the two pendula move exactly out of phase so that $\theta_1 = -\theta_2$ at all times. Now the effect of the spring torque is a maximum and the frequency increases.

General Motion

Any general motion of the pendula is the superposition of Mode I and Mode II. In the appendix we derive an equation,

$$\theta_1 = 2A\cos\left[\frac{\omega_B t}{2}\right]\cos[\omega_0 t] \qquad (9)$$

The motion described by Eq.(9) can be regarded as an oscillation at angular frequency ω_0 with an amplitude that varies at the frequency, $\frac{\omega_B}{2}$. The frequency ω_B is called the "beat frequency". The graph shows the typical behavior for the coupled oscillator. The average oscillation grows and falls in amplitude periodically. The time between nulls is the beat period. Note that there are two nulls in the time period determined by $\frac{\omega_B}{2}$.



Beat Frequency

In the appendix, we first define the angular beat frequency as

$$\omega_B = \omega_{\mathrm{II}} - \omega_{\mathrm{I}}$$

where ω_{II} and ω_{I} are the angular frequencies Mode II and Mode I, respectively. This also means that the beat frequency f_B equals the difference between the Mode II and Mode I frequencies.

$$f_B = f_{\mathrm{II}} - f_{\mathrm{I}}$$

Then we show how the beat frequency depends on other parameters in the coupled pendula. The results of this derivation are

$$\omega_B = \frac{kd^2}{\mathrm{I}\omega_0} \qquad (10)$$

or in terms of the frequency

$$f_B = \frac{kd^2}{4\pi^2 \mathrm{I} f_0} \qquad (11)$$

To arrive at Eq. (10) & (11), we also used the approximation that $\omega_{\mathrm{II}} + \omega_{\mathrm{I}} = 2\omega_0$. The validity of this approximation is examined in the appendix.

DETAILED PROCEDURE:

1. The masses have been measured and the location of the center of mass marked for each pendula. Record the masses. Measure the distance of the center of mass from the pivot point. Also, use the Hooke's law apparatus to determine the value of the spring constant.

2. Turn on the computer. Start the DataStudio program by double click the DataStudio icon.

3. Drag the Motion Sensor from Sensors list to Digital Channel 1. Double click sensor icon. At the "Sensor Properties" window, click Tab "Motion Sensor" and then select **30** for Trigger Rate. Click OK to return to the Experiment Setup window.

4. Use motion sensor to measure the period of each pendulum before attaching a spring. You will use this information in your calculations to compute the moments of inertia of the individual physical pendula.

- Start pendulum swinging, click Start button to record the data. Continually record the data for about 20 seconds. Click Stop button to end the recording.
- Use Graph Display to show position vs. time. Click "Smart Tool" button at Tool bar. The cross hair appears, the X coordinate is shown time, the Y coordinate is shown position. Measure the times of 10 completed cycles with the cross hair. Calculate the period T_0 using $T_0 = t/10$.

  (Note: The minimum distance from pendulum to the motion sensor is greater than 40 cm.)

5. Next, attach a spring to the pin positions farthest from the pivot point, i.e., the bottom pins. Measure and record this distance. The pendula should be separated so that the spring is slightly stretched.

6. Start one pendulum swinging, hold the second pendulum still and then release it also. Use motion sensor to measure the total time required for the second pendulum to reach maximum amplitude and then return to rest. At the Graph your measured time between nulls is the beat period and its reciprocal is the beat frequency.

7. Set the pendula into motion with both swinging in the same direction. We call this Mode I. Use motion sensor to measure the Mode I period as step 4. Then compute the Mode I frequency.

8. Set the pendula into motion each moving in the opposite direction. This is Mode II. Use motion sensor to measure the Mode II period as step 4. Then compute the Mode II frequency.

9. Repeat steps 6, 7, 8 for four additional pin positions. When you finish, you should have data for the beat frequencies and Mode I and Mode II frequencies at pin locations 50 cm, 45 cm, 40 cm, 35 cm and 30 cm.

CALCULATIONS:

1. Calculate the moment of inertia of the uncoupled pendula from Eq. (2) in the appendix using your measurement of the frequency of the single pendulum and distance L from center of mass to the pivot point.

2. Tabulate your beat frequency measurement data, f_B, and distances d for all spring positions. In this same table calculate d^2.

3. Plot your measured values of f_B vs. d^2. Calculate the slope of this line. Discuss and compare this slope to a corresponding value calculated from Eq. (11). Use your measurement of the spring constant, moment of inertia, and your measured value of f_0 in Eq. (11). We suggest that using DataStudio program plot f_B vs. d^2, as follows.

- Choose “New Emptydata Table…” from “Experiment” menu to open “Editable Data Table”.
- Type the data of d^2 in X column, data of f_B in Y column.
- Click and Drag the “data” to Graph display.
- Click “Fit” button on the Graph window, and then select “Linear Fit”. The slope will be showing on the graph.

4. Use your frequency measurements of Mode I and II to see if the relation $f_B = f_{II} - f_I$ is true.

5. Examine the approximation that $f_{II} + f_I = 2f_0$. Do this by calculating the beat frequency, f_B, as predicted by Eq. (14) in the appendix.

DATA ORGANIZATION SHEET
EXP COUPLED OSCILLATORS

M = _________ (kg)   L = __________ (m)

1. Spring Constant

ΔF (g)	10	15	20	25	30	35
Δx (m)						

Spring Constant, k = _________ N/m

2. Free Oscillation (no spring)

t	$T_0 = \frac{t}{10}$	$f_0 = \frac{1}{T_0}$	$I = \frac{MgL}{4\pi^2 f_0^2}$

3. Coupled Pendulum

d (m)	d^2 (m^2)	Mode I		Mode II		Calculated beat	Measured beat		difference %
		T	f_I	T	f_{II}	$f_{II} - f_I$	T	f_B	

MATHEMATICAL APPENDIX

Single Pendulum

You will recall that a single, physical pendulum has a torque equation that looks like,

$$\mathrm{I}\alpha = -Mg\mathrm{I}\theta \tag{1}$$

and the angular frequency of this single pendulum is

$$\omega_0^{\ 2} = Mg\ell/\mathrm{I} \tag{2}$$

where I = moment of inertia of the body about the pivot point, ℓ = distance from the pivot point to the center of mass.

Mechanically Coupled Oscillator

In a coupled pendulum, two identical pendula are coupled by a weak spring each connected to the pendula a distance d below the pivot point. In addition to the torque produced by gravity, each pendula then experiences a torque from the spring. This spring torque is a function of the spring constant, k, the displacement of the spring and the distance, d, from the pivot point. Each pendulum then experiences a torque equal to $kd^2(\theta_2 - \theta_1)$, where θ_2 and θ_1 are the angular displacements of each pendulum from equilibrium.

Normal Modes

The description of motion is simplified if we describe the motion in terms of two so-called normal modes of oscillation. We shall call these Mode I and Mode II for our pendula in this experiment. The general motion will be expressed as a sum of these two normal modes.

Mode I

In Mode I, the two pendula move together. That is, $\theta_1 = \theta_2$ at all times. In this case the effect of the spring is zero since it does not stretch or compress. Then the angular frequency in Mode I is the same as that of the single pendulum. That is,

$$\omega_{\mathrm{I}}^{\ 2} = \omega_0^{\ 2} = Mg\ell/\mathrm{I} \tag{3}$$

and the angular motion of pendulum 1 in Mode I is then

$$\theta_1 = A\cos(\omega_{\mathrm{I}} t) \tag{4}$$

Mode II

In Mode II, the two pendula move exactly out of phase so that $\theta_1 = -\theta_2$ at all times. Now the effect of the spring torque is a maximum. The differential equation of motion is

$$\mathrm{I}\alpha = -\left[Mg\ell + 2kd^2\right]\theta_1 \qquad (5)$$

and the angular frequency of Mode II is

$$\omega_{\mathrm{II}}{}^2 = Mg\ell/\mathrm{I} + 2kd^2/\mathrm{I} \qquad (6)$$

The angular motion of pendulum 1 in Mode II is

$$\theta_1 = B\cos(\omega_{\mathrm{II}} t) \qquad (7)$$

General Motion

Any general motion of the pendula is the superposition of Mode I and Mode II. In the case where the amplitudes are equal,

$$\theta_1 = A\cos(\omega_{\mathrm{I}} t) + A\cos(\omega_{\mathrm{II}} t) \qquad (8)$$

Using the formula for the sum of cosine, Eq. (8) can be written as

$$\theta_1 = 2A\cos\left[(\omega_{\mathrm{II}} - \omega_{\mathrm{I}})t/2\right]\cos\left[(\omega_{\mathrm{II}} + \omega_{\mathrm{I}})t/2\right]$$

or

$$\theta_1 = 2A\cos\left[\omega_B t/2\right]\cos\left[\omega_0 t\right] \qquad (9)$$

where we define $\omega_{\mathrm{II}} - \omega_{\mathrm{I}} = \omega_B$ and approximate $(\omega_{\mathrm{II}} + \omega_{\mathrm{I}})/2 = \omega_0$. The motion described by Eq. (9) can be regarded as an oscillation at angular frequency with amplitude that varies at the frequency, $\omega_B/2$. The frequency ω_B is called the "beat frequency". The average oscillation grows and falls in amplitude periodically. The time between nulls is the beat period. Note that there are two nulls in the time period determined by $\omega_B/2$.

Beat Frequency

We can calculate how the beat frequency depends on other parameters in the coupled pendula. Using Eqs. (3) and (6) for the angular frequencies of the two normal modes, we note that

$$\omega_{\mathrm{II}}{}^2 - \omega_{\mathrm{I}}{}^2 = 2kd^2/\mathrm{I}$$

By factoring we write

$$(\omega_{\mathrm{II}} - \omega_{\mathrm{I}})(\omega_{\mathrm{II}} + \omega_{\mathrm{I}}) = 2kd^2/\mathrm{I}$$

and then approximate

$$\omega_B (2\omega_0) = 2kd^2/\mathrm{I}$$

which becomes

$$\omega_B = kd^2/\mathrm{I}\omega_0 \qquad (10)$$

or in terms of the frequency

$$f_B = kd^2/4\pi^2 \mathrm{I} f_0 \qquad (11)$$

To arrive at Eqs. (10) and (11), we used the definition $\omega_{\mathrm{II}} - \omega_{\mathrm{I}} = \omega_B$. We also used the approximation that $\omega_{\mathrm{II}} + \omega_{\mathrm{I}} = 2\omega_0$. Next, let's examine this approximation in more detail.

The definition $\omega_B = \omega_{\mathrm{II}} - \omega_{\mathrm{I}}$ implies that we can write

$$\omega_{\mathrm{II}} + \omega_{\mathrm{I}} = (\omega_B + \omega_{\mathrm{I}}) + \omega_{\mathrm{I}}$$

or that

$$\omega_{\mathrm{II}} + \omega_{\mathrm{I}} = 2\omega_{\mathrm{I}} + \omega_B \qquad (12)$$

Therefore, our approximation that $\omega_{\mathrm{II}} + \omega_{\mathrm{I}} = 2\omega_0$ in effect is that ω_B is negligible with respect to $2\omega_0$. We still use the fact that $\omega_{\mathrm{I}} = \omega_0$. We can use Eq. (12) to find a more accurate expression for ω_B (and f_B). Substituting Eq. (12) into

$$(\omega_{\mathrm{II}} - \omega_{\mathrm{I}})(\omega_{\mathrm{II}} + \omega_{\mathrm{I}}) = 2kd^2/\mathrm{I}$$

we get

$$\omega_B (2\omega_{\mathrm{I}} + \omega_B) = 2kd^2/\mathrm{I}$$

Simplifying results in

$$\omega_B{}^2 + 2\omega_{\mathrm{I}}\omega_B - 2kd^2/\mathrm{I} = 0 \qquad (13)$$

In terms of the frequency

$$f_B{}^2 + 2 f_0 f_B - kd^2/2\pi^2 \mathrm{I} = 0 \qquad (14)$$

Given the values for k, d, f_0, and I, Eq. (14) can be solved using the quadratic formula, giving a prediction for the beat frequency, f_B. For many applications, the value of f_B when squared is negligible, and Eq. (14) can usefully be approximated by Eq. (11).