

Signals and Systems Using MATLAB

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Chapter 5 - Frequency Analysis The Fourier Transform

What is in this chapter?

- From Fourier series to Fourier transform
- Existence of Fourier transform
- Fourier and Laplace transforms
- Time frequency relations and Fourier transform
- Spectral representation of periodic and aperiodic signals
 - Modulation and signal transmission
 - Convolution and Filtering

From the Fourier Series to the Fourier Transform

Aperiodic signal $x(t)$ is a periodic signal $\tilde{x}(t)$ with infinite period. From Fourier series $\tilde{x}(t)$ and limiting process we obtain:

$$x(t) \Leftrightarrow X(\Omega)$$

where $x(t)$ is transformed into $X(\Omega)$ in the frequency-domain by

Fourier transform:
$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

while $X(\Omega)$ is transformed into $x(t)$ in the time-domain by

Inverse Fourier Transform:
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{X(\Omega)} e^{j\Omega t} d\Omega$$

Aperiodic: $x(t) = \lim_{T_0 \rightarrow \infty} \tilde{x}(t)$ periodic of period T_0

Fourier Series:
$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \underline{X_n} e^{jn\Omega_0 t} \quad \Omega_0 = \frac{2\pi}{T_0}$$

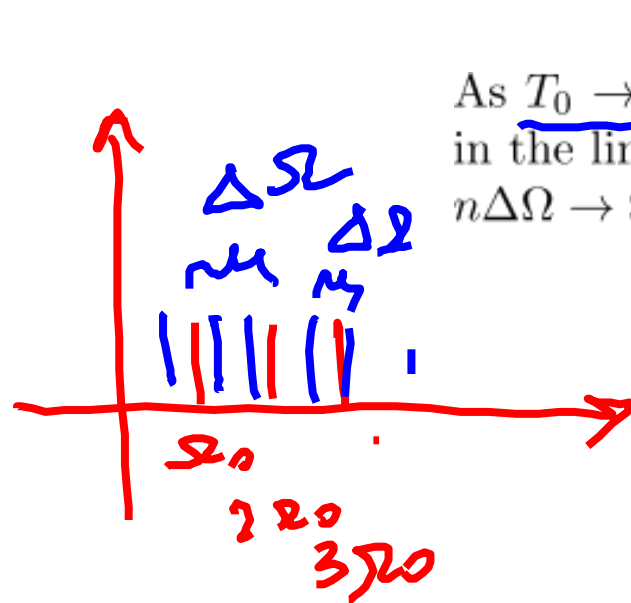
$$X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jn\Omega_0 t} dt$$

$T_0 \rightarrow \infty \quad \frac{1}{T_0} \rightarrow 0$

Define $X(\Omega_n) = T_0 X_n$ where $\{\Omega_n = n\Omega_0\}$, harmonic frequencies and $\Delta\Omega = 2\pi/T_0 = \Omega_0$ be frequency interval between harmonics then

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \frac{X(\Omega_n)}{T_0} e^{j\Omega_n t} = \sum_n X(\Omega_n) e^{j\Omega_n t} \frac{\Delta\Omega}{2\pi}$$

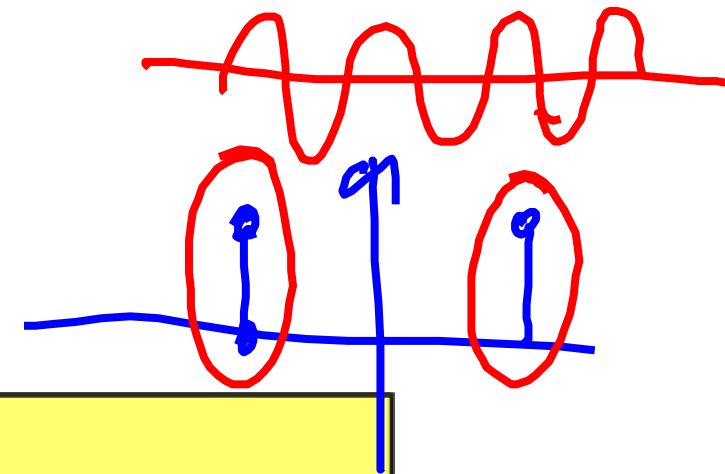
$$X(\Omega_n) = \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-j\Omega_n t} dt$$



As $T_0 \rightarrow \infty$ then $\Delta\Omega \rightarrow d\Omega$, the line spectrum becomes denser, i.e. the lines in the line spectrum get closer, the sum becomes an integral, and $\Omega_n = n\Omega_0 = n\Delta\Omega \rightarrow \Omega$ so that in the limit we obtain

$$\text{Inverse FT: } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

$$\text{FT: } X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$



Existence of the Fourier Transform *The Fourier transform*

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

of a signal $x(t)$ exists (i.e., we can calculate its Fourier transform via this integral) provided

- $x(t)$ is absolutely integrable or the area under $|x(t)|$ is finite,
- $x(t)$ has only a finite number of discontinuities as well as maxima and minima.

“It appears that almost nothing has a Fourier transform —nothing except practical communication signals. No signal amplitude goes to infinity and no signal lasts forever; therefore, no practical signal can have infinite area under it, and hence all have Fourier Transforms.” (Prof. E. Craig)

Signals of practical interest have Fourier transforms and their spectra can be displayed using a **spectrum analyzer**

Fourier Transforms from the Laplace Transform

If ROC of $X(s) = \mathcal{L}[x(t)]$ contains the $j\Omega$ -axis, then

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{L}[x(t)]|_{s=j\Omega} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \\ &= X(s)|_{s=j\Omega}\end{aligned}$$

Rules of Thumb for Computing the Fourier Transform of a Signal $x(t)$:

- If $x(t)$ has a finite time support and in that support $x(t)$ is finite, $\mathcal{F}[x(t)]$ exists
- If $x(t)$ has a Laplace transform $X(s)$ with ROC containing $j\Omega$ -axis, $X(\Omega) = X(s)|_{s=j\Omega}$.
- If $x(t)$ is periodic its Fourier transform is obtained from its Fourier series using delta functions
- If $x(t)$ is none of the above, if it has discontinuities (e.g., $x(t) = u(t)$), or it has discontinuities and it is not finite energy (e.g., $x(t) = \cos(\Omega_0 t)u(t)$) or it has possible discontinuities in the frequency domain even though it has finite energy (e.g., $x(t) = \text{sinc}(t)$) use properties of the Fourier transform.

Example Fourier transform using Laplace for

$$(a) \quad x_1(t) = u(t)$$

$$(b) \quad x_2(t) = e^{-2t}u(t)$$

$$(c) \quad x_3(t) = e^{-|t|}$$

(a) $\mathcal{L}[x_1(t)] = X_1(s) = 1/s$, ROC = $\{s = \sigma + j\Omega : \sigma > 0, -\infty < \Omega < \infty\}$,

Laplace transform cannot be used to find $\mathcal{F}[x_1(t)]$

(b) $\mathcal{L}[x_2(t)] = X_2(s) = 1/(s+2)$, ROC = $\{s = \sigma + j\Omega : \sigma > -2, -\infty < \Omega < \infty\}$

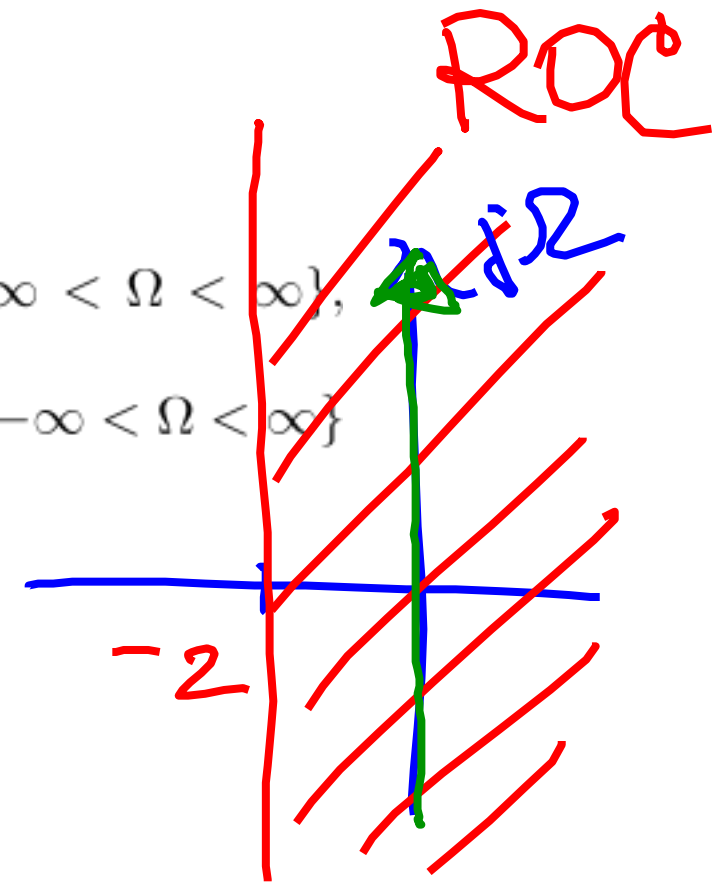
containing $j\Omega$ -axis, then

$$X_2(\Omega) = \frac{1}{s+2} \Big|_{s=j\Omega} = \frac{1}{j\Omega+2}$$

$$(c) \quad \mathcal{L}[x_3(t)] = X_3(s) = \frac{1}{s+1} + \frac{1}{-s+1} = \frac{2}{1-s^2}$$

ROC = $\{s = \sigma + j\Omega : -1 < \sigma < 1, -\infty < \Omega < \infty\}$ containing $j\Omega$ -axis, then

$$X_3(\Omega) = X_3(s) \Big|_{s=j\Omega} = \frac{2}{1-(j\Omega)^2} = \frac{2}{1+\Omega^2}$$



Inverse Proportionality of Time and Frequency

The support of $X(\Omega)$ is inversely proportional to the support of $x(t)$.

If $x(t)$ has a Fourier transform $X(\Omega)$ and $\alpha \neq 0$ is a real number, then $x(\alpha t)$ is

- is contracted ($\alpha > 1$),
- is contracted and reflected ($\alpha < -1$),
- is expanded ($0 < \alpha < 1$),
- is expanded and reflected ($-1 < \alpha < 0$) or
- is reflected ($\alpha = -1$)

$$x(\alpha t) \Leftrightarrow \frac{1}{|\alpha|} X\left(\frac{\Omega}{\alpha}\right)$$

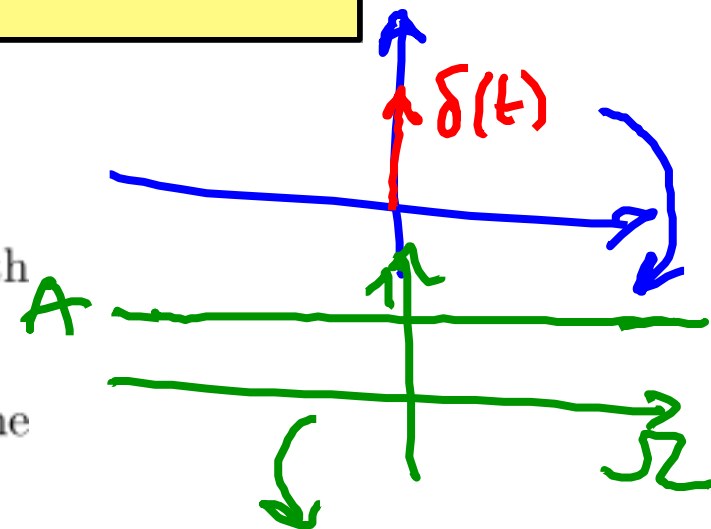
$\mathcal{F}(x(t)) = X(\Omega)$

$\alpha < 1$
 $\frac{1}{\alpha} > 1$

Frequency is inversely proportional to time:

- $x_1(t) = \delta(t)$, its support is only at $t = 0$, $X_1(\Omega) = 1, -\infty < \Omega < \infty$ with infinite support
- Opposite case: $x_2(t) = A, -\infty < t < \infty$, $X_2(\Omega) = 2\pi A\delta(\Omega)$ since the inverse Fourier transform is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\Omega) e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi A\delta(\Omega) e^{j\Omega t} d\Omega = A$$



- Transition $x_2(t)$ to $x_1(t)$: consider $x_3(t) = A[u(t + \tau/2) - u(t - \tau/2)]$ with

$$X_3(s) = \frac{A}{s} [e^{s\tau/2} - e^{-s\tau/2}]$$

so that

$$X_3(\Omega) = X(s)|_{s=j\Omega} = A\tau \frac{\sin(\Omega\tau/2)}{\Omega\tau/2}$$

$A = 1/\tau$ as $\tau \rightarrow 0$ the pulse $x_3(t)$ becomes $\delta(t)$ and $X_3(\Omega)$ becomes unity
 $\tau \rightarrow \infty, x_3(t) \rightarrow A X_3(\Omega) \rightarrow \delta(\Omega)$

NOTE: $x(t) \Leftrightarrow X(\Omega)$ means to $x(t)$ in time domain corresponds a FT $X(\Omega)$ in the frequency domain. This is NOT an equality, far from it!

Example If $x(t) = u(t) - u(t - 1)$, find FT of $x_1(t) = x(2t)$

$$X(s) = \frac{1 - e^{-s}}{s} \quad \text{ROC: whole s-plane}$$

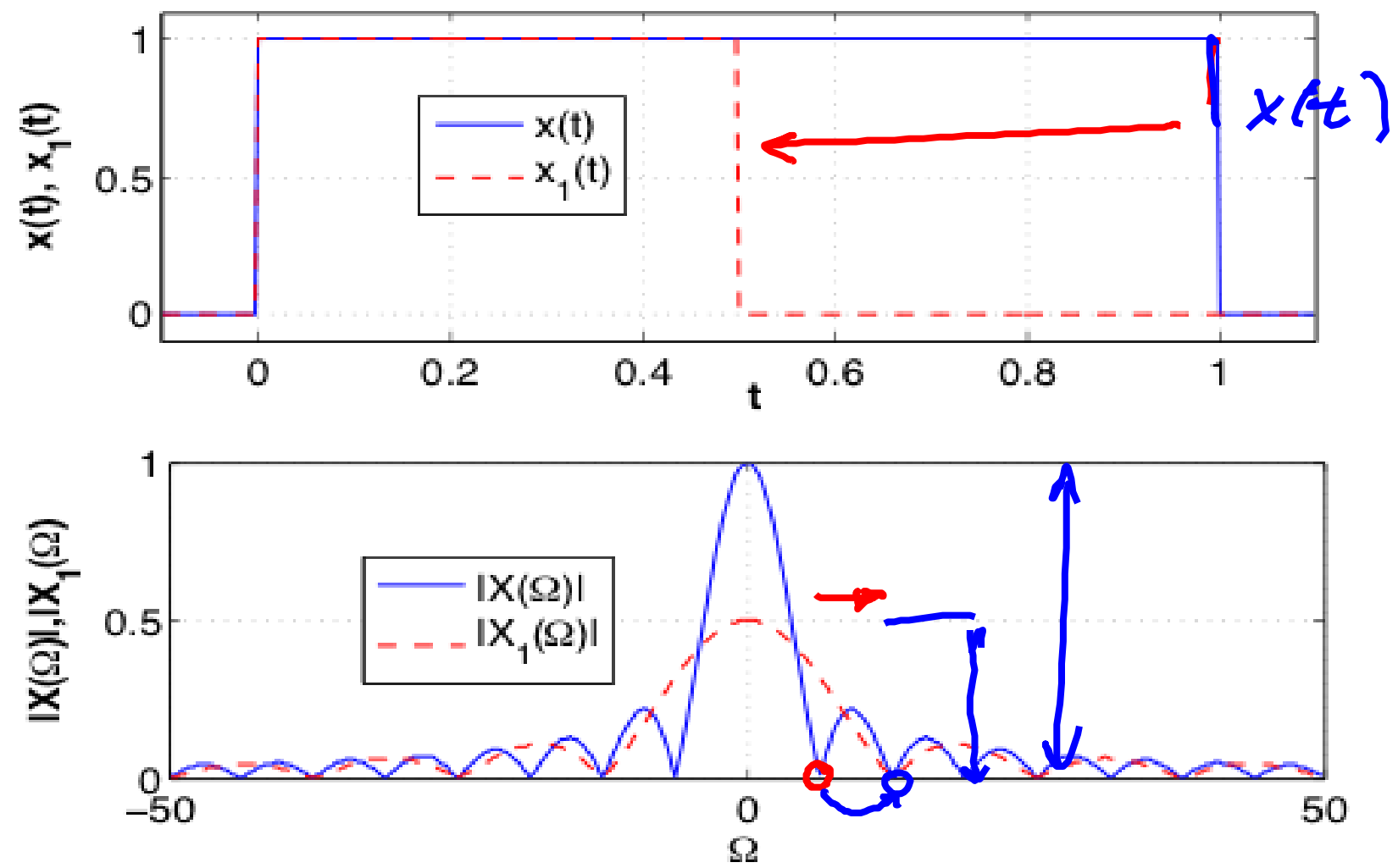
$$X(\Omega) = \frac{1 - e^{-j\Omega}}{j\Omega} = \frac{e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})}{2j\Omega/2} = \frac{\sin(\Omega/2)}{\Omega/2} e^{-j\Omega/2}$$

$$x_1(t) = x(2t) = u(2t) - u(2t - 1) = u(t) - u(t - 0.5)$$

$$X_1(\Omega) = \frac{1 - e^{-j\Omega/2}}{j\Omega} = \frac{e^{-j\Omega/4}(e^{j\Omega/4} - e^{-j\Omega/4})}{j\Omega}$$

$$= \frac{1}{2} \frac{\sin(\Omega/4)}{\Omega/4} e^{-j\Omega/4} = \frac{1}{2} X(\Omega/2)$$

$$\frac{1}{2} X\left(\frac{\Omega}{2}\right)$$



Duality

To the Fourier transform pair

$$x(t) \Leftrightarrow X(\Omega)$$

corresponds the following dual Fourier transform pair

$$X(t) \Leftrightarrow 2\pi x(-\Omega)$$

Inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\rho) e^{j\rho t} d\rho$$

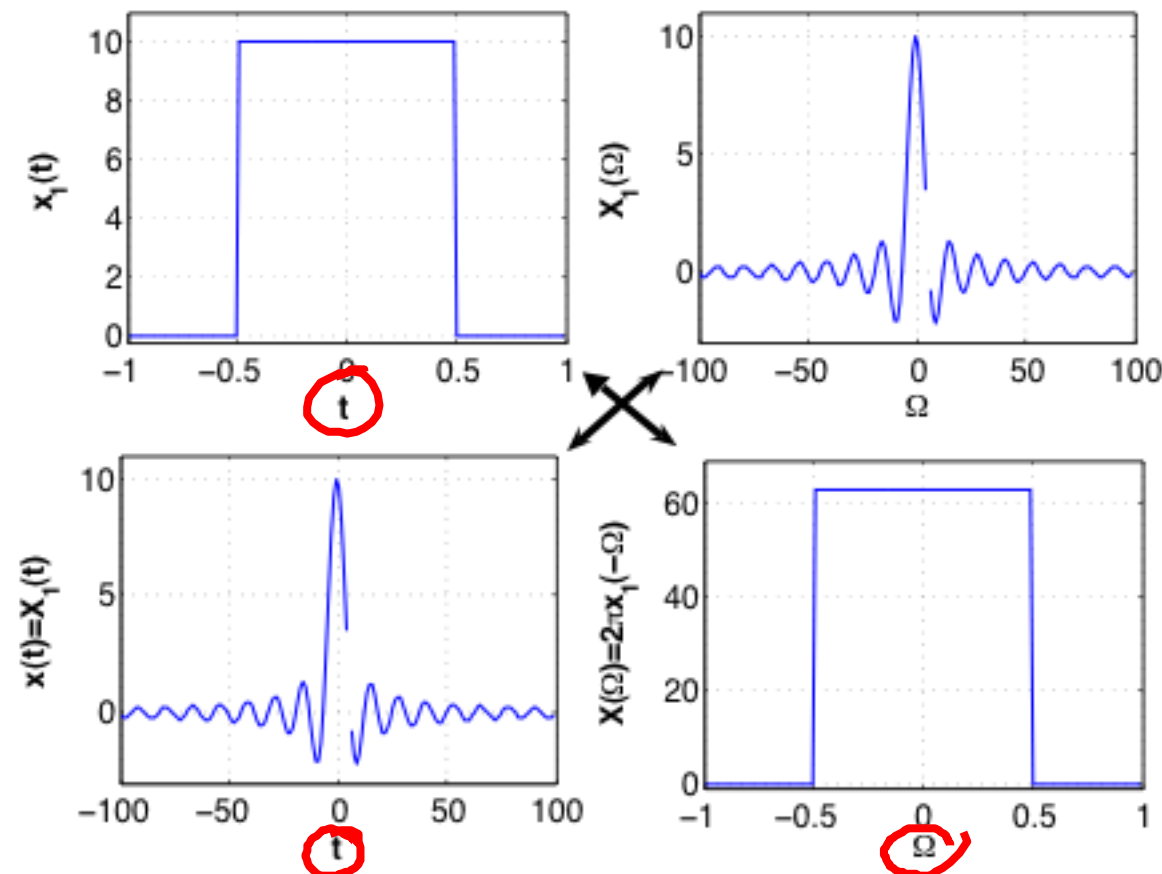
replace t by $-\Omega$ and multiply by 2π :

$$2\pi x(-\Omega) = \int_{-\infty}^{\infty} X(\rho) e^{-j\rho\Omega} d\rho = \int_{-\infty}^{\infty} X(t) e^{-j\Omega t} dt = \mathcal{F}[X(t)]$$

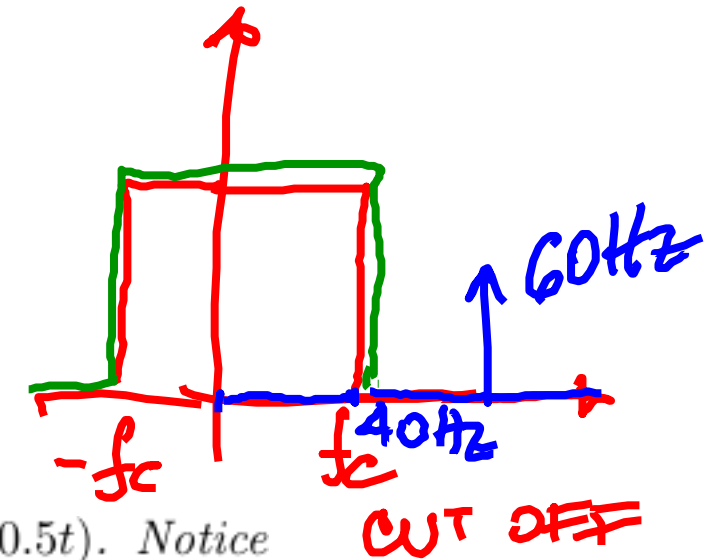
ρ and t are dummy variables inside the integral

- Example: Fourier transform pair

$$\begin{array}{ccc} A\delta(t) & \Leftrightarrow & A \\ A & \Leftrightarrow & 2\pi A\delta(-\Omega) = 2\pi A\delta(\Omega) \end{array}$$



sinc



Application of duality to find Fourier transform of $x(t) = 10\text{sinc}(0.5t)$. Notice that $X(\Omega) = 2\pi x_1(\Omega) \approx 6.28x_1(\Omega) = 63.8[u(\Omega + 0.5) - u(\Omega - 0.5)]$

Example Find the Fourier transform of $x(t) = A \cos(\Omega_0 t)$ using duality
Consider the following Fourier pair

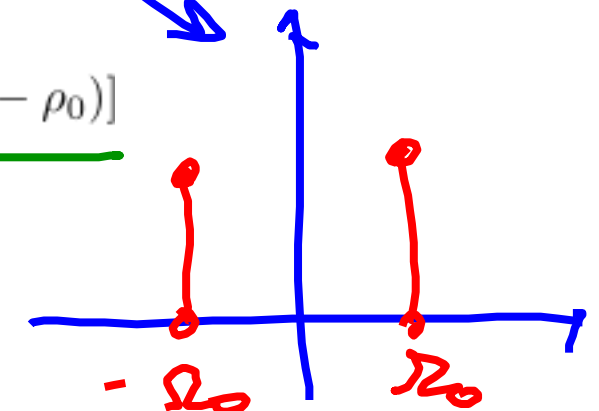
$$\delta(t - \rho_0) + \delta(t + \rho_0) \Leftrightarrow e^{-j\rho_0\Omega} + e^{j\rho_0\Omega} = 2 \cos(\rho_0\Omega)$$

$$2 \cos(\rho_0 t) \Leftrightarrow 2\pi[\delta(-\Omega - \rho_0) + \delta(-\Omega + \rho_0)] = 2\pi[\delta(\Omega + \rho_0) + \delta(\Omega - \rho_0)]$$

Replacing ρ_0 by Ω_0 and canceling the 2 in both sides we have

$$x(t) = \cos(\Omega_0 t) \Leftrightarrow X(\Omega) = \pi[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)]$$

indicating that it only exists at $\pm\Omega_0$



Spectral Representation — Unification of the spectral representation of both periodic and aperiodic signals

Signal Modulation

Frequency shift: If $X(\Omega)$ is the Fourier transform of $x(t)$, then we have the pair

$$x(t)e^{j\Omega_0 t} \Leftrightarrow X(\Omega - \Omega_0)$$

Modulation: The Fourier transform of the modulated signal

$$x(t) \cos(\Omega_0 t)$$

is given by

$$0.5 [X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

i.e., $X(\Omega)$ is shifted to frequencies Ω_0 and $-\Omega_0$, and multiplied by 0.5.

$\rightarrow 88.3 \text{ MHz}$

Remarks

- Amplitude modulation: consists in multiplying message $x(t)$ by a sinusoid of frequency higher than the maximum frequency of the incoming signal

$$\text{Modulated signal } x(t) \cos(\Omega_0 t) = 0.5[x(t)e^{j\Omega_0 t} + x(t)e^{-j\Omega_0 t}]$$

$$\mathcal{F}[x(t) \cos(\Omega_0 t)] = 0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

Modulation shifts the frequencies of $x(t)$ to frequencies around $\pm\Omega_0$

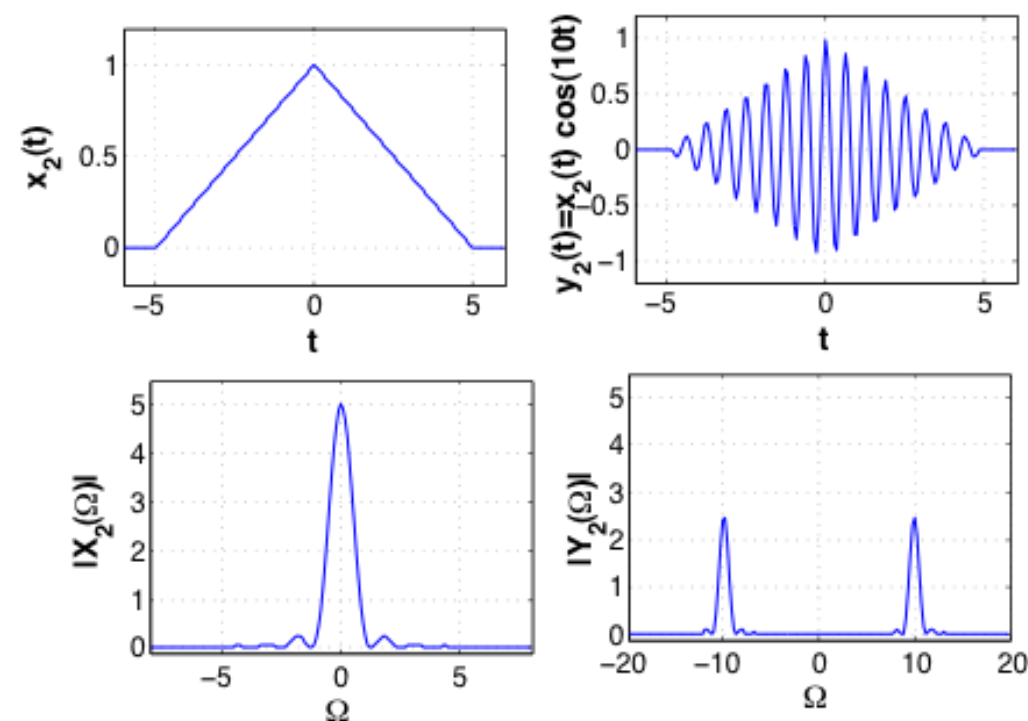
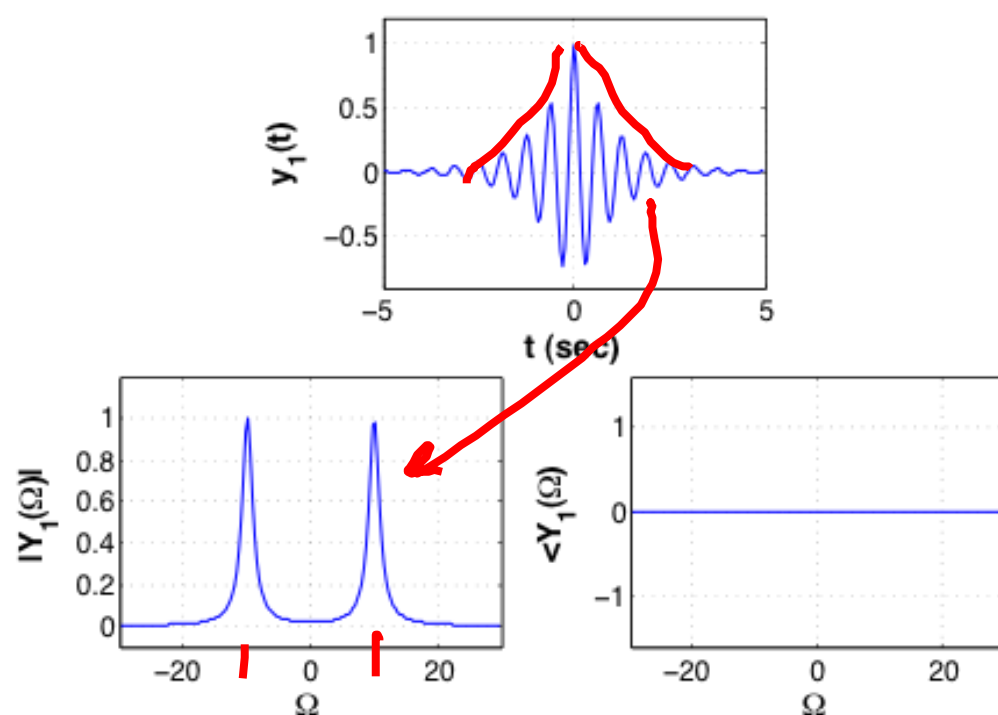
- Modulation using a sine, instead of a cosine, changes the phase of the Fourier transform of the incoming signal besides performing the frequency shift
- According to eigenfunction property of LTI systems, modulation systems are not LTI.

Example Modulate a carrier $\cos(10t)$ with:

1. $x_1(t) = e^{-|t|}$, $-\infty < t < \infty$. $x_1(t)$ is low-pass signal
see spectrum before
2. $x_2(t) = 0.2[r(t+5) - 2r(t) + r(t-5)]$.

The modulated signals are

- (i) $y_1(t) = x_1(t) \cos(10t) = e^{-|t|} \cos(10t)$, $-\infty < t < \infty$
- (ii) $y_2(t) = x_2(t) \cos(10t) = 0.2[r(t+5) - 2r(t) + r(t-5)] \cos(10t)$



Why Modulation? Modulation changes frequency content of a message from its baseband frequencies to higher frequencies making its transmission over the airwaves possible

Music ($0 \leq f \leq 22\text{KHz}$), and speech ($100 \leq f \leq 5\text{KHz}$) relatively low frequency signals requiring an antenna of length

$$\frac{\lambda}{4} = \frac{3 \times 10^8}{4f} \quad \text{meters}$$

if $f = 30\text{KHz} \Rightarrow$ length of antenna $2.5\text{km} \approx 1.5\text{miles}$

thus need to increase baseband frequencies.

Fourier Transform of Periodic Signals

A periodic signal $x(t)$ of period T_0 :

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \quad \Leftrightarrow \quad \underline{X(\Omega)} = \sum_k \underline{2\pi X_k \delta(\Omega - k\Omega_0)}$$

obtained by representing $x(t)$ by its Fourier series.

Fourier series of $x(t)$:

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \quad \Omega_0 = 2\pi/T_0$$

$$X(\Omega) = \sum_k \mathcal{F}[X_k e^{jk\Omega_0 t}] = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

Remarks

- $|X(\Omega)|$ vs Ω , the Fourier magnitude spectrum of periodic $x(t)$ is analogous to its line spectrum
- Direct computation

$$\mathcal{F}[\cos(\Omega_0 t)] = \mathcal{F}[0.5e^{j\Omega_0 t} + 0.5e^{-j\Omega_0 t}] = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

$$\begin{aligned} \mathcal{F}[\sin(\Omega_0 t)] &= \mathcal{F}\left[\frac{0.5}{j}e^{j\Omega_0 t} - \frac{0.5}{j}e^{-j\Omega_0 t}\right] = \frac{\pi}{j}\delta(\Omega - \Omega_0) - \frac{\pi}{j}\delta(\Omega + \Omega_0) \\ &= \pi e^{-j\pi/2}\delta(\Omega - \Omega_0) + \pi e^{j\pi/2}\delta(\Omega + \Omega_0) \end{aligned}$$

Parseval's Energy Conservation A finite-energy signal $x(t)$, with Fourier transform $X(\Omega)$, its energy is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega \quad \left. \vphantom{\int_{-\infty}^{\infty}} \right\} \text{ } \underbrace{\hspace{10em}}$$

$|X(\Omega)|^2$ energy density : energy per frequency

$|X(\Omega)|^2$ vs Ω : energy spectrum of $x(t)$, energy of the signal distributed over frequency

Example Is $\delta(t)$ a finite energy signal?

Using Parseval's result: $\mathcal{F}\delta(t) = 1$ for all frequencies then its energy is infinite

In time-domain:

$$p_{\Delta}(t) = \frac{1}{\Delta} [u(t + \Delta/2) - u(t - \Delta/2)] \Rightarrow \delta(t) \text{ as } \Delta \rightarrow 0, \text{ unit area}$$

$$p_{\Delta}^2(t) = \frac{1}{\Delta^2} [u(t + \Delta/2) - u(t - \Delta/2)] \Rightarrow \delta^2(t) \text{ as } \Delta \rightarrow 0, \text{ infinite area } 1/\Delta$$

$\delta(t)$ is not finite energy

Symmetry of Spectral Representations

If $X(\Omega)$ is FT of real-valued signal $x(t)$, periodic or aperiodic then

- Magnitude $|X(\Omega)|$ even function of Ω :

$$|X(\Omega)| = |X(-\Omega)|$$

- Phase $\angle X(\Omega)$ odd function of Ω :

$$\angle X(\Omega) = -\angle X(-\Omega)$$

$ X(\Omega) $ vs Ω	Magnitude Spectrum
$\angle X(\Omega)$ vs Ω	Phase Spectrum
$ X(\Omega) ^2$ vs Ω	Energy/Power Spectrum.

Remarks

- If signal is complex, the above symmetry will not hold
- Meaning of “negative” frequencies:
 - only positive frequencies exist and can be measured,
 - spectrum of a real signal requires negative frequencies

Example MATLAB to compute Fourier transform of

$$(a) \quad x(t) = u(t) - u(t - 1)$$

$$(b) \quad x(t) = e^{-t}u(t)$$

For $x(t) = e^{-t}u(t)$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Example 5.11
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sym t
x2=exp(-t)*heaviside(t);
X2=fourier(x2)
X2m=sqrt((real(X2))^2+(imag(X2))^2); % magnitude
X2p=imag(log(X2)); % phase
```

Magnitude:

$$|X_2(\Omega)| = \sqrt{\mathcal{R}e[X_2(\Omega)]^2 + \mathcal{I}m[X_2(\Omega)]^2}.$$

$$\log(X_2(\Omega)) = \log(|X_2(\Omega)|e^{j\angle X_2(\Omega)}) = \log(|X_2(\Omega)|) + j\angle X_2(\Omega)$$

so that

$$\angle X_2(\Omega) = \mathcal{I}m[\log(X_2(\Omega))].$$

$$z(t) = x_1(t + 0.5) = u(t + 0.5) - u(t - 0.5)$$

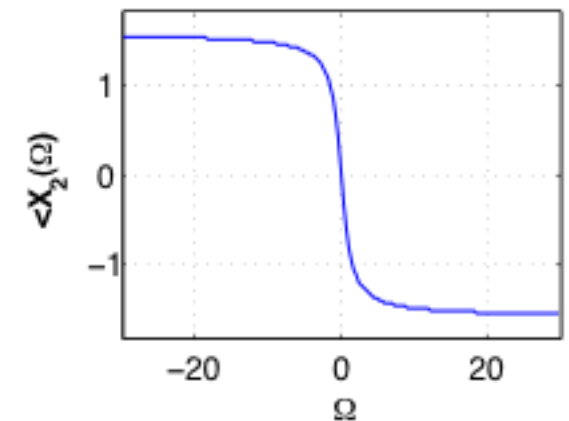
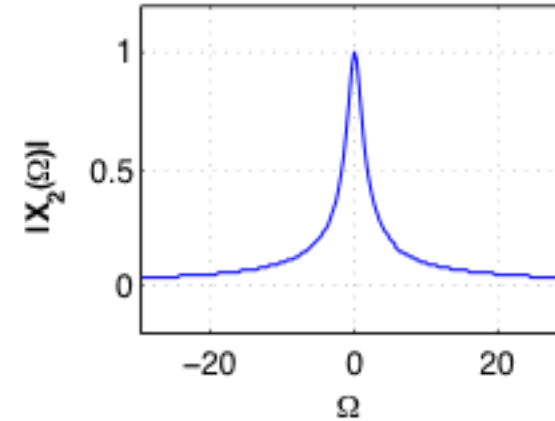
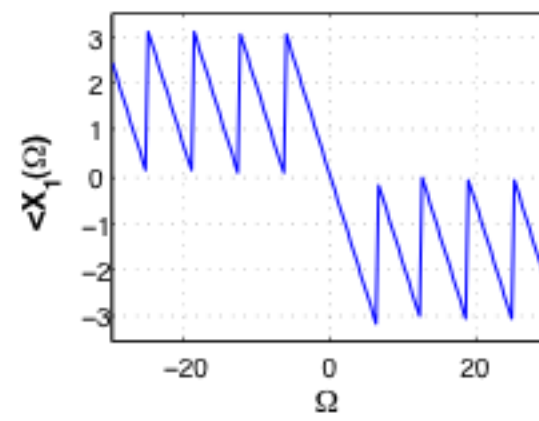
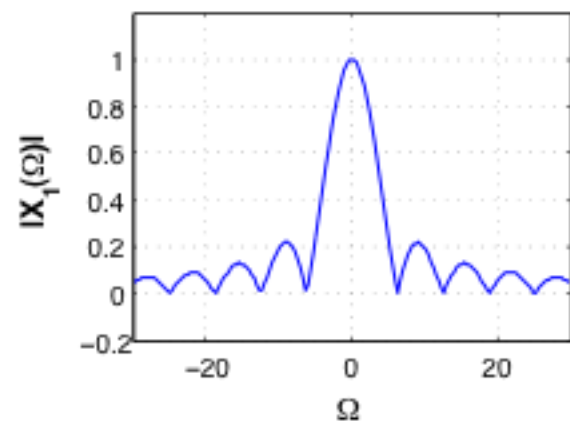
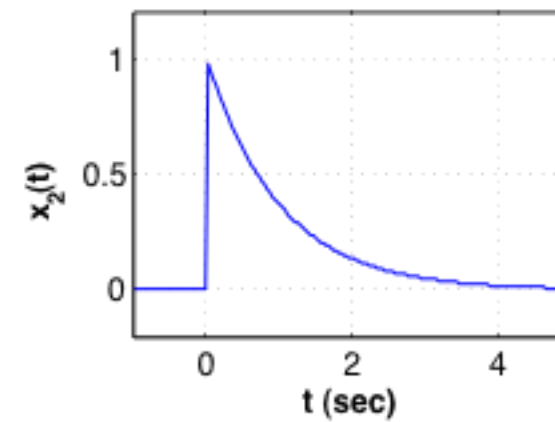
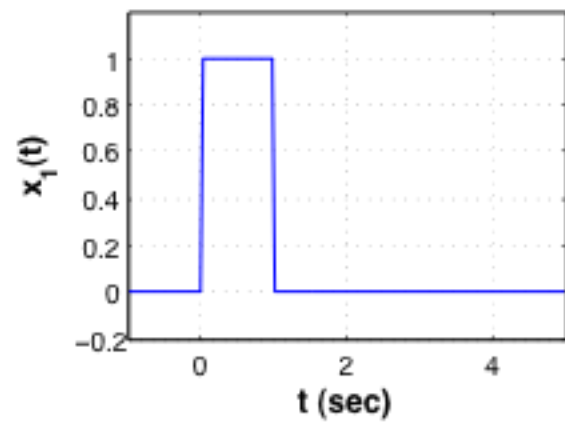
$$Z(\Omega) = \frac{\sin(\Omega/2)}{\Omega/2} \quad \text{real-valued}$$

$$\angle Z(\Omega) = \begin{cases} 0 & Z(\Omega) \geq 0 \\ \pm\pi & Z(\Omega) < 0 \end{cases}$$

$$z(t) = x_1(t + 0.5) \Rightarrow Z(\Omega) = X_1(\Omega)e^{j0.5\Omega}$$

$$X_1(\Omega) = e^{-j0.5\Omega} Z(\Omega)$$

$$\angle X_1(\Omega) = \angle Z(\Omega) - 0.5\Omega = \begin{cases} -0.5\Omega & Z(\Omega) \geq 0 \\ \pm\pi - 0.5\Omega & Z(\Omega) < 0 \end{cases}$$



Convolution and Filtering If $x(t)$ (periodic or aperiodic) is input to a stable LTI system with a frequency response $H(j\Omega) = \mathcal{F}[h(t)]$, $h(t)$ impulse response of the system, the output of the LTI system is the convolution integral $y(t) = (x * h)(t)$, with Fourier transform

$$Y(\Omega) = \mathcal{F}[(x * h)] = X(\Omega) H(j\Omega)$$

If $x(t)$ is periodic the output is also periodic with Fourier transform

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\Omega_0) \delta(\Omega - k\Omega_0)$$

where X_k are the Fourier series coefficients of $x(t)$ and Ω_0 its fundamental frequency.

Eigenfunction property of LTI systems:
Aperiodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \Rightarrow$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [X(\Omega) H(j\Omega)] e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\Omega) e^{j\Omega t} d\Omega$$

Handwritten diagram and notes:

$$\begin{array}{c} x(t) \xrightarrow{\mathcal{F}} X(\Omega) \xrightarrow{H(j\Omega)} Y(\Omega) \xrightarrow{\mathcal{F}^{-1}} y(t) \\ \text{CONVOLUTION} \\ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \end{array}$$

Periodic signal of period T_0

$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - k\Omega_0) \Rightarrow Y(\Omega) = X(\Omega) H(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\Omega_0) \delta(\Omega - k\Omega_0)$$

output $y(t)$ is periodic

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{X_k H(jk\Omega_0)}_{Y_k} e^{jk\Omega_0 t}$$

Basics of Filtering

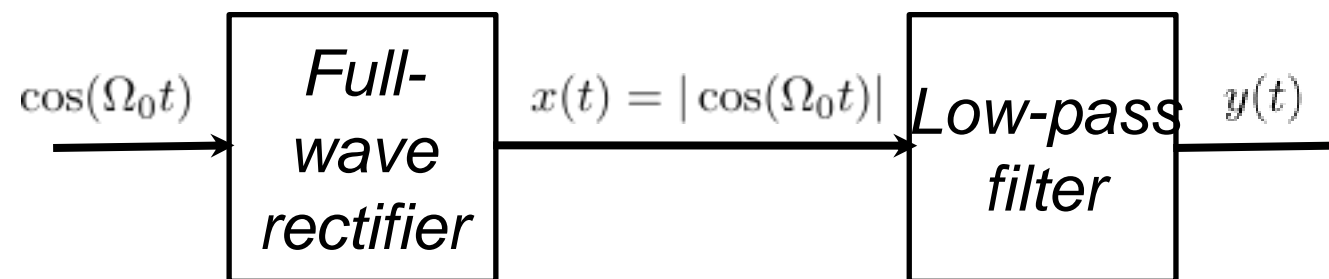
- Filtering consists in getting rid of undesirable components of a signal, e.g., noise $\eta(t)$ is added to a desired signal $x(t)$

$$y(t) = x(t) + \eta(t)$$

Filter design: find $H(s) = B(s)/A(s)$ satisfying certain specifications to get rid of noise.

- Frequency discriminating filters keep the frequency components of a signal in a certain frequency band and attenuate the rest.

Example Obtain dc source of unity amplitude using a full-wave rectifier and a low-pass filter (it keeps only the low-frequency components)



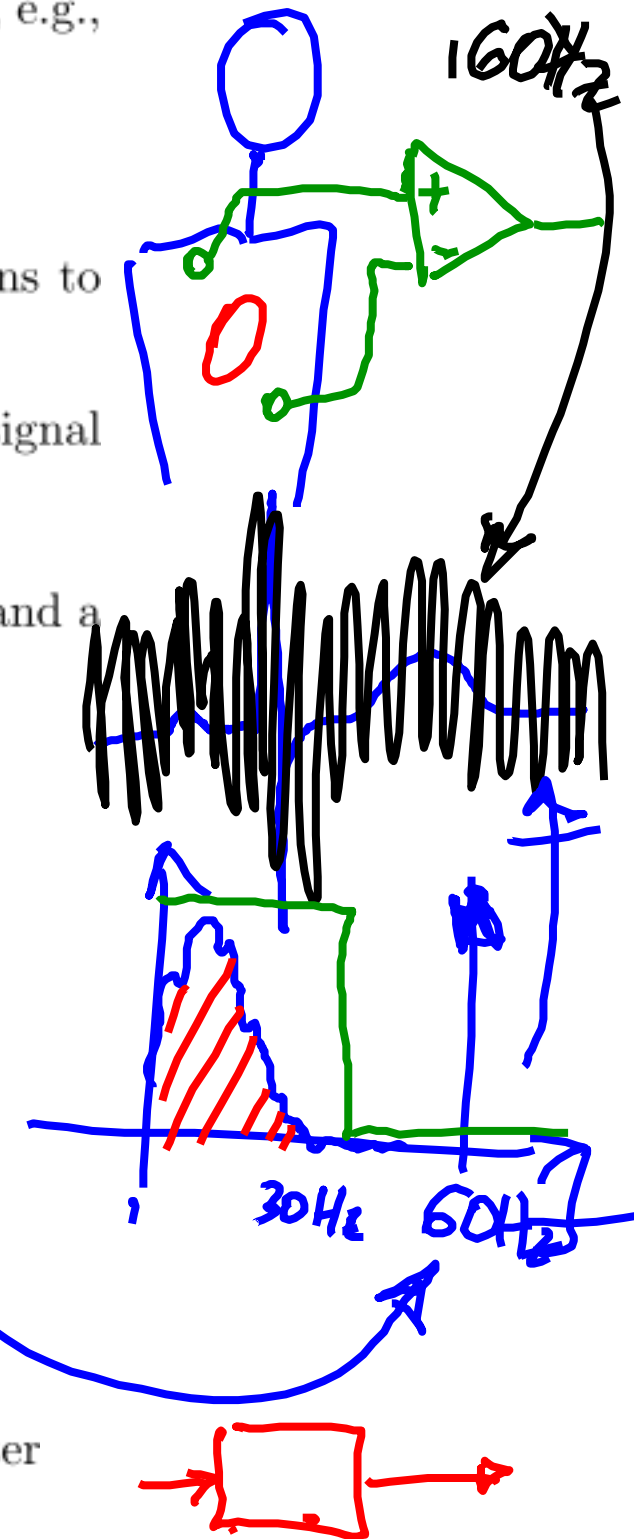
FS coefficients:

$$X_0 = \frac{2}{\pi}$$

$$X_k = \frac{2(-1)^k}{\pi(1 - 4k^2)} \quad k \neq 0$$

Filter out all harmonics and leave average component: ideal low-pass filter

$$H(j\Omega) = \begin{cases} A & -\Omega_0 < \Omega_c < \Omega_0, \text{ where } \Omega_0 = 2\pi/T_0 = 2\pi \\ 0 & \text{otherwise} \end{cases}$$

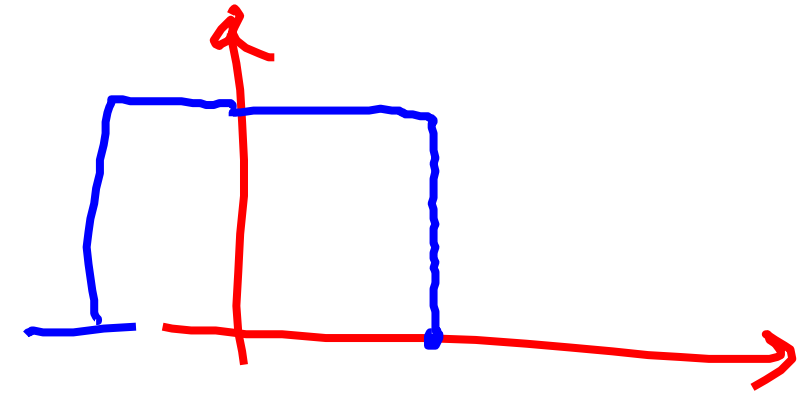


Ideal Filters

- Low-pass filter (keeps low-frequency components)

$$|H_{lp}(j\Omega)| = \begin{cases} 1 & -\Omega_1 \leq \Omega \leq \Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

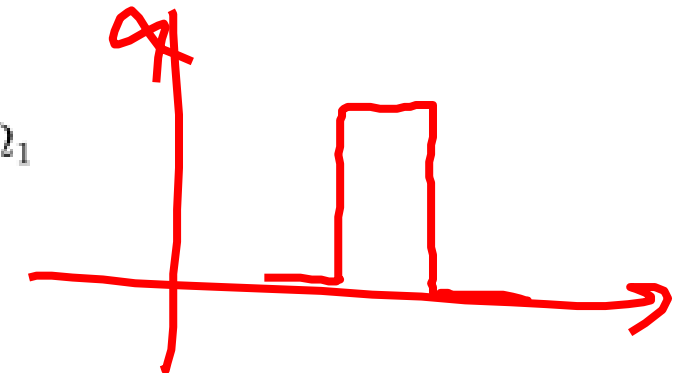
$$\angle H_{lp}(j\Omega) = -\alpha\Omega$$



- Band-pass filter (keeps middle frequency components)

$$|H_{bp}(j\Omega)| = \begin{cases} 1 & \Omega_1 \leq \Omega \leq \Omega_2 \quad \text{and} \quad -\Omega_2 \leq \Omega \leq -\Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

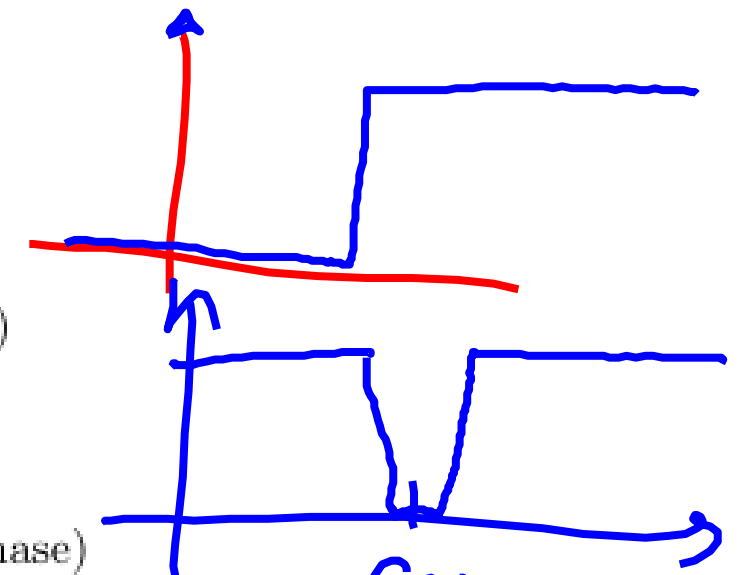
linear phase in the passband



- High-pass filter (keeps high-frequency components)

$$|H_{hp}(j\Omega)| = \begin{cases} 1 & \Omega \geq \Omega_2 \quad \text{and} \quad \Omega \leq -\Omega_2 \\ 0 & \text{otherwise} \end{cases}$$

linear phase in the passband



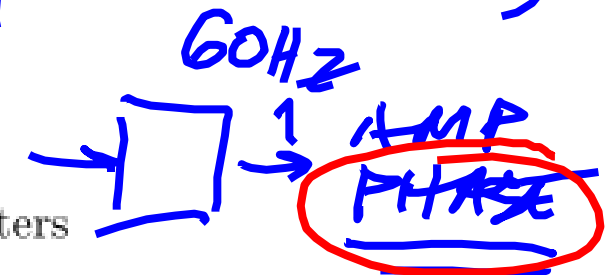
- Band-stop filter (attenuates middle frequency components)

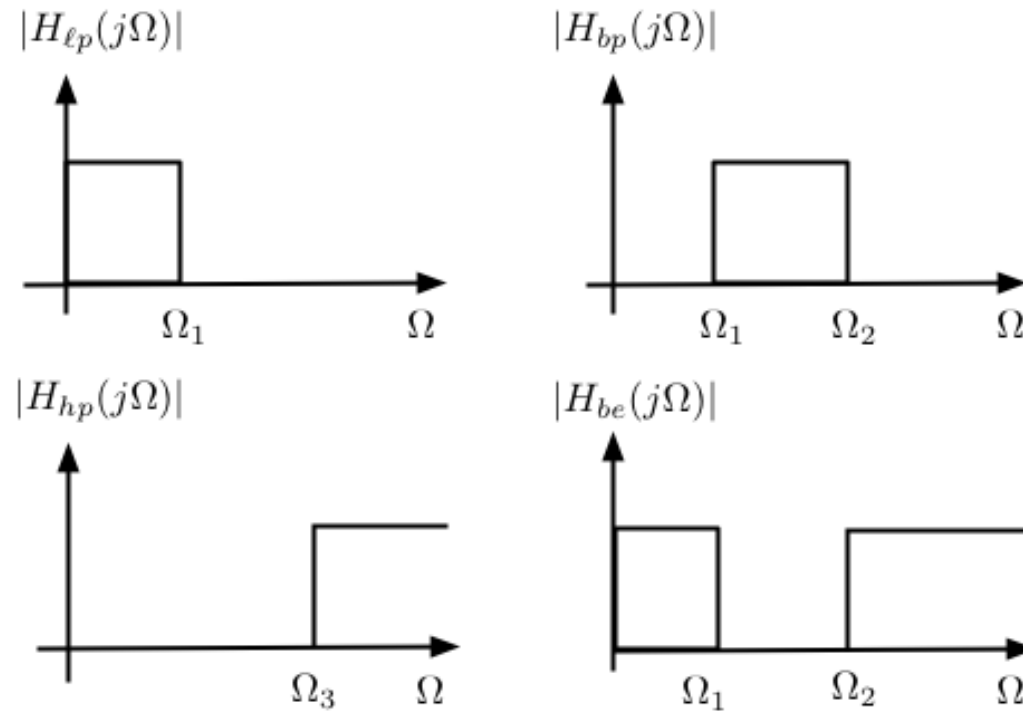
$$|H_{bs}(j\Omega)| = 1 - |H_{bp}(j\Omega)|$$

- All-pass filter (keeps all frequency components, changes phase)

$$|H_{ap}(j\Omega)| = |H_{lp}(j\Omega)| + |H_{bp}(j\Omega)| + |H_{hp}(j\Omega)| = 1$$

- Multi-band filter: combination of the low-, band-, and high-pass filters





Example Gibb's phenomenon of Fourier series: ringing around discontinuities of periodic signals. Consider a periodic train of square pulses $x(t)$ of period T_0 displaying discontinuities at $kT_0/2$, for $k = \pm 1, \pm 2, \dots$. Show Gibb's phenomenon is due to ideal low-pass filtering.

Ideal low-pass filter

$$H(j\Omega) = \begin{cases} 1 & -\Omega_c \leq \Omega \leq \Omega_c \\ 0 & \text{otherwise} \end{cases}$$

Periodic signal $x(t)$, of fundamental frequency $\Omega_0 = 2\pi/T_0$, is

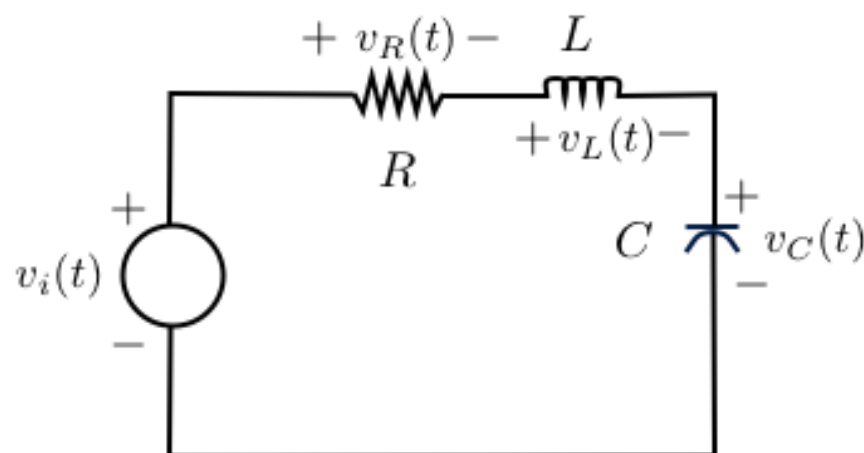
$$X(\Omega) = \mathcal{F}[x(t)] = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - k\Omega_0)$$

Output of the filter:

$$\begin{aligned} x_N(t) &= \mathcal{F}^{-1}[X(\Omega)H(j\Omega)] = \mathcal{F}^{-1}\left[\sum_{k=-N}^N 2\pi X_k \delta(\Omega - k\Omega_0)\right] \\ &= [x * h](t) \quad \text{cutoff frequency: } N\Omega_0 < \Omega_c < (N+1)\Omega_0 \end{aligned}$$

convolution around the discontinuities of $x(t)$ causes ringing before and after them, independent of N

Example Obtain different filters from an RLC circuit by choosing different outputs. Let $R = 1 \Omega$, $L = 1 \text{ H}$, and $C = 1 \text{ F}$, and $IC = 0$



Low-pass Filter: Output $V_C(s)$

$$H_{lp}(s) = \frac{V_C(s)}{V_i(s)} = \frac{1}{s^2 + s + 1}$$

- input a dc source (frequency $\Omega = 0$), inductor short circuit, capacitor open circuit, so $V_C(s) = V_i(s)$
- input of very high frequency, $\Omega \rightarrow \infty$, inductor open circuit, capacitor short circuit $V_C(s) = V_i(s) = 0$

High-pass Filter: Output $V_L(s)$

$$H_{hp}(s) = \frac{V_L(s)}{V_i(s)} = \frac{s^2}{s^2 + s + 1}$$

- dc input (frequency zero), inductor is short circuit $V_L(s) = 0$
- input of very high frequency, $\Omega \rightarrow \infty$, inductor is open circuit $V_L(s) = V_i(s)$

Band-pass Filter: Output $V_R(s)$

$$H_{bp}(s) = \frac{V_R(s)}{V_i(s)} = \frac{s}{s^2 + s + 1}$$

- For zero frequency, capacitor is open circuit so voltage across the resistor is zero
- For very high frequency, inductor is open circuit, voltage across resistor is zero
- For some middle frequency, serial LC combination resonates (zero impedance) maximum voltage across resistor

Band-stop Filter: Output voltage across inductor and the capacitor

$$H_{bs}(s) = \frac{s^2 + 1}{s^2 + s + 1}$$

- At low and high frequencies, LC is open-circuit, $V_{LC}(s) = V_i(s)$
- At the resonance frequency $\Omega_r = 1$ the impedance of the LC connection is zero, so the output voltage is zero

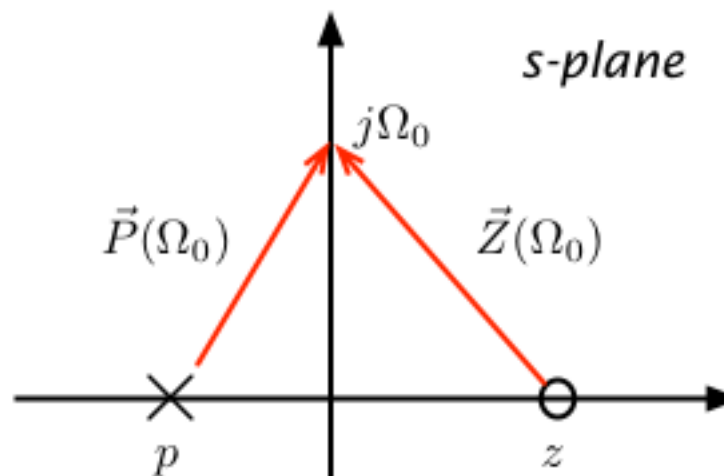
Frequency Response from Poles and Zeros

For a filter with transfer function

$$H(s) = \frac{\prod_i (s - z_i)}{\prod_k (s - p_k)} \quad \text{where } z_i, p_k \text{ are zeros and poles of } H(s)$$

with vectors $\vec{Z}_i(\Omega)$ and $\vec{P}_k(\Omega)$, going from each of the zeros and poles to the frequency at which we are computing the magnitude and phase response in the $j\Omega$ -axis, gives

$$\begin{aligned} H(j\Omega) = H(s)|_{s=j\Omega} &= \frac{\prod_i \vec{Z}_i(\Omega)}{\prod_k \vec{P}_k(\Omega)} \\ &= \underbrace{\frac{\prod_i |\vec{Z}_i(\Omega)|}{\prod_k |\vec{P}_k(\Omega)|}}_{|H(j\Omega)|} \underbrace{e^{j[\sum_i \angle(\vec{Z}_i(\Omega)) - \sum_k \angle(\vec{P}_k(\Omega))]}_{e^{j\angle H(j\Omega)}} \end{aligned}$$



Example RC circuit in series with voltage source $v_i(t)$. Choose the output to obtain low-pass and high-pass filters and use the poles and zeros of the transfer functions to determine their frequency responses. Let $R = 1 \Omega$, $C = 1 \text{ F}$ and the initial conditions be zero.

Low-pass filter:

$$H(s) = \frac{V_C(s)}{V_i(s)} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1 + s}$$

$$H(j\Omega) = \frac{1}{1 + j\Omega} = \frac{1}{\vec{P}(\Omega)}$$

Geometrically

$$\begin{aligned} \Omega = 0 & \quad \vec{P}(0) = 1e^{j0} H(j0) = 1e^{j0} \\ \Omega = 1 & \quad \vec{P}(1) = \sqrt{2}e^{j\pi/4} \Rightarrow H(j1) = 0.707e^{-j\pi/4} \\ \Omega = \infty & \quad \vec{P}(\infty) = \infty e^{j\pi/2} \Rightarrow H(j\infty) = 0e^{-j\pi/2} \end{aligned}$$

High-pass filter:

$$H(s) = \frac{V_r(s)}{V_s(s)} = \frac{CRs}{CRs + 1} = \frac{s}{s + 1}$$

$$H(j\Omega) = \frac{j\Omega}{1 + j\Omega} = \frac{\vec{Z}(\Omega)}{\vec{P}(\Omega)}$$

$$\begin{aligned} \Omega = 0 & \quad \vec{P}(0) = 1e^{j0} \quad \vec{Z}(0) = 0e^{j\pi/2} \quad H(j0) = \frac{\vec{Z}(0)}{\vec{P}(0)} = 0e^{j\pi/2} \\ \Omega = 1 & \quad \vec{P}(1) = \sqrt{2}e^{j\pi/4} \quad \vec{Z}(1) = 1e^{j\pi/2} \quad H(j1) = \frac{\vec{Z}(1)}{\vec{P}(1)} = 0.707e^{j\pi/4} \\ \Omega = \infty & \quad \vec{P}(\infty) = \infty e^{j\pi/2} \quad \vec{Z}(\infty) = \infty e^{j\pi/2} \quad H(j\infty) = \frac{\vec{Z}(\infty)}{\vec{P}(\infty)} = 1e^{j0} \end{aligned}$$

Remarks

- Poles create “hills” at frequencies in the $j\Omega$ -axis in front of their imaginary parts.
- Zeros create “valleys” at the frequencies in the $j\Omega$ -axis in front of their imaginary parts

Example Use MATLAB to find and plot the poles and zeros and the corresponding magnitude and phase frequency responses of:

- (a) A second-order band-pass and a high-pass filters realized using an RLC series connection ($R=1$, $L=1$, $C=1$)
- (b) An all-pass filter with a transfer function

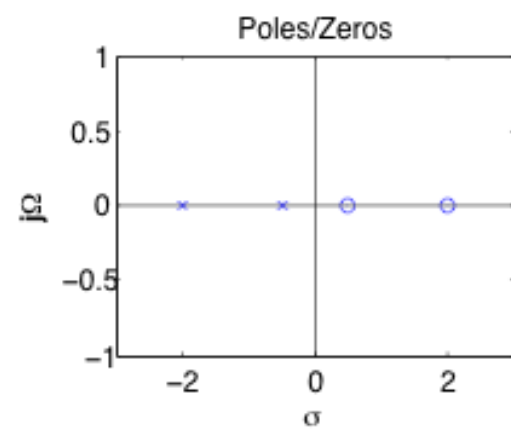
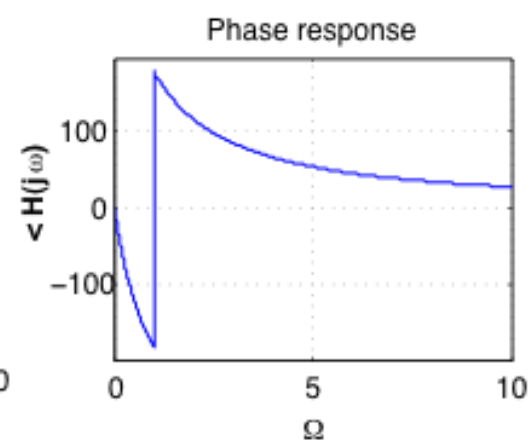
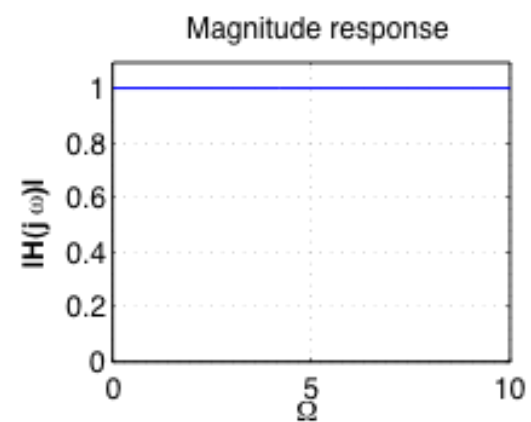
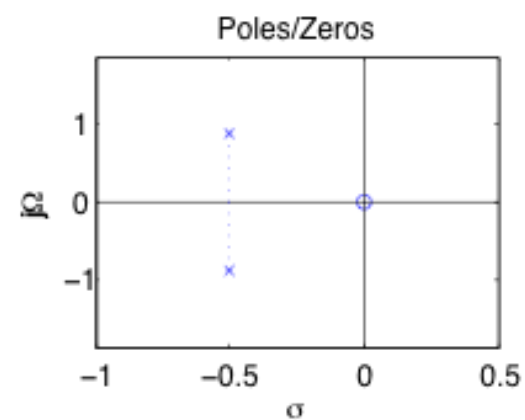
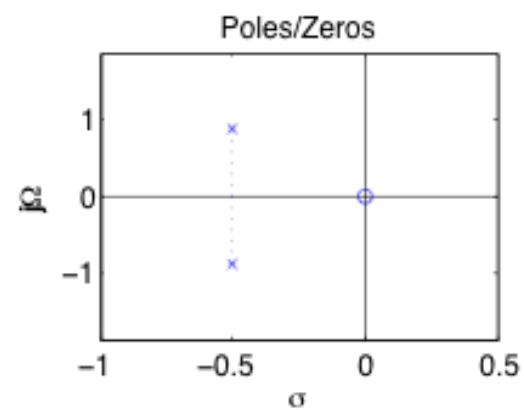
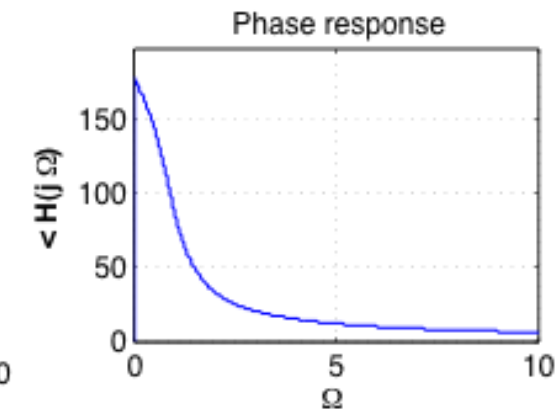
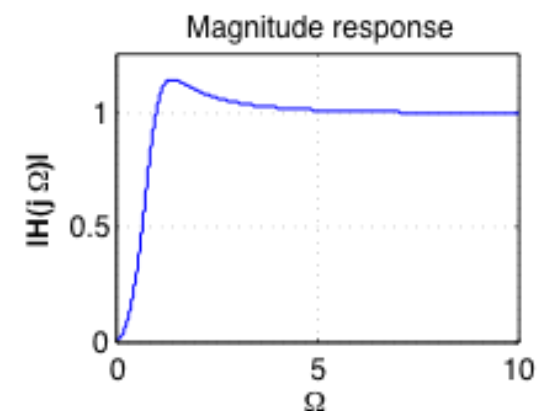
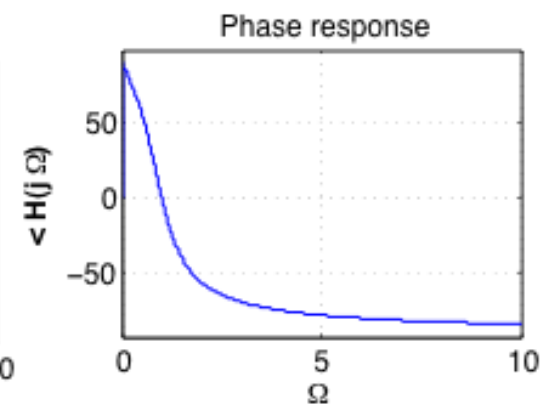
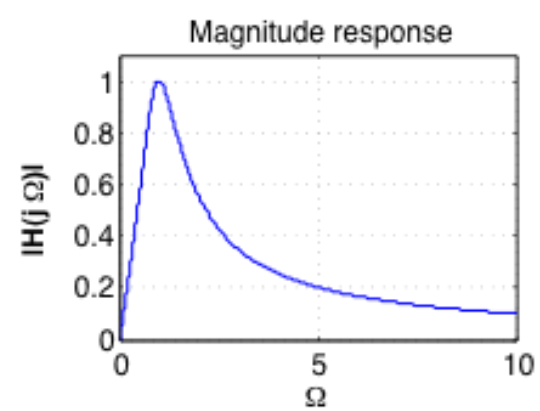
$$H(s) = \frac{s^2 - 2.5s + 1}{s^2 + 2.5s + 1}$$

(a) From previous example,

$$H_{bp}(s) = \frac{s}{s^2 + s + 1}$$
$$H_{hp}(s) = \frac{s^2}{s^2 + s + 1}$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Example 5.18 --- Frequency response
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n=[0 1 0]; % numerator coefficients -- bandpass
% n=[1 0 0]; % numerator coefficients -- highpass
d=[1 1 1]; % denominator coefficients
wmax=10; % maximum frequency
[w,Hm,Ha]=freqresp_s(n,d,wmax); % frequency response
splane(n,d) % plotting of poles and zeros
```

```
function [w,Hm,Ha]=freqresp_s(b,a,wmax)
w=0:0.01:wmax;
H=freqs(b,a,w);
Hm=abs(H); % magnitude
Ha=angle(H)*180/pi; % phase in degrees
```



The Spectrum Analyzer— Device that measures the spectral characteristics of a signal

- Implemented as a bank of narrow-band band-pass filters with fixed bandwidths covering the desired frequencies (used for the audio range of the spectrum)
Input $x(t)$, output of one of bandpass filter with very narrow bandwidth $\Delta\Omega$:

$$y(t) = \frac{1}{2\pi} \int_{\Omega_0 - 0.5\Delta\Omega}^{\Omega_0 + 0.5\Delta\Omega} X(\Omega) e^{j\Omega t} d\Omega$$

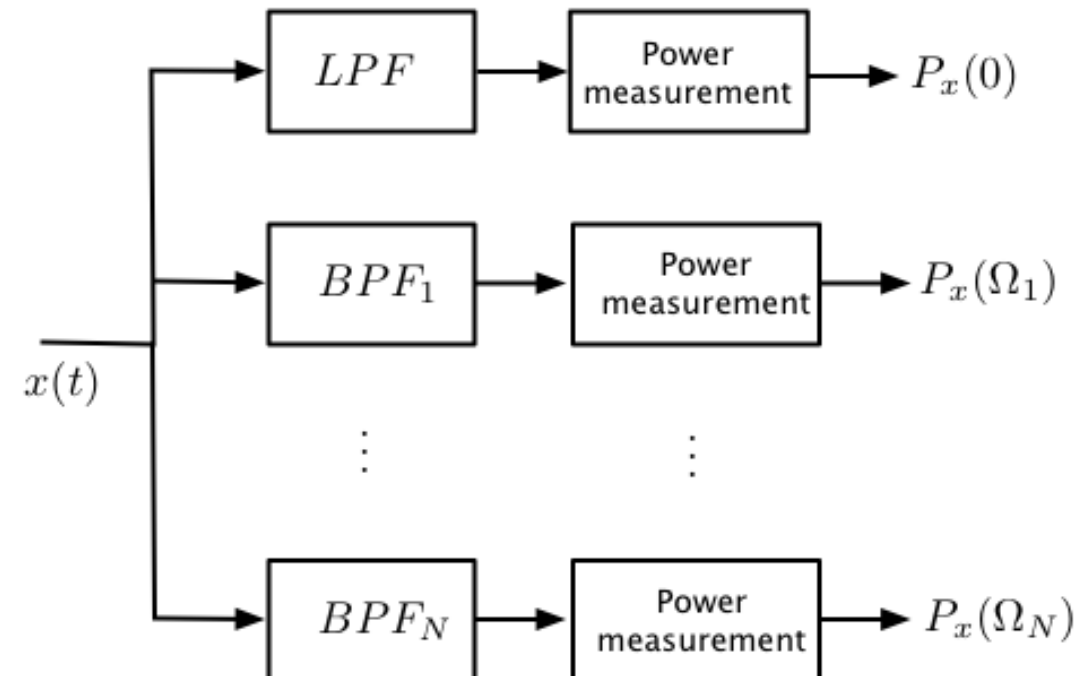
$$\approx \frac{1}{2\pi} \Delta\Omega X(\Omega_0) e^{j\Omega_0 t}$$

mean square of this signal

$$\frac{1}{T} \int_T |y(t)|^2 dt = \left(\frac{\Delta\Omega}{2\pi} \right)^2 |X(\Omega_0)|^2$$

proportional to the power or the energy of the signal in $\Omega_0 \pm \Delta\Omega$. A similar computation can be done at each of the frequencies of the input signal

- Radio frequency spectrum analyzers resemble an AM demodulator



What have we accomplished?

- Unification of frequency representation of periodic and aperiodic signals
- Frequency response of LTI systems
- Duality in time and frequency
- Convolution and Filtering
 - Connection of Fourier series and Laplace transform
 - Inverse time frequency relation

Where do we go from here?

- Application of Laplace analysis and transient response
- Application of Fourier analysis and steady state response
 - Filter design
- Application of time-frequency relation in sampling theory