SIGNALS AND SYSTEMS USING MATLAB
Chapter 11 — Fourier Analysis of Discrete-time Signals and Systems

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$$\boxed{\begin{array}{ll} \text{DTFT} & X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}, \qquad -\pi \leq \omega < \pi \\ \text{IDTFT} & x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{array}}$$

- Periodic

$$X(e^{j(\omega+2\pi k)}) = \sum_n x[n]e^{-j(\omega+2\pi k)n} = X(e^{j\omega}) \qquad k \text{ integer}$$

- Sampling and DTFT

$$X_s(e^{j\omega}) = \mathcal{F}[x_s(t)] = \sum_n x(nT_s)\mathcal{F}[\delta(t-nT_s)] = \sum_n x(nT_s)e^{-jn\Omega T_s}$$

- Z-transform and the DTFT

$$X_s(e^{j\omega}) = X(z)|_{z=e^{j\omega}}, \quad UC \subset ROC$$

- Eigenvalues and the DTFT
  LTI system, input $x[n] = e^{j\omega_0 n}$, the steady-state output

$$y[n] = \sum_k h[k]x[n-k] = \sum_k h[k]e^{j\omega_0(n-k)} = e^{j\omega_0 n}H(e^{j\omega_0})$$

$$H(e^{j\omega_0}) = \sum_k h[k]e^{-j\omega_0 k}, \quad \text{DTFT}[h[n]]$$

Example: Non-causal $x[n] = \alpha^{|n|}$ with $|\alpha| < 1$

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{m=0}^{\infty} \alpha^m z^m - 1 \\ &= \frac{1-\alpha^2}{1-\alpha(z+z^{-1})+\alpha^2}, \quad \text{ROC: } |\alpha| < |z| < \frac{1}{|\alpha|} \\ X(e^{j\omega}) &= X(z)|_{z=e^{j\omega}} = \frac{1-\alpha^2}{(1+\alpha^2) - 2\alpha\cos(\omega)} \end{aligned}$$

The DTFT at $\omega = 0$ gives

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] e^{j0n} = \sum_{n=-\infty}^{\infty} \alpha^{|n|} = \frac{2}{1-\alpha} - 1 = \frac{1+\alpha}{1-\alpha}$$

equivalently

$$X(e^{j0}) = \frac{1-\alpha^2}{1-2\alpha+\alpha^2} = \frac{1-\alpha^2}{(1-\alpha)^2} = \frac{1+\alpha}{1-\alpha}$$

Dual pairs

$$\boxed{\begin{array}{rcl} \delta[n-k], \quad \text{integer } k & \Leftrightarrow & e^{-j\omega k} \\[2em] e^{-j\omega_0 n}, \quad -\pi \le \omega_0 < \pi & \Leftrightarrow & 2\pi\delta(\omega+\omega_0) \\[2em] \sum_k X[k]e^{-j\omega_k n} & \Leftrightarrow & \sum_k 2\pi X[k]\delta(\omega+\omega_k) \end{array}}$$

DTFT of

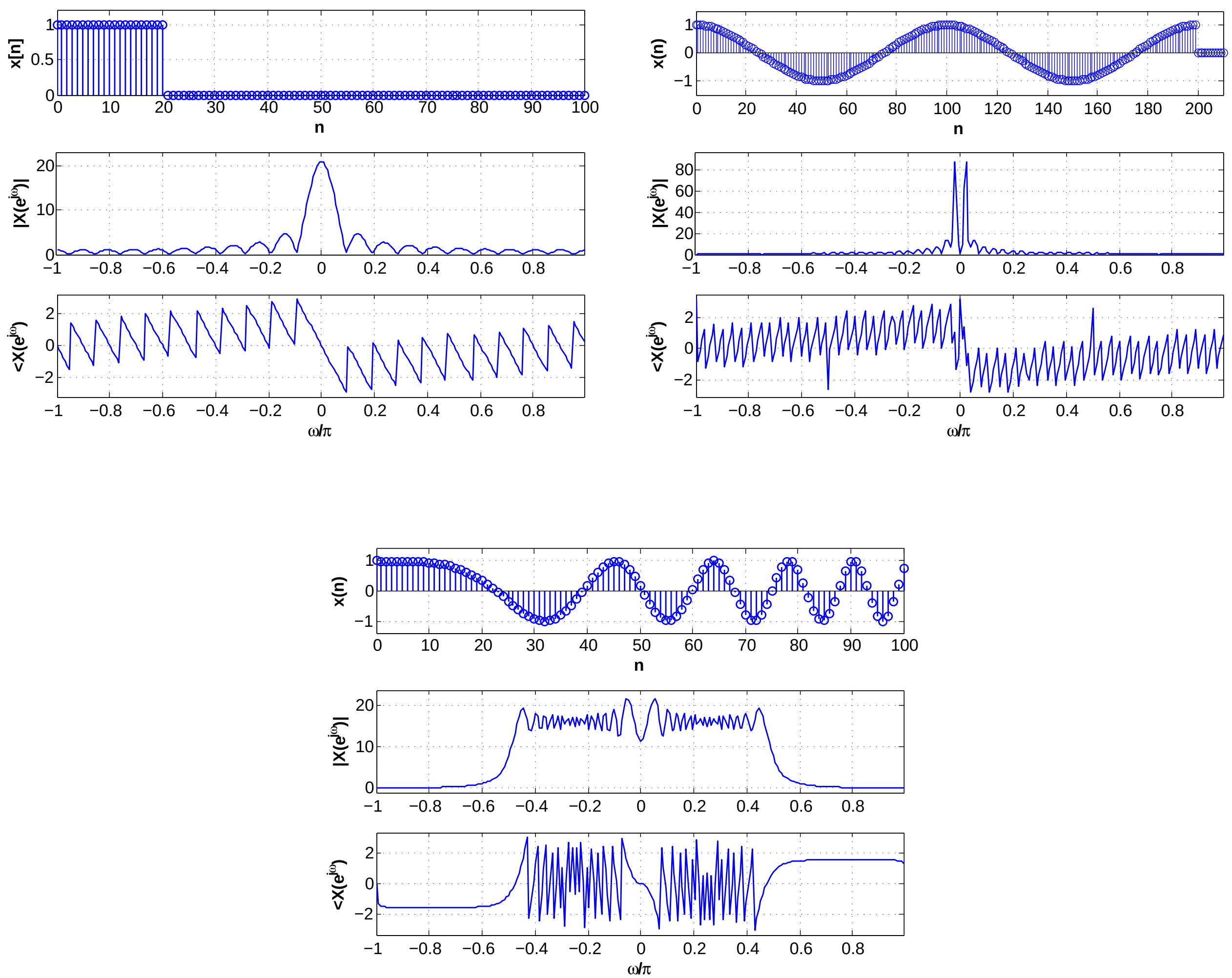
$$x[n] = \sum_\ell A_\ell \cos(\omega_\ell n + \theta_\ell) = \sum_\ell 0.5A_\ell(e^{j(\omega_\ell n+\theta_\ell)} + e^{-j(\omega_\ell n+\theta_\ell)})$$

$$X(e^{j\omega}) = \sum_\ell \pi A_\ell \left[e^{j\theta_\ell}\delta(\omega-\omega_\ell) + e^{-j\theta_\ell}\delta(\omega+\omega_\ell)\right] \qquad -\pi \le \omega < \pi$$

Example:

$$X(e^{j\omega}) = 1 + \delta(\omega-4) + \delta(\omega+4) + 0.5\delta(\omega-2) + 0.5\delta(\omega+2) \quad \Rightarrow$$

$$x[n] = \frac{1}{2\pi}\delta[n] + \frac{1}{0.5\pi}\cos(4n) + \frac{1}{\pi}\cos(2n)$$



DTFT of a pulse, a windowed sinusoid and a chirp: magnitude and phase spectra for each

- $x[n]$, band–limited to π/M in $[-\pi, \pi)$ or $|X(e^{j\omega})| = 0$, $|\omega| > \pi/M$ for an integer $M > 1$, can be down–sampled by a factor of M to generate a discrete-time signal

$$x_d[n] = x[Mn] \quad \text{with} \quad X_d(e^{j\omega}) = \frac{1}{M} X(e^{j\omega/M})$$

  an expanded version of $X(e^{j\omega})$.
- A signal $x[n]$ is up-sampled by a factor of $L > 1$ to generate a signal $x_u[n] = x[n/L]$ for $n = \pm kL$, $k = 0, 1, 2, \cdots$ and zero otherwise. The DTFT of $x_u[n]$ is $X(e^{jL\omega})$ or a compressed version of $X(e^{j\omega})$.

Example: Ideal low-pass filter with frequency response

$$H(e^{j\omega}) = \begin{cases} 1 & -\pi/2 \leq \omega \leq \pi/2 \\ 0 & -\pi \leq \omega < -\pi/2 \quad \text{and} \quad \pi/2 < \omega \leq \pi \end{cases}$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \begin{cases} 0.5 & n = 0 \\ \sin(\pi n/2)/(\pi n) & n \neq 0 \end{cases}$$

Down–sampled impulse response

$$h_d[n] = h[2n] = \begin{cases} 0.5 & n = 0 \\ \sin(\pi n)/(2\pi n) = 0 & n \neq 0 \end{cases} = 0.5\delta[n]$$

$$H_d(e^{j\omega}) = \frac{1}{2} H(e^{j\omega/2}) = \frac{1}{2}, \qquad -\pi \leq \omega < \pi$$

Example: Pulse $x[n] = u[n] - u[n-4]$ down–sampled by $M = 2$ gives

$$x_d[n] = x[2n] = u[2n] - u[2n-4] = u[n] - u[n-2]$$

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} \;\; \text{ROC: whole Z–plane (except for the origin)}$$

$$X(e^{j\omega}) = e^{-j(\frac{3}{2}\omega)}\left[e^{j(\frac{3}{2}\omega)} + e^{j(\frac{1}{2}\omega)} + e^{-j(\frac{1}{2}\omega)} + e^{-j(\frac{3}{2}\omega)}\right]$$

$$= 2e^{-j(\frac{3}{2}\omega)}\left[\cos\left(\frac{\omega}{2}\right) + \cos\left(\frac{3\omega}{2}\right)\right]$$

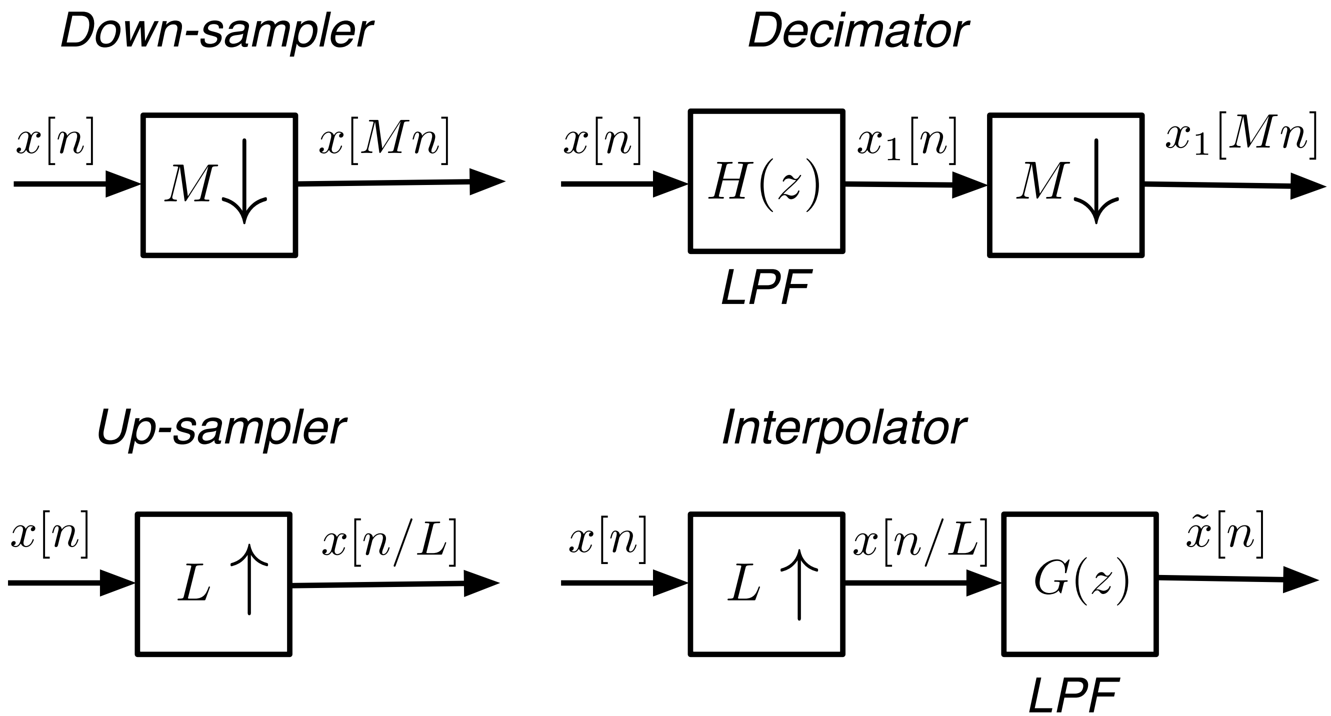
$$X_d(z) = 1 + z^{-2} \;\Rightarrow\; X_d(e^{j\omega}) = e^{-j\omega}\left[e^{j\omega} + e^{-j\omega}\right]$$

$$= 2e^{-j\omega}\cos(\omega)$$

$$X_d(e^{j\omega}) \neq \;\; 0.5X(e^{j\omega/2})$$

Aliasing: maximum frequency of $x[n]$ is not $\pi/M = \pi/2$

Passing $x[n]$ through ideal low-pass filter $H(e^{j\omega})$ with cut-off frequency $\pi/2$, output $x_1[n]$ has maximum frequency of $\pi/2$ and down–sampling it with $M = 2$ would give a signal with a DTFT $0.5X_1(e^{j\omega/2})$



Down–sampler and decimator (top) and up-sampler and interpolator (bottom)

$$H(e^{j\omega}) = \begin{cases} 1 & -\pi/M \le \omega \le \pi/M \\ 0 & \text{otherwise in } [-\pi, \pi) \end{cases}$$

$$G(e^{j\omega}) = \begin{cases} L & -\pi/L \le \omega \le \pi/L \\ 0 & \text{otherwise in } [-\pi, \pi) \end{cases}$$

Properties of the DTFT

Z-transform:	$x[n],\ X(z),\ \lvert z\rvert = 1 \in ROC$	$X(e^{j\omega}) = X(z)\rvert_{z=e^{j\omega}}$
Periodicity:	$x[n]$	$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)}), \quad k \ integer$
Linearity:	$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
Time-shifting:	$x[n-N]$	$e^{-j\omega N} X(e^{j\omega})$
Frequency-shift:	$x[n]e^{j\omega_o n}$	$X(e^{j(\omega-\omega_0)})$
Convolution:	$(x * y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Symmetry:	$x[n]$ real-valued	$\lvert X(e^{j\omega})\rvert$ even function of ω
		$\angle X(e^{j\omega})$ odd function of ω
Parseval's relation:	$\sum_{n=\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega}\rvert^2 d\omega$	

Example: DTFT of sinusoids cannot be found from the Z-transform or from the sum defining the DTFT

- Cosine – using frequency–shift property

$$
\begin{aligned}
x[n] &= \cos(\omega_0 n) = 0.5(e^{j\omega_0 n} + e^{-j\omega_0 n}) \\
X(e^{j\omega}) &= DTFT[0.5]_{\omega-\omega_0} + DTFT[0.5]_{\omega+\omega_0} = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]
\end{aligned}
$$

- Sine – using time–shift property

$$
\begin{aligned}
y[n] &= \sin(\omega_0 n) = \cos(\omega_0(n - \pi/(2\omega_0)) = x[n - \pi/(2\omega_0)] \\
Y(e^{j\omega}) &= X(e^{j\omega})e^{-j\omega\pi/(2\omega_0)} = \pi\left[\delta(\omega-\omega_0)e^{-j\omega\pi/(2\omega_0)} + \delta(\omega+\omega_0)e^{-j\omega\pi/(2\omega_0)}\right] \\
&= \pi\left[\delta(\omega-\omega_0)e^{-j\pi/2} + \delta(\omega+\omega_0)e^{j\pi/2}\right] = -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]
\end{aligned}
$$

Example: For $x[n] = \cos(\omega_0 n + \phi), \quad -\pi \le \phi < \pi$,

$$X(e^{j\omega}) = \pi\left[e^{-j\phi}\delta(\omega-\omega_0) + e^{j\phi}\delta(\omega+\omega_0)\right]$$

magnitude $|X(e^{j\omega})| = |X(e^{-j\omega})| = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

phase $\theta(\omega) = \begin{cases} \phi & \omega = -\omega_0 \\ -\phi & \omega = \omega_0 \\ 0 & \text{otherwise} \end{cases}$

Example: FIR filters

$$(i)\quad h_1[n] = \sum_{k=0}^{9} \frac{1}{10}\delta[n-k]$$

$$H_1(z) = \frac{1}{10}\sum_{n=0}^{9} z^{-n} = 0.1\frac{1-z^{-10}}{1-z^{-1}} = 0.1\frac{z^{10}-1}{z^9(z-1)} = 0.1\frac{\prod_{k=1}^{9}(z-e^{j2\pi k/10})}{z^9}$$

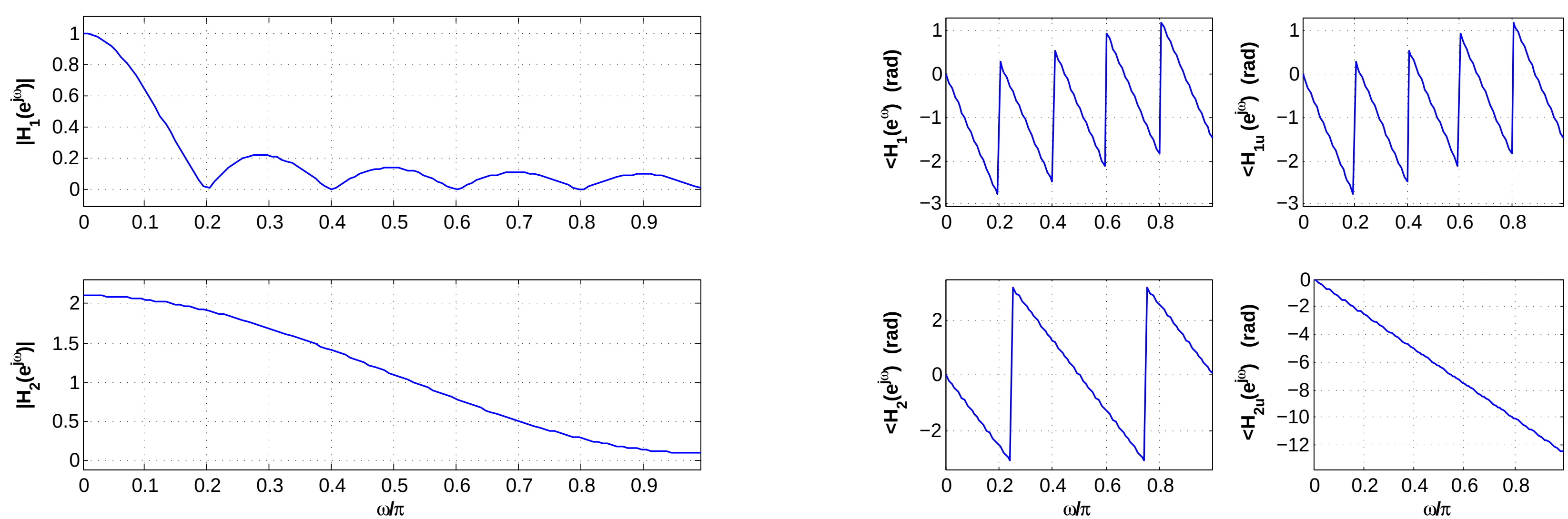
Because zeros on UC, its phase is not defined at the frequencies of the zeros (not continuous) and it cannot be unwrapped

$$(ii)\quad h_2[n] = 0.5\delta[n-3] + 1.1\delta[n-4] + 0.5\delta[n-5] \;\; \text{symmetric about} \, n = 4$$

$$H_2(z) = 0.5z^{-3} + 1.1z^{-4} + 0.5z^{-5} = z^{-4}(0.5z + 1.1 + 0.5z^{-1})$$

$$\text{frequency response } H_2(e^{j\omega}) = e^{-j4\omega}(1.1 + \cos(\omega))$$

Since $1.1 + \cos(\omega) > 0$ for $-\pi \leq \omega < \pi$, the phase $\angle H_2(e^{j\omega}) = -4\omega$, i.e., a linear phase.



$h[n]$ impulse response of stable LTI system, output

$$y[n] = \sum_k x[k]\ h[n-k], \quad x[n] \text{ (input)}$$

$$Y(z) = H(z)X(z) \qquad ROC : \mathcal{R}_Y = \mathcal{R}_H \cap \mathcal{R}_X$$

$$\text{UC} \subset \mathcal{R}_Y \quad \Rightarrow \quad Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad \text{or}$$

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Example: All–pass system or cascade systems with transfer functions

$$H_i(z) = K_i \frac{z - 1/\alpha_i}{z - \alpha_i^*} \qquad |z| > |\alpha_i|, \quad i = 1, \cdots N-1, \quad |\alpha_i| < 1, \ K_i > 0$$

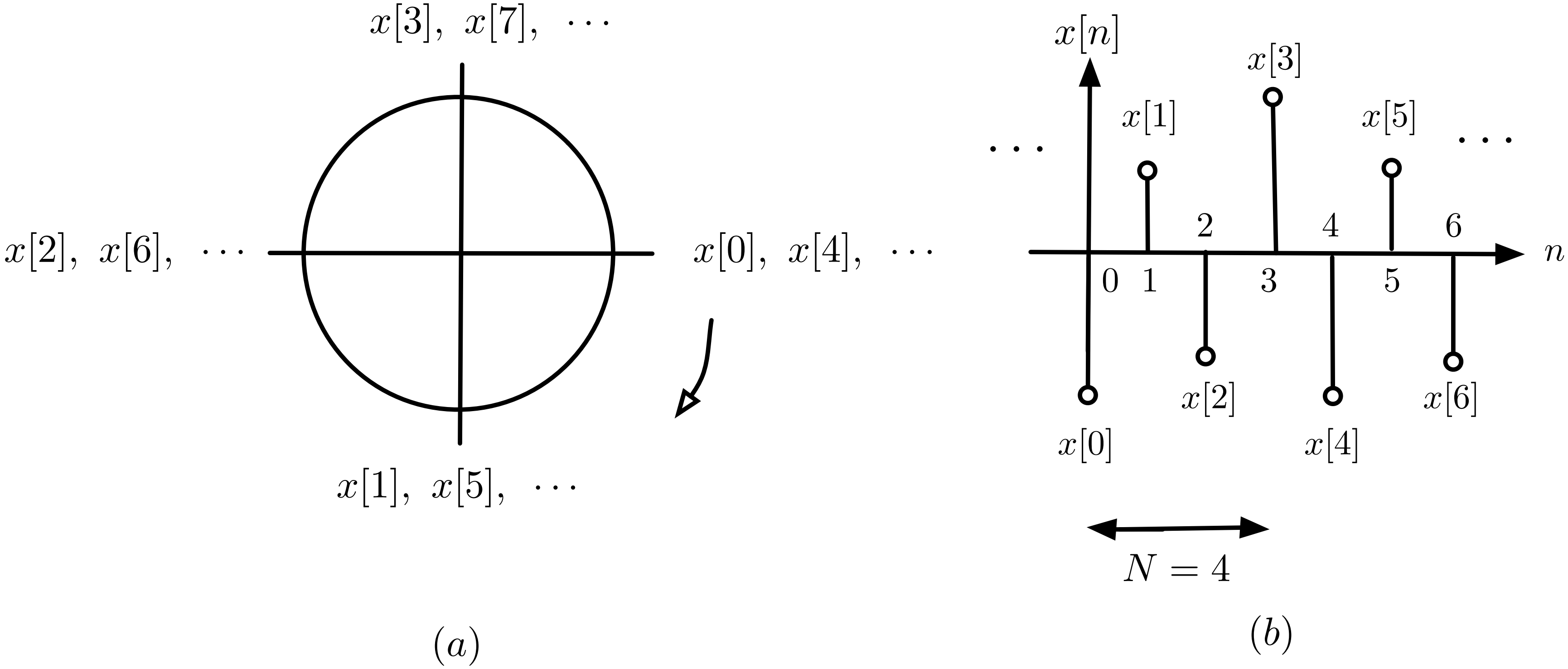
For zero $1/\alpha_i$ of $H_i(z)$, a pole α_i^* exists

$$|H_i(e^{j\omega})|^2 \ = \ H_i(e^{j\omega})H_i^*(e^{j\omega}) = K_i^2 \frac{e^{j\omega}(e^{-j\omega} - \alpha_i)e^{-j\omega}(e^{j\omega} - \alpha_i^*)}{\alpha_i\alpha_i^*(e^{j\omega} - \alpha_i^*)(e^{-j\omega} - \alpha_i)} = \frac{K_i^2}{|\alpha_i|^2}$$

$$H(e^{j\omega}) = \prod_i H_i(e^{j\omega}) = \prod_i |\alpha_i| \frac{e^{j\omega} - 1/\alpha_i}{e^{j\omega} - \alpha_i}, \quad \Rightarrow$$

$$|H(e^{j\omega})| = \prod_i |H_i(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = \sum_i \angle H_i(e^{j\omega})$$

$$Y(e^{j\omega}) = |X(e^{j\omega})| e^{j(\angle X(e^{j\omega}) + \angle H(e^{j\omega}))}$$



Circular (a) and linear (b) representations of a periodic discrete-time signal x[n]

Periodic signal $x[n]$ of fundamental period N

$$\text{Fourier series } x[n] = \sum_{k=k_0}^{k_0+N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

Fourier series coefficients

$$X[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

- Connection with the Z-transform

$$x_1[n] = x[n](u[n] - u[n-N]) \text{ period of } x[n]$$

$$\mathcal{Z}(x_1[n]) = \sum_{n=0}^{N-1} x[n] z^{-n} \quad \text{ROC: whole Z-plane, except for origin}$$

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} \mathcal{Z}(x_1[n]) \Big|_{z=e^{j\frac{2\pi}{N}k}}$$

Example: Periodic $x[n]$, fundamental period $N = 20$, and period $x_1[n] = u[n] - u[n-10]$

$$X[k] = \frac{z^{-5}(z^5 - z^{-5})}{20 z^{-0.5}(z^{0.5} - z^{-0.5})} \Big|_{z=e^{j\frac{2\pi}{20}k}} = \frac{e^{-j9\pi k/20} \sin(\pi k/2)}{20 \sin(\pi k/20)}$$

$$\boxed{\begin{aligned} &\text{Fourier series} \quad x[n] = \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N} \\ &X(e^{j\omega}) = \sum_{k=0}^{N-1} 2\pi X[k]\delta(\omega - 2\pi k/N) \qquad -\pi \leq \omega < \pi \end{aligned}}$$

Example: Periodic signal

$$\delta_M[n] = \sum_{m=-\infty}^{\infty} \delta[n - mM], \text{ fundamental period} M$$

$$\text{DTFT:} \quad \Delta_M(e^{j\omega}) = \sum_{m=-\infty}^{\infty} e^{-j\omega mM}$$

$$\text{Fourier series coefficients:} \quad \Delta_M[k] = \frac{1}{M} \sum_{n=0}^{M-1} \delta[n] e^{-j2\pi nk/M} = \frac{1}{M}$$

$$\text{Fourier series} \quad \delta_M[n] = \sum_{k=0}^{M-1} \frac{1}{M} e^{j2\pi nk/M}$$

$$\text{DTFT:} \quad \Delta_M(e^{j\omega}) = \frac{2\pi}{M} \sum_{k=0}^{M-1} \delta\left(\omega - \frac{2\pi k}{M}\right) \qquad -\pi \leq \omega < \pi$$

$x[n]$ periodic of fundamental period N input of LTI system

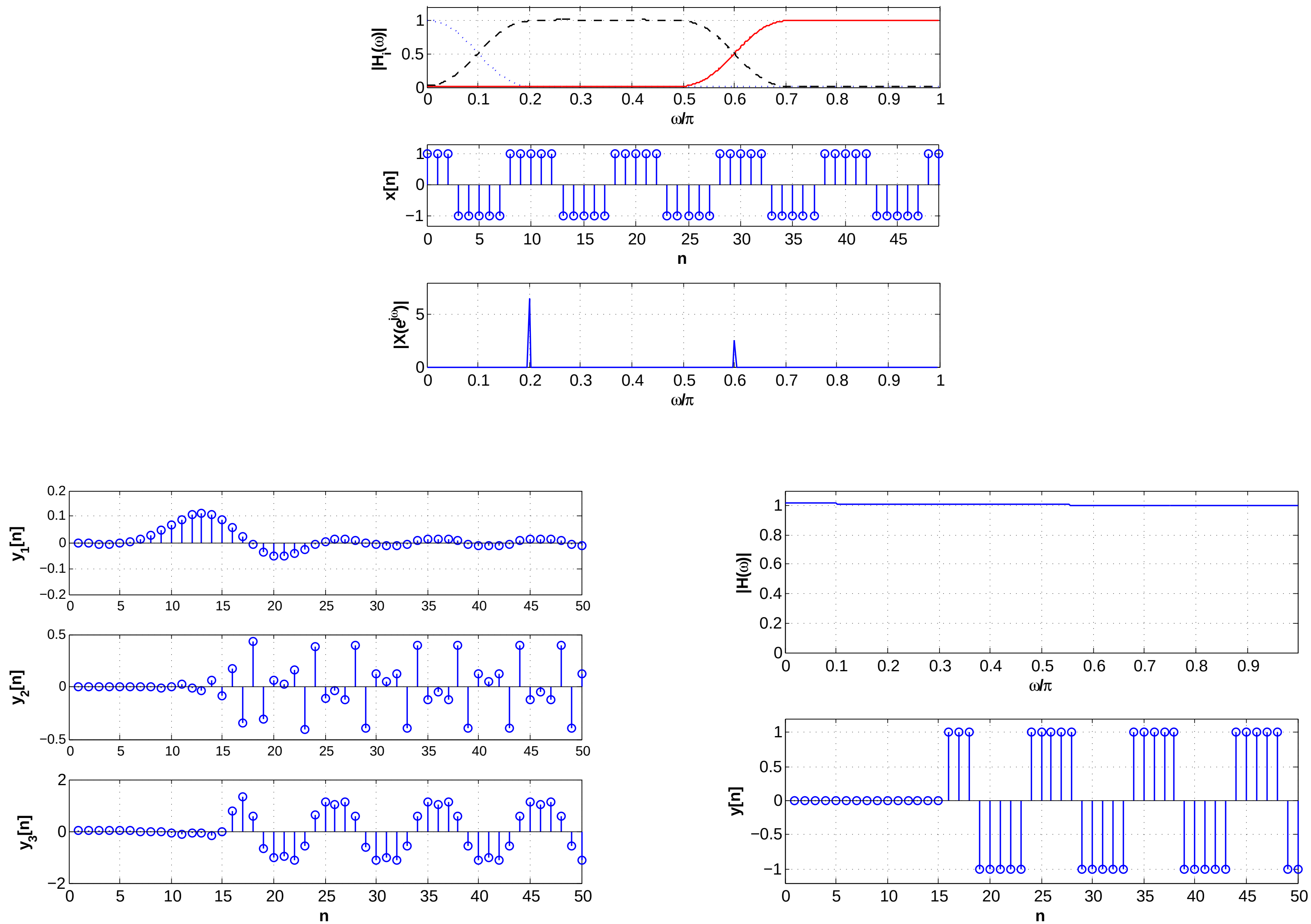
$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j(k\omega_0)n} \qquad \omega_0 = \frac{2\pi}{N}$$

eigenfunction property of LTI systems: periodic output

$$y[n] = \sum_{k=0}^{N-1} X[k] H(e^{jk\omega_0}) e^{jk\omega_0 n} \qquad \omega_0 = \frac{2\pi}{N}$$ fundamental frequency

coefficients $Y[k] = X[k]H(e^{jk\omega_0})$

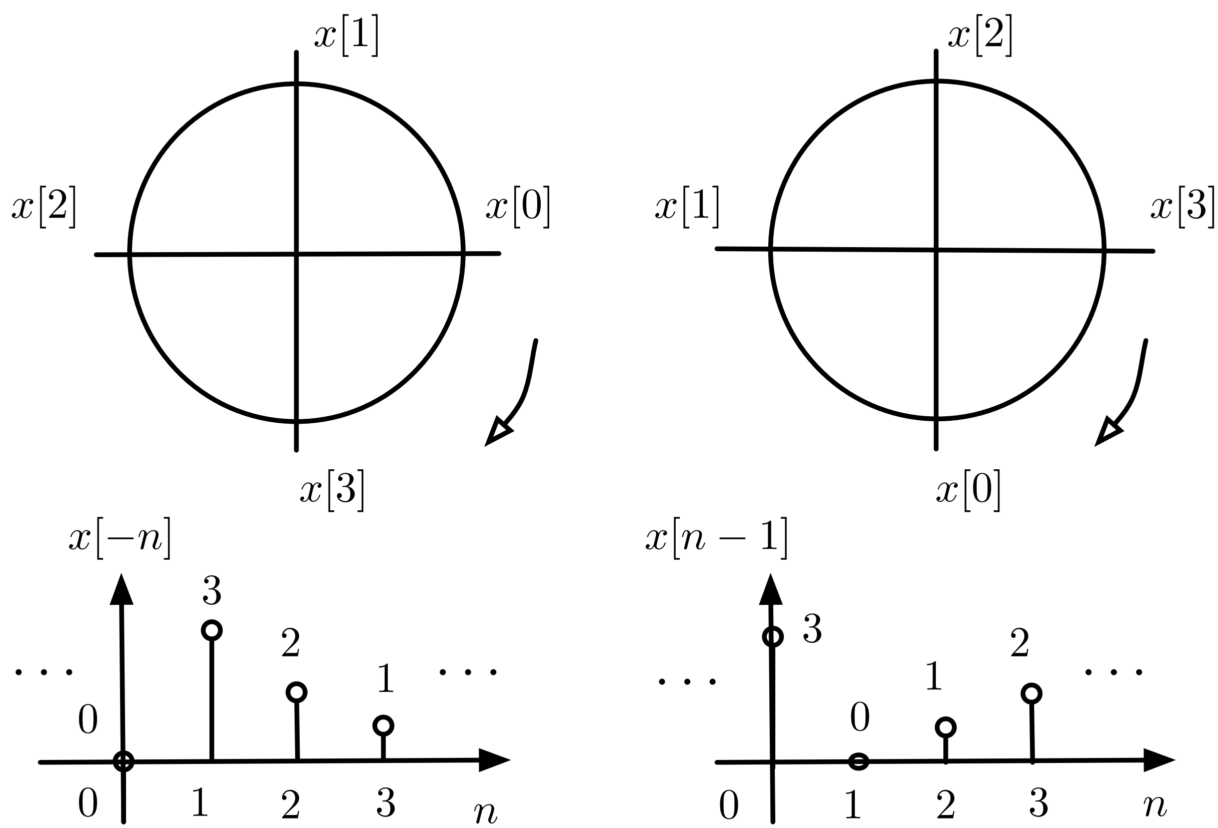
frequency response $H(e^{jk\omega_0}) = H(z)\,|_{z=e^{jk\omega_0}}$



Spectral analyzer: magnitude response of low-pass, band-pass and high-pass filters, input signal and its magnitude spectrum (top, center figure). Outputs of filters (bottom left), overall magnitude response of the bank of filters (all-pass filter) and overall response.

$$\boxed{x[n-M] \quad \Leftrightarrow \quad X[k]e^{-j2\pi Mk/N} \ \text{FS coefficients}}$$

Example: Linear shift vs circular shift
$x[n]$ periodic, of fundamental period $N = 4$ with period $x_1[n] = n, \ n = 0, \cdots, 3$.



Circular representation of $x[-n]$ and $x[n-1]$

Periodic signals $x[n]$ and $y[n]$ of the same fundamental period N

$$x[n]y[n] \quad \Leftrightarrow \quad \sum_{m=0}^{N-1} X[m]Y[k-m], \;\; 0 \le k \le N-1 \;\; \text{(periodic convolution)}$$

$$\text{(periodic convolution)} \;\; \sum_{m=0}^{N-1} x[m]y[n-m], \; 0 \le n \le N-1 \quad \Leftrightarrow \quad NX[k]Y[k]$$

Example: Multiplication of the Fourier series

$$x[n] = X[0] + X[1]e^{j\omega_0 n}, \quad y[n] = Y[0] + Y[1]e^{j\omega_0 n} \qquad \omega_0 = 2\pi/N = \pi$$

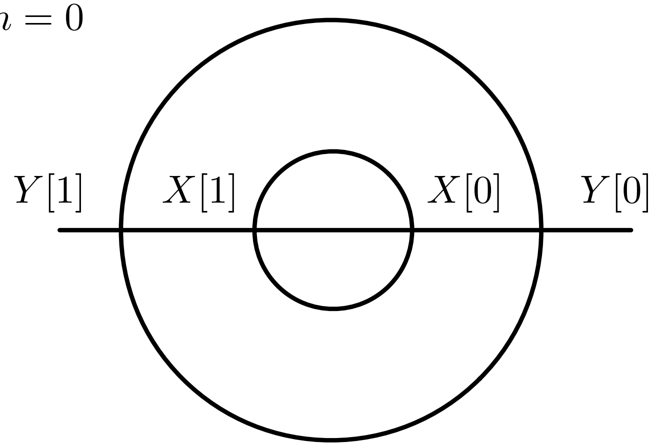
$$x[n]y[n] = \underbrace{(X[0]Y[0] + X[1]Y[1])}_{V[0]} + \underbrace{(X[0]Y[1] + X[1]Y[0])}_{V[1]} e^{j\omega_0 n}$$

Using the periodic convolution formula we have that

$$V[0] = \sum_{k=0}^{1} X[k]Y[-k] = X[0]Y[0] + X[1]Y[-1] = X[0]Y[0] + X[1]Y[2-1]$$

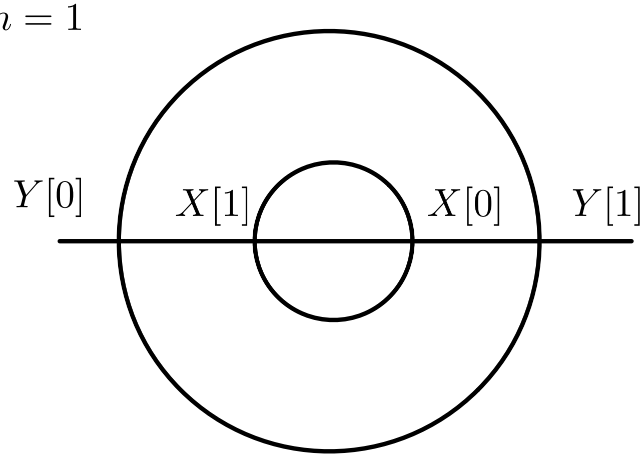
$$V[1] = \sum_{k=0}^{1} X[k]Y[1-k] = X[0]Y[1] + X[1]Y[0]$$

$m = 0$



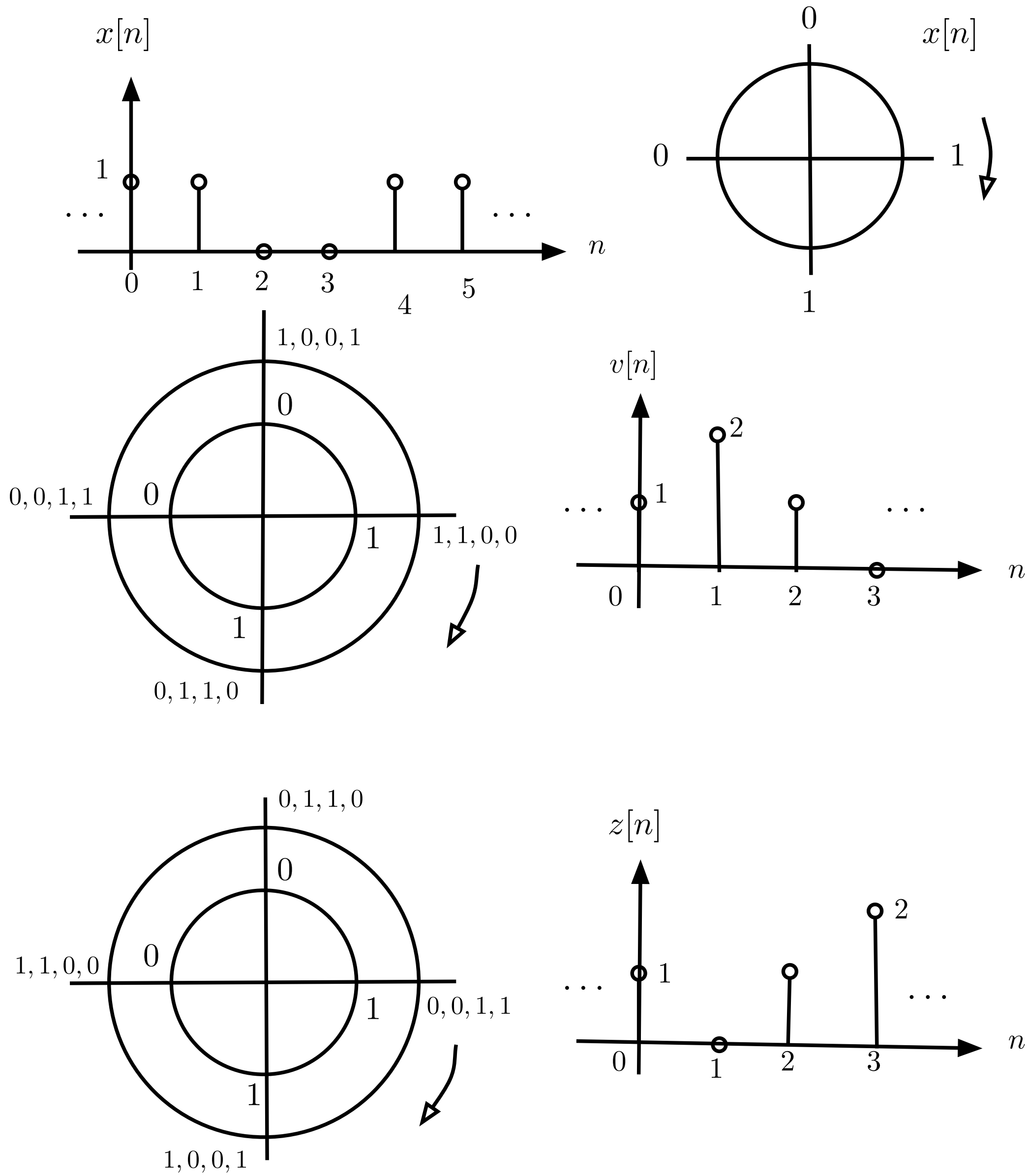
$$V[0] = X[0]Y[0] + X[1]Y[1]$$

$m = 1$



$$V[1] = X[0]Y[1] + X[1]Y[0]$$

Periodic convolution of the Fourier series coefficients $\{X[k]\}$ *and* $\{Y[k]\}$.



Linear and circular representations of $x[n]$ (top). Periodic convolution of $x[n]$ with itself to get $v[n]$, and linear representation of $v[n]$ (middle); periodic convolution of $x[n]$ and $y[n] = x[n-2]$, the result is $z[n] = v[n-2]$ represented linearly (bottom).

- Periodic signals

$$\boxed{\begin{aligned} &x[n] \text{ periodic, of fundamental period } N \\ &\text{DFT } X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \qquad 0 \leq k \leq N-1 \\ &\text{IDFT } x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N} \qquad 0 \leq n \leq N-1 \end{aligned}}$$

$X[k]$, $x[n]$ periodic of the same fundamental period N

- Fourier series and DFT

$$\text{periodic signal } \tilde{x}[n] \;\Rightarrow\; \text{FS: } \tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\omega_0 nk} \qquad 0 \leq n \leq N-1$$

FS coefficients

$$\tilde{X}[k] = \frac{1}{N} \mathcal{Z}[\tilde{x}_1[n]]\,|_{z=e^{jk\omega_0}} = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\omega_0 nk}, \;\; 0 \leq k \leq N-1, \;\; \omega_0 = 2\pi/N$$

$$\text{period } \tilde{x}_1[n] = \tilde{x}[n] W[n], \;\; W[n] = u[n] - u[n-N]$$

$$\text{DFT } X[k] = N\tilde{X}[k]$$

Aperiodic signal $y[n]$ of finite length N:

- Choose $L \geq N$, the length of the DFT, to be fundamental period of periodic extension $\tilde{y}[n]$ having $y[n]$ as a period with padded zeros if necessary

- Find DFT of $\tilde{y}[n]$,

$$\tilde{y}[n] = \frac{1}{L}\sum_{k=0}^{L-1} \tilde{Y}[k] e^{j2\pi nk/L} \qquad 0 \leq n \leq L-1$$

  and IDFT

$$\tilde{Y}[k] = \sum_{n=0}^{L-1} \tilde{y}[n] e^{-j2\pi nk/L} \qquad 0 \leq k \leq L-1$$

- DFT of $y[n]$: $Y[k] = \tilde{Y}[k]$ for $0 \leq k \leq L-1$, and

  IDFT of $Y[k]$: $y[n] = \tilde{y}[n]W[n]$, $0 \leq n \leq L-1$,

  $W[n] = u[n] - u[n-L]$ is a rectangular window of length L.

Given finite length $x[n]$ or period of periodic signal

- DFT is efficiently computed using FFT algorithm
- Causal aperiodic signal:inputting $\{x[n],\ n = 0, 1, \cdots, N-1\}$ into FFT gives $\{X[k],,\ k = 0, 1, \cdots, N-1\}$ or DFT of $x[n]$ using FFT of length $L = N$
  For $L > N$ DFT attach $L - N$ zeros at the end of the above sequence
- Non-causal aperiodic signal: $\{x[n],\ n = -n_0, \cdots 0, 1, \cdots, N - n_0 - 1\}$ use periodic extension to get

$$\underbrace{x[0]\ x[1]\ \cdots x[N-n_0-1]}_{\text{causal samples}}\ \underbrace{x[-n_0]\ x[-n_0+1]\cdots x[-1]}_{\text{non-causal samples}}$$

  $L > N$ DFT: zeros between the causal and non-causal components can be attached

$$\underbrace{x[0]\ x[1]\ \cdots x[N-n_0-1]}_{\text{causal samples}}\ 0\ 0\ \cdots\ 0\ 0\ \underbrace{x[-n_0]\ x[-n_0+1]\cdots x[-1]}_{\text{non-causal samples}}$$

- Periodic signal:$x[n]$ periodic of fundamental period N choose $L = N$ (or a multiple of N) to find DFT $X[k]$ using FFT
  If $L = MN$ (several periods) divide the obtained DFT by M

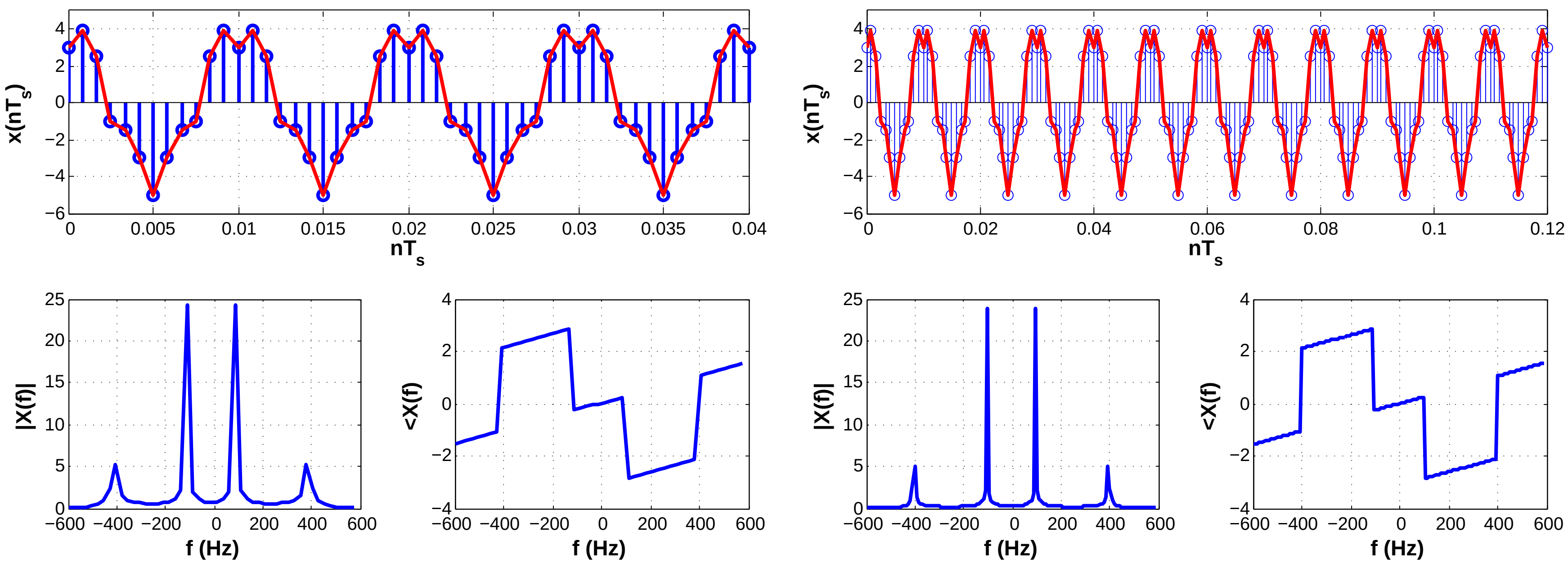
- Frequency resolution
  - $x[n]$ periodic of fundamental period N, non–zero frequency components exist only at harmonic frequencies $\{2\pi k/N\}$
  - $x[n]$ aperiodic, the number of frequency components depend on length L of DFT

  To increase number of frequencies considered or frequency resolution of the DFT
  - Aperiodic signal: increase number of samples in signal without distorting the signal by padding with zeros
  - Periodic signal: consider several periods and divide the DFT by number of periods used
- Frequency scales N-DFT of $x[n]$ of length N, is sequence of complex values $X[k]$ for $k = 0, 1, \cdots, N-1$, or the following equivalent frequency scale

$$
\begin{array}{l}
[0, 2\pi/N, \cdots, 2\pi(N-1)/N] \text{ (rad)} \\
[-\pi, -(N-2)\pi/N, \cdots, \pi - 2\pi/N] \text{ (rad)} \\
[-1, -(N-2)/N, \cdots, 1 - 2/N]
\end{array}
$$

For sampled signals

$$
\begin{array}{l}
T_s, \text{ sampling period}, \ f_s \text{ sampling frequency} \\
\Omega = \dfrac{\omega}{T_s} = \omega f_s \ \text{ (rad/sec)} \quad \text{or} \quad f = \dfrac{\omega}{2\pi T_s} = \dfrac{\omega f_s}{2\pi} \ \text{ (Hz)} \\
\text{giving scales} \\
[-\pi f_s, \cdots, \pi f_s] \ \text{ (rad/sec) and } \ [-f_s/2, \cdots, f_s/2] \ \text{ (Hz)}
\end{array}
$$



Computation of the FFT of a periodic signal using 4 and 12 periods to improve the frequency resolution of the FFT. Notice that both magnitude and phase responses look alike, but when we use 12 periods these spectra look sharper due to the increase in the number of frequency components added.

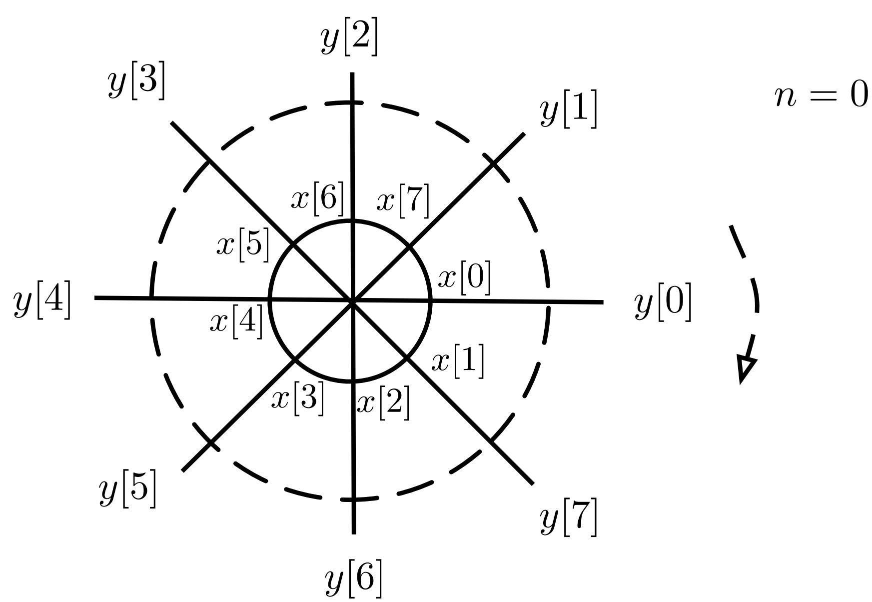
$x[n]$, of length M, input of an LTI system with impulse response $h[n]$ of length K

- Linear convolution
  - Find DFTs $X[k]$ and $H[k]$ of length $L \geq M + K - 1$ for $x[n]$ and $h[n]$
  - Multiply them to get $Y[k] = X[k]H[k]$.
  - Find the inverse DFT of $Y[k]$ of length L to obtain $y[n]$.

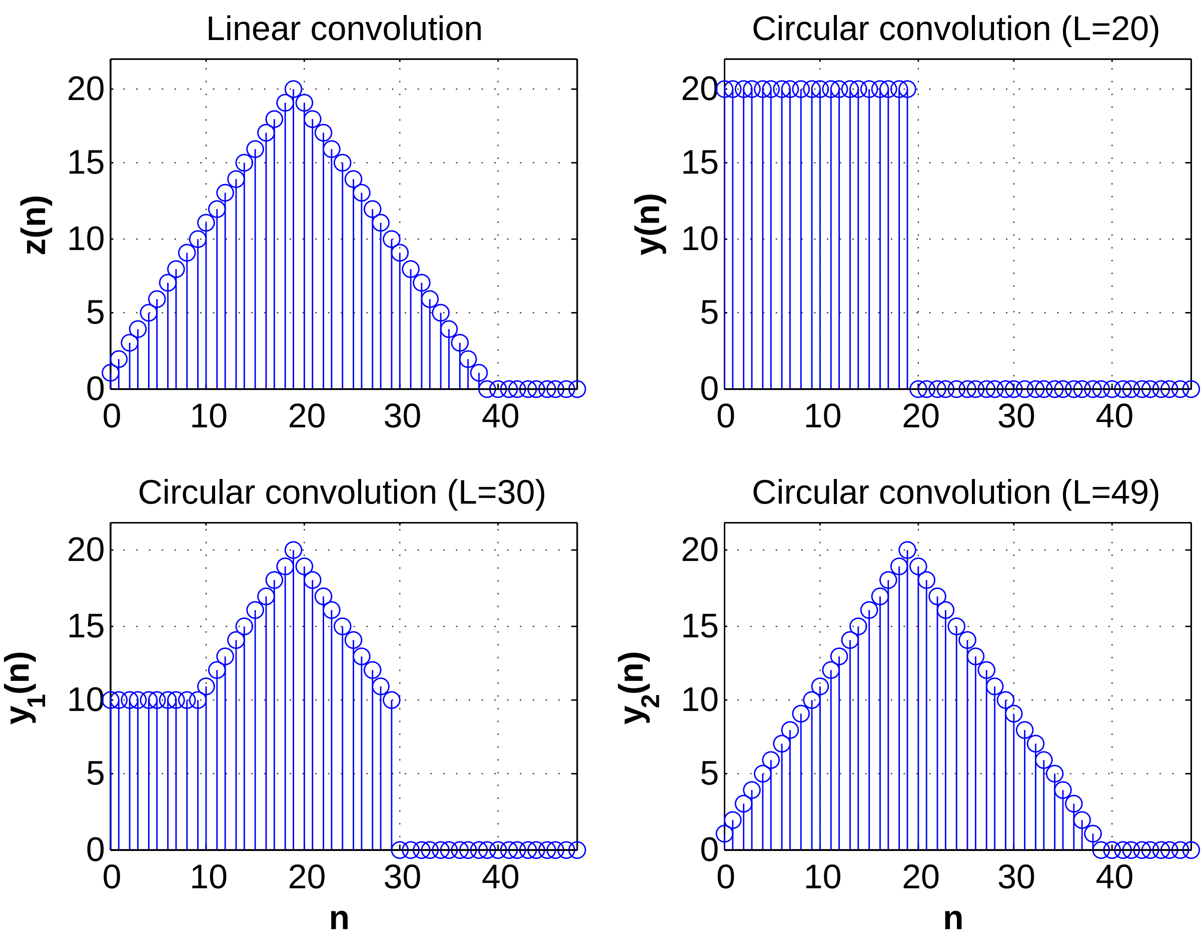
  Computationally efficient using FFT

- Linear vs circular convolutions

$$\boxed{\begin{array}{lcl} Y[k] = X[k]H[k] & \Leftrightarrow & y[n] = (x \otimes_L h)[n] \text{ circular convolution} \\ \text{If } L \geq M + K - 1 & \Rightarrow & y[n] = (x \otimes_L h)[n] = (x * h)[n] \end{array}}$$

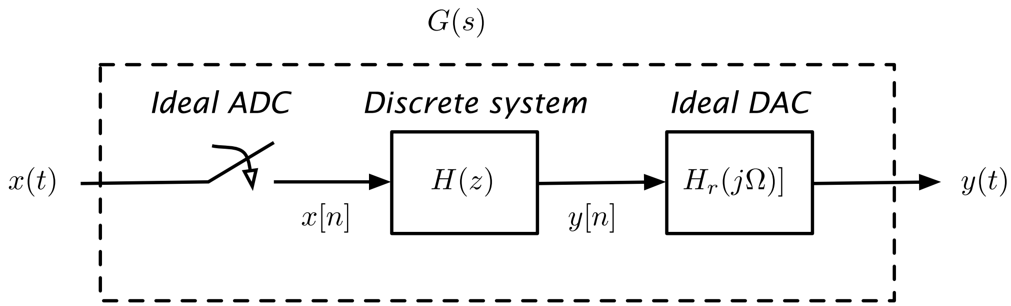


Circular convolution of length $L = 8$ of $x[n]$ and $y[n]$



Circular vs linear convolutions: Top-left: linear convolution of $x[n]$ with itself. Top-right and bottom-left: circular convolutions of $x[n]$ with itself of length $L < 2N - 1$. Bottom-right: circular convolution of $x[n]$ with itself of length $L > 2N - 1$ coinciding with the linear convolution.

- Discrete and continuous–time signals can be processed discretely using FFT



Equivalent analog system

Discrete processing of analog signals using A/D and D/A converters. $G(s)$ is the transfer function of the overall system, while $H(z)$ is the transfer function of the discrete-time system.

- Duality of DFT and IDFT – consider $x[n]$ complex

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad k = 0, \cdots, N-1, \quad W_N = e^{-j2\pi/N} \\
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0, \cdots, N-1
\end{aligned}
$$

- Complexity of algorithm:
  - Total number of additions and multiplications: direct calculation of $X[k]$, $k = 0, \cdots, N-1$, from DFT requires $N \times N$ complex multiplications, and $N \times (N-1)$ complex additions.
  - Storage: $\{X[k]\}$ are complex requiring $2N^2$ locations in memory

- Fundamental principle of "Divide and Conquer"
- Periodicity: W_N^{nk} periodic in n and k of fundamental period N

$$W_N^{nk} = \begin{cases} W_N^{(n+N)k} \\ W_N^{n(k+N)} \end{cases}$$

- Symmetry:

$$\left[W_N^{nk}\right]^* = W_N^{(N-n)k} = W_N^{n(N-k)}$$

- Decimation–in–time

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N/2-1} \left[x[2n] W_N^{k(2n)} + x[2n+1] W_N^{k(2n+1)}\right]$$

$$W_N^{k(2n)} = e^{-j2\pi(2kn)/N} = e^{-j2\pi kn/(N/2)} = W_{N/2}^{kn}$$

$$W_N^{k(2n+1)} = W_N^k W_{N/2}^{kn}$$

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{kn} = Y[k] + W_N^k Z[k]$$

Computation of $X[k]$:

$$\begin{aligned} X[k] &= Y[k] + W_N^k Z[k] \qquad k = 0, \cdots, (N/2) - 1 \\ X[k + N/2] &= Y[k + N/2] + W_N^{k+N/2} Z[k + N/2] \\ &= Y[k] - W_N^k Z[k] \qquad k = 0, \cdots, N/2 - 1 \end{aligned}$$

Matrix form

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{\Omega}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{\Omega}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_{N/2} \\ \mathbf{Z}_{N/2} \end{bmatrix} = \mathbf{A_1} \begin{bmatrix} \mathbf{Y}_{N/2} \\ \mathbf{Z}_{N/2} \end{bmatrix}$$

$\mathbf{I}_{N/2}$ unit matrix, $\mathbf{\Omega}_{N/2}$ diagonal matrix with entries $\{W_N^k,\ k = 0, \cdots, N/2 - 1\}$

If $N = 2^\gamma$, repeating above process

$$\mathbf{X}_N = \left[\prod_{i=1}^{\gamma} \mathbf{A}_i\right] \mathbf{P}_N\, \mathbf{x} \qquad \mathbf{x} = [x[0], \cdots, x[N-1]]^T$$

$\mathbf{P}_N$ permutation matrix

Number of operations of the order of $N \log_2 N = \gamma N <<$ the original number of order N^2.

Example: Decimation-in-time FFT algorithm for $N = 4$
Direct computation of DFT in matrix form

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & 1 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$\begin{aligned} W_4^4 &= W_4^{4+0} = e^{-j2\pi 0/4} = W_4^0 = 1 \\ W_4^6 &= W_4^{4+2} = e^{-j2\pi 2/4} = W_4^2 \\ W_4^9 &= W_4^{4+4+1} = e^{-j2\pi 1/4} = W_4^1 \end{aligned}$$

Number of real multiplications is 16×4 and of real additions is $12 \times 2 + 16 \times 2$ giving a total of 120 operations
In matrix form

$$\begin{bmatrix} X[0] \\ X[1] \\ \cdots \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 0 & W_4^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \vdots & -1 & 0 \\ 0 & 1 & \vdots & 0 & -W_4^1 \end{bmatrix} \begin{bmatrix} Y[0] \\ Y[1] \\ \cdots \\ Z[0] \\ Z[1] \end{bmatrix} = \mathbf{A}_1 \begin{bmatrix} Y[0] \\ Y[1] \\ Z[0] \\ Z[1] \end{bmatrix}$$

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Repeating process

$$\begin{bmatrix} Y[0] \\ Y[1] \\ \cdots \\ Z[0] \\ Z[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & \vdots & 0 & 0 \\ 1 & -1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 & 1 \\ 0 & 0 & \vdots & 1 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ \cdots \\ x[1] \\ x[3] \end{bmatrix} = \mathbf{A}_2 \begin{bmatrix} x[0] \\ x[2] \\ x[1] \\ x[3] \end{bmatrix}.$$

The scrambled $\{x[n]\}$ entries can be written

$$\begin{bmatrix} x[0] \\ x[2] \\ x[1] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \mathbf{P}_4 \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

finally giving

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \mathbf{A}_1 \, \mathbf{A}_2 \, \mathbf{P_4} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

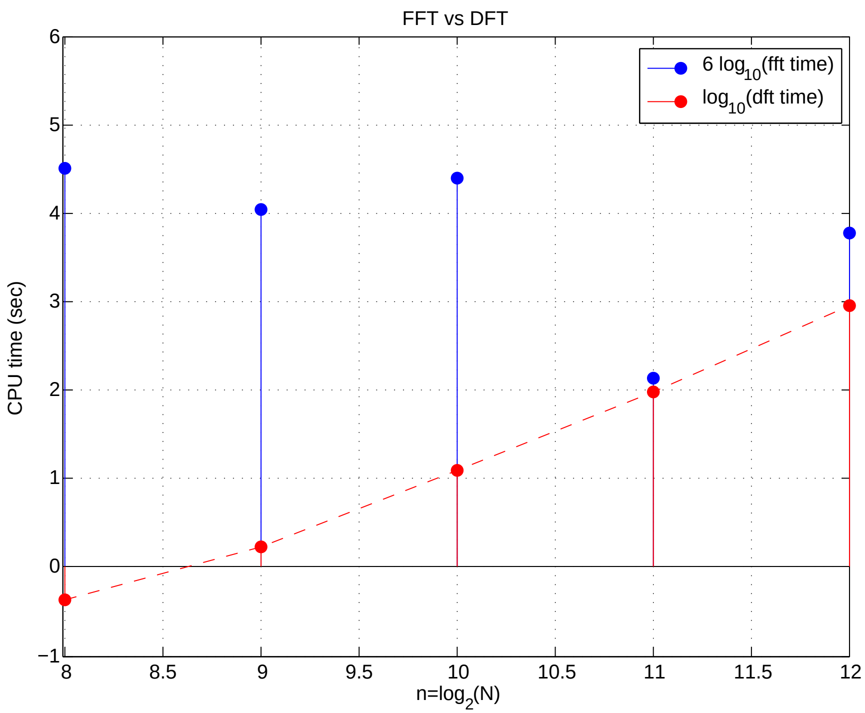
Number of complex additions and multiplications is now 10 (2 complex multiplications and 8 complex additions) not counting multiplications by 1 or -1 Computation of the Inverse DFT

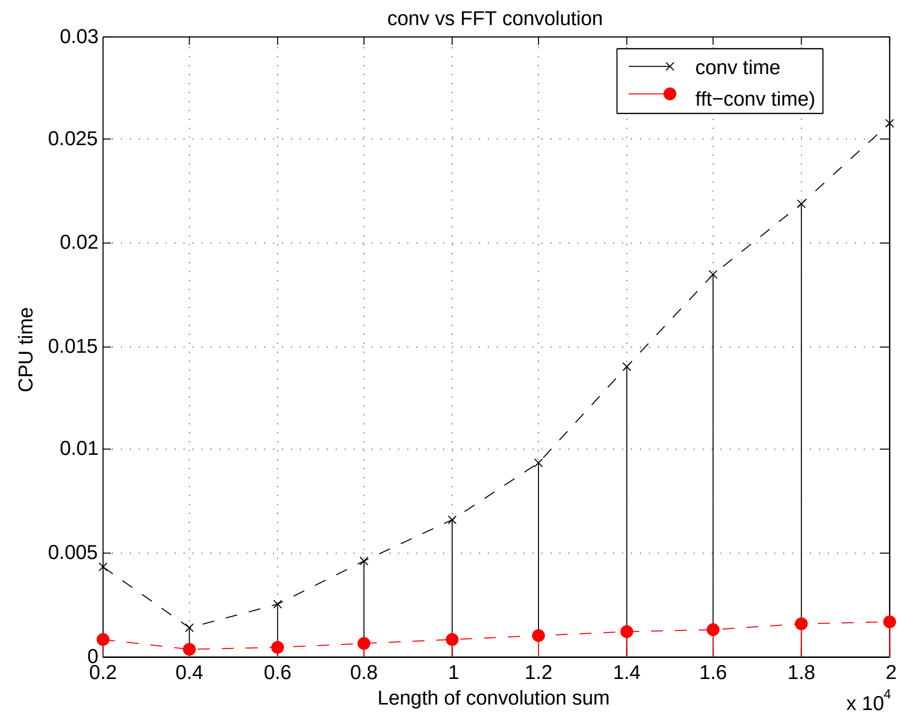
Assuming $x[n]$ complex

$$Nx^*[n] = \sum_{k=0}^{N-1} X^*[k] W^{nk}$$

use FFT algorithm of $\{X^*[k]\}$ to find $Nx^*[n]$, compute its complex conjugate and divide by N

Example: FFT and direct computation, convolution





Left: execution times for the fft *and the* dft *functions, in logarithmic scale, used in computing the DFT of sequences of ones of increasing lenghth $N = 256$ to 4096 (corresponding to $n = 8, \cdots 12$). The CPU time for the FFT is multiplied by 10^6. Right: comparison of execution times of convolution of sequence of ones with itself using MATLAB's conv function and a convolution sum implemented with FFT.*