Signals and Systems Using MATLAB

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Chapter 4 - Frequency Analysis

The Fourier Series

What is in this chapter?

- Eigenfunctions and LTI systems
- Complex and trigonometric Fourier Series
- Line spectrum: distribution of power over frequency
- Laplace and Fourier Series
- Properties of Fourier Series
- Response of LTI systems to periodic signals

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Eigenfunctions revisited

$x(t) = e^{j\Omega_0 t}$, $-\infty < t < \infty$, input to causal, stable LTI system with impulse response $h(t)$, output in steady state is

$$y(t) = \int_0^\infty h(\tau)x(t-\tau)d\tau = e^{j\Omega_0 t} \underbrace{\int_0^\infty h(\tau)e^{-j\Omega_0 \tau}d\tau}_{H(j\Omega_0)} = e^{j\Omega_0 t}H(j\Omega_0)$$

$$\textit{frequency response of the system at } \Omega_0 : \quad H(j\Omega_0) = \int_0^\infty h(\tau)e^{-j\Omega_0\tau}d\tau$$

$x(t) = e^{j\Omega_0 t}$ is eigenfunction of the LTI system as it appears at both input and output.

Generalization

- Periodic

$$x(t) = \sum_k X_k e^{j\Omega_k t} \quad \Rightarrow y(t) \quad = \quad \sum_k X_k e^{j\Omega_k t} H(j\Omega_k)$$

- Aperiodic

$$x(t) = \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \quad \Rightarrow \quad y(t) = \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}H(j\Omega)d\Omega$$

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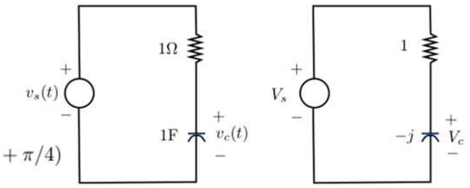
Eigenfunction property of a stable LTI system:
For a stable LTI with transfer function $H(s)$:

Input $x(t) = A\cos(\Omega_0 t + \theta)$
steady state output $y(t) = A|H(j\Omega_0)|\cos(\Omega_0 t + \theta + \angle H(j\Omega_0))$
$H(j\Omega_0) = H(s)|_{s=j\Omega_0}$

Example Find steady-state voltage across capacitor in RC circuit



$v_s(t) = 4\cos(t + \pi/4)$

Phasor Approach: $V_s = 4\angle\pi/4$ phasor for $v_s(t)$, V_c phasor for $v_c(t)$

$$\text{voltage division: } \frac{V_c}{V_s} = \frac{-j}{1-j} = \frac{-j(1+j)}{2} = \frac{\sqrt{2}}{2}\angle -\pi/4$$

$$V_c = 2\sqrt{2}\angle 0 \quad \Rightarrow \quad \text{steady state response } \; v_c(t) = 2\sqrt{2}\cos(t)$$

Eigenfunction Approach.

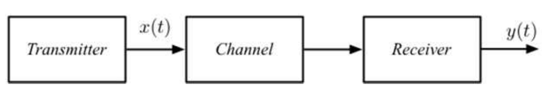
$$H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1/s}{1+1/s} = \frac{1}{s+1}$$

$$\text{frequency response at } \Omega_0 = 1 \quad H(j1) = \frac{\sqrt{2}}{2}\angle -\pi/4$$

eigenfunction property: in steady-state

$$v_c(t) = 4|H(j1)|\cos(t + \pi/4 + \angle H(j1)) = 2\sqrt{2}\cos(t)$$

Example Ideal communication system $y(t) = x(t - \tau)$



Impulse response: $h(t) = \delta(t - \tau)$, τ delay of the transmission

$$\text{output: } \; y(t) = \int_0^\infty \underbrace{\delta(\rho - \tau)}_{h(\rho)} x(t-\rho)d\rho = x(t - \tau)$$

Frequency response of the ideal communication system

$$\text{input } \; x(t) = e^{j\Omega_0 t} \quad \Rightarrow \quad \text{output} y(t) = e^{j\Omega_0 t}H(j\Omega_0)$$

but also

$$y(t) = x(t - \tau) = e^{j\Omega_0(t-\tau)}$$

so frequency response at Ω_0:

$$H(j\Omega_0) = 1e^{-j\tau\Omega_0}$$

Complex Exponential Fourier Series

- Complex functions $\{\psi_k(t)\}$, $t \in [a,b]$, are orthonormal (orthogonal and normalized) if for $\psi_\ell(t)$, $\psi_m(t)$, $\ell \neq m$, inner product

$$\int_a^b \psi_\ell(t)\psi_m^*(t)dt = \begin{cases} 0 & \ell \neq m \\ 1 & \ell = m. \end{cases}$$

- Finite energy $x(t)$, $t \in [a,b]$ approximated by

$$\hat{x}(t) = \sum_k a_k \psi_k(t)$$

by minimizing energy of the error function

$$\int_a^b |\varepsilon(t)|^2 dt = \int_a^b \left| x(t) - \sum_k a_k \psi_k(t) \right|^2 dt$$

with respect to coefficients $\{a_k\}$.

- Fourier proposed sinusoids as the functions $\{\psi_k(t)\}$ to represent periodic signals
- A periodic signal $x(t)$
  - is defined for $-\infty < t < \infty$,
  - for integer k, $x(t+kT_0) = x(t)$, where T_0 is the **fundamental period**

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The **Fourier Series representation** *of a periodic signal $x(t)$, of period T_0, is given by*

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \qquad \Omega_0 = \frac{2\pi}{T_0} \qquad \textit{fundamental frequency: } \Omega_0 = 2\pi/T_0 (rad/sec)$$

where Fourier coefficients X_k are

$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}dt \qquad k = 0, \pm 1, \pm 2, \cdots, \quad \textit{any } t_0$$

Information needed for the Fourier series obtained from any period of $x(t)$

Line Spectra

- Fourier Series determines frequency components of periodic signal and how power is distributed over the different frequencies present in the signal
- Power spectrum computed and displayed by **spectrum analyzer**

Parseval's Theorem –Power Distribution over Frequency
Power P_x of periodic $x(t)$, of period T_0, is

$$P_x = \underbrace{\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt}_{\textit{in time domain}} = \underbrace{\sum_k |X_k|^2}_{\textit{in frequency domain}}$$

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Periodic $x(t)$ is represented in the frequency by its

Magnitude line spectrum $|X_k|$ *vs* $k\Omega_0$

Phase line spectrum $\angle X_k$ *vs* $k\Omega_0$

Power line spectrum, $|X_k|^2$ *vs.* $k\Omega_0$ *of* $x(t)$ *displays distribution of the power of the signal over frequency*

Symmetry of Line Spectra

Real-valued periodic signal $x(t)$, *of period* T_0, *with Fourier coefficients* $\{X_k = |X_k|e^{j\angle X_k}\}$ *at harmonic frequencies* $\{k\Omega_0 = 2\pi k/T_0\}$:

(i) $|X_k| = |X_{-k}|$, *i.e., magnitude* $|X_k|$ *is even function of* $k\Omega_0$

(ii) $\angle X_k = -\angle X_{-k}$ *i.e., phase* $\angle X_k$ *is odd function of* $k\Omega_0$

For real-valued signals display $k \geq 0$

Magnitude line spectrum: *plot of* $|X_k|$ *vs* $k\Omega_0$

Phase line spectrum: *plot of* $\angle X_k$ *vs* $k\Omega_0$

Trigonometric Fourier Series *A real-valued, periodic signal* $x(t)$, *of period* T_0, *is equivalently represented by*

$$\begin{aligned} x(t) &= X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \theta_k) \\ &= \underbrace{c_0}_{dc} + \sum_{k=1}^{\infty} \underbrace{2[c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)]}_{k^{th}-harmonic} \qquad \Omega_0 = \frac{2\pi}{T_0} \\ c_k &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\Omega_0 t)\, dt \qquad k = 0, 1, \cdots \\ d_k &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\Omega_0 t)\, dt \qquad k = 1, 2, \cdots \end{aligned}$$

The coefficients $X_k = |X_k|e^{j\theta_k}$ *are connected with the coefficients* c_k *and* d_k *by*

$$|X_k| = \sqrt{c_k^2 + d_k^2}$$

$$\theta_k = -\tan^{-1}\left[\frac{d_k}{c_k}\right]$$

The functions $\{\cos(k\Omega_0 t),\ \sin(k\Omega_0 t)\}$ *are orthonormal.*

Example Find FS of raised cosine signal ($B \geq A$),

$$x(t) = B + A\cos(\Omega_0 t + \theta)$$

periodic of period T_0 and fundamental frequency $\Omega_0 = 2\pi/T_0$. For $y(t) = 1 + \sin(100t)$ use symbolic MATLAB.
Using Euler's identity

$$\begin{aligned} x(t) &= B + \frac{A}{2}\left[e^{j(\Omega_0 t+\theta)} + e^{-j(\Omega_0 t+\theta)}\right] \\ &= B + \frac{Ae^{j\theta}}{2}e^{j\Omega_0 t} + \frac{Ae^{-j\theta}}{2}e^{-j\Omega_0 t} \end{aligned}$$

which gives

$$\begin{aligned} X_0 &= B \\ X_1 &= \frac{Ae^{j\theta}}{2} \\ X_{-1} &= X_1^* \end{aligned}$$

If $\theta = -\pi/2$

$$y(t) = B + A\sin(\Omega_0 t)$$

Fourier series coefficients $Y_0 = B$ and $Y_1 = Ae^{-j\pi/2}/2$ so that $|Y_1| = |Y_{-1}| = A/2$ and $\angle Y_1 = -\angle Y_{-1} = -\pi/2$

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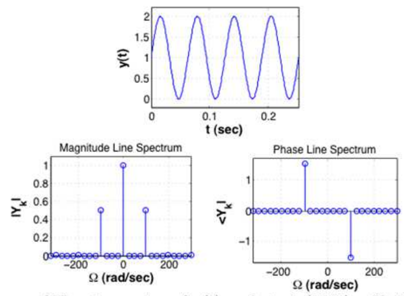

Line spectrum of Fourier series of $y(t) = 1 + \sin(100t)$. Notice the even and the odd symmetries of the magnitude and phase spectra. The phase is $-\pi/2$ at $\Omega = 100$ (rad/sec).

```
function [X, w]=fourierseries(x,T0,N)
%%%%%
% symbolic Fourier Series computation
% x: periodic signal
% T0: period
% N: number of harmonics
% X,w: Fourier series coefficients at harmonic frequencies
%%%%%
syms t
% computation of N Fourier series coefficients
for k=1:N,
    X1(k)=int(x*exp(-j*2*pi*(k-1)*t/T0),t,0,T0)/T0;
    X(k)=subs(X1(k));
    w(k)=(k-1)*2*pi/T0; % harmonic frequencies
end
```

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Fourier Coefficients from Laplace
For periodic $x(t)$, of period T_0, if

$$\textit{period of } x(t): \quad x_1(t) = x(t)[u(t_0) - u(t - t_0 - T_0)] \qquad \textit{for any } t_0$$

Fourier coefficients of $x(t)$

$$X_k = \frac{1}{T_0}\mathcal{L}\left[x_1(t)\right]_{s=jk\Omega_0} \qquad \Omega_0 = \frac{2\pi}{T_0} \quad \textit{fundamental frequency}$$

Example Periodic pulse train $x(t)$, of period $T_0 = 1$. Find its Fourier series.
This signal

- has dc component of 1,
- $x(t) - 1$ (zero-average signal) is well represented by cosine so the Fourier coefficients will be real

Integral expression for Fourier coefficients:

$$X_k = \frac{1}{T_0}\int_{-1/4}^{3/4} x(t)e^{-j\Omega_0 kt}dt = \int_{-1/4}^{1/4} 2e^{-j2\pi kt}dt = \frac{2}{\pi k}\left[\frac{e^{j\pi k/2} - e^{-j\pi k/2}}{2j}\right] = \frac{\sin(\pi k/2)}{(\pi k/2)}$$

With the Laplace transform,

$$x_1(t) = x(t), \quad -0.5 \le t \le 0.5$$
$$x_1(t - 0.25) = 2[u(t) - u(t - 0.5)]$$

Laplace transform: $X_1(s) = (2/s)[e^{0.25s} - e^{-0.25s}] \quad \Rightarrow X_k = \dfrac{2}{jk\Omega_0 T_0} 2j\sin(k\Omega_0/4)$

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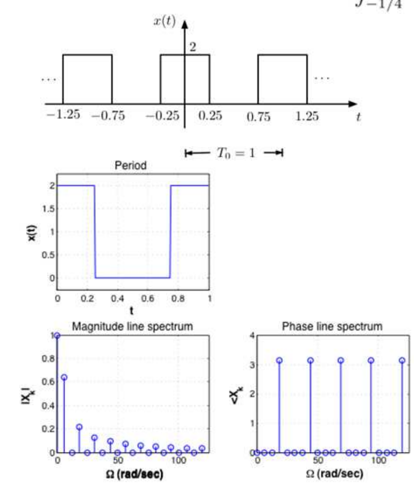
For $\Omega_0 = 2\pi$, $T_0 = 1$,

$$X_k = \frac{\sin(\pi k/2)}{\pi k/2} \qquad k \neq 0$$

$$\text{dc value or average } X_0 = \int_{-1/4}^{1/4} 2dt = 1$$



Period of train of rectangular pulses and its magnitude and phase line spectra.

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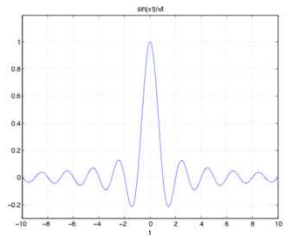
Notice:

1. X_k in terms of the **sinc function** $\sin(x)/x$ which is
   - even, i.e., $\sin(x)/x = \sin(-x)/(-x)$,
   - value at $x = 0$ found by f L'Hopital's rule

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{d\sin(x)/dx}{dx/dx} = 1$$

   - is bounded,

$$\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$



2. Zero-mean signal

$$x(t) - 1 = \sum_{k=-\infty, k\neq 0}^{\infty}\frac{\sin(\pi k/2)}{(\pi k/2)}e^{jk2\pi t} = 2\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{(\pi k/2)}\cos(2\pi kt)$$

3. In general, Fourier coefficients are complex and need to be represented by their magnitudes and phases. In this case, the X_k coefficients are real-valued.

4. dc value and 5 harmonics, provide a very good approximation of the pulse train

k	$X_k = X_{-k}$	X_k^2
0	1	1
1	0.64	0.41
2	0	0
3	− 0.21	0.041
4	0	0
5	0.13	0.016

Convergence of the Fourier Series

Fourier series of a piecewise smooth (continuous or discontinuous) periodic signal $x(t)$ converges for all values of t.

Dirichlet conditions: Fourier series converges to the periodic signal $x(t)$, if signal is

1. *absolutely integrable,*
2. *has a finite number of maxima, minima and discontinuities.*

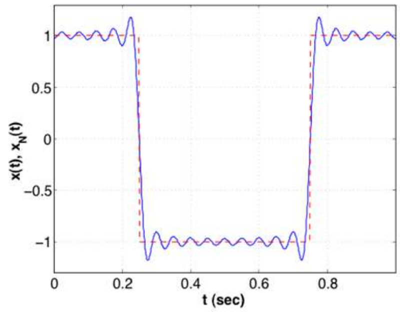
FS equals $x(t)$ at every continuity point and equals the average

$$0.5[x(t+0+) + x(t+0-)]$$

of the right-hand limit $x(t+0+)$ and the left-hand limit $x(t+0-)$ at every discontinuity point. If $x(t)$ is continuous everywhere, then the series converges absolutely and uniformly.

Example Gibb's phenomenon: approximate train of pulses $x(t)$ with zero mean and period $T_0 = 1$ with a Fourier series $x_N(t)$ with $N = 1, \cdots, 20$

- Discontinuities cause Gibb's phenomenon
- Even if N is increased, there is an overshoot around the discontinuities



Approximate Fourier series $x_N(t)$ of the pulse train $x(t)$ (discontinuous) using the dc component and 20 harmonics. The approximate $x_N(t)$ displays the Gibb's phenomenon around the discontinuities.

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Time and Frequency Shifting

Time-shifting: A periodic signal $x(t)$, of period T_0, remains periodic of the same period when shifted in time. If X_k are the Fourier coefficients of $x(t)$, the Fourier coefficients for $x(t-t_0)$ are

$$\{X_k e^{-jk\Omega_0 t_0} = |X_k| e^{j(\angle X_k - k\Omega_0 t_0)}\}$$

(only a change in phase is caused by the time shift) since

$$x(t) = \sum_k X_k e^{jk\Omega_0 t}$$

$$x(t-t_0) = \sum_k X_k e^{jk\Omega_0 (t-t_0)} = \sum_k \left[X_k e^{-jk\Omega_0 t_0}\right] e^{jk\Omega_0 t}$$

Frequency-shifting: A periodic signal $x(t)$, of period T_0, modulates a complex exponential $e^{j\Omega_1 t}$,

- *the modulated signal $x(t)e^{j\Omega_1 t}$ is periodic of period T_0 if $\Omega_1 = M\Omega_0$, for an integer $M \geq 1$,*
- *the Fourier coefficients X_k are shifted to frequencies $k\Omega_0 + \Omega_1$*
- *the modulated signal is real-valued by multiplying $x(t)$ by $\cos(\Omega_1 t)$.*

Application in communications

Example Modulate a sinusoid $\cos(20\pi t)$ with a train of square pulses

$$x_1(t) = 0.5[1 + sign(\sin(\pi t))]$$

and with a sinusoid

$$x_2(t) = \cos(\pi t)$$

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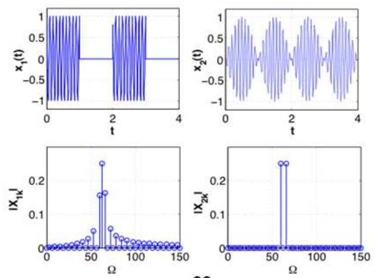
Use *fourierseries* to find FS of modulated signals and to plot their magnitude line spectra

$$\text{sign}(x(t)) = \begin{cases} -1 & x(t) < 0 \\ 1 & x(t) \geq 0 \end{cases}$$

i.e., it determines the sign of the signal

```
% Example 4.12 --- Modulation
%
syms t
T0=2;
m=heaviside(t)-heaviside(t-T0/2);
m1=heaviside(t)-heaviside(t-T0);
x=m*cos(20*pi*t);
x1=m1*cos(pi*t)*cos(20*pi*t);
[X,w]=fourierseries(x,T0,60);
[X1,w1]=fourierseries(x1,T0,60);
```



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Response of LTI Systems to Periodic Signals

Eigenfunction Property of LTI Systems: *Steady state response to a complex exponential (or a sinusoid) of a certain frequency is the same complex exponential (or sinusoid), but its amplitude and phase are affected by the frequency response of the system at that frequency.*

Steady State Response

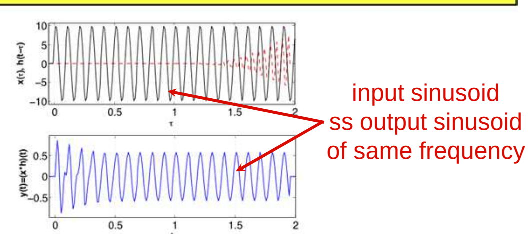
If input $x(t)$ of a causal and stable LTI system, with impulse response $h(t)$, is periodic of period T_0 and FS

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \angle X_k) \qquad \Omega_0 = \frac{2\pi}{T_0}$$

steady-state response of the system is

$$y(t) = X_0|H(j0)|\cos(\angle H(j0)) + 2\sum_{k=1}^{\infty} |X_k||H(jk\Omega_0)|\cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0))$$

$$H(jk\Omega_0) = \int_0^{\infty} h(\tau)e^{-jk\Omega_0\tau}d\tau \quad \textit{frequency response at} k\Omega_0$$



Convolution simulation. Top figure: input $x(t)$ (solid line) and $h(t-\tau)$ (dashed line); bottom figure: output $y(t)$ transient and steady-state response.

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- If $x(t)$ is a combination of sinusoids of frequencies not harmonically related, thus not periodic, the eigenfunction property still holds

$$x(t) = \sum_k A_k \cos(\Omega_k t + \theta_k) \;\Rightarrow\; y_{ss}(t) = \sum_k A_k |H(j\Omega_k)| \cos(\Omega_k t + \theta_k + \angle H(j\Omega_k))$$

- If LTI system is represented by a differential equation and the input is a sinusoid, or combination of sinusoids, it is not necessary to use the Laplace transform to obtain the complete response and then let $t \to \infty$ to find the sinusoidal steady-state response. Laplace transform only needed to find the transfer function of the system, which can then be used in steady state equation

Example A zero-mean pulse train

$$x(t) = \sum_{k=-\infty, \neq 0}^{\infty} \frac{\sin(k\pi/2)}{k\pi/2} e^{j2k\pi t}$$

is source of an RC circuit (a low-pass filter, i.e., a system that keeps the low-frequency harmonics and get rid of the high-frequency harmonics of the input)

$$H(s) = \frac{1}{1 + s/100}$$

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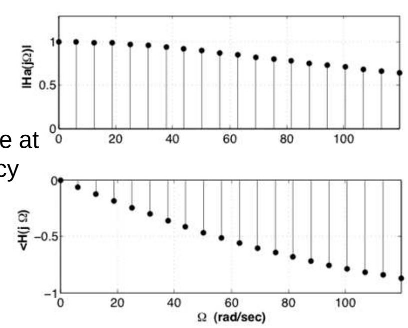
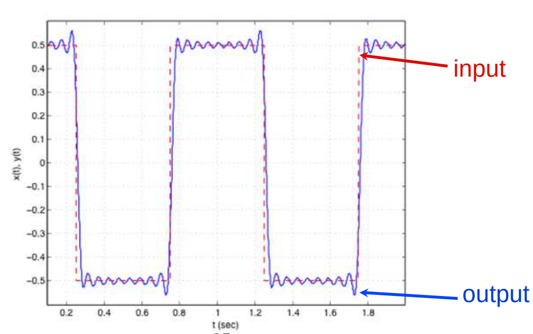
magnitude and phase at harmonic frequency





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What have we accomplished?

- Sinusoidal representation of periodic signals
- Eigenfunction property of LTI systems
- Response of LTI systems to periodic signals
  - Connection of Fourier series and Laplace transform
  - Inverse time frequency relation

Where do we go from here?

- Extension of Fourier representation for aperiodic signals
- Unification of spectral theory for periodic and aperiodic signals
- Convolution and frequency response of LTI systems
- Connection of Laplace and Fourier transforms

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