SIGNALS AND SYSTEMS USING MATLAB
Chapter 12 — Introduction to the Design of Discrete Filters

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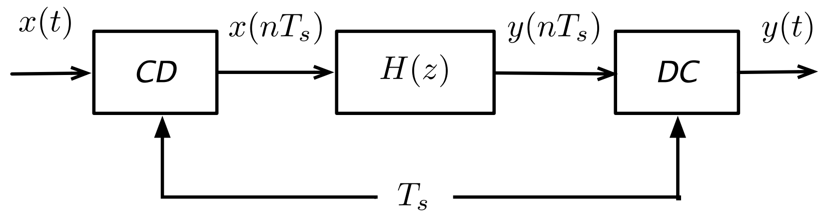
- Filtering of continuous and discrete–time signals



Discrete filtering of analog signals using ideal continuous to discrete (C/D), or a sampler, and discrete to continuous (D/C) converter, or a reconstruction filter

- Filtering of periodic or aperiodic signals

$H(z)$ transfer function of discrete-time LTI system

periodic input

$$x[n] = \sum_k A_k \cos(\omega_k n + \phi_k)$$

steady-state response

$$y_{ss}[n] = \sum_k A_k |H(e^{j\omega_k})| \cos(\omega_k n + \phi_k + \theta(\omega_k))$$

$x[n]$ aperiodic signal with $X(z)$

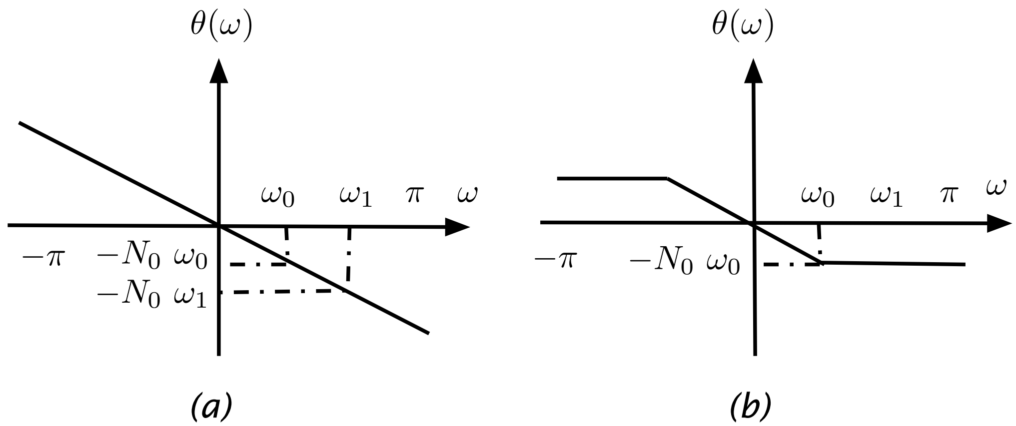
$Y(z) = H(z)X(z)$ or on the unit circle when $z = e^{j\omega}$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Filtering of

$$x[n] = 1 + \cos(\omega_0 n) + \cos(\omega_1 n) \qquad \omega_1 = 2\omega_0 \qquad n \geq 0$$

all–pass filter transfer function: $H(z) = \alpha z^{-N_0}$



(a) Linear, (b) nonlinear phase

Steady–state response for linear phase

$$y_{ss}[n] = \alpha\,[1 + \cos(\omega_o(n - N_0)) + \cos(\omega_1(n - N_0))] = \alpha x[n - N_0]$$

Steady–state response for non–linear phase

$$y_{ss}[n] = \alpha[1 + \cos(\omega_0(n - N_0)) + \cos(\omega_1(n - 0.5N_0))] \neq \alpha x[n - N_0].$$

- Group delay

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega}.$$

General expression of linear phase: If $\tau(\omega) = \tau$ (constant)

$$\theta(\omega) = \begin{cases} -\tau\omega - \theta_0 & -\pi \le \omega < 0 \\ 0 & \omega = 0 \\ -\tau\omega + \theta_0 & 0 < \omega \le \pi \end{cases}$$

line through origin if $\theta_0 = 0$

Example: IIR and FIR filters with input $x[n]$ and output $y[n]$

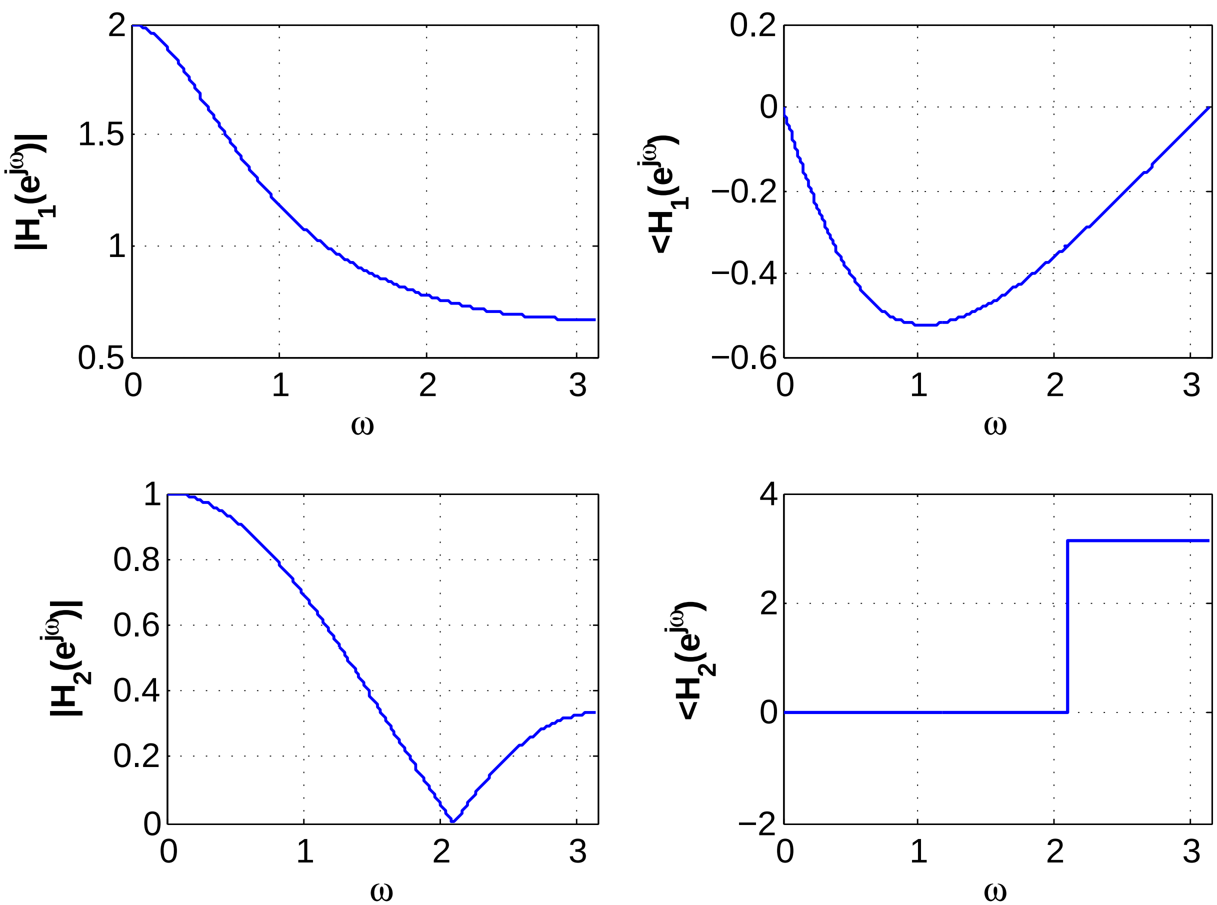
$$(i) \;\; y[n] = 0.5y[n-1] + x[n], \;\; (ii) \;\; y[n] = \tfrac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

$$(i) \; H_1(z) = \frac{1}{1 - 0.5z^{-1}}$$

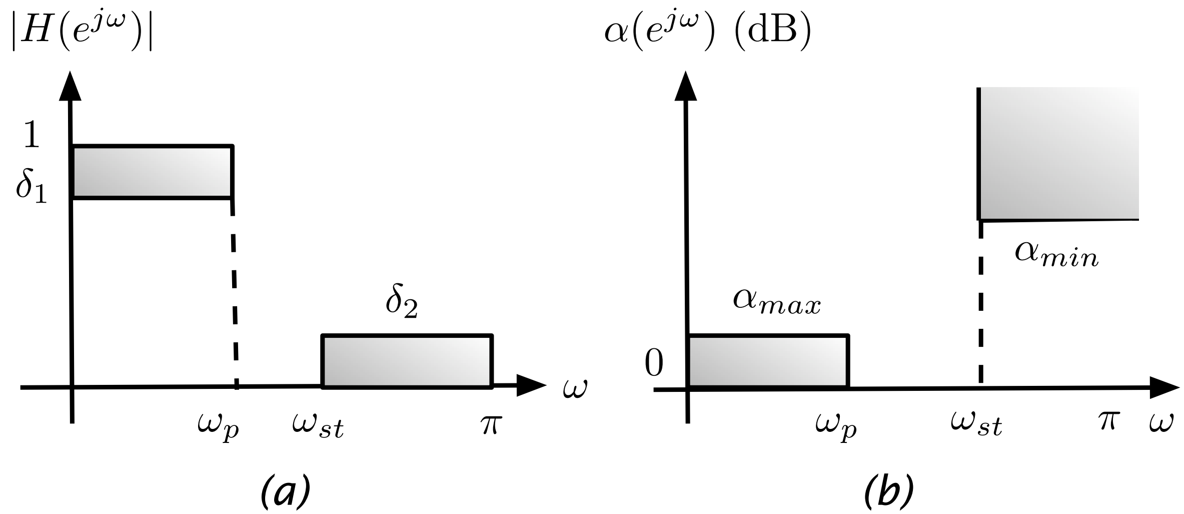
$$(ii) \; H_2(z) = \frac{1}{3}[z^{-1} + 1 + z] = \frac{1 + z + z^2}{3z} = \frac{(z - 1e^{j2.09})(z - 1e^{-j2.09})}{3z}$$

$$\angle H_1(e^{j\omega}) = -\;\tan^{-1}\left(\frac{0.5\sin(\omega)}{1 - 0.5\cos(\omega)}\right) \quad \text{non–linear}$$

$$\angle H_2(e^{j\omega}) = \begin{cases} 0 & \text{when } 1 + 2\cos(\omega) \ge 0 \\ -\pi & \text{when } 1 + 2\cos(\omega) < 0 \end{cases} \quad \text{non–linear}$$



Magnitude and phase responses of $H_1(z)$ (top), and $H_2(z)$ (bottom)



Low-pass magnitude specifications for IIR filter: (a) linear scale, (b) loss scale

Passband: $\delta_1 \leq |H(e^{j\omega})| \leq 1 \qquad 0 \leq \omega \leq \omega_p$

Stopband: $0 < |H(e^{j\omega})| \leq \delta_2 \qquad \omega_{st} \leq \omega \leq \pi$

Loss function $\alpha(e^{j\omega}) = -10 \log_{10} |H(e^{j\omega})|^2 = -20 \log_{10} |H(e^{j\omega})|$ *dBs*

Passband: $0 \leq \alpha(e^{j\omega}) \leq \alpha_{max} \qquad 0 \leq \omega \leq \omega_p$

Stopband: $\alpha_{min} \leq \alpha(e^{j\omega}) < \infty \qquad \omega_{st} \leq \omega \leq \pi$

$$\alpha_{max} = -20 \log_{10} \delta_1, \quad \alpha_{min} = -20 \log_{10} \delta_2$$

Example: Not–normalized loss specifications

$$\text{Passband} \quad 10 \leq \hat{\alpha}(e^{j\omega}) \leq 11 \qquad 0 \leq \omega \leq \frac{\pi}{2}$$

$$\text{Stopband} \quad \hat{\alpha}(e^{j\omega}) \geq 50 \qquad \frac{3\pi}{4} \leq \omega \leq \pi$$

Let $\hat{\alpha}(e^{j\omega}) = 10 + \alpha(e^{j\omega}) \quad \Rightarrow$

$$0 \leq \alpha(e^{j\omega}) \leq 1 \qquad 0 \leq \omega \leq \frac{\pi}{2}$$

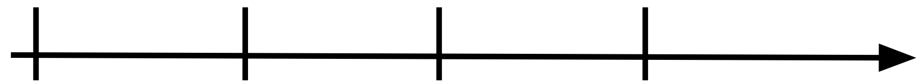
$$\alpha(e^{j\omega}) \geq 40 \qquad \frac{3\pi}{4} \leq \omega \leq \pi$$

$$\hat{\alpha}(e^{j0}) = 10, \;\; \alpha_{max} = 11 - 10 = 1, \;\; \alpha_{min} = 50 - 10 = 40$$

$H(z)$ normalized filter $\Rightarrow \; \hat{H}(z) = KH(z)$

dc frequency: $-20\log_{10}|\hat{H}(e^{j0})| = -20\log_{10}K - 20\log_{10}|H(e^{j0})|$

or $10 = -20\log_{10}K + 0 \quad \Rightarrow \; K = 10^{-0.5} = \dfrac{1}{\sqrt{10}}$



0	f_p	f_{st}	$0.5 f_s$	f (Hz)
0	Ω_p	Ω_{st}	$0.5\Omega_s$	Ω (rad/sec)
0	ω_p	ω_{st}	π	ω (rad)
0	$\frac{\omega_p}{\pi}$	$\frac{\omega_{st}}{\pi}$	1	$\frac{\omega}{\pi}$

Frequency scales in discrete filter design

- Time-domain specifications: desired impulse response $h_d[n]$

low-pass filter desired frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1e^{-j\omega N} & 0 \leq \omega \leq \omega_c \\ 0 & \omega_c < \omega \leq \pi \end{cases}$$

desired impulse response

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{-j\omega N}e^{j\omega n}d\omega$$

Approaches

- Optimization techniques
- Analog filter design and transformations
  - Sampling transform $z = e^{sT_s}$ (impulse invariant method)— possible aliasing
  - Bilinear transformation— results from trapezoidal rule approximation

$$\text{Let } H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}$$

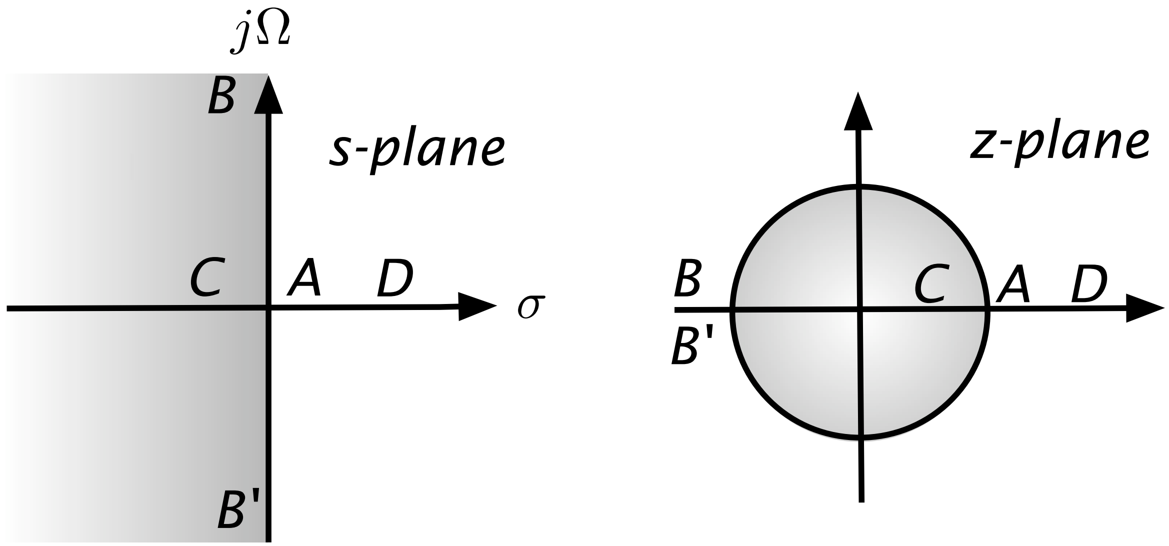
$$y(nT_s) = \int_{(n-1)T_s}^{nT_s} x(\tau)d\tau + y((n-1)T_s)$$

$$y(nT_s) \approx \frac{[x(nT_s) + x((n-1)T_s)]\ T_s}{2} + y((n-1)T_s)$$

$$Y(z) = \frac{T_s(1+z^{-1})}{2(1-z^{-1})}X(z) \ \Rightarrow \ H(z) = \frac{Y(z)}{X(z)} = \frac{T_s}{2}\,\frac{1+z^{-1}}{1-z^{-1}}$$

$$\text{Directly from } H(s) \text{ by letting } s = \frac{2}{T_s}\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$\text{inverse transformation} \quad z = \frac{1+(T_s/2)s}{1-(T_s/2)s}$$



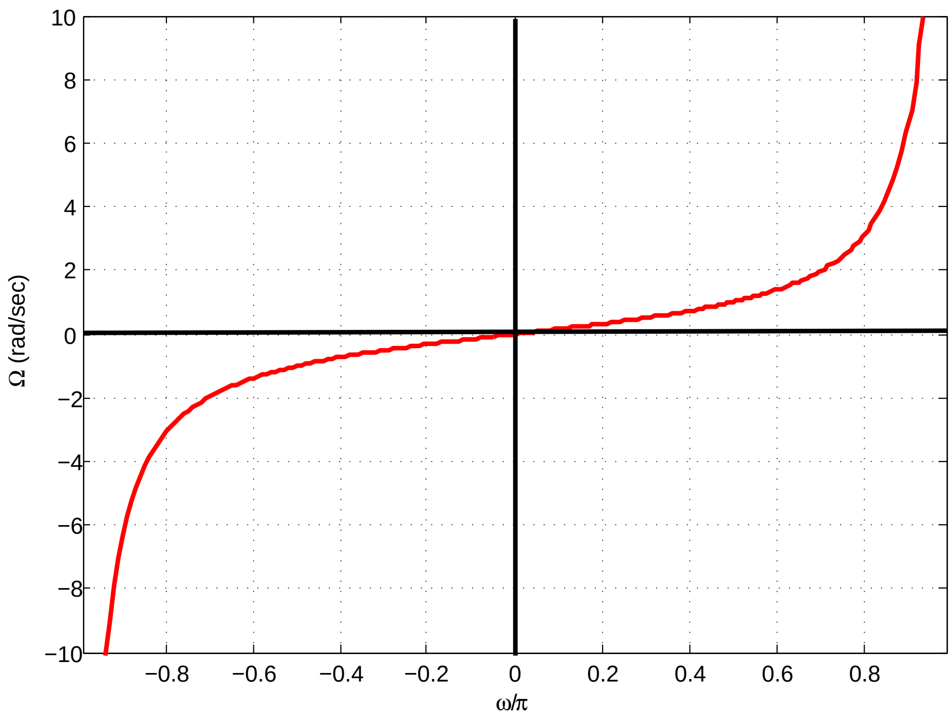
Mapping of s-plane into Z-plane by bilinear transformation

$$
\begin{aligned}
r &= \sqrt{\frac{(1+\sigma/K)^2 + (\Omega/K)^2}{(1-\sigma/K)^2 + (\Omega/K)^2}} \\
\omega &= \tan^{-1}\left(\frac{\Omega/K}{1+\sigma/K}\right) + \tan^{-1}\left(\frac{\Omega/K}{1-\sigma/K}\right)
\end{aligned}
$$

Warping effects of BT

$$\Omega = K\tan(\omega/2) = K\left[\frac{\omega}{2} + \frac{\omega^3}{24} + \cdots\right]$$

$$\Omega \approx \frac{\omega}{T_s} \text{ for small } \omega$$



Linear relation between Ω and ω $(K = 1)$ for small ω, warping as $\omega \rightarrow \pm\pi$

Applying $\Omega = K\tan(\omega/2)$ to analog Butterworth equation

$$|H_N(j\Omega')|^2 = \frac{1}{1+(\Omega')^{2N}} \quad \Omega' = \frac{\Omega}{\Omega_{hp}}$$ gives

$$|H_N(e^{j\omega})|^2 = \frac{1}{1+\left[\frac{\tan(0.5\omega)}{\tan(0.5\omega_{hp})}\right]^{2N}}$$

Using frequency transformation $\dfrac{\Omega_{st}}{\Omega_p} = \dfrac{\tan(\omega_{st}/2)}{\tan(\omega_p/2)} \Rightarrow$

$$N \geq \frac{\log_{10}[(10^{0.1\alpha_{min}}-1)/(10^{0.1\alpha_{max}}-1)]}{2\log_{10}\left[\frac{\tan(\omega_{st}/2)}{\tan(\omega_p/2)}\right]}$$

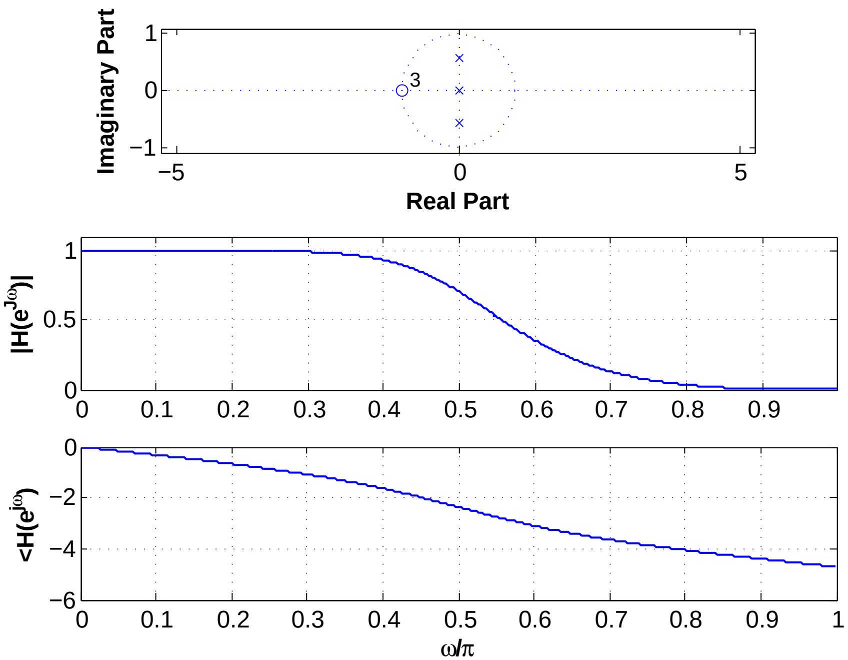
$$2\tan^{-1}\left[\frac{\tan(\omega_p/2)}{(10^{0.1\alpha_{max}}-1)^{1/2N}}\right] \leq \omega_{hp} \leq 2\tan^{-1}\left[\frac{\tan(\omega_{st}/2)}{(10^{0.1\alpha_{min}}-1)^{1/2N}}\right]$$

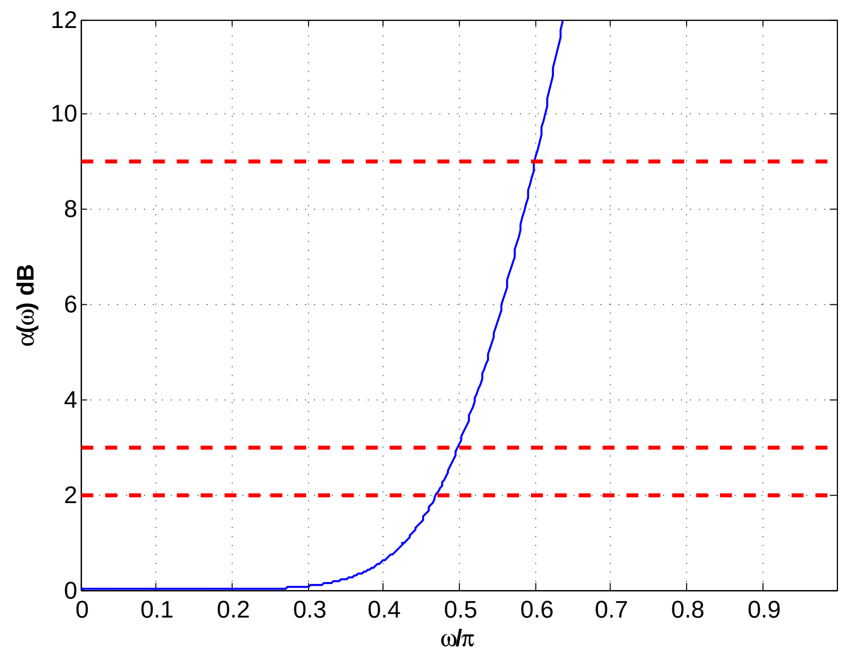
$$\Omega'_{hp} = 1 \ \rightarrow \ \omega_{hp} \ \Rightarrow \ K_b = \frac{\Omega'}{\tan(0.5\omega)}\Big|_{\Omega'=1,\omega=\omega_{hp}} = \frac{1}{\tan(0.5\omega_{hp})}$$

$$H_N(s) \ \rightarrow \ H_N(z) = H_N(s)\Big|_{s=K_b(1-z^{-1})/(1+z^{-1})}$$

Example: Low–pass filter specifications

$$\omega_p = 0.47\pi \text{ (rad)} \qquad \alpha_{max} = 2\ dB$$
$$\omega_{st} = 0.6\pi \text{ (rad)} \qquad \alpha_{min} = 9\ dB$$
$$\alpha(e^{j0}) = 0 \text{ dB}$$



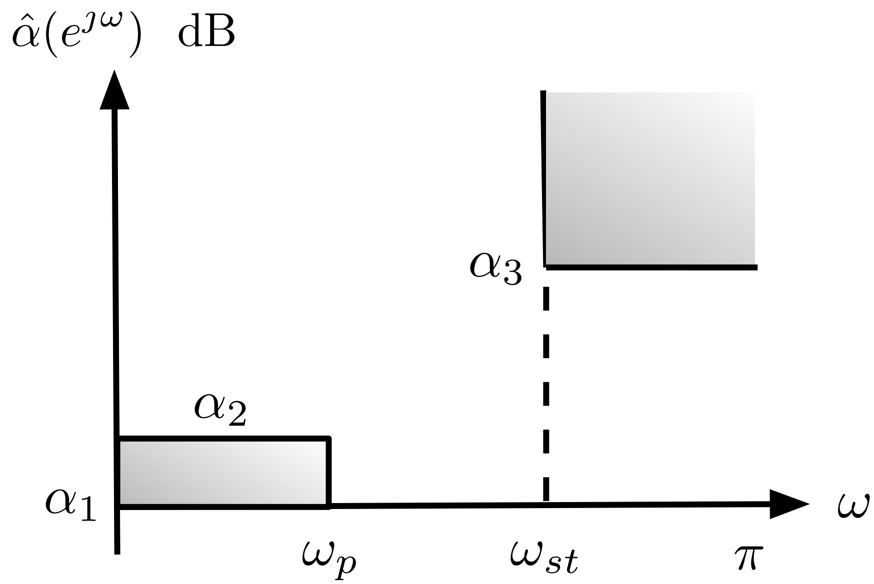


Design of low-pass Butterworth filter using MATLAB: poles and zeros, magnitude and phase response (left); verification of specifications using loss function $\alpha(\omega)$

$$H(z) = \frac{0.166 + 0.497z^{-1} + 0.497z^{-2} + 0.166z^{-3}}{1 - 0.006z^{-1} + 0.333z^{-2} - 0.001z^{-3}}$$

Example: Design of Butterworth LPF for processing analog signal

$$
\begin{array}{lll}
f_p = 2,250 \text{ Hz} & \text{passband frequency}, & \alpha_1 = -18 \text{ dB dc loss} \\
f_{st} = 2,700 \text{ Hz} & \text{stopband frequency}, & \alpha_2 = -15 \text{ dB loss in passband} \\
f_s = 9,000 \text{ Hz} & \text{sampling frequency}, & \alpha_3 = -9 \text{ dB loss in stopband}
\end{array}
$$



Normalized specs

$\hat{\alpha}(e^{j0}) = -18$ dB, dc gain

$\alpha_{max} = \alpha_2 - \alpha_1 = 3$ dB

$\alpha_{min} = \alpha_3 - \alpha_1 = 9$ dB,

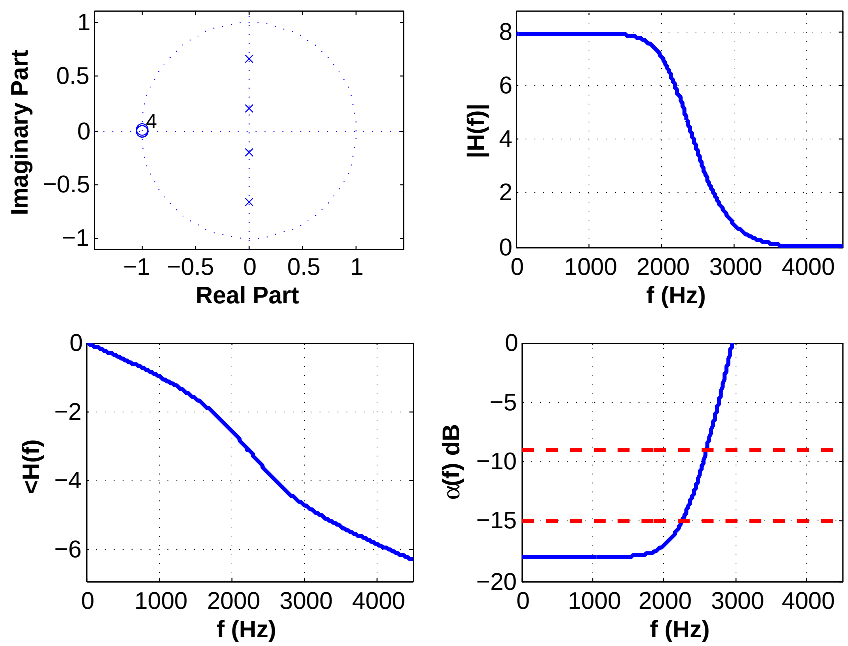
$\omega_p = \dfrac{2\pi f_{hp}}{f_s} = 0.5\pi$ (half-power freq.)

$\omega_{st} = \dfrac{2\pi f_{st}}{f_s} = 0.6\pi$

$$T_s = 1/f_s = (1/9) \times 10^{-3} \ \text{sec/sample} \quad \Rightarrow \quad K_b = \cot(\pi f_{hp} T_s) = 1$$

$$0.5\omega_{hp} = \pi/4 \ \Rightarrow \ \alpha(e^{j\omega}) = 10\log_{10}(1 + (\tan(0.5\omega))^{2N})$$

$$\text{at } \omega = \omega_{st} \ \alpha(e^{j\omega_{st}}) \geq \alpha_{min} \ \Rightarrow \ N = \left\lceil \frac{\log_{10}(10^{0.1\alpha_{min}} - 1)}{2\log_{10}(\tan(0.5\omega_{st}))} \right\rceil = 4$$



$$H'(z) = GH(z) \ \text{ with dc loss of } -18dB$$

$$\alpha(e^{j0}) = -18 = -20\log_{10} G \ \Rightarrow \ G = 7.94$$

$$H'(z) = GH(z) = \frac{(z+1)^4}{(z^2 + 0.45)(z^2 + 0.04)}$$

Mapping $\Omega'_p = 1 \;\rightarrow\; \omega_p \;\Rightarrow K_c = \dfrac{1}{\tan(0.5\omega_p)}$

$$\frac{\Omega}{\Omega_p} = \frac{\tan(0.5\omega)}{\tan(0.5\omega_p)} \;\Rightarrow\; |H_N(e^{j\omega})|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\tan(0.5\omega)/\tan(0.5\omega_p))}$$

$C(.)$ Chebyshev polynomials

$$\varepsilon = (10^{0.1\alpha\,\mathrm{max}} - 1)^{1/2}$$

$$N \geq \frac{\cosh^{-1}\left([(10^{0.1\alpha\,\mathrm{min}} - 1)/(10^{0.1\alpha\,\mathrm{max}} - 1)]^{1/2}\right)}{\cosh^{-1}[\tan(0.5\omega_{st})/\tan(0.5\omega_p)]}$$

$$\omega_{hp} = 2\tan^{-1}\left[\tan(0.5\omega_p)\cosh\left(\frac{1}{N}\cosh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\right]$$

$$H_N(z) = H_N(s)|_{s=K_c(1-z^{-1})/(1+z^{-1})}$$

Example: Difference between even and odd order lowpass Chebyshev filters

Specs for first filter

$$\alpha(e^{j0}) = 0 \;\; dB$$
$$\omega_p = 0.47\pi \;\; rad, \;\; \alpha_{max} = 2 \;\; \text{dBs}$$
$$\omega_{st} = 0.6\pi \;\; rad, \;\;\; \alpha_{min} = 6 \;\; \text{dBs}$$

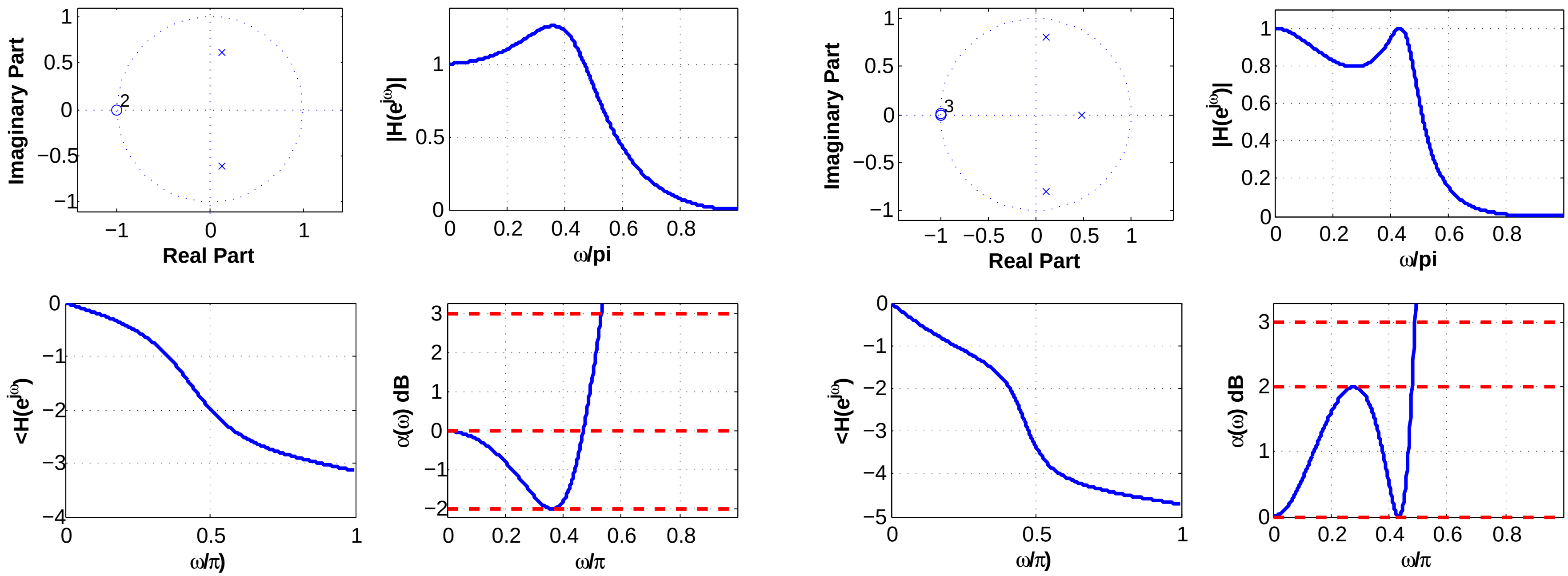
Specs for second filter

$$\omega_p = 0.48 \;\; rad, \;\; \text{others the same}$$

Transfer functions

$$H_1(z) = \frac{0.224 + 0.449z^{-1} + 0.224z^{-2}}{1 - 0.264z^{-1} + 0.394z^{-2}}, \;\;\; \omega_{hp} = 0.493\pi \;\; rad.$$

$$H_2(z) = \frac{0.094 + 0.283z^{-1} + 0.283z^{-2} + 0.094z^{-3}}{1 - 0.691z^{-1} + 0.774z^{-2} - 0.327z^{-3}}$$



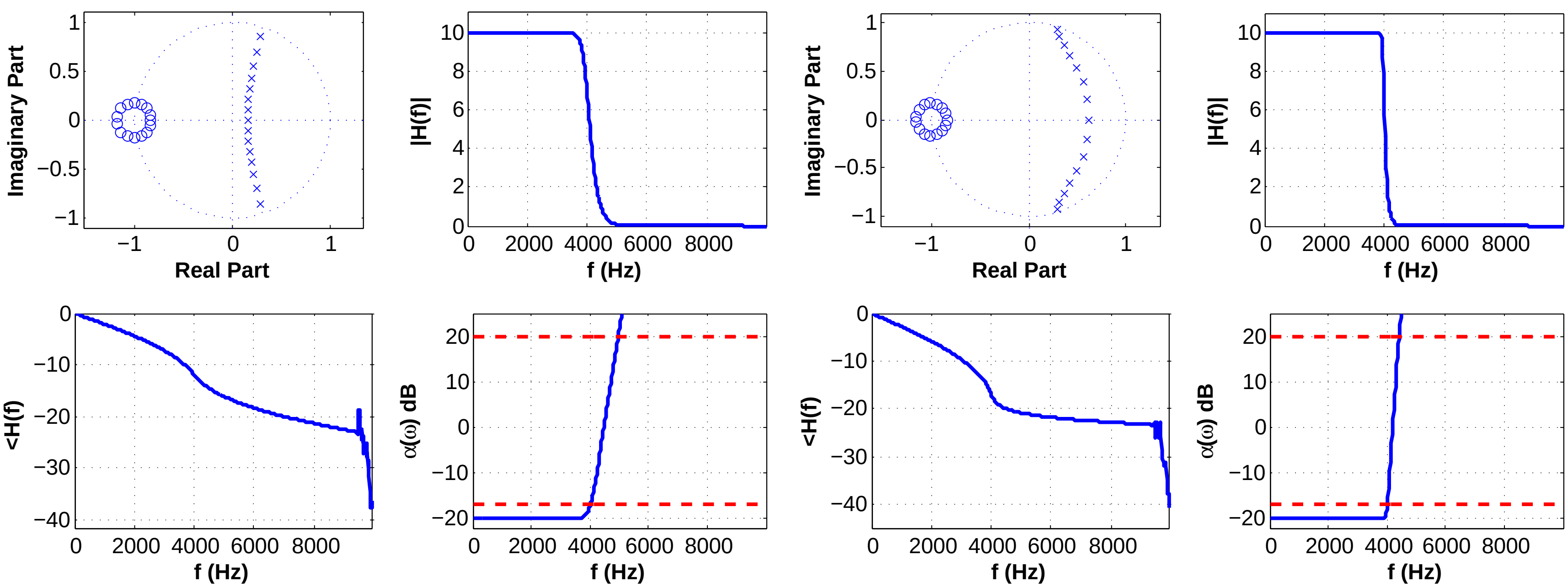
Two Chebyshev filters with different transition bands: even-order filter for $\omega_p = 0.47\pi$ *on the left, and odd-order filter for* $\omega_p = 0.48\pi$ *(narrower transition band) on the right.*

Example: Butterworth and Chebyshev filter designs to filter an acoustic signal

$$\text{dc gain} = 10, \quad \text{half-power frequency} \quad f_{hp} = 4 \; KHz$$
$$\text{band-stop frequency} \quad f_{st} = 5 \; Khz \quad \alpha_{min} = 60 \text{ dBs}$$
$$\text{sampling frequency} \quad f_s = 20KHz$$

Normalized specs

$$\text{dc gain} = 10 \Rightarrow \alpha(e^{j0}) = -20 \text{ dBs}$$
$$\text{half-power frequency:} \;\; \omega_{hp} = 2\pi f_{hp}(1/f_s) = 0.4\pi \;\; \text{rad}$$
$$\text{band-stop frequency:} \;\; \omega_{st} = 2\pi f_{st}(1/f_s) = 0.5\pi \;\; \text{rad}$$



Equal-order ($N = 15$) Butterworth (left) and Chebyshev filters acoustic signal

Given prototype LPF we wish to transform it to desired filter using

$$G(z^{-1}) = Z^{-1} \text{ that}$$

- is rational
- maps inside of the unit circle in the Z-plane into the inside of the unit circle in the z-plane
- maps $|Z| = 1$ into $|z| = 1$

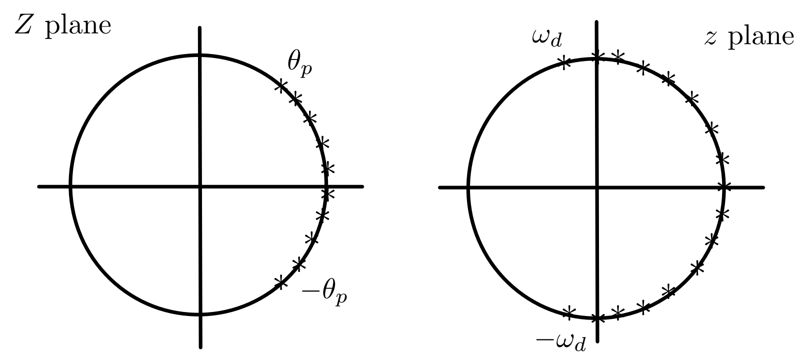
If $Z = Re^{j\theta}$, $z = re^{j\omega}$
$G(e^{-j\omega}) = |G(e^{-j\omega})|e^{j\angle(G(e^{-j\omega}))} = 1\ e^{-j\theta}$ all-pass characteristics

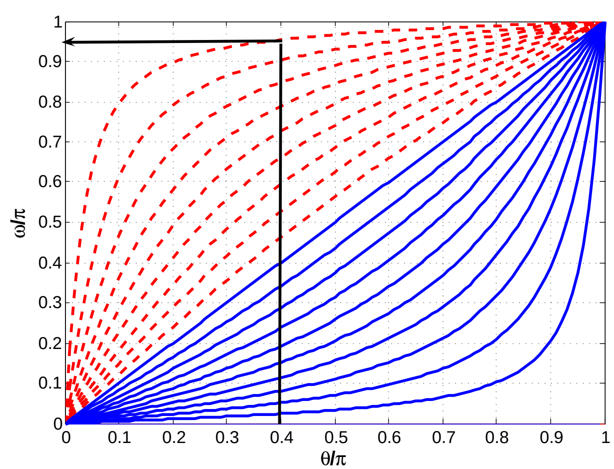
$$Z^{-1} = G(z^{-1}) = A\prod_k \frac{z^{-1} - \alpha_k}{1 - \alpha_k^* z^{-1}}$$

$$Z^{-1} = A\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

$$A = 1, \quad \alpha = \frac{\sin((\theta_p - \omega_d)/2)}{\sin((\theta_p + \omega_d)/2)}$$

- Zero frequencies in two planes are mapped into each other, then A
- Letting $Z = 1e^{j\theta}$ and $z = 1e^{j\omega}$ in transformation we get α
- Non-linear relation between frequencies θ and ω





- Low-pass to high-pass transformation

$$\text{LP-HP} \quad Z^{-1} = -\left(\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}\right) \quad \text{(linear)}$$

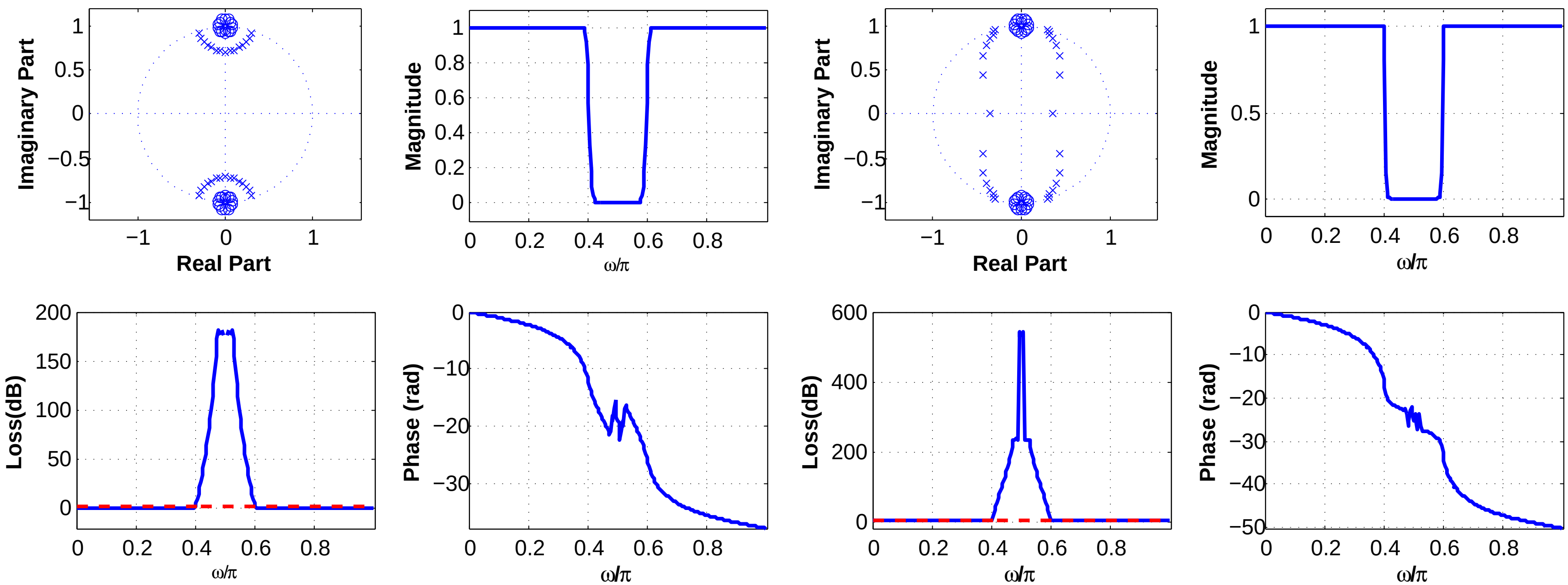
$$\alpha = \frac{\cos((\theta_p + \omega_d)/2)}{\cos((\theta_p - \omega_d)/2)}$$

- Low–pass to band–pass and band–eliminating transformation

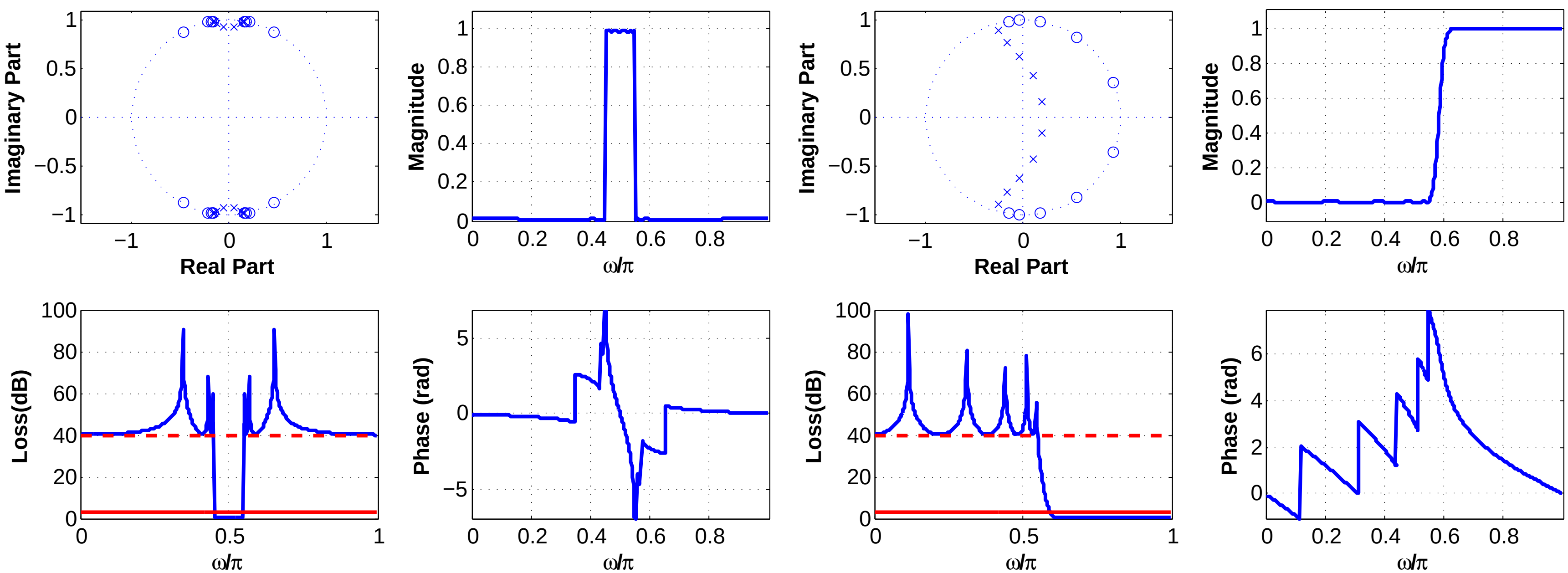
$$\text{LP-BP} \quad Z^{-1} = -\left(\frac{z^{-2} - bz^{-1} + c}{cz^{-2} - bz^{-1} + 1}\right) \quad \text{(quadratic)}$$

$$\text{LP-BE} \quad Z^{-1} = \frac{z^{-2} - (b/k)z^{-1} - c}{-cz^{-2} - (b/k)z^{-1} + 1} \quad \text{(quadratic)}$$

$$b = 2\alpha k/(k+1)$$
$$c = (k-1)/(k+1)$$
$$\alpha = (\cos((\omega_{du} + \omega_{d\ell})/2))/(\cos((\omega_{du} - \omega_{d\ell})/2))$$
$$k = \cot((\omega_{du} - \omega_{d\ell})/2)\tan(\theta_p/2)$$



Bandstop Butterworth (left) and Chebyshev (right) filters



Elliptic band-pass filter (left) and high-pass using cheby2*:(clockwise for each side from top left) poles/zeros, magnitude, phase frequency responses, loss.*

- Window Method
  Desired low-pass frequency response

$$|H_d(e^{j\omega})| = \begin{cases} 1 & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\text{linear phase} \quad \theta(\omega) = -\omega M/2$$

  impulse response

$$h_d[n] = \begin{cases} \sin(\omega_c(n - M/2)/(\pi(n - M/2)) & n \ne M/2 \\ \omega_c/\pi & n = M/2 \end{cases}$$

  Window w[n] of length M and centered at $M/2$

$$\text{windowed impulse response} \quad h[n] = h_d[n]\mathrm{w}[n]$$

$$\text{designed FIR filter} \quad H(z) = \sum_{n=0}^{M-1} h[n] z^{-n}$$

- Window method is a trial–and–error procedure
- Symmetry of $h[n]$ with respect to $M/2$, independent of whether this is an integer or not, guarantees the linear phase of the filter

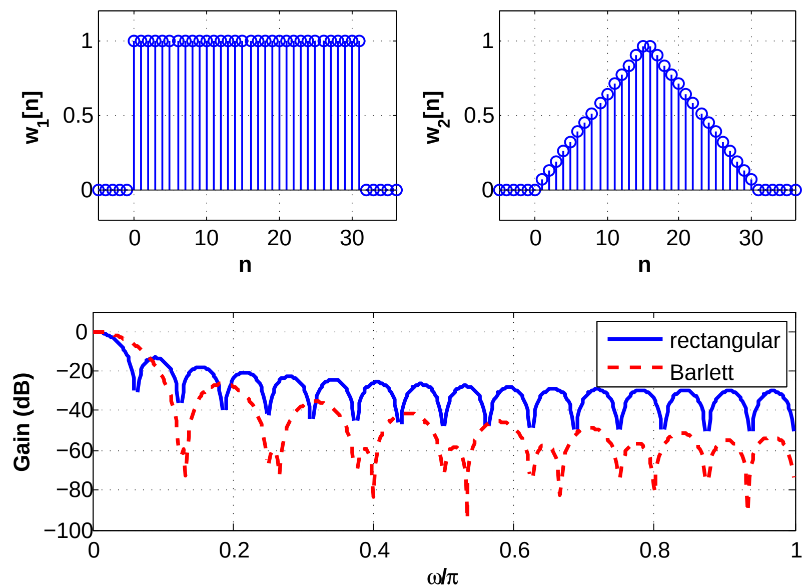
Rectangular window of length N

$$\mathrm{w}[n] = \begin{cases} 1 & -(N-1)/2 \leq n \leq (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

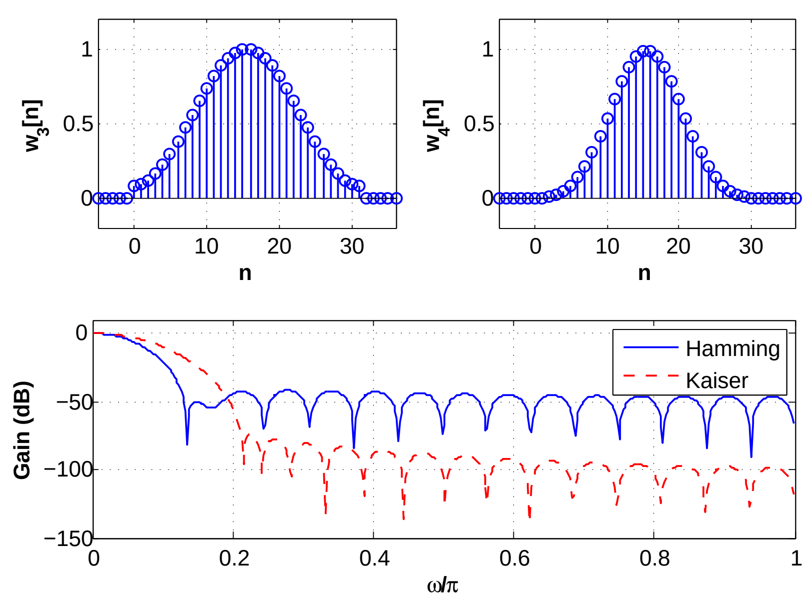
- $H_{\mathrm{w}}(e^{j\omega}) = H_d(e^{j\omega})$, requires rectangular window of infinite length
- $h_{\mathrm{w}}[n] = h_d[n]\mathrm{w}[n]$, in frequency

$$\begin{aligned} H_{\mathrm{w}}(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \\ &= \int_{-\pi}^{\pi} H_d(e^{j\theta})\delta(\omega-\theta)d\theta = H_d(e^{j\omega}) \end{aligned}$$

- If N is finite, convolution gives distorted $H_d(e^{j\omega})$.
- Good approximation of $H_d(e^{j\omega})$ using a finite window $\mathrm{w}[n]$ requires the window must have
  - spectrum approximating an impulse in frequency
  - most of the energy concentrated in the low frequencies



Rectangular and Bartlett causal windows and their spectra



Hamming and Kaiser causal windows and their spectra

Example: Design low-pass FIR filter of length $M = 21$ for filtering analog signals

$$H_d(e^{jf}) = \begin{cases} 1 & -125 \leq f \leq 125 \text{ Hz} \\ 0 & \text{elsewhere in } -f_s/2 < f \leq f_s/2 \end{cases}$$
$$f_s = 1000 \text{ Hz}$$

Use rectangular Hamming windows

Using $\omega = 2\pi f/f_s$, the discrete frequency response is given by

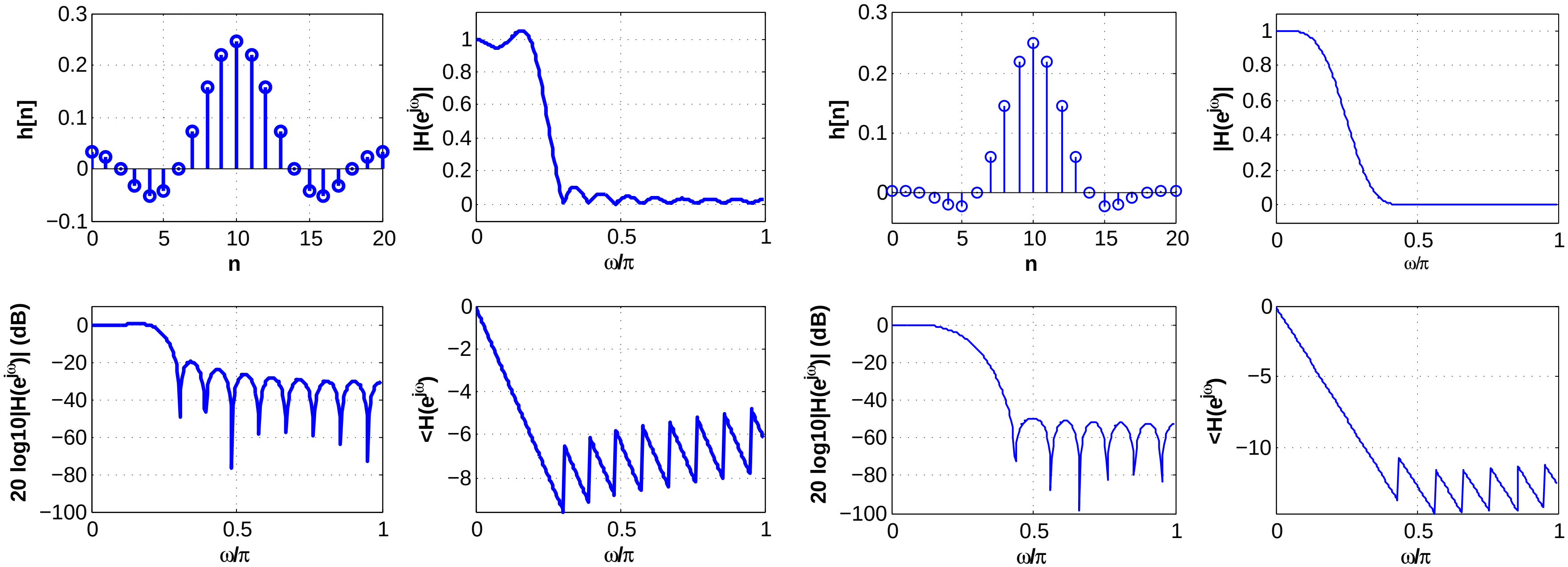
$$H_d(e^{j\omega}) = \begin{cases} 1 & -\pi/4 \leq \omega \leq \pi/4 \text{ rad} \\ 0 & \text{elsewhere in } -\pi < \omega \leq \pi \end{cases}$$

The desired impulse response:

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \begin{cases} \sin(\pi n/4)/(\pi n) & n \neq 0 \\ 0.25 & n = 0 \end{cases}$$

Rectangular window:

$$\hat{H}(z) = H_{\mathrm{W}}(z)z^{-10} = 0.25z^{-10} + \sum_{n=0, n\neq 10}^{20} \frac{\sin(\pi(n-10)/4)}{\pi(n-10)} z^{-n}$$



Low-pass FIR filters using rectangular (left) and Hamming windows.

Design high-pass FIR filter of order $M - 1 = 14$, and cut-off frequency 0.2π using the Kaiser window.

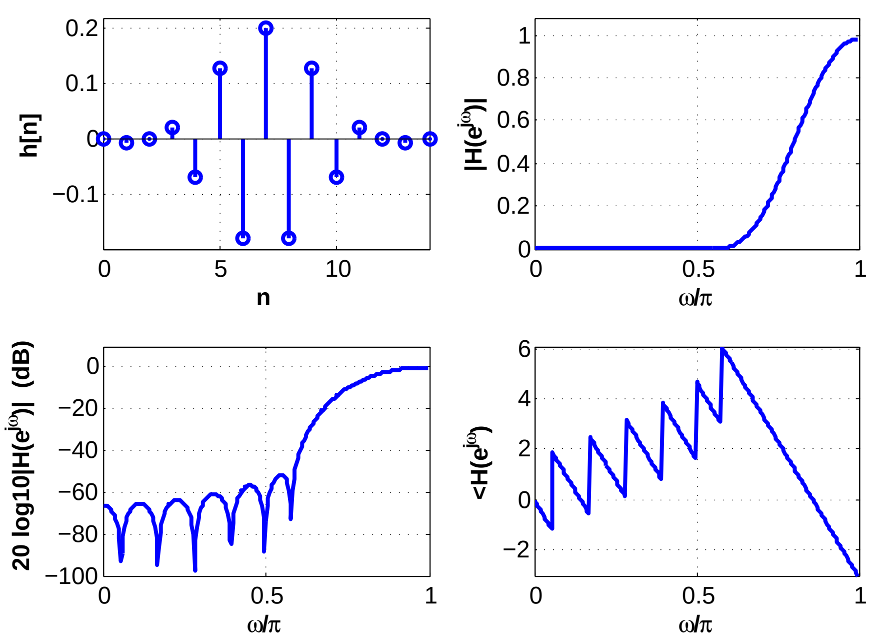
$h_{lp}[n]$ impulse response of ideal low-pass filter

$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise in } [-\pi,\ \pi) \end{cases}$$

Modulation property of the DTFT:

$$2h_{lp}[n]\cos(\omega_0 n) \quad \Leftrightarrow \quad H_{lp}(e^{j(\omega+\omega_0)}) + H_{lp}(e^{j(\omega-\omega_0)})$$

If $\omega_0 = \pi$, $h_{hp}[n] = 2h_{lp}[n]\cos(\pi n) = 2(-1)^n h_{lp}[n]$ is the desired impulse response of the high-pass filter



High-pass FIR filter design using Kaiser window.

- hardware or software realization
- issues to consider
  - computational complexity, minimal realizations
  - quantization
- common realizations
  - Direct Form
  - Cascade
  - Parallel
- Direct minimal realization
  - Constant numerator

$$\text{transfer function} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} \frac{b_0}{1 + \sum_{k=1}^{N-1} a_k z^{-k}}$$

$$\text{input/output relation} \quad y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + b_0 x[n]$$

requires $N-1$ delays for the output, and none for the input. This is a minimum realization of $H(z)$

- Polynomial denominator

$$\text{numerator} \quad B(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

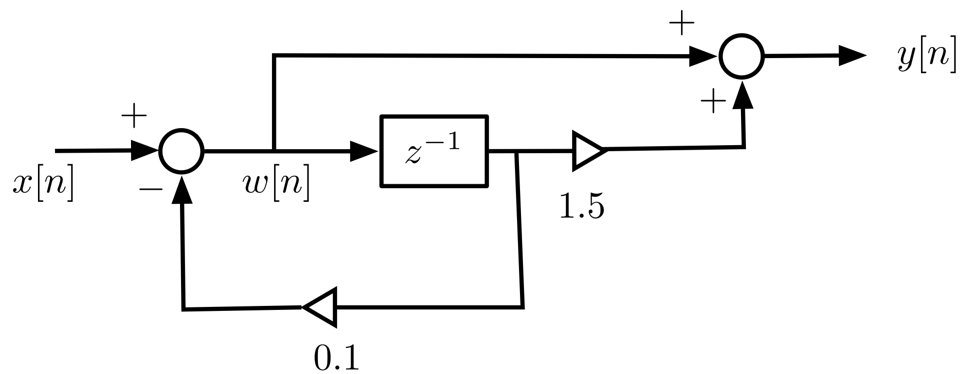
$$Y(z) = H(z)X(z) = B(z)\left[\frac{X(z)}{A(z)}\right]$$

$$\frac{W(z)}{X(z)} = \frac{1}{A(z)} \quad \Rightarrow \quad w[n] = -\sum_{k=1}^{N-1} a_k w[n-k] + b_0 x[n]$$

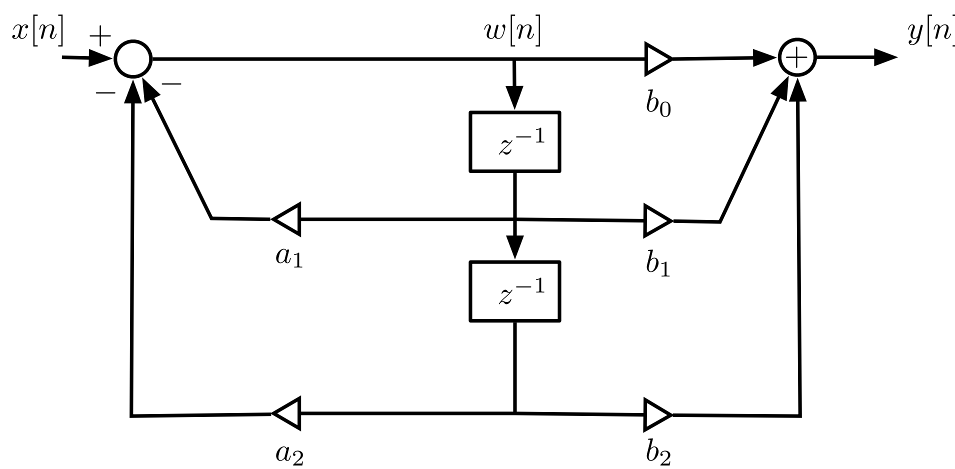
Output $y[n]$ obtained from

$$Y(z) = B(z)W(z) \quad \Rightarrow \quad y[n] = \sum_{k=0}^{M-1} b_k w[n-k]$$

an input–output equation which uses the delayed signals $\{w[n-k]\}$ from above. Number of delays used corresponds to the order of the denominator $A(z)$ which is the order of the filter, minimal realization



Minimal direct realization of $H(z) = (1 + 1.5z^{-1})/(1 + 0.1z^{-1})$

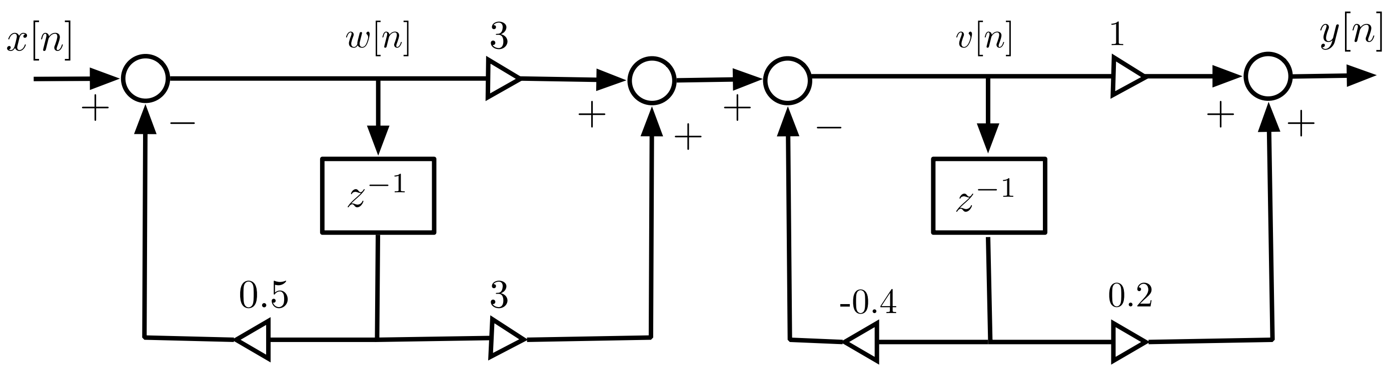


Minimal direct realization of first- and second-order filters

- Cascade realization

$$H(z) = \frac{B(z)}{A(z)} = \prod_i H_i(z)$$

$H_i(z)$ realized using minimal direct form

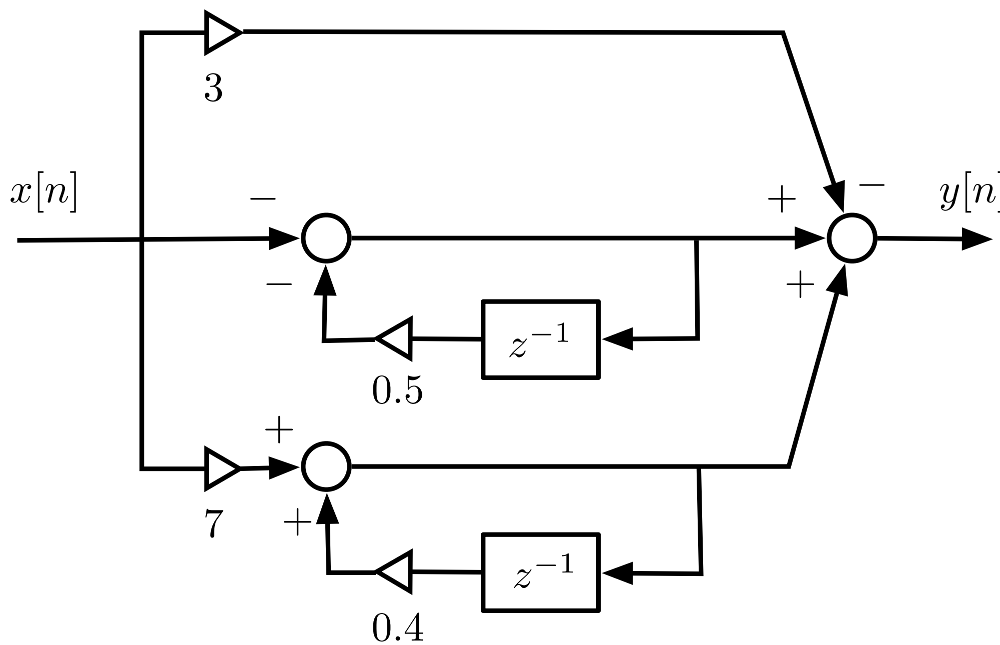


Cascade realization of $H(z) = \left[3(1 + z^{-1})/(1 + 0.5z^{-1})\right]\left[(1 + 0.2z^{-1})/(1 - 0.4z^{-1})\right]$

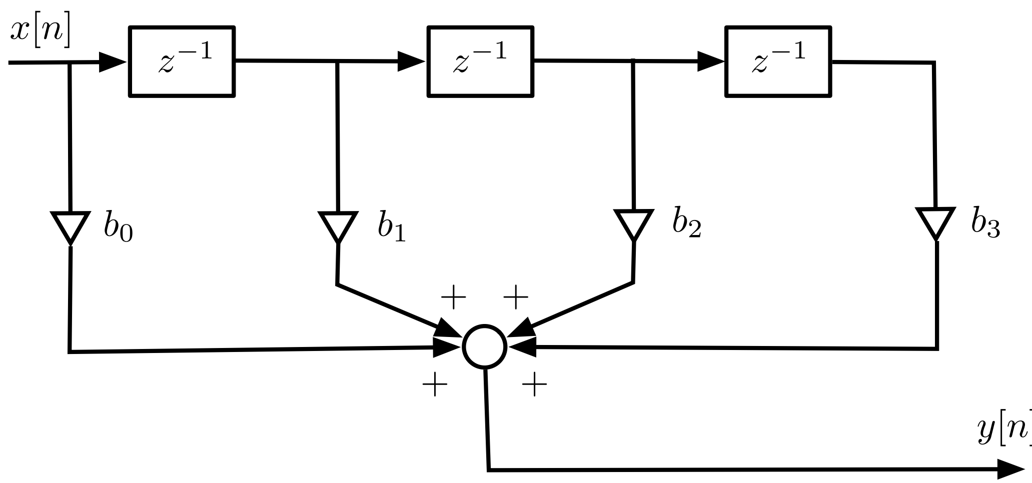
- Parallel realization

$$H(z) = \frac{B(z)}{A(z)} = C + \sum_{i=1}^{r} H_i(z)$$

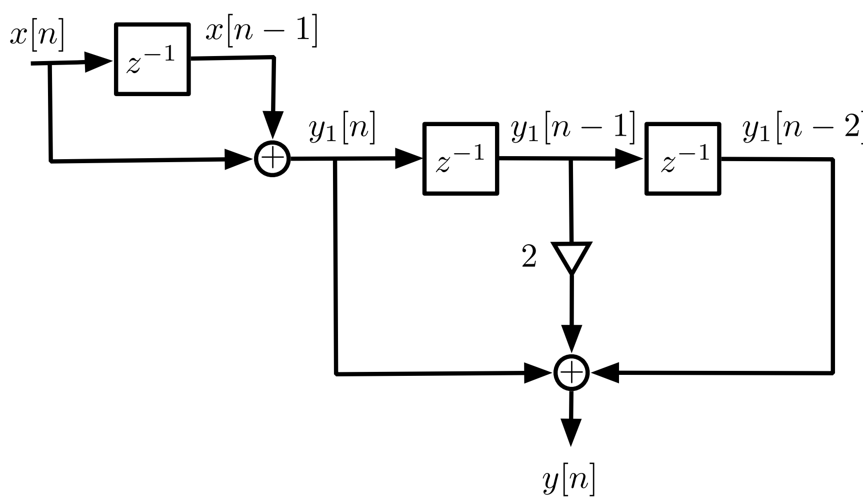
C constant, $H_i(z)$ first- or second-order filters with real coefficients



Parallel realization for $H(z) = (3 + 3.6z^{-1} + 0.6z^{-2})/(1 + 0.1z^{-1} - 0.2z^{-2})$



Direct form realization of FIR filter of order $M = 3$.



Cascade realization of FIR filter.