Lecture 10

Limit of Statistical Learnability (Detection) in Spike-Wigner Model
(a.k.a. Symmetric Vector-Spin Glass Model)

Spike-Wigner Model

Signal to Noise Ratio (SNR)

$$\mathbf{Y} = \sqrt{\frac{\lambda}{N}}\,\mathbf{x}^*\mathbf{x}^{*T} + \boldsymbol{\xi}$$

symmetric rank 1 matrix

symmetric i.i.d. noise

$\mathbf{x}^* \in \mathbb{R}^N$ with $x_i^* \sim P_X(x)$

$\xi_{ij} = \xi_{ji} \sim \mathcal{N}(0,1)$

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The Posterior is

$$P(\mathbf{x} \mid \mathbf{Y}) = \frac{1}{Z(\mathbf{Y})} \left[\prod_{i=1}^{N} P_X\left(x_i\right) \right] \left[\prod_{i \le j} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(y_{ij} - \sqrt{\frac{\lambda}{N}} x_i x_j \right)^2 \right) \right]$$

$$\propto \frac{1}{Z(\mathbf{Y})} \left[\prod_{i}^{N} P_X\left(x_i\right) \right] \exp\left(\sum_{i \le j} \left[-\frac{\lambda}{2N} x_i^2 x_j^2 + \sqrt{\frac{\lambda}{N}} y_{ij} x_i x_j \right] \right)$$

From Posterior to Minimum Mean Square Error Estimator

$$\hat{\mathbf{x}}_{\mathrm{MSE}}(\mathbf{Y}) = \begin{bmatrix} \hat{x}_{\mathrm{MSE},1}(\mathbf{Y}) \\ \vdots \\ \hat{x}_{\mathrm{MSE},N}(\mathbf{Y}) \end{bmatrix} \qquad \hat{x}_{\mathrm{MSE},i}(\mathbf{Y}) = \left\langle x_i \right\rangle_{\mathbf{Y}} = \int \mathrm{d}\mathbf{x} P(\mathbf{x} \mid \mathbf{Y}) x_i$$

How does this estimator perform on average?

$$\mathrm{MMSE}(\lambda) = \mathbb{E}_{\mathbf{Y},x^*}\left[\left(\langle x \rangle_{\mathbf{Y}} - x^*\right)^2\right] = \mathbb{E}_{\mathbf{Y},x^*}\left[\langle x \rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^*\langle x \rangle_{\mathbf{Y}}\right]$$

From Posterior to Minimum Mean Square Error Estimator

$$\hat{\mathbf{x}}_{\mathrm{MSE}}(\mathbf{Y}) = \begin{bmatrix} \hat{x}_{\mathrm{MSE},1}(\mathbf{Y}) \\ \vdots \\ \hat{x}_{\mathrm{MSE},N}(\mathbf{Y}) \end{bmatrix}$$

$$\hat{x}_{\mathrm{MSE},i}(\mathbf{Y}) = \left\langle x_i \right\rangle_{\mathbf{Y}} = \int \mathrm{d}\mathbf{x} P(\mathbf{x} \mid \mathbf{Y}) x_i$$

How does this estimator perform on average?

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(Nishimori)

$$= \mathbb{E}_{x^*}\left[(x^*)^2\right] - \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left\langle \frac{1}{N}\sum_i x_i^* x_i \right\rangle_{\mathbf{x}^*,\boldsymbol{\xi}}\right]$$

$N \to \infty$

$$\approx \mathbb{E}_{x^*}\left[(x^*)^2\right] - m^*(\lambda)$$

How does this error depend on the SNR ratio?

Need to evaluate the *typical value* of Posterior (quenched random variable in the exponent)

$$P(\mathbf{x} \mid \mathbf{Y}) = \frac{1}{Z(\mathbf{Y})}\left[\prod_i P_X(x_i)\right] \exp\left(\sum_{i \le j}\left[-\frac{\lambda}{2N}x_i^2 x_j^2 + \sqrt{\frac{\lambda}{N}} y_{ij} x_i x_j\right]\right)$$

$$= \frac{1}{Z(\mathbf{Y})}\left[\prod_i P_X(x_i)\right] \exp\left(\sum_{i \le j}\left[-\frac{\lambda}{2N}x_i^2 x_j^2 + \frac{\lambda}{N} x_i x_j x_i^* x_j^* + \sqrt{\frac{\lambda}{N}} \xi_{ij} x_i x_j\right]\right)$$

We'll be interested in the average free energy per site

$$\lim_{N\to\infty} \mathbb{E}_{\mathbf{Y}}\left[\frac{1}{N}\log(Z(\mathbf{Y}))\right] = \lim_{N\to\infty} \mathbb{E}_{\mathbf{x}^*, \boldsymbol{\xi}}\left[\frac{1}{N}\log(Z(\mathbf{Y}))\right]$$

Replica Trick to the rescue!

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left(\int \mathrm{d}\mathbf{x} \left[\prod_i P_X\left(x_i\right)\right] \exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2 + \frac{\lambda}{N}x_ix_jx_i^*x_j^* + \sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_i P_X(x_i)\right]\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2+\frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

$$= \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\prod_{\alpha=1}^{n}\int \mathrm{d}\mathbf{x}^{(\alpha)}\prod_i P_X\left(x_i^{(\alpha)}\right)\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\frac{\lambda}{N}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_i^{(\alpha)}x_j^{(\alpha)}\right]\right)\right]$$

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_i P_X\left(x_i\right)\right]\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2+\frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

$$= \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\prod_{\alpha=1}^{n}\int \mathrm{d}\mathbf{x}^{(\alpha)}\prod_i P_X\left(x_i^{(\alpha)}\right)\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\frac{\lambda}{N}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_i^{(\alpha)}x_j^{(\alpha)}\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\sum_{\alpha}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2\right]+\frac{\lambda}{N}\sum_{\alpha}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*\right)\prod_{i\le j}\mathbb{E}_{\xi_{ij}}\left[\exp\left(\xi_{ij}\sqrt{\frac{\lambda}{N}}\sum_{\alpha}x_i^{(\alpha)}x_j^{(\alpha)}\right)\right]\right]$$

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_i P_X\left(x_i\right)\right]\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2+\frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

$$= \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\prod_{\alpha=1}^{n}\int \mathrm{d}\mathbf{x}^{(\alpha)}\prod_i P_X\left(x_i^{(\alpha)}\right)\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\frac{\lambda}{N}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_i^{(\alpha)}x_j^{(\alpha)}\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\sum_\alpha\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2\right]+\frac{\lambda}{N}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*\right)\boxed{\prod_{i\le j}\mathbb{E}_{\xi_{ij}}\left[\exp\left(\xi_{ij}\sqrt{\frac{\lambda}{N}}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}\right)\right]}\right]$$

$$= \exp\left(\frac{\lambda}{2N}\sum_{i\le j}\sum_{\alpha,\beta}x_i^{(\alpha)}x_j^{(\alpha)}x_i^{(\beta)}x_j^{(\beta)}\right)$$

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\xi}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\xi}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_i P_X(x_i)\right]\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2+\frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

$$= \mathbb{E}_{\mathbf{x}^*,\xi}\left[\prod_{\alpha=1}^{n}\int \mathrm{d}\mathbf{x}^{(\alpha)}\prod_i P_X\left(x_i^{(\alpha)}\right)\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\frac{\lambda}{N}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_i^{(\alpha)}x_j^{(\alpha)}\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\sum_\alpha\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2\right]+\frac{\lambda}{N}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*\right)\boxed{\prod_{i\le j}\mathbb{E}_{\xi_{ij}}\left[\exp\left(\xi_{ij}\sqrt{\frac{\lambda}{N}}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}\right)\right]}\right]$$

$$= \exp\left(\frac{\lambda}{2N}\sum_{i\le j}\sum_{\alpha,\beta}x_i^{(\alpha)}x_j^{(\alpha)}x_i^{(\beta)}x_j^{(\beta)}\right)$$

$$\sum_{i\le j}\frac{a_i}{N}\frac{a_j}{N}=\frac{1}{2}\left(\sum_i\frac{a_i}{N}\right)^2+\cancel{\frac{1}{2}\sum_i\frac{a_i^2}{N^2}}\quad O(N^{-1})$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\frac{\lambda N}{2}\sum_\alpha\left(\sum_i\frac{x_i^*x_i^{(\alpha)}}{N}\right)^2+\frac{\lambda N}{2}\sum_{\alpha<\beta}\left(\sum_i\frac{x_i^{(\alpha)}x_i^{(\beta)}}{N}\right)^2\right)\right]$$

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\xi}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\xi}\left[\left(\int d\mathbf{x}\left[\prod_i P_X(x_i)\right]\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2+\frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

$$= \mathbb{E}_{\mathbf{x}^*,\xi}\left[\prod_{\alpha=1}^{n}\int d\mathbf{x}^{(\alpha)}\prod_i P_X\left(x_i^{(\alpha)}\right)\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\frac{\lambda}{N}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_i^{(\alpha)}x_j^{(\alpha)}\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)dx_i^{(\alpha)}\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\sum_\alpha\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2\right]+\frac{\lambda}{N}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*\right)\boxed{\prod_{i\le j}\mathbb{E}_{\xi_{ij}}\left[\exp\left(\xi_{ij}\sqrt{\frac{\lambda}{N}}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}\right)\right]}\right]$$

$$= \exp\left(\frac{\lambda}{2N}\sum_{i\le j}\sum_{\alpha,\beta}x_i^{(\alpha)}x_j^{(\alpha)}x_i^{(\beta)}x_j^{(\beta)}\right)$$

$$\sum_{i\le j}\frac{a_i}{N}\frac{a_j}{N} = \frac{1}{2}\left(\sum_i\frac{a_i}{N}\right)^2+\cancel{\frac{1}{2}\sum_i\frac{a_i^2}{N^2}}\quad O(N^{-1})$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)dx_i^{(\alpha)}\exp\left(\frac{\lambda N}{2}\sum_\alpha\left(\sum_i\frac{x_i^*x_i^{(\alpha)}}{N}\right)^2+\frac{\lambda N}{2}\sum_{\alpha<\beta}\left(\sum_i\frac{x_i^{(\alpha)}x_i^{(\beta)}}{N}\right)^2\right)\right]$$

$m_\alpha = \frac{1}{N}\sum_i x_i^{(\alpha)}x_i^*$ "overlap with the truth"

$q_{\alpha\beta} = \frac{1}{N}\sum_i x_i^{(\alpha)}x_i^{(\beta)}$ "replicas overlap"

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)dx_i^{(\alpha)}\int\prod_\alpha\delta\left(m_\alpha-\frac{1}{N}\sum_i x_i^{(\alpha)}x_i^*\right)dm_\alpha\int\prod_{\alpha<\beta}\delta\left(q_{\alpha\beta}-\frac{1}{N}\sum_i x_i^{(\alpha)}x_i^{(\beta)}\right)dq_{\alpha\beta}\exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2+\sum_{\alpha<\beta}q_{\alpha\beta}^2\right]\right)\right]$$

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\xi}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\xi}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_i P_X\left(x_i\right)\right]\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2+\frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

$$= \mathbb{E}_{\mathbf{x}^*,\xi}\left[\prod_{\alpha=1}^n\int \mathrm{d}\mathbf{x}^{(\alpha)}\prod_i P_X\left(x_i^{(\alpha)}\right)\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\frac{\lambda}{N}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_i^{(\alpha)}x_j^{(\alpha)}\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\sum_\alpha\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2\right]+\frac{\lambda}{N}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*\right)\prod_{i\le j}\mathbb{E}_{\xi_{ij}}\left[\exp\left(\xi_{ij}\sqrt{\frac{\lambda}{N}}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}\right)\right]\right]$$

$$= \exp\left(\frac{\lambda}{2N}\sum_{i\le j}\sum_{\alpha,\beta}x_i^{(\alpha)}x_j^{(\alpha)}x_i^{(\beta)}x_j^{(\beta)}\right)$$

$$\sum_{i\le j}\frac{a_i}{N}\frac{a_j}{N} = \frac{1}{2}\left(\sum_i\frac{a_i}{N}\right)^2+\frac{1}{2}\cancel{\sum_i\frac{a_i^2}{N^2}}^{\,O(N^{-1})}$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\frac{\lambda N}{2}\sum_\alpha\left(\sum_i\frac{x_i^*x_i^{(\alpha)}}{N}\right)^2+\frac{\lambda N}{2}\sum_{\alpha<\beta}\left(\sum_i\frac{x_i^{(\alpha)}x_i^{(\beta)}}{N}\right)^2\right)\right]$$

$m_\alpha = \frac{1}{N}\sum_i x_i^{(\alpha)}x_i^*$ “overlap with the truth”

$q_{\alpha\beta} = \frac{1}{N}\sum_i x_i^{(\alpha)}x_i^{(\beta)}$ “replicas overlap”

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\int\prod_\alpha\delta\left(m_\alpha-\frac{1}{N}\sum_i x_i^{(\alpha)}x_i^*\right)\mathrm{d}m_\alpha\int\prod_{\alpha<\beta}\delta\left(q_{\alpha\beta}-\frac{1}{N}\sum_i x_i^{(\alpha)}x_i^{(\beta)}\right)\mathrm{d}q_{\alpha\beta}\exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2+\sum_{\alpha<\beta}q_{\alpha\beta}^2\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\int\prod_\alpha\exp\left(\hat{m}_\alpha\left[Nm_\alpha-\sum_i x_i^{(\alpha)}x_i^*\right]\right)\mathrm{d}\hat{m}_\alpha\mathrm{d}m_\alpha\int\prod_{\alpha<\beta}\exp\left(\hat{q}_{\alpha\beta}\left[Nq_{\alpha\beta}-\sum_i x_i^{(\alpha)}x_i^{(\beta)}\right]\right)\mathrm{d}\hat{q}_{\alpha\beta}\mathrm{d}q_{\alpha\beta}\exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2+\sum_{\alpha<\beta}q_{\alpha\beta}^2\right]\right)\right]$$

Replica Trick in Spike-Wigner Model

$$\mathbb{E}_{\mathbf{x}^*,\xi}[Z^n] = \mathbb{E}_{\mathbf{x}^*,\xi}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_i P_X(x_i)\right]\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}x_i^2x_j^2+\frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_ix_j\right]\right)\right)^n\right]$$

$$= \mathbb{E}_{\mathbf{x}^*,\xi}\left[\prod_{\alpha=1}^n\int \mathrm{d}\mathbf{x}^{(\alpha)}\prod_i P_X\left(x_i^{(\alpha)}\right)\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\frac{\lambda}{N}x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*+\sqrt{\frac{\lambda}{N}}\xi_{ij}x_i^{(\alpha)}x_j^{(\alpha)}\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\sum_{i\le j}\left[-\frac{\lambda}{2N}\sum_\alpha\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2\right]+\frac{\lambda}{N}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}x_i^*x_j^*\right)\boxed{\prod_{i\le j}\mathbb{E}_{\xi_{ij}}\left[\exp\left(\xi_{ij}\sqrt{\frac{\lambda}{N}}\sum_\alpha x_i^{(\alpha)}x_j^{(\alpha)}\right)\right]}\right]$$

$$= \exp\left(\frac{\lambda}{2N}\sum_{i\le j}\sum_{\alpha,\beta}x_i^{(\alpha)}x_j^{(\alpha)}x_i^{(\beta)}x_j^{(\beta)}\right)$$

$$\sum_{i\le j}\frac{a_i}{N}\frac{a_j}{N} = \frac{1}{2}\left(\sum_i\frac{a_i}{N}\right)^2+\frac{1}{2}\cancel{\sum_i\frac{a_i^2}{N^2}}\quad O(N^{-1})$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\exp\left(\frac{\lambda N}{2}\sum_\alpha\left(\sum_i\frac{x_i^*x_i^{(\alpha)}}{N}\right)^2+\frac{\lambda N}{2}\sum_{\alpha<\beta}\left(\sum_i\frac{x_i^{(\alpha)}x_i^{(\beta)}}{N}\right)^2\right)\right]$$

$m_\alpha = \frac{1}{N}\sum_i x_i^{(\alpha)}x_i^*$ "overlap with the truth"

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$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\int\prod_\alpha\delta\left(m_\alpha-\frac{1}{N}\sum_i x_i^{(\alpha)}x_i^*\right)\mathrm{d}m_\alpha\int\prod_{\alpha<\beta}\delta\left(q_{\alpha\beta}-\frac{1}{N}\sum_i x_i^{(\alpha)}x_i^{(\beta)}\right)\mathrm{d}q_{\alpha\beta}\exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2+\sum_{\alpha<\beta}q_{\alpha\beta}^2\right]\right)\right]$$

$$= \mathbb{E}_{\mathbf{x}^*}\left[\int\prod_{\alpha,i}P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\int\prod_\alpha\exp\left(\hat{m}_\alpha\left[Nm_\alpha-\sum_i x_i^{(\alpha)}x_i^*\right]\right)\mathrm{d}\hat{m}_\alpha\mathrm{d}m_\alpha\int\prod_{\alpha<\beta}\exp\left(\hat{q}_{\alpha\beta}\left[Nq_{\alpha\beta}-\sum_i x_i^{(\alpha)}x_i^{(\beta)}\right]\right)\mathrm{d}\hat{q}_{\alpha\beta}\mathrm{d}q_{\alpha\beta}\exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2+\sum_{\alpha<\beta}q_{\alpha\beta}^2\right]\right)\right]$$

$$= \int\prod_\alpha\mathrm{d}\hat{m}_\alpha\mathrm{d}m_\alpha\int\prod_{\alpha<\beta}\mathrm{d}\hat{q}_{\alpha\beta}\mathrm{d}q_{\alpha\beta}\exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2+\sum_{\alpha<\beta}q_{\alpha\beta}^2\right]+N\left[\sum_\alpha m_\alpha\hat{m}_\alpha+\sum_{\alpha<\beta}q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int\prod_\alpha P_X(x_\alpha)\,\mathrm{d}x_\alpha\exp\left(-\sum_\alpha\hat{m}_\alpha x_* x_\alpha-\sum_{\alpha<\beta}\hat{q}_{\alpha\beta}x_\alpha x_\beta\right)\right]\right\}^N$$

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right]$$

$$= \int \prod_{\alpha} \mathrm{d}\hat{m}_\alpha \mathrm{d}m_\alpha \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2}\left[\sum_{\alpha} m_\alpha^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N\left[\sum_{\alpha} m_\alpha \hat{m}_\alpha + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\sum_{\alpha} \hat{m}_\alpha x_* x_\alpha - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta} x_\alpha x_\beta\right)\right]\right\}^N$$

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] \\
&= \int \prod_{\alpha} \mathrm{d}\hat{m}_\alpha \mathrm{d}m_\alpha \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2}\left[\sum_{\alpha} m_\alpha^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N\left[\sum_{\alpha} m_\alpha \hat{m}_\alpha + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\sum_{\alpha} \hat{m}_\alpha x_* x_\alpha - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta} x_\alpha x_\beta\right)\right]\right\}^N
\end{aligned}
$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$
= \int \mathrm{d}\hat{m}\ \mathrm{d}m \int \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(\frac{\lambda N}{2}\left[nm^2 + \frac{n^2-n}{2}q^2\right] + N\left[nm\hat{m} + \frac{n^2-n}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x_*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\hat{m}\sum_{\alpha} x_* x_\alpha - \hat{q}\sum_{\alpha<\beta} x_\alpha x_\beta\right)\right]\right\}^N
$$

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right]$$

$$= \int \prod_{\alpha} \mathrm{d}\hat{m}_\alpha \mathrm{d}m_\alpha \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2}\left[\sum_{\alpha} m_\alpha^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N\left[\sum_{\alpha} m_\alpha \hat{m}_\alpha + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\sum_{\alpha} \hat{m}_\alpha x_* x_\alpha - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta} x_\alpha x_\beta\right)\right]\right\}^N$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m \int \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(\frac{\lambda N}{2}\left[nm^2 + \frac{n^2-n}{2}q^2\right] + N\left[nm\hat{m} + \frac{n^2-n}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x_*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\hat{m}\sum_{\alpha} x_* x_\alpha - \hat{q}\sum_{\alpha<\beta} x_\alpha x_\beta\right)\right]\right\}^N$$

$$\int \mathscr{D}z = \int dz \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})$$

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m\ \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(nN\left[\frac{\lambda}{2}m^2 + \frac{\lambda}{4}(n-1)q^2 + m\hat{m} + \frac{n-1}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(\frac{\hat{q}}{2}\sum_{\alpha} x_\alpha^2 - \hat{m}\sum_{\alpha} x_* x_\alpha\right)\int \mathscr{D}z \exp\left(-\mathrm{i}z\sqrt{\hat{q}}\sum_{\alpha} x_\alpha\right)\right]\right\}^N$$

Hubbard-Stratonovich Transformation

$$\exp\left(-\frac{a}{2}x^2\right) = \frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2a} - ixz\right)\mathrm{d}z, \quad \forall a > 0$$

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right]$$

$$= \int \prod_{\alpha} \mathrm{d}\hat{m}_\alpha \mathrm{d}m_\alpha \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta}\mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2}\left[\sum_{\alpha} m_\alpha^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N\left[\sum_{\alpha} m_\alpha \hat{m}_\alpha + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\sum_{\alpha} \hat{m}_\alpha x_* x_\alpha - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta} x_\alpha x_\beta\right)\right]\right\}^N$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m \int \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(\frac{\lambda N}{2}\left[nm^2 + \frac{n^2-n}{2}q^2\right] + N\left[nm\hat{m} + \frac{n^2-n}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x_*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\hat{m}\sum_{\alpha} x_* x_\alpha - \hat{q}\sum_{\alpha<\beta} x_\alpha x_\beta\right)\right]\right\}^N$$

$$\int \mathscr{D}z = \int dz \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})$$

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m\ \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(nN\left[\frac{\lambda}{2}m^2 + \frac{\lambda}{4}(n-1)q^2 + m\hat{m} + \frac{n-1}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(\frac{\hat{q}}{2}\sum_{\alpha} x_\alpha^2 - \hat{m}\sum_{\alpha} x_* x_\alpha\right)\int \mathscr{D}z \exp\left(-\mathrm{i}z\sqrt{\hat{q}}\sum_{\alpha} x_\alpha\right)\right]\right\}^N$$

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right]$$

$$= \int \prod_{\alpha} \mathrm{d}\hat{m}_\alpha \mathrm{d}m_\alpha \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta}\mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N\left[\sum_\alpha m_\alpha \hat{m}_\alpha + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_\alpha P_X(x_\alpha)\,\mathrm{d}x_\alpha \exp\left(-\sum_\alpha \hat{m}_\alpha x_* x_\alpha - \sum_{\alpha<\beta}\hat{q}_{\alpha\beta}x_\alpha x_\beta\right)\right]\right\}^N$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m \int \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(\frac{\lambda N}{2}\left[nm^2 + \frac{n^2-n}{2}q^2\right] + N\left[nm\hat{m} + \frac{n^2-n}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x_*}\left[\int \prod_\alpha P_X(x_\alpha)\,\mathrm{d}x_\alpha \exp\left(-\hat{m}\sum_\alpha x_* x_\alpha - \hat{q}\sum_{\alpha<\beta} x_\alpha x_\beta\right)\right]\right\}^N$$

$$\int \mathscr{D}z = \int dz \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})$$

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$$\left\{\mathbb{E}_{x_*}\left[\int \mathscr{D}z \prod_\alpha \left\{\int P_X(x_\alpha)\,\mathrm{d}x_\alpha \exp\left(\frac{\hat{q}}{2}x_\alpha^2 - \hat{m}x_* x_\alpha - \mathrm{i}z\sqrt{\hat{q}}x_\alpha\right)\right\}\right]\right\}^N$$

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Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m \int \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(\frac{\lambda N}{2}\left[nm^2 + \frac{n^2-n}{2}q^2\right] + N\left[nm\hat{m} + \frac{n^2-n}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x_*}\left[\int \prod_\alpha P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(-\hat{m}\sum_\alpha x_* x_\alpha - \hat{q}\sum_{\alpha<\beta} x_\alpha x_\beta\right)\right]\right\}^N$$

$$\int \mathscr{D}z = \int dz \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})$$

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$$\left\{\mathbb{E}_{x_*}\left[\int \mathscr{D}z \prod_\alpha \left\{\int P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \exp\left(\frac{\hat{q}}{2}x_\alpha^2 - \hat{m}x_* x_\alpha - \mathrm{i}z\sqrt{\hat{q}}x_\alpha\right)\right\}\right]\right\}^N$$

Now take the limit $n \to 0$ $\quad \mathbb{E}\left[X^n\right] = \mathbb{E}\left[e^{n\log(X)}\right] \approx \mathbb{E}[1 + n\log(X)] = 1 + n\mathbb{E}[\log(X)] = \exp(\log(1 + n\mathbb{E}[\log(X)])) \approx \exp(n\mathbb{E}[\log(X)])$

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m\ \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(nN\left[\frac{\lambda}{2}m^2 - \frac{\lambda}{4}q^2 + m\hat{m} - \frac{1}{2}q\hat{q}\right]\right)\exp\left\{nN\mathbb{E}_{x_*}\left[\int \mathscr{D}z \log\left(\int P_X(x)\mathrm{d}x \exp\left(\frac{\hat{q}}{2}x^2 - \hat{m}x_* x - \mathrm{i}z\sqrt{q}x\right)\right)\right]\right\}$$

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right]$$

$$= \int \prod_\alpha \mathrm{d}\hat{m}_\alpha \mathrm{d}m_\alpha \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta}\mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2}\left[\sum_\alpha m_\alpha^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N\left[\sum_\alpha m_\alpha \hat{m}_\alpha + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right)\left\{\mathbb{E}_{x^*}\left[\int \prod_\alpha P_X\left(x_\alpha\right)\mathrm{d}x_\alpha \exp\left(-\sum_\alpha \hat{m}_\alpha x_* x_\alpha - \sum_{\alpha<\beta}\hat{q}_{\alpha\beta}x_\alpha x_\beta\right)\right]\right\}^N$$

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$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m \int \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(\frac{\lambda N}{2}\left[nm^2 + \frac{n^2-n}{2}q^2\right] + N\left[nm\hat{m} + \frac{n^2-n}{2}q\hat{q}\right]\right)\left\{\mathbb{E}_{x_*}\left[\int \prod_\alpha P_X\left(x_\alpha\right)\mathrm{d}x_\alpha \exp\left(-\hat{m}\sum_\alpha x_* x_\alpha - \hat{q}\sum_{\alpha<\beta}x_\alpha x_\beta\right)\right]\right\}^N$$

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$$\left\{\mathbb{E}_{x_*}\left[\int \mathscr{D}z \prod_\alpha \left\{\int P_X\left(x_\alpha\right)\mathrm{d}x_\alpha \exp\left(\frac{\hat{q}}{2}x_\alpha^2 - \hat{m}x_* x_\alpha - \mathrm{i}z\sqrt{\hat{q}}x_\alpha\right)\right\}\right]\right\}^N$$

Now take the limit $n \to 0$ $\quad \mathbb{E}\left[X^n\right] = \mathbb{E}\left[e^{n\log(X)}\right] \approx \mathbb{E}[1 + n\log(X)] = 1 + n\mathbb{E}[\log(X)] = \exp(\log(1 + n\mathbb{E}[\log(X)])) \approx \exp(n\mathbb{E}[\log(X)])$

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m\ \mathrm{d}\hat{q}\ \mathrm{d}q \exp\left(nN\left[\frac{\lambda}{2}m^2 - \frac{\lambda}{4}q^2 + m\hat{m} - \frac{1}{2}q\hat{q}\right]\right)\exp\left\{nN\mathbb{E}_{x_*}\left[\int \mathscr{D}z \log\left(\int P_X(x)\mathrm{d}x \exp\left(\frac{\hat{q}}{2}x^2 - \hat{m}x_* x - \mathrm{i}z\sqrt{q}x\right)\right)\right]\right\}$$

$$= \int \mathrm{d}\hat{m}\ \mathrm{d}m\ \mathrm{d}\hat{q}\ \mathrm{d}q \exp(nN\Phi(m,q,\hat{m},\hat{q}))$$

$$\Phi(m,q,\hat{m},\hat{q}) = \frac{\lambda}{4}\left(2m^2 - q^2\right) + m\hat{m} - \frac{1}{2}q\hat{q} + \mathbb{E}_{x^*}\left[\int \mathscr{D}z \log\left(\int P_X(x)\mathrm{d}x \exp\left[\frac{\hat{q}}{2}x^2 + \left(\sqrt{\hat{q}}z - \hat{m}x_*\right)x\right]\right)\right]$$

$$\Phi(m,q,\hat{m},\hat{q}) = \frac{\lambda}{4}\left(2m^2 - q^2\right) + m\hat{m} - \frac{1}{2}q\hat{q} + \mathbb{E}_{x^*}\left[\int \mathscr{D}z \log\left(\int P_X(x)\mathrm{d}x \exp\left[\frac{\hat{q}}{2}x^2 + \left(\sqrt{\hat{q}}z - \hat{m}x_*\right)x\right]\right)\right]$$

$m_\alpha = \frac{1}{N}\sum_i x_i^{(\alpha)} x_i^*$ “overlap with the truth"

$q_{\alpha\beta} = \frac{1}{N}\sum_i x_i^{(\alpha)} x_i^{(\beta)}$ “replicas overlap"

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Nishimori Identity (for two Replicas)

Consider a model $y = f(x^*) + \eta$ where f is a continuous bounded function, and η is the additive noise.
Denote $\langle\, . \,\rangle_{x^*,\eta}$ the expectation with respect to $P(X = .\,|\,Y = f(x^*) + \eta)$, then

$$\mathbb{E}_{X^*,\eta}\left[\langle g\left(X, X^*\right)\rangle_{X^*,\eta}\right] = \mathbb{E}_{X^*,\eta}\left[\left\langle g\left(X^{(1)}, X^{(2)}\right)\right\rangle_{X^*,\eta}\right]$$

where $X^{(1)}, X^{(2)}$ are two independent replicas distributed as $P(X = .\,|\,Y = f(x^*) + \eta)$

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Nishimori identity implies $m = q, \;\; \hat{m} = \hat{q}$

$$\Phi_{\text{Nishi}}(m,\hat{m}) = \Phi(m,q,\hat{m},\hat{q})\,|_{q=m,\hat{q}=\hat{m}} = \frac{\lambda}{4}m^2 + \frac{1}{2}m\hat{m} + \mathbb{E}_{x^*}\left[\int \mathscr{D}z \log\left(\int P_X(x)\mathrm{d}x e^{\frac{\hat{m}}{2}x^2 - \left(\mathrm{i}\sqrt{\hat{m}}z + \hat{m}x_*\right)x}\right)\right]$$

$$\Phi(m,q,\hat{m},\hat{q}) = \frac{\lambda}{4}\left(2m^2 - q^2\right) + m\hat{m} - \frac{1}{2}q\hat{q} + \mathbb{E}_{x^*}\left[\int \mathscr{D}z \log\left(\int P_X(x)\mathrm{d}x \exp\left[\frac{\hat{q}}{2}x^2 + \left(\sqrt{\hat{q}}z - \hat{m}x_*\right)x\right]\right)\right]$$

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$$\Phi_{\text{Nishi}}\left(m,\hat{m}\right) = \Phi(m,q,\hat{m},\hat{q})\,|_{q=m,\hat{q}=\hat{m}} = \frac{\lambda}{4}m^2 + \frac{1}{2}m\hat{m} + \mathbb{E}_{x^*}\left[\int \mathscr{D}z \log\left(\int P_X(x)\mathrm{d}x e^{\frac{\hat{m}}{2}x^2 - \left(\mathrm{i}\sqrt{\hat{m}}z + \hat{m}x_*\right)x}\right)\right]$$

Extremization implies $\hat{m} = -\lambda m$

$$\boxed{\Phi_{\text{RS}}(m) \triangleq \Phi_{\text{Nishi}}(m,\hat{m})\Big|_{\hat{m}=-\lambda m} = -\frac{\lambda}{4}m^2 + \mathbb{E}_{x_*,z}\left[\log\left(\int P_X(x)\mathrm{d}x e^{\frac{-\lambda m}{2}x^2 + \left(\lambda m x_* + \sqrt{\lambda m}z\right)x}\right)\right]}$$

Recall on what we care about

$$\mathrm{MMSE}(\lambda) = \mathbb{E}_{\mathbf{Y},x^*}\left[\left(\langle x\rangle_{\mathbf{Y}} - x^*\right)^2\right] = \mathbb{E}_{\mathbf{Y},x^*}\left[\langle x\rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^*\langle x\rangle_{\mathbf{Y}}\right]$$

Recall on what we care about

$$\mathrm{MMSE}(\lambda) = \mathbb{E}_{\mathbf{Y},x^*}\left[\left(\langle x\rangle_{\mathbf{Y}} - x^*\right)^2\right] = \mathbb{E}_{\mathbf{Y},x^*}\left[\langle x\rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^*\langle x\rangle_{\mathbf{Y}}\right]$$

(Nishimori)

$$= \mathbb{E}_{x^*}\left[(x^*)^2\right] - \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left\langle \frac{1}{N}\sum_i x_i^* x_i \right\rangle_{\mathbf{x}^*,\boldsymbol{\xi}}\right]$$

$N \to \infty$

$$\approx \mathbb{E}_{x^*}\left[(x^*)^2\right] - m^*(\lambda)$$

Recall on what we care about

$$\mathrm{MMSE}(\lambda) = \mathbb{E}_{\mathbf{Y},x^*}\left[\left(\langle x\rangle_{\mathbf{Y}} - x^*\right)^2\right] = \mathbb{E}_{\mathbf{Y},x^*}\left[\langle x\rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^*\langle x\rangle_{\mathbf{Y}}\right]$$

(Nishimori)

$$= \mathbb{E}_{x^*}\left[(x^*)^2\right] - \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left\langle \frac{1}{N}\sum_i x_i^* x_i \right\rangle_{\mathbf{x}^*,\boldsymbol{\xi}}\right]$$

$N \to \infty$

$$\approx \mathbb{E}_{x^*}\left[(x^*)^2\right] - m*(\lambda)$$

A Simple Solvable Case

Assume now that each $x_i^* = \pm 1$ are distributed with equal probabilities, i.e. $P_X(x) = \frac{1}{2}[\delta(x-1) + \delta(x+1)]$ (Radamacher's Distributed), in homework 3, you'll show that

$$\Phi_{\mathrm{RS}}(m,\lambda) = -\frac{\lambda m^2}{4} - \frac{\lambda m}{2} + \frac{1}{2}\mathbb{E}_z\left[\log\left\{\frac{1}{2}(\cosh(2\lambda m) + \cosh(2z\sqrt{\lambda m}))\right\}\right]$$

and that

$$m^*(\lambda) = \mathbb{E}_z\left[\frac{2\sinh(2\lambda m) + \frac{z}{\sqrt{\lambda m}}\sinh(2z\sqrt{\lambda m})}{\cosh(2\lambda m) + \cosh(2z\sqrt{\lambda m})}\right] - 1$$

Example of Computational Phase Transition (for Statistical Estimation)