SIGNALS AND SYSTEMS USING MATLAB
Chapter 9 — Discrete-time Signals and Systems

Luis F. Chaparro

A discrete-time signal $x[n]$ is a function of an integer sample index n:

$$\boxed{\begin{array}{rl} x[.] : \mathcal{I} \to \mathcal{R} & (\mathcal{C}) \\ n & x[n] \end{array}}$$

Example: Continuous–time signal

$$x(t) = 3\cos(2\pi t + \pi/4), \quad -\infty < t < \infty$$

Sampling: Nyquist sampling rate condition

$$T_s \leq \frac{\pi}{\Omega_{max}} = \frac{\pi}{2\pi} = 0.5 \text{ sec/sample}$$

For $T_s = 0.5$ sec/sample we obtain

$$x[n] = 3\cos(2\pi t + \pi/4)|_{t=0.5n} = 3\cos(\pi n + \pi/4) \qquad -\infty < n < \infty$$

a function of the integer n

- Signal $x[n]$ is periodic if
  - defined in $-\infty < n < \infty$, and
  - there is integer $N > 0$, the fundamental period of $x[n]$ such that

$$x[n + kN] = x[n] \qquad \text{for any integer } k$$

- Aperiodic signal does not satisfy one or both of the above conditions
- Periodic discrete-time sinusoids, of fundamental period N:

$$x[n] = A\cos\left(\frac{2\pi m}{N} n + \theta\right) \qquad -\infty < n < \infty$$

Example: Continuous–time vs discrete–time sinusoids

$$x(t) = \cos(t + \pi/4), \quad -\infty < t < \infty, \text{ periodic of fundamental period } T_0 = 2\pi$$
$$\text{Nyquist condition } \ (i)\ T_s \leq \frac{\pi}{\Omega_0} = \pi$$
$$\text{periodic sampled signal } x(t)|_{t=nT_s} = \cos(nT_s + \pi/4), \text{ fundamental period } N \text{ if}$$
$$\cos((n+N)T_s + \pi/4) = \cos(nT_s + \pi/4) \quad \Rightarrow \quad (ii)\ NT_s = 2k\pi$$

For sinusoid with fundamental period $N = 10$, then

$$(ii)\ T_s = k\pi/5, \text{ for } \ k \text{ satisfying}$$
$$(i)\ 0 < T_s = k\pi/5 \leq \pi \quad \text{ so that } \quad 0 < k \leq 5$$

$k = 1, 3$ so that $N = 10$, $\omega = 2\pi k/10$, k, N not divisible by each other

$k = 2$, 4 give $\omega = 2\pi/5, 2\pi 2/5$, 5 as fundamental period

$k = 5$ give 2 as fundamental period, $\omega = 2\pi/2$

- Sampling

$$x(t) = A\cos(\Omega_0 t + \theta) \qquad -\infty < t < \infty, \quad \text{fundamental period } T_0$$

results in periodic discrete sinusoid

$$x[n] = A\cos(\Omega_0 T_s n + \theta) = A\cos\left(\frac{2\pi T_s}{T_0} n + \theta\right)$$

provided that

$$(i)\ \frac{T_s}{T_0} = \frac{m}{N}, \quad (ii)\ T_s \leq \frac{\pi}{\Omega_0} = \frac{T_0}{2} \ \text{(Nyquist condition)}$$

- $z[n] = x[n] + y[n]$, periodic $x[n]$ with fundamental period N_1, periodic $y[n]$ with fundamental period N_2

  $z[n]$ is periodic if

$$\frac{N_2}{N_1} = \frac{p}{q} \quad p,\ q \text{ integers not divisible by each other}$$

  Fundamental period of $z[n]$ is $qN_2 = pN_1$

Finite–energy and finite–power signals

Discrete-time signal $x[n]$

$$\boxed{\begin{array}{ll} \text{Energy:} & \mathcal{E}_x = \displaystyle\sum_{n=-\infty}^{\infty} |x[n]|^2 \\ \text{Power:} & P_x = \displaystyle\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \end{array}}$$

- $x[n]$ is finite energy or square summable if $\mathcal{E}_x < \infty$
- $x[n]$ is absolutely summable if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- $x[n]$ is finite power if $P_x < \infty$

Example: "Causal" sinusoid

$$x(t) = \begin{cases} 2\cos(\Omega_0 t - \pi/4) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is sampled using $T_s = 0.1$ to obtain

$$x[n] = x(t)|_{t=0.1n} = 2\cos(0.1\Omega_0 n - \pi/4) \qquad n \geq 0$$

and zero otherwise

- $\Omega_0 = \pi$

$$x[n] \ = \ 2\cos(2\pi n/20 - \pi/4), \quad n \geq 0, \ 0 \text{ otherwise} \ \ \text{repeats every } N_0 = 20, \ \text{for } n \geq 0$$

$$P_x \ = \ \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{N}{2N+1} \underbrace{\left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2\right]}_{\text{power of period } n\geq 0} = \frac{1}{2N_0} \sum_{n=0}^{N_0-1} |x[n]|^2$$

$$= \ \frac{4}{40} 0.5 \left[\sum_{n=0}^{19} 1 + \sum_{n=0}^{19} \cos\left(\frac{2\pi n}{10} - \pi/2\right)\right] = \frac{2}{40}[20+0] = 1$$

- $\Omega_0 = 3.2$ rad/sec (an upper approximation of π), $x[n]$ does not repeat periodically after $n = 0$, frequency $3.2/10 \neq 2\pi m/N$ for integers m and N

$$P_x \ = \ \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Discrete-time signal $x[n]$ is

- delayed by N (an integer) samples if $x[n-N]$ is $x[n]$ shifted to the right N samples,
- advanced by M(an integer) samples if $x[n+M]$ is $x[n]$ shifted to the left M samples,
- reflected if the variable n in $x[n]$ is negated, i.e., $x[-n]$.

$$\boxed{\begin{array}{l} x[n] \ \text{ is even if } x[n] = x[-n] \\ x[n] \ \text{ is odd if } x[n] = -x[-n] \end{array}}$$

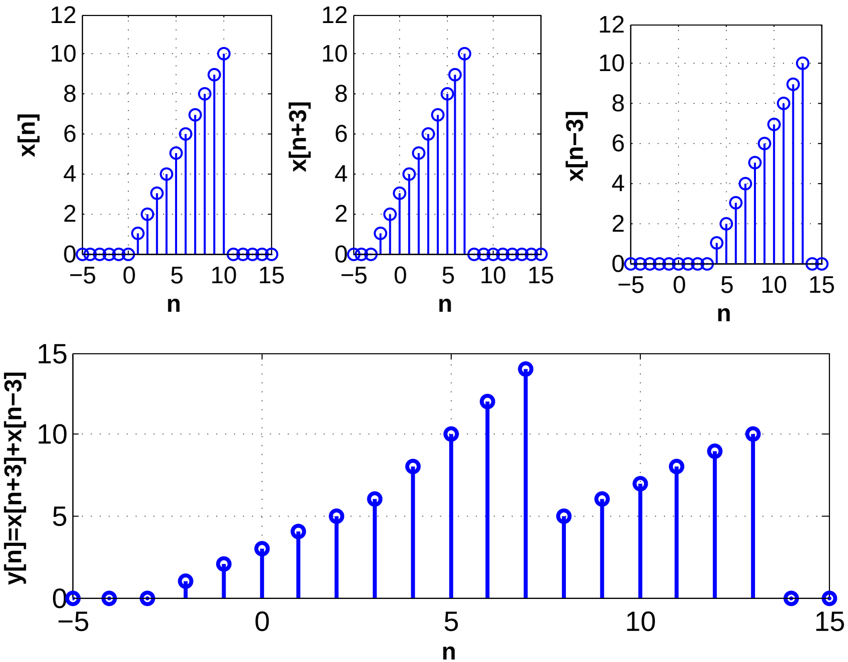
Any signal $x[n]$ can be represented as

$$\boxed{\begin{aligned} x[n] &= \underbrace{\frac{1}{2}\left(x[n] + x[-n]\right)}_{x_e[n]} + \underbrace{\frac{1}{2}\left(x[n] - x[-n]\right)}_{x_o[n]} \\ &= x_e[n] + x_o[n] \end{aligned}}$$

Example: Triangular discrete pulse

$$x[n] = \begin{cases} n & 0 \leq n \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

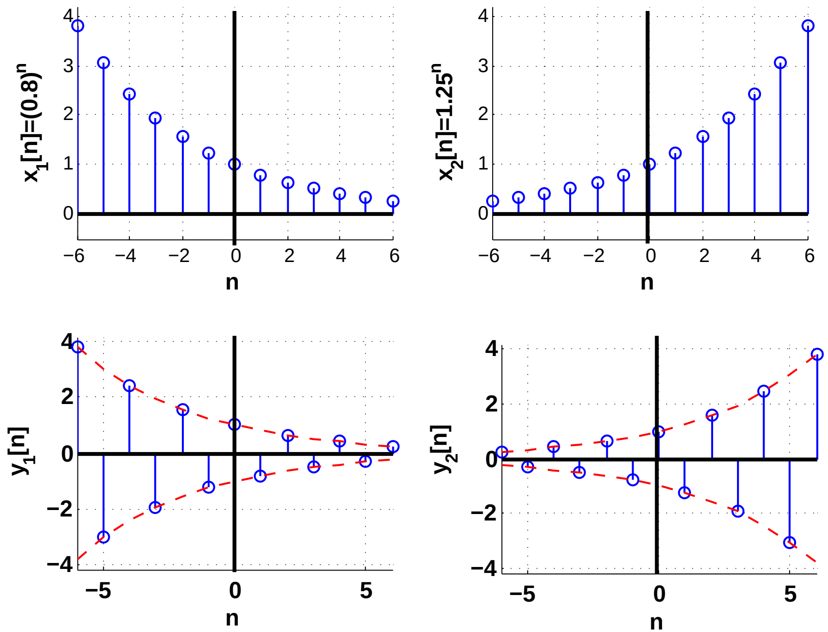
$$y[n] = x[n+3] + x[n-3] = \begin{cases} n+3 & -3 \leq n \leq 2 \\ 2n & 3 \leq n \leq 7 \\ n-3 & 8 \leq n \leq 13 \\ 0 & \text{otherwise} \end{cases}$$



Basic discrete-time signals

- Complex exponential

$$x[n] = |A|e^{j\theta}(|\alpha|e^{j\omega_0})^n = |A||\alpha|^n e^{j(\omega_0 n+\theta)}$$
$$= |A||\alpha|^n\left[\cos(\omega_0 n+\theta)+j\sin(\omega_0 n+\theta)\right] \quad \omega_0 : \text{ discrete frequency in radians}$$



Real exponential $x_1[n] = 0.8^n$, $x_2[n] = 1.25^n$ *(top) and modulated* $y_1[n] = x_1[n]\cos(\pi n)$ *and* $y_2[n] = x_2[n]\cos(\pi n)$

Example: $x(t) = e^{-at}\cos(\Omega_0 t)u(t)$, determine $a > 0$, Ω_0 and T_s to get

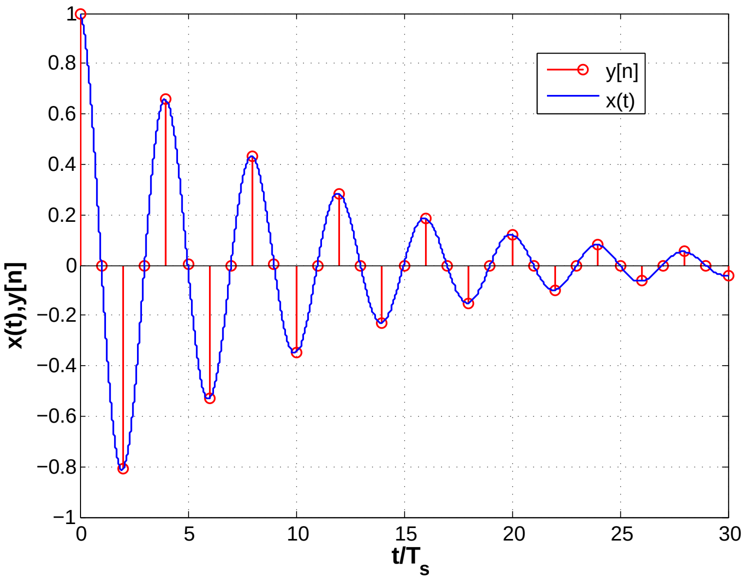
$$y[n] = 0.9^n \cos(\pi n/2) \qquad n \geq 0 \text{ and zero otherwise}$$

$$(i)\ 0.9 = e^{-aT_s},\ (ii)\ \pi/2 = \Omega_0 T_s$$

$$(iii)\ T_s \leq \frac{\pi}{\Omega_{max}}, \quad \Omega_{max} = N\Omega_0, \quad N \geq 2 \ \ x(t) \text{ not band-limited}$$

$$N = 2 \quad \Rightarrow \quad T_s = 0.25, \quad \Omega_0 = 2\pi$$

$$0.9 = e^{-a/4} \quad \Rightarrow \quad a = -4\log 0.9$$

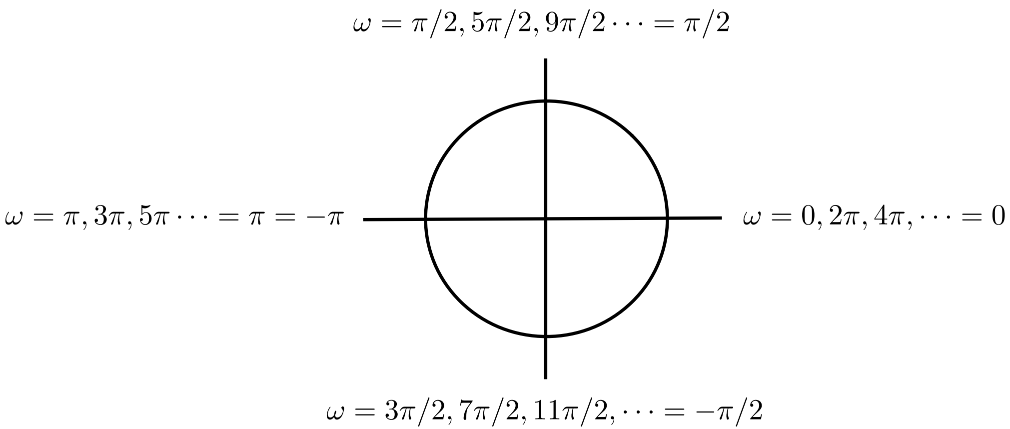


Discrete-time sinusoids

Special case of complex exponential

$$x[n] = |A|e^{j(\omega_0 n+\theta)} = |A|\cos(\omega_0 n + \theta) + j|A|\sin(\omega_0 n + \theta)$$

- Periodic if $w_0 = 2\pi m/N$ (rad), integers m and $N > 0$ not divisible
- ω (radians) repeats periodically with 2π as fundamental period



- To avoid this ambiguity let $-\pi < \omega \leq \pi$

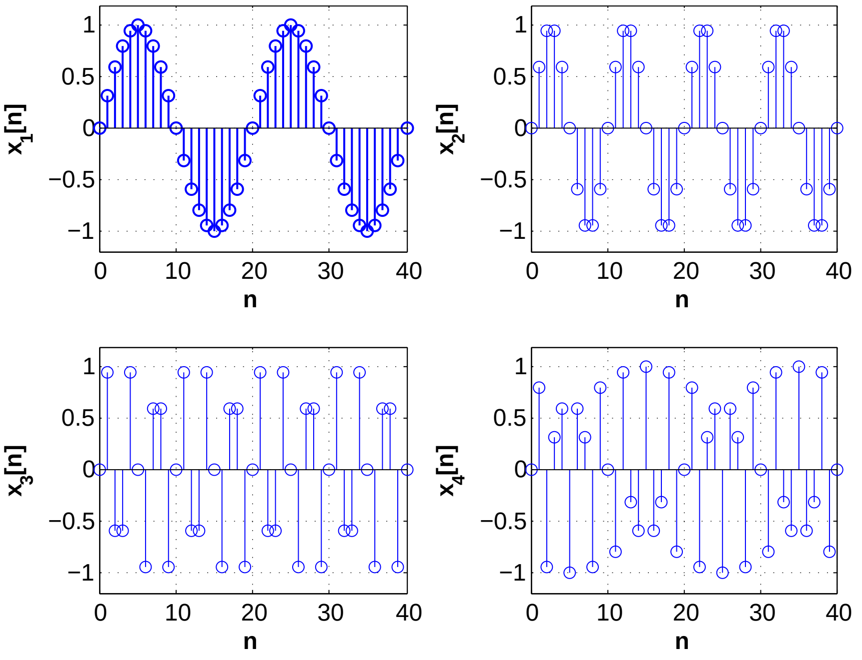
Example: Signal $\sin(3\pi n) = \sin(\pi n)$; $\sin(1.5\pi n) = \sin(-0.5\pi n) = -\sin(0.5\pi n)$

Example: Consider the four sinusoids

$$x_1[n] = \sin(0.1\pi n) = \sin\left(\frac{2\pi}{20}n\right), \quad x_2[n] = \sin(0.2\pi n) = \sin\left(\frac{2\pi}{20}2n\right),$$

$$x_3[n] = \sin(0.6\pi n) = \sin\left(\frac{2\pi}{20}6n\right), \quad x_4[n] = \sin(0.7\pi n) = \sin\left(\frac{2\pi}{20}7n\right)$$

periodic of fundamental periods 20



Discrete-time unit-step and unit-sample signals

- Definitions

$$\boxed{\begin{aligned} \text{Unit–step} \quad u[n] &= \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \\ \text{Unit–sample} \quad \delta[n] &= \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}}$$

- Connection

$$\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m] \end{aligned}$$

- Generic representation of discrete-time signals

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}$$

Dynamic systems $\mathcal{S}\{.\}$

$$\boxed{\boxed{y[n] = \mathcal{S}\{x[n]\}}}$$

- Linear
- Time-invariant
- Stable
- Causal

System $\mathcal{S}$ is

- Linear: for inputs $x[n]$ and $v[n]$, and constants a and b, superposition applies

$$\mathcal{S}\{ax[n] + bv[n]\} = a\mathcal{S}\{x[n]\} + b\mathcal{S}\{v[n]\}$$

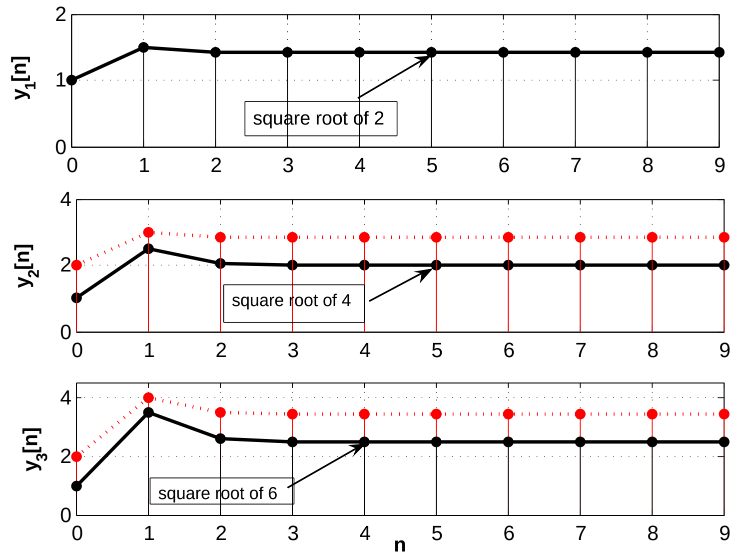
- Time-invariant:

$$\begin{array}{rcl} \text{input } x[n] & \rightarrow & \text{output } y[n] = \mathcal{S}\{x[n]\} \\ x[n \pm M] & \rightarrow & y[n \pm M] = \mathcal{S}\{x[n \pm M]\} \end{array}$$

Example: A square-root computation System

$$y[n] = 0.5\left[y[n-1] + \frac{\alpha}{y[n-1]}\right] \qquad n > 0$$

$$y[1] = 0.5\left[y[0] + \frac{\alpha}{y[0]}\right], \quad y[2] = 0.5\left[y[1] + \frac{\alpha}{y[1]}\right], \quad y[3] = 0.5\left[y[2] + \frac{\alpha}{y[2]}\right], \quad \cdots$$



Non-linear system: square root of 2 (top); square root of 4 compared with twice the square root of 2 (middle); sum of previous responses compared with response of square root of $2+4$ (bottom)

- Recursive/infinite impulse response (IIR) system

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m] \qquad n \geq 0 \ \text{ initial conditions } \ y[-k],\ k = 1, \cdot$$

- Non-recursive/finite impulse response (FIR) system

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m]$$

Example: Moving-average (MA) discrete system

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]), \ \text{ input } x[n], \ \text{output } y[n]$$

- Linearity: system is linear

$$\{x_i[n], \ \ i = 1, 2\} \ \ \rightarrow \ \{y_i[n], \ i = 1, 2\}$$

$$ax_1[n] + bx_2[n] = \frac{1}{3}[(ax_1[n] + bx_2[n]) + (ax_1[n-1] + bx_2[n-1]) + (ax_1[n-2] + bx_2[n$$

- Time invariance system is time–invariant

$$\text{input } x_1[n] = x[n-N] \ \rightarrow \ \frac{1}{3}(x[n-N] + x[n-N-1] + x[n-N-2]) = y[n-N]$$

Example: Autoregressive moving average (ARMA) system

$$y[n] = 0.5y[n-1] + x[n] + x[n-1] \qquad n \geq 0,\ y[-1]$$

Initial condition $y[-1] = -2$, and the input $x[n] = u[n]$, recursively

$$\begin{aligned} &y[0] = 0.5y[-1] + x[0] + x[-1] = 0, \quad y[1] = 0.5y[0] + x[1] + x[0] = 2, \\ &y[2] = 0.5y[1] + x[2] + x[1] = 3, \quad \cdots \end{aligned}$$

Initial condition $y[-1] = -2$, and input $x[n] = 2u[n]$ (doubled), response

$$\begin{aligned} &y_1[0] = 0.5y_1[-1] + x[0] + x[-1] = 1, \quad y_1[1] = 0.5y_1[0] + x[1] + x[0] = 4.5, \\ &y_1[2] = 0.5y_1[1] + x[2] + x[1] = 6.25, \quad \cdots \end{aligned}$$

$$y_1[n] \neq 2y[n] \ \ \text{(system non–linear)}$$

If initial condition $y[-1] = 0$ the system is linear

Steady–state: if $x[n] = u[n]$, any $y[-1]$, assuming as $n \to \infty$ $Y = y[n] = y[n-1]$ and since $x[n] = x[n-1] = 1$, then

$$Y = 0.5Y + 2 \quad \text{or} \quad Y = 4 \ \ \text{independent of IC}$$

$h[n]$ impulse response of LTI system: input $\delta[n]$, zero IC

generic representation $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

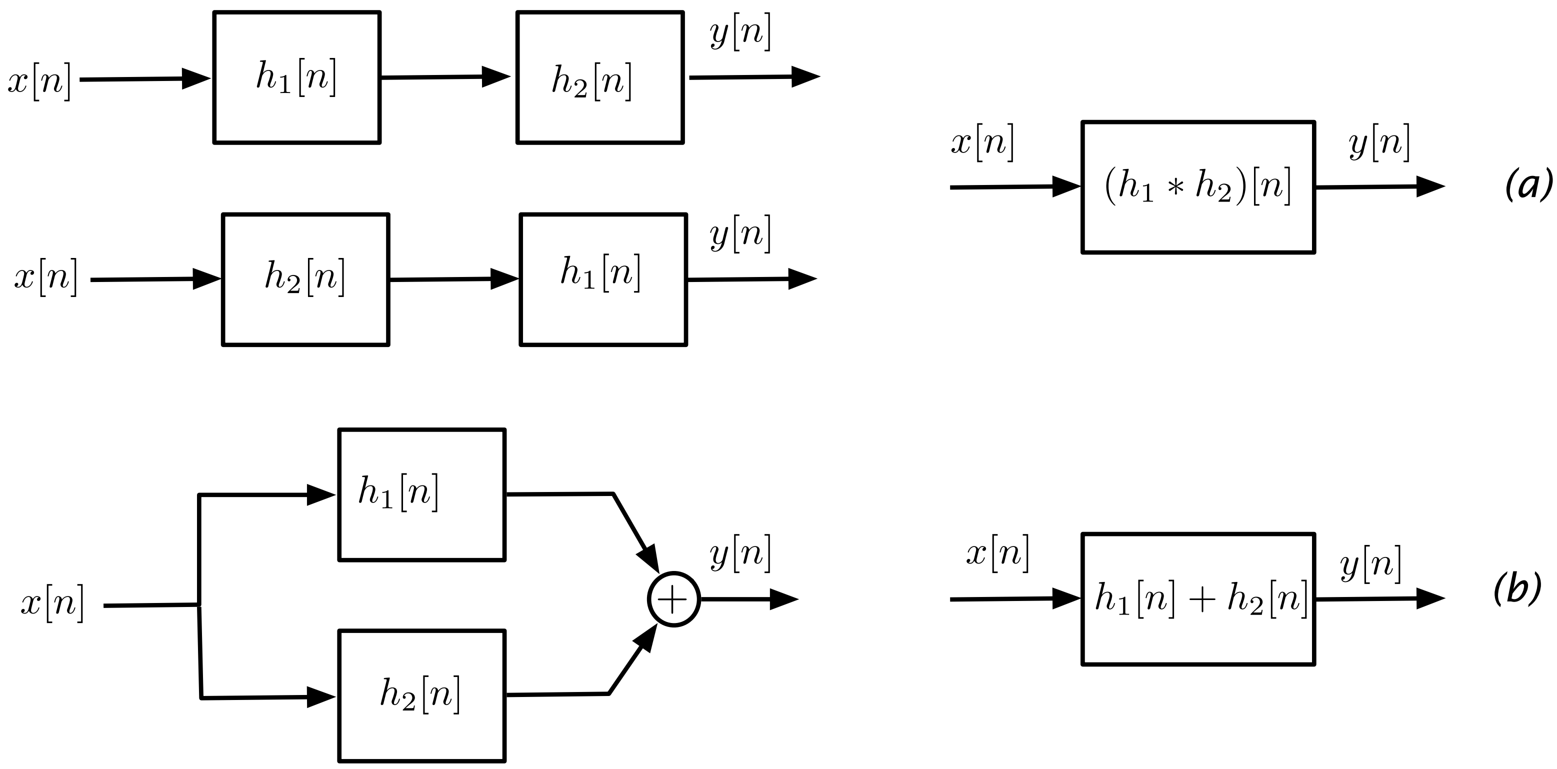
output $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} x[n-m]h[m]$ convolution sum

- Non-recursive or FIR system: output obtained by convolution sum

$$y[n] = \sum_{k=0}^{N-1} b_k x[n-k]$$

$$h[n] = \sum_{k=0}^{N-1} b_k \delta[n-k] = b_0\delta[n] + b_1\delta[n-1] + \cdots + b_{N-1}\delta[n-(N-1)]$$

$$h[n] = b_n, \quad n = 0, \cdots, N-1 \quad \Rightarrow \quad y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

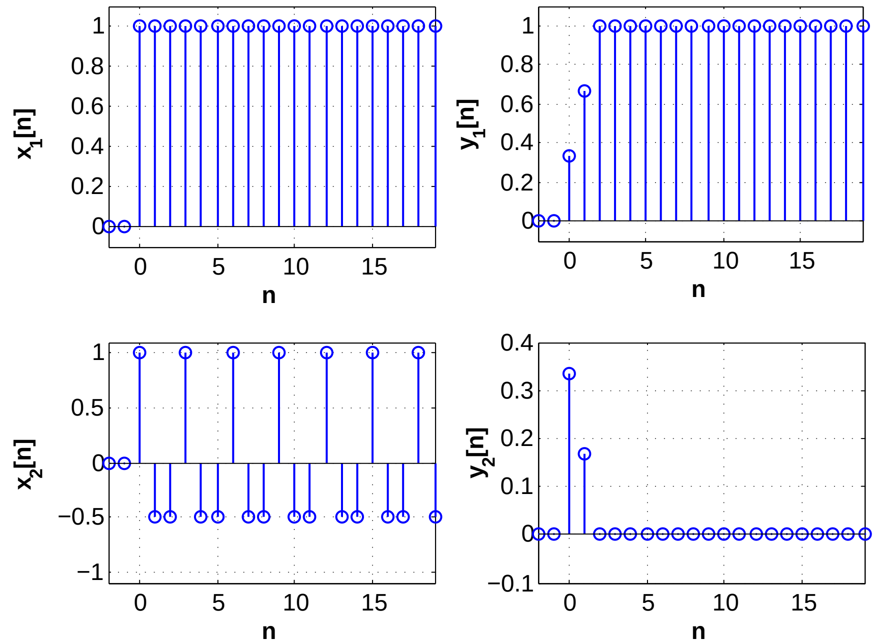


Cascade (a) and parallel (b) connections of LTI systems with impulse responses $h_1[n]$ and $h_2[n]$. Equivalent systems on the right. Notice the interchange of systems in the cascade connection.

Example: FIR system

$$\text{I/O equation} \quad y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

$$\text{impulse response} \quad h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2]) \quad \Rightarrow \quad y[n] = (h * x)[n]$$



Convolution sums for a moving averaging system $y[n] = (x[n] + x[n-1] + x[n-2])/3$ *with inputs* $x_1[n] = u[n]$ *(top) and* $x_2[n] = \cos(2\pi n/3)u[n]$ *(bottom) .*

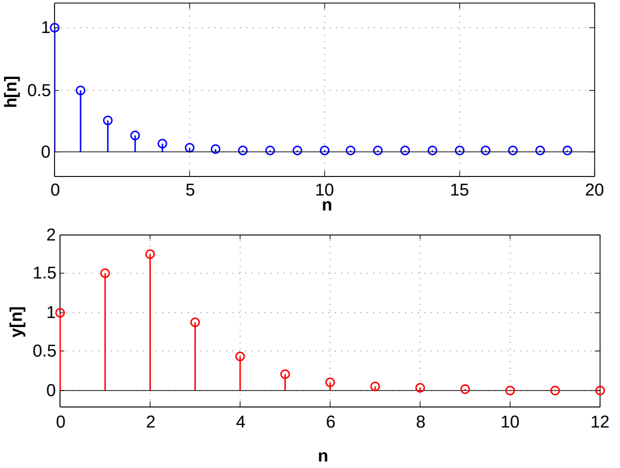
Example: Autoregressive system

$$y[n] = 0.5y[n-1] + x[n] \qquad n \geq 0 \text{ first order difference equation}$$

$$\text{impulse response } x[n] = \delta[n],\ y[n] = h[n], \text{initial condition } y[-1] = h[-1] = 0$$
$$h[0] = 0.5h[-1] + \delta[0] = 1, \qquad h[1] = 0.5h[0] + \delta[1] = 0.5,$$
$$h[2] = 0.5h[1] + \delta[2] = 0.5^2, \qquad h[3] = 0.5h[2] + \delta[3] = 0.5^3, \cdots \quad h[n] = 0.5^n$$

Input $x[n] = u[n] - u[n-3]$ using convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} (u[k] - u[k-3])0.5^{n-k}u[n-k] = \begin{cases} 0 & n < 0 \\ 0.5^n(2^{n+1} - 1) & n = 0, 1, 2 \\ 7(0.5^n) & n \geq 3. \end{cases}$$



Impulse response $h[n]$ (top), and response $y[n]$ due to $x[n] = u[n] - u[n-3]$ (bottom)

Causality and BIBO stability

- Causality: System $\mathcal{S}$ is causal if:
  - input $x[n] = 0$, and no initial conditions, the output is $y[n] = 0$,
  - present output $y[n]$ does not depend on future inputs
- BIBO stability:for bounded $x[n]$, $|x[n]| < M < \infty$, the output of BIBO stable system $y[n]$ is also bounded, $|y[n]| < L < \infty$, for all n.
- LTI systems:
  - Causality: $h[n] = 0$ for $n < 0$
  - BIBO stability

$$\sum_k |h[k]| < \infty, \qquad \text{absolutely summable}$$

Example:

- Causal non–linear time–invariant system

$$y[n] = x^2[n],$$

- Non–causal LTI system

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1]).$$

Example: Deconvolution

Assume input $x[n]$ and output $y[n]$ of a causal LTI system are given, find impulse response $h[n]$ of the system

$$y[n] = \sum_{m=0}^{n} h[n-m]x[m] = h[n]x[0] + \sum_{m=1}^{n} h[n-m]x[m]$$

$$h[n] = \frac{1}{x[0]}\left[y[n] - \sum_{m=1}^{n} h[n-m]x[m]\right], \quad x[0] \neq 0$$

$$h[0] = \frac{1}{x[0]}y[0], \quad h[1] = \frac{1}{x[0]}\left(y[1] - h[0]x[1]\right), \quad h[2] = \frac{1}{x[0]}\left(y[2] - h[0]x[2] - h[1]x[1]\right) \quad \cdots$$

Let $y[n] = \delta[n], x[n] = u[n]$:

$h[0] = 1, \quad h[1] = -1, \quad h[2] = 0, \quad h[3] = 0, \quad \cdots$

$h[n] = \delta[n] - \delta[n-1]$ length of $h[n] =$ (length of $y[n]$) $-$ (length of $x[n]$) $+ 1$

Example: Autoregressive system

$$y[n] = 0.5y[n-1] + x[n]$$

Impulse response

$$\begin{aligned}
h[0] &= 0.5h[-1] + \delta[0] = 1, \\
h[1] &= 0.5h[0] + \delta[1] = 0.5, \\
h[2] &= 0.5h[1] + \delta[2] = 0.5^2 \\
h[3] &= 0.5h[2] + \delta[3] = 0.5^3, \quad \cdots \\
&\Rightarrow h[n] = 0.5^n u[n] \\
\sum_{n=-\infty}^{\infty} |h[n]| &= \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2
\end{aligned}$$

$h[n]$ is absolutely summable, so system is BIBO stable