Department of Electrical and Computer Engineering
The University of Alabama in Huntsville

CPE 381: Fundamentals of Signals and Systems for Computer Engineers

Homework #3 Solution

1. (20 points) A system with input x(t) and output y(t) is defined by the following differential equation:

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Find the impulse response $h(t)$ and the unit-step response $s(t)$.

If $Y(s) = \mathcal{L}[y(t)]$ and $X(s) = \mathcal{L}[x(t)]$, then

$$Y(s)\,[s^2 + 3s + 2] = X(s)$$

To find impulse response, we let x(t) = (t), and X(s) =1, then Y(s) is equal to H(s):

$$Y(s) = H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

We find

$$A = H(s)(s+1)\big|_{s=-1} = \frac{1}{-1+2} = 1$$

and

$$B = H(s)(s+2)\big|_{s=-2} = \frac{1}{-2+1} = -1$$

therefore:

$$h(t) = \left[e^{-t} - e^{-2t}\right] \cdot u(t)$$

Similarly, unit step response is:

$$S(s) = \frac{H(s)}{s} = \frac{1}{s \cdot (s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

and A=0.5, B= -1, C=0.5, therefore:

$$s(t) = 0.5 \cdot u(t) - e^{-t} \cdot u(t) + 0.5e^{-2t} \cdot u(t)$$

2. (10 points)

The Laplace transform of the response is:

$$S(s) = H(s)X(s) = \frac{s}{s(s^2 + s + 1)} = \frac{1}{(s + 1/2)^2 + 3/4}$$

since (take a look at page 199)

$$\mathcal{L}\left[\mathrm{Ae}^{-\alpha t} \sin(\Omega_0 t \cdot u(t))\right] = \frac{\mathrm{A}\Omega_0}{(\mathrm{s} + \alpha)^2 + \Omega_0^2}$$

Therefore, the Inverse Laplace transform of the response is:

$$s(t) = \frac{2}{\sqrt{3}} e^{-0.5t} \sin\left(\sqrt{3}t/2\right)u(t)$$

$y_1(t) = s(t) - s(t-1)$

3. (10 points) Pr. 3.30

3.30. Feedback stabilization

An unstable system can be stabilized by using negative feedback with a gain K in the feedback loop. For instance, consider an unstable system with transfer function

$$H(s) = \frac{2}{s-1}$$

which has a pole in the right-hand s-plane, making the impulse response of the system $h(t)$ grow as t increases. Use negative feedback with a gain $K > 0$ in the feedback loop, and put $H(s)$ in the forward loop. Draw a block diagram of the system. Obtain the transfer function $G(s)$ of the feedback system and determine the value of K that makes the overall system BIBO stable (i.e., its poles in the open left-hand s-plane).

General solution:

$$Y(s) = \big(X(s) - G(s)\,Y(s)\big)F(s) = \frac{F(s)}{1 + F(s)G(s)}X(s)$$

$$H(s) = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

In this particular case system output is:

Y(s) = (X(s) – K Y(s)) H(s)

= X(s) H(s) – K H(s) Y(s)

and

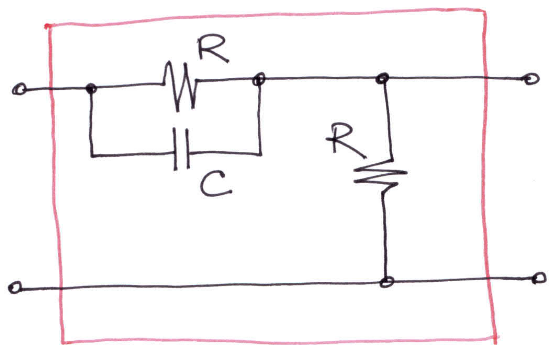
$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + K \cdot H(s)} = \frac{2}{s + 2K - 1}$$

In order to have the pole in the left-hand s-plane we need 2K – 1 > 0 → K > 0.5

For example, K = 1 → pole at s = -1 and impulse response

$$g(t) = 2e^{-t}u(t)$$

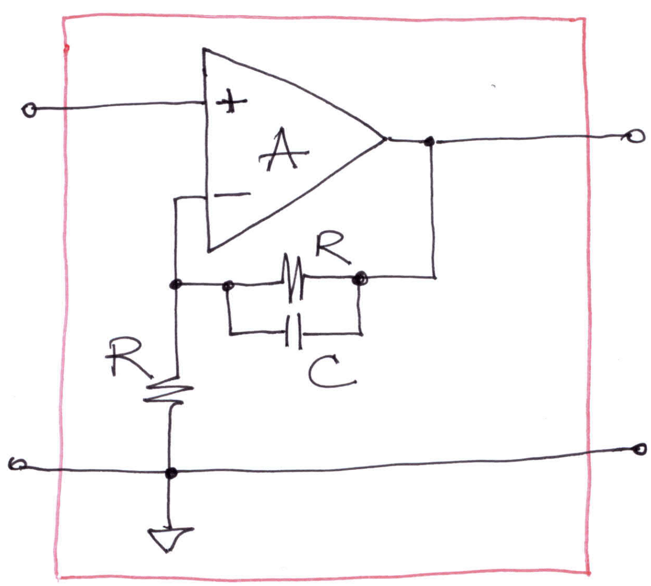
4. a) (10 points) What is the transfer function of the following circuit:



$$H(s) = \frac{R}{R + R \,||\, \frac{1}{Cs}} = \frac{R}{R + \frac{R}{RCs+1}} = \frac{RCs+1}{RCs+2} = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$$

b) (5 points) What is the transfer function of the following
Hints:

- you can use solutions of problem #3 and #4a
- to simplify the result you can assume that A → ∞



Since

$$H(s) = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

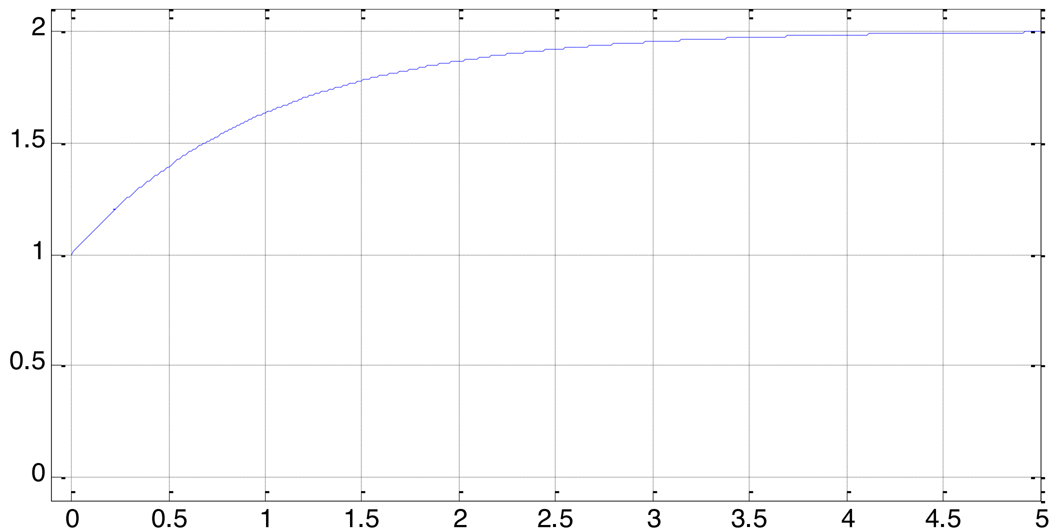
$F(s) = A$ and $G(s) = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$

$$H(s) = \frac{A}{1 + A\left(\frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}\right)} \quad for\ A \to \infty\ H(s) = \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}}$$

c) (10 points) Find and plot the unit-step response *s*(t) of the system?

$$S(s) = \frac{1}{s} \cdot \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}} = \frac{A}{s} + \frac{B}{s+\frac{1}{RC}} = \frac{2}{s} - \frac{1}{s+\frac{1}{RC}} =$$

$$s(t) = (2 - e^{-\frac{t}{RC}}) \cdot u(t)$$



5. (10 points) Find the inverse Laplace transform of the following function:

$$X(s) = \frac{1}{(s+4)(s+3)}$$

What is the ROC of this function?

$$X(s) = \frac{1}{(s+4)(s+3)} = \frac{A_1}{s+4} + \frac{A_2}{s+3}$$

$$A_1 = X(s)\cdot(s+4)\Big|_{s=-4} = -1$$

$$A_2 = X(s)\cdot(s+3)\Big|_{s=-3} = 1$$

$$x(t) = \left[e^{-3t} - e^{-4t}\right]\cdot u(t)$$

ROC: s > max(p_i) = max([-3 -4]) = -3

6. (15 points) Consider a second order (N = 2) differential equation

y''(t) + 5y'(t) + 4y(t) = x(t)

Assume the above equation represents a system with input x(t) and output y(t).

$$s^2Y(s) - sy(0-) - y(0) + 5sY(s) - 5y(0-) + 4Y(s) = X(s)$$

All initial conditions are equal to zero.

$$s^2Y(s) + 5sY(s) + 4Y(s) = X(s)$$

$$H(s) = \frac{1}{(s+1)(s+4)} = \frac{\frac{1}{3}}{s+1} - \frac{\frac{1}{3}}{s+4}$$

The impulse response h(t):

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+4)}\right) = \left(\frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}\right)u(t)$$

The unit-step response s(t) of the system.

$$s(t) = \mathcal{L}^{-1}\left(\frac{1}{s(s+1)(s+4)}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s} + \frac{\frac{1}{3}}{s+1} + \frac{\frac{1}{12}}{s+4}\right) = \left(\frac{1}{4} + \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\right)u(t)$$

7. (10 points) Evaluate formula for power distribution over frequency of a periodic signal *x(t)* (Parseval's theorem). Describe Magnitude Line Spectrum and Phase Line Spectrum and their symmetry.

The power of a periodic signal $x(t)$ of period T_0 is given by

$$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

Replacing the Fourier series of $x(t)$ in the power equation we have that

$$\begin{aligned}
\frac{1}{T_0} \int_{T_0} |x(t)|^2 dt &= \frac{1}{T_0} \int_{T_0} \sum_k \sum_m X_k X_m^* e^{j\Omega_0 kt} e^{-j\Omega_0 mt} dt \\
&= \sum_k \sum_m X_k X_m^* \frac{1}{T_0} \int_{T_0} e^{j\Omega_0 kt} e^{-j\Omega_0 mt} dt \\
&= \sum_k |X_k|^2
\end{aligned}$$

after we apply the orthonormality of the Fourier exponentials. Even though $x(t)$ is real, we let $|x(t)|^2 = x(t)x^*(t)$ in the above equations, permitting us to express them in terms of X_k and its conjugate. The above indicates that the power of $x(t)$ can be computed in either the time or the frequency domain giving exactly the same result.

Moreover, considering the signal to be a sum of harmonically related components or

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} = \sum_k x_k(t)$$

the power of each of these components is given by

$$\frac{1}{T_0} \int_{T_0} |x_k(t)|^2 dt = \frac{1}{T_0} \int_{T_0} |X_k e^{jk\Omega_0 t}|^2 dt = \frac{1}{T_0} \int_{T_0} |X_k|^2 dt = |X_k|^2$$

and the power of $x(t)$ is the sum of the powers of the Fourier series components. This indicates that the power of the signal is distributed over the harmonic frequencies $\{k\Omega_0\}$. A plot of $|X_k|^2$ versus the harmonic frequencies $k\Omega_0$, $k = 0, \pm 1, \pm 2, \ldots$, displays how the power of the signal is distributed over the harmonic frequencies. Given the discrete nature of the harmonic frequencies $\{k\Omega_0\}$ this plot consists of a line at each frequency and as such it is called the *power line spectrum* (that is, a periodic signal has no power in nonharmonic frequencies). Since $\{X_k\}$ are complex, we define two additional spectra, one that displays the magnitude $|X_k|$ versus $k\Omega_0$, called the *magnitude line spectrum*, and the *phase line spectrum* or $\angle X_k$ versus $k\Omega_0$ showing the phase of the coefficients $\{X_k\}$ for $k\Omega_0$. The power line spectrum is simply the magnitude spectrum squared.

A periodic signal $x(t)$, of period T_0, is represented in the frequency by its

$$\text{Magnitude line spectrum}: \quad |X_k| \ vs\ k\Omega_0 \tag{4.15}$$

$$\text{Phase line spectrum}: \quad \angle X_k \ vs\ k\Omega_0 \tag{4.16}$$

8. (5 points) Represent the trigonometric Fourier series of a real-valued periodic signal *x(t)*. How do you calculate coefficients?

The *trigonometric Fourier series* of a real-valued, periodic signal $x(t)$, of period T_0, is an equivalent representation that uses sinusoids rather than complex exponentials as the basis functions. It is given by

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \theta_k)$$

$$= c_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)] \qquad \Omega_0 = \frac{2\pi}{T_0} \tag{4.19}$$

where $X_0 = c_0$ is called the *DC component*, and $\{2|X_k| \cos(k\Omega_0 t + \theta_k)\}$ are the kth *harmonics* for $k = 1, 2 \ldots$. The frequencies $\{k\Omega_0\}$ are said to be harmonically related. The coefficients $\{c_k, \ d_k\}$ are obtained from $x(t)$ as follows:

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\Omega_0 t)\, dt \qquad k = 0, 1, \ldots$$

$$d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\Omega_0 t)\, dt \qquad k = 1, 2, \ldots \tag{4.20}$$

The coefficients $X_k = |X_k| e^{j\theta_k}$ are connected with the coefficients c_k and d_k by

$$|X_k| = \sqrt{c_k^2 + d_k^2}$$

$$\theta_k = -\tan^{-1}\left[\frac{d_k}{c_k}\right]$$

The functions $\{\cos(k\Omega_0 t), \ \sin(k\Omega_0 t)\}$ are orthonormal.