Machine Learning for Physical Scientists

Lecture 10

Limit of Statistical Learnability (Detection) in Spike-Wigner Model (a.k.a. Symmetric Vector-Spin Glass Model)

Spike-Wigner Model

Signal to Noise Ratio (SNR)

$$\mathbf{Y} = \sqrt{\frac{\lambda}{N}} \mathbf{x} * \mathbf{x}^{*T} + \boldsymbol{\xi}$$

symmetric rank 1 matrix

symmetric i.i.d. noise

$$\mathbf{x}^* \in \mathbb{R}^N$$
 with $x_i^* \sim P_X(x)$

$$\xi_{ij} = \xi_{ji} \sim \mathcal{N}(0,1)$$

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The Posterior is

$$P(\mathbf{x} \mid \mathbf{Y}) = \frac{1}{Z(\mathbf{Y})} \left[\prod_{i=1}^{N} P_X(x_i) \right] \left[\prod_{i \le j} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(y_{ij} - \sqrt{\frac{\lambda}{N}} x_i x_j \right)^2 \right) \right]$$

$$\propto \frac{1}{Z(\mathbf{Y})} \left[\prod_{i}^{N} P_X(x_i) \right] \exp\left(\sum_{i \le j} \left[-\frac{\lambda}{2N} x_i^2 x_j^2 + \sqrt{\frac{\lambda}{N}} y_{ij} x_i x_j \right] \right)$$

From Posterior to Minimum Mean Square Error Estimator

$$\hat{\mathbf{x}}_{\mathrm{MSE}}(\mathbf{Y}) = \begin{bmatrix} \hat{x}_{\mathrm{MSE},1}(\mathbf{Y}) \\ \vdots \\ \hat{x}_{\mathrm{MSE},N}(\mathbf{Y}) \end{bmatrix} \qquad \hat{x}_{\mathrm{MSE},i}(\mathbf{Y}) = \left\langle x_i \right\rangle_{\mathbf{Y}} = \int d\mathbf{x} P(\mathbf{x} \mid \mathbf{Y}) x_i$$

How does this estimator perform on average?

$$\mathrm{MMSE}(\lambda) = \mathbb{E}_{\mathbf{Y}, x^*} \left[\left(\langle x \rangle_{\mathbf{Y}} - x^* \right)^2 \right] = \mathbb{E}_{\mathbf{Y}, x^*} \left[\langle x \rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^* \langle x \rangle_{\mathbf{Y}} \right]$$

From Posterior to Minimum Mean Square Error Estimator

$$\hat{\mathbf{x}}_{\mathrm{MSE}}(\mathbf{Y}) = \begin{bmatrix} \hat{x}_{\mathrm{MSE},1}(\mathbf{Y}) \\ \vdots \\ \hat{x}_{\mathrm{MSE},N}(\mathbf{Y}) \end{bmatrix} \qquad \hat{x}_{\mathrm{MSE},i}(\mathbf{Y}) = \left\langle x_i \right\rangle_{\mathbf{Y}} = \int d\mathbf{x} P(\mathbf{x} \mid \mathbf{Y}) x_i$$

How does this estimator perform on average?

$$\begin{aligned} \text{MMSE}(\lambda) &= \mathbb{E}_{\mathbf{Y},x^*} \left[\left(\langle x \rangle_{\mathbf{Y}} - x^* \right)^2 \right] = \mathbb{E}_{\mathbf{Y},x^*} \left[\langle x \rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^* \langle x \rangle_{\mathbf{Y}} \right] \\ &= \mathbb{E}_{x^*} \left[(x^*)^2 \right] - \mathbb{E}_{\mathbf{x}^*,\xi} \left[\left\langle \frac{1}{N} \sum_i x_i^* x_i \right\rangle_{\mathbf{x}^*,\xi} \right] \\ & N \to \infty \\ &\approx \mathbb{E}_{x^*} \left[(x^*)^2 \right] - m^* (\lambda) \end{aligned}$$

How does this error depend on the SNR ratio?

Need to evaluate the *typical value* of Posterior (quenched random variable in the exponent)

$$P(\mathbf{x} \mid \mathbf{Y}) = \frac{1}{Z(\mathbf{Y})} \left[\prod_{i} P_{X}(x_{i}) \right] \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} x_{i}^{2} x_{j}^{2} + \sqrt{\frac{\lambda}{N}} y_{ij} x_{i} x_{j} \right] \right)$$

$$= \frac{1}{Z(\mathbf{Y})} \left[\prod_{i} P_{X}(x_{i}) \right] \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} x_{i}^{2} x_{j}^{2} + \frac{\lambda}{N} x_{i} x_{j} x_{i}^{*} x_{j}^{*} + \sqrt{\frac{\lambda}{N}} \xi_{ij} x_{i} x_{j} \right] \right)$$

We'll be interested in the average free energy per site

$$\lim_{N \to \infty} \mathbb{E}_{\mathbf{Y}} \left[\frac{1}{N} \log(Z(\mathbf{Y})) \right] = \lim_{N \to \infty} \mathbb{E}_{\mathbf{x}^*, \boldsymbol{\xi}} \left[\frac{1}{N} \log(Z(\mathbf{Y})) \right]$$

Replica Trick to the rescue!

$$\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] = \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_i P_X\left(x_i\right)\right] \exp\left(\sum_{i\leq j}\left[-\frac{\lambda}{2N}x_i^2x_j^2 + \frac{\lambda}{N}x_ix_jx_i^*x_j^* + \sqrt{\frac{\lambda}{N}}\boldsymbol{\xi}_{ij}x_ix_j\right]\right)\right]^n\right]$$

$$\begin{split} \mathbb{E}_{\mathbf{x}^{*},\boldsymbol{\xi}}\left[Z^{n}\right] &= \mathbb{E}_{\mathbf{x}^{*},\boldsymbol{\xi}}\left[\left(\int \mathrm{d}\mathbf{x}\left[\prod_{i}P_{X}\left(x_{i}\right)\right]\exp\left(\sum_{i\leq j}\left[-\frac{\lambda}{2N}x_{i}^{2}x_{j}^{2} + \frac{\lambda}{N}x_{i}x_{j}x_{i}^{*}x_{j}^{*} + \sqrt{\frac{\lambda}{N}}\xi_{ij}x_{i}x_{j}\right]\right)\right)^{n}\right] \\ &= \mathbb{E}_{\mathbf{x}^{*},\boldsymbol{\xi}}\left[\prod_{\alpha=1}^{n}\int \mathrm{d}\mathbf{x}^{(\alpha)}\prod_{i}P_{X}\left(x_{i}^{(\alpha)}\right)\exp\left(\sum_{i\leq j}\left[-\frac{\lambda}{2N}\left(x_{i}^{(\alpha)}\right)^{2}\left(x_{i}^{(\alpha)}\right)^{2} + \frac{\lambda}{N}x_{i}^{(\alpha)}x_{j}^{(\alpha)}x_{i}^{*}x_{j}^{*} + \sqrt{\frac{\lambda}{N}}\xi_{ij}x_{i}^{(\alpha)}x_{j}^{(\alpha)}\right]\right)\right] \end{split}$$

$$\mathbb{E}_{\mathbf{x} \in \mathcal{E}} \left[Z^{n} \right] = \mathbb{E}_{\mathbf{x} \in \mathcal{E}} \left[\left(\int d\mathbf{x} \left[\prod_{i} P_{X} \left(x_{i} \right) \right] \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} x_{i}^{2} x_{j}^{2} + \frac{\lambda}{N} x_{i} x_{j} x_{i}^{*} x_{j}^{*} + \sqrt{\frac{\lambda}{N}} \xi_{ij} x_{i} x_{j}} \right] \right) \right)^{n} \right]$$

$$= \mathbb{E}_{\mathbf{x} \in \mathcal{E}} \left[\prod_{\alpha = 1}^{n} \int d\mathbf{x}^{(\alpha)} \prod_{i} P_{X} \left(x_{i}^{(\alpha)} \right) \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \left(x_{i}^{(\alpha)} \right)^{2} \left(x_{j}^{(\alpha)} \right)^{2} + \frac{\lambda}{N} x_{i}^{(\alpha)} x_{j}^{(\alpha)} x_{i}^{*} x_{j}^{*} + \sqrt{\frac{\lambda}{N}} \xi_{ij} x_{i}^{(\alpha)} x_{j}^{(\alpha)} \right] \right) \right]$$

$$= \mathbb{E}_{\mathbf{x} \in \mathcal{E}} \left[\int \prod_{\alpha, i} P_{X} \left(x_{i}^{(\alpha)} \right) dx_{i}^{(\alpha)} \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \sum_{\alpha} \left(x_{i}^{(\alpha)} \right)^{2} \left(x_{j}^{(\alpha)} \right)^{2} \right] + \frac{\lambda}{N} \sum_{\alpha} x_{i}^{(\alpha)} x_{j}^{(\alpha)} x_{i}^{*} x_{j}^{*} \right) \prod_{i \leq j} \mathbb{E}_{\mathbf{x} \in \mathcal{E}} \left[\exp \left(\xi_{ij} \sqrt{\frac{\lambda}{N}} \sum_{\alpha} x_{i}^{(\alpha)} x_{j}^{(\alpha)} \right) \right] \right]$$

$$\mathbb{E}_{\boldsymbol{x},\boldsymbol{\xi}}\left[Z^{n}\right] = \mathbb{E}_{\boldsymbol{x},\boldsymbol{\xi}}\left[\left(\int d\mathbf{x}\left[\prod_{i}P_{X}\left(x_{i}\right)\right]\exp\left(\sum_{i\leq j}\left[-\frac{\lambda}{2N}x_{i}^{2}x_{j}^{2} + \frac{\lambda}{N}x_{i}x_{j}x_{i}^{*}x_{j}^{*} + \sqrt{\frac{\lambda}{N}}\xi_{ij}x_{i}x_{j}}\right]\right)\right)^{n}\right]$$

$$= \mathbb{E}_{\boldsymbol{x},\boldsymbol{\xi}}\left[\prod_{\alpha=1}^{n}\int d\mathbf{x}^{(\alpha)}\prod_{i}P_{X}\left(x_{i}^{(\alpha)}\right)\exp\left(\sum_{i\leq j}\left[-\frac{\lambda}{2N}\left(x_{i}^{(\alpha)}\right)^{2}\left(x_{j}^{(\alpha)}\right)^{2} + \frac{\lambda}{N}x_{i}^{(\alpha)}x_{j}^{(\alpha)}x_{i}^{*}x_{j}^{*} + \sqrt{\frac{\lambda}{N}}\xi_{ij}x_{i}^{(\alpha)}x_{j}^{(\alpha)}\right]\right)\right]$$

$$= \mathbb{E}_{\boldsymbol{x}}\left[\int\prod_{\alpha,i}P_{X}\left(x_{i}^{(\alpha)}\right)dx_{i}^{(\alpha)}\exp\left(\sum_{i\leq j}\left[-\frac{\lambda}{2N}\sum_{\alpha}\left(x_{i}^{(\alpha)}\right)^{2}\left(x_{j}^{(\alpha)}\right)^{2}\right] + \frac{\lambda}{N}\sum_{\alpha}x_{i}^{(\alpha)}x_{j}^{(\alpha)}x_{i}^{*}x_{j}^{*}\right)\prod_{i\leq j}\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{y}}\left[\exp\left(\xi_{ij}\sqrt{\frac{\lambda}{N}}\sum_{\alpha}x_{i}^{(\alpha)}x_{i}^{(\alpha)}x_{j}^{(\alpha)}\right)\right]\right]$$

$$= \exp\left(\frac{\lambda}{2N}\sum_{i\leq j}\sum_{\alpha}\sum_{\alpha}x_{i}^{(\alpha)}x_{i}^{(\alpha)}x_{i}^{(\beta)}x_{j}^{(\beta)}\right)$$

$$\begin{split} \mathbb{E}_{\mathbf{x},\mathbf{x}}[Z^n] &= \mathbb{E}_{\mathbf{x},\mathbf{x}} \left[\left(\int \mathrm{d}\mathbf{x} \left[\prod_i P_X\left(x_i\right) \right] \exp\left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} x_i^2 x_j^2 + \frac{\lambda}{N} x_i x_j x_i^{**} x_j^* + \sqrt{\frac{\lambda}{N}} \xi_{ij} x_i x_j} \right] \right) \right)^n \right] \\ &= \mathbb{E}_{\mathbf{x},\mathbf{x}} \left[\prod_{a=1}^n \int \mathrm{d}\mathbf{x}^{(a)} \prod_i P_X\left(x_i^{(a)}\right) \exp\left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \left(x_i^{(a)} \right)^2 \left(x_j^{(a)} \right)^2 + \frac{\lambda}{N} x_i^{(a)} x_j^{(a)} x_i^{**} x_j^* + \sqrt{\frac{\lambda}{N}} \xi_{ij} x_i^{(a)} x_j^{(a)} \right] \right] \right] \\ &= \mathbb{E}_{\mathbf{x}} \left[\left[\prod_{a,i} P_X\left(x_i^{(a)}\right) \mathrm{d}x_i^{(a)} \exp\left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \sum_a \left(x_i^{(a)} \right)^2 \left(x_j^{(a)} \right)^2 \right] + \frac{\lambda}{N} \sum_a x_i^{(a)} x_j^{(a)} x_i^{**} x_j^* \right) \prod_{i \leq j} \mathbb{E}_{\mathbf{x},\mathbf{y}} \left[\exp\left(\sum_{i \neq j} \sqrt{\frac{\lambda}{N}} \sum_a x_i^{(a)} x_j^{(a)} \right) \right] \right] \\ &\sum_{i \leq j} \frac{a_i}{N} \frac{a_j}{N} = \frac{1}{2} \left(\sum_i \frac{a_i}{N} \right)^2 + \frac{1}{2} \sum_i \frac{a_i^2}{N^2} \frac{O(N^{-1})}{N^2} \\ &= \mathbb{E}_{\mathbf{x}} \left[\prod_{a,i} P_X\left(x_i^{(a)}\right) \mathrm{d}x_i^{(a)} \exp\left(\frac{\lambda N}{2} \sum_a \left(\sum_i \frac{x_i^{(a)} x_i^{(a)}}{N} \right)^2 + \frac{\lambda N}{2} \sum_{a \leq \beta} \left(\sum_i \frac{x_i^{(a)} x_i^{(\beta)}}{N} \right)^2 \right] \end{aligned}$$

$$\begin{split} \mathbb{E}_{\boldsymbol{w} \in [Z^n]} &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\left(\int \mathrm{d}\mathbf{x} \left[\prod_i P_X(x_i) \right] \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} x_i^2 x_j^2 + \frac{\lambda}{N} x_i x_j x_j^* x_j^* + \sqrt{\frac{\lambda}{N}} x_i x_j x_j^* x_j^* \right] \right) \right]^n \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n \int \mathrm{d}\mathbf{x}^{(\alpha)} \prod_i P_X\left(x_i^{(\alpha)}\right) \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \left(x_i^{(\alpha)} \right)^2 \left(x_j^{(\alpha)} \right)^2 + \frac{\lambda}{N} \sum_i x_i^{(\alpha)} x_j^{(\alpha)} x_j^* x_j^* \right) + \sqrt{\frac{\lambda}{N}} \sum_i x_i^{(\alpha)} x_i^{(\alpha)} x_j^* \right) \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \sum_{\alpha} \left(x_i^{(\alpha)} \right)^2 \left(x_j^{(\alpha)} \right)^2 \right] + \frac{\lambda}{N} \sum_i x_i^{(\alpha)} x_j^{(\alpha)} x_j^{(\alpha)} x_j^{(\alpha)} \right) \right] \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\frac{\lambda}{2N} \sum_{\alpha} \left(\sum_i \frac{\mathbf{w}_i x_i^{(\alpha)}}{N} \right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N} \right)^2 \right) \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_i \frac{\mathbf{w}_i x_i^{(\alpha)}}{N} \right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N} \right)^2 \right) \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_i \frac{\mathbf{w}_i x_i^{(\alpha)}}{N} \right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N} \right)^2 \right) \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_i \frac{\mathbf{w}_i x_i^{(\alpha)}}{N} \right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N} \right)^2 \right] \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_i \frac{\mathbf{w}_i x_i^{(\alpha)}}{N} \right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N} \right)^2 \right] \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_i \frac{\mathbf{w}_i x_i^{(\alpha)}}{N} \right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N} \right) \right] \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\sum_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) + \frac{\lambda}{2} \sum_{\alpha \leq \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\alpha)}}{N} \right) \right] \right] \\ &= \mathbb{E}_{\boldsymbol{w} \in [Z^n]} \left[\prod_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \exp \left(\sum_{\alpha = 1}^n P_X\left(x_i^{(\alpha)}$$

$$\begin{split} \mathbb{E}_{\mathbf{a},\mathbf{g}}[Z^{n}] &= \mathbb{E}_{\mathbf{a},\mathbf{g}} \left[\left[\int \mathrm{d}\mathbf{x} \left[\prod_{i} P_{X}(\mathbf{x}_{i}) \right] \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \mathbf{x}_{i}^{2} \mathbf{x}_{j}^{2} + \frac{\lambda}{N} \mathbf{x}_{i} \mathbf{x}_{j}^{*} \mathbf{x}_{i}^{*} \mathbf{x}_{j}^{*} + \sqrt{\frac{\lambda}{N}} \mathbf{x}_{j}^{*} \mathbf{x}_{i}^{*} \mathbf{x}_{j}^{*}} \right] \right) \right]^{n} \\ &= \mathbb{E}_{\mathbf{a},\mathbf{g}} \left[\prod_{\alpha = 1}^{n} \int \mathrm{d}\mathbf{x}^{(\alpha)} \prod_{i} P_{X}\left(\mathbf{x}_{i}^{(\alpha)}\right) \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \left(\mathbf{x}_{i}^{(\alpha)} \right)^{2} \left(\mathbf{x}_{j}^{(\alpha)} \right)^{2} + \frac{\lambda}{N} \mathbf{x}_{i}^{(\alpha)} \mathbf{x}_{j}^{(\alpha)} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \right) + \sqrt{\frac{\lambda}{N}} \mathbf{x}_{j}^{*} \mathbf{x}_{i}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \right] \right) \\ &= \mathbb{E}_{\mathbf{a},\mathbf{b}} \left[\prod_{\alpha,i} P_{X}\left(\mathbf{x}_{i}^{(\alpha)}\right) d\mathbf{x}_{i}^{(\alpha)} \exp \left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \sum_{\alpha} \left(\mathbf{x}_{i}^{(\alpha)} \right)^{2} \left(\mathbf{x}_{j}^{(\alpha)} \right)^{2} + \frac{\lambda}{N} \sum_{\alpha} \mathbf{x}_{i}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \mathbf{x}_{j}^{*} \right) \right] \\ &= \mathbb{E}_{\mathbf{a},\mathbf{b}} \left[\prod_{\alpha,i} P_{X}\left(\mathbf{x}_{i}^{(\alpha)}\right) d\mathbf{x}_{i}^{(\alpha)} \exp \left(\frac{\lambda N}{2} \sum_{\alpha} \left(\sum_{i} \frac{\mathbf{x}_{i}^{*} \mathbf{x}_{i}^{*} \mathbf{x}_{i}^{*} \mathbf{x}_{i}^{*} \mathbf{x}_{j}^{*} \mathbf{x}$$

$$\begin{split} \mathbb{E}_{\mathbf{Q},\mathbf{Z}}[Z^a] &= \mathbb{E}_{\mathbf{Q},\mathbf{Z}}\left[\left[\int \mathrm{d}\mathbf{x} \left[\prod_{i} P_X\left(\mathbf{x}_i^{(a)}\right) \exp\left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \mathbf{x}_i^2 \mathbf{x}_j^2 + \frac{\lambda}{N} \mathbf{x}_j \mathbf{x}_j^{(a)} \mathbf{x}_j^{(a)} + \sqrt{\frac{\lambda}{N}} \mathbf{x}_j^{(a)} \mathbf{x}_j^{(a)}\right]\right)\right]^{s} \\ &= \mathbb{E}_{\mathbf{Q},\mathbf{Z}}\left[\prod_{\alpha = 1}^{n} \int \mathrm{d}\mathbf{x}^{(a)} \prod_{i} P_X\left(\mathbf{x}_i^{(a)}\right) \exp\left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \left(\mathbf{x}_i^{(a)}\right)^2 \left(\mathbf{x}_j^{(a)}\right)^2 + \frac{\lambda}{N} \mathbf{x}_i^{(a)} \mathbf{x}_j^{(a)} \mathbf{x}_j^{(a)} + \sqrt{\frac{\lambda}{N}} \mathbf{x}_j^{(a)} \mathbf{x}_j^{(a)} \mathbf{x}_j^{(a)}\right]\right)\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_j^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\sum_{i \leq j} \left[-\frac{\lambda}{2N} \sum_{\alpha} \left(\mathbf{x}_i^{(a)}\right)^2 \left(\mathbf{x}_j^{(a)}\right)^2 + \frac{\lambda}{N} \sum_{\alpha} \mathbf{x}_i^{(a)} \mathbf{x}_j^{(a)} \mathbf{x}_j^{(a)} \mathbf{x}_j^{(a)}\right)\right]\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_j^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\frac{\lambda}{2N} \sum_{\alpha} \left(\mathbf{x}_j^{(a)}\right)^2 + \frac{\lambda}{N} \sum_{\alpha \leq j} \left(\sum_{i} \frac{\mathbf{x}_j^{(a)} \mathbf{x}_i^{(a)} \mathbf{x}_j^{(a)}}{N}\right)\right]\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_{i} \frac{\mathbf{x}_j^{(a)} \mathbf{x}_i^{(a)}}{N}\right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq j} \left(\sum_{i} \frac{\mathbf{x}_j^{(a)} \mathbf{x}_i^{(a)}}{N}\right)^2\right]\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_{i} \frac{\mathbf{x}_j^{(a)} \mathbf{x}_i^{(a)}}{N}\right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq j} \left(\sum_{i} \frac{\mathbf{x}_j^{(a)} \mathbf{x}_i^{(a)}}{N}\right)^2\right]\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\frac{\lambda}{2} \sum_{\alpha} \left(\sum_{i} \frac{\mathbf{x}_j^{(a)} \mathbf{x}_i^{(a)}}{N}\right)^2 + \frac{\lambda}{2} \sum_{\alpha \leq j} \left(\sum_{i} \frac{\mathbf{x}_j^{(a)} \mathbf{x}_i^{(j)}}{N}\right)^2\right]\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\frac{\lambda}{2} \sum_{\alpha} \mathbf{x}_i^{(a)} \mathbf{x}_i^{(a)}\right)\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\frac{\lambda}{2} \sum_{\alpha} \mathbf{x}_i^{(a)} \mathbf{x}_i^{(a)}\right)\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} + \sum_{\alpha} \mathbf{x}_i^{(a)} \mathbf{x}_i^{(a)}\right)\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} \exp\left(\frac{\lambda}{2} \sum_{\alpha} \mathbf{x}_i^{(a)} \mathbf{x}_i^{(a)}\right)\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right) \mathrm{d}\mathbf{x}_i^{(a)} + \sum_{\alpha} \mathbf{x}_i^{(a)} \mathbf{x}_i^{(a)} \mathbf{x}_i^{(a)}\right)\right] \\ &= \mathbb{E}_{\mathbf{Q}}\left[\prod_{\alpha \leq j} P_X\left(\mathbf{x}_i^{(a)}\right)$$

$$\begin{split} & \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] \\ & = \int \prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \int \prod_{\alpha < \beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2} \left[\sum_{\alpha} m_{\alpha}^2 + \sum_{\alpha < \beta} q_{\alpha\beta}^2\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha < \beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right) \left\{ \mathbb{E}_{x^*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\sum_{\alpha} \hat{m}_{\alpha} x_* x_{\alpha} - \sum_{\alpha < \beta} \hat{q}_{\alpha\beta} x_{\alpha} x_{\beta}\right)\right]\right\}^N \end{split}$$

$$\begin{split} &\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] \\ &= \int\!\prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \!\int\!\prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2} \left[\sum_{\alpha} m_{\alpha}^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right) \left\{\mathbb{E}_{x^*}\left[\int\!\prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\sum_{\alpha} \hat{m}_{\alpha}x_*x_{\alpha} - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta}x_{\alpha}x_{\beta}\right)\right]\right\}^{N} \\ &= \int\!\prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \int\!\prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(-\sum_{\alpha} \hat{m}_{\alpha}x_*x_{\alpha} - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta}x_{\alpha}x_{\beta}\right)\right] \\ + N\left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right] + N\left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right] \\ + N\left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right] + N\left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right] \\ + N\left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha}\right] \\ + N\left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha}$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \! \mathrm{d}\hat{m} \, \mathrm{d}m \int \! \mathrm{d}\hat{q} \, \mathrm{d}q \exp \left(\frac{\lambda N}{2} \left[nm^2 + \frac{n^2 - n}{2} q^2 \right] + N \left[nm\hat{m} + \frac{n^2 - n}{2} q\hat{q} \right] \right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X \left(x_{\alpha} \right) \mathrm{d}x_{\alpha} \exp \left(-\hat{m} \sum_{\alpha} x_* x_{\alpha} - \hat{q} \sum_{\alpha < \beta} x_{\alpha} x_{\beta} \right) \right] \right\}^{N}$$

$$\begin{split} &\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] \\ &= \int\!\prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \!\int\!\prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2} \left[\sum_{\alpha} m_{\alpha}^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right) \left\{\mathbb{E}_{x^*}\left[\int\!\prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\sum_{\alpha} \hat{m}_{\alpha}x_*x_{\alpha} - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta}x_{\alpha}x_{\beta}\right)\right]\right\}^N \right\} \\ &= \int\!\prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \int\!\prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(-\sum_{\alpha} \hat{m}_{\alpha}x_*x_{\alpha} - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta}x_{\alpha}x_{\beta}\right)\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha}\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha}\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha}\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha} m_{\alpha}\hat{m}_{\alpha}\right] + N \left[\sum_{\alpha} m_{\alpha$$

Assuming replica symmetric ansatz $m_{\alpha}\equiv m, \hat{m}_{\alpha}\equiv \hat{m}, q_{\alpha\beta}\equiv q, \hat{q}_{\alpha\beta}\equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \int \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(\frac{\lambda N}{2} \left[nm^2 + \frac{n^2 - n}{2}q^2\right] + N \left[nm\hat{m} + \frac{n^2 - n}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\hat{m}\sum_{\alpha} x_* x_{\alpha} - \hat{q}\sum_{\alpha < \beta} x_{\alpha} x_{\beta}\right)\right] \right\}_{\mathcal{D}Z}^{N}$$

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \ \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(nN \left[\frac{\lambda}{2}m^2 + \frac{\lambda}{4}(n-1)q^2 + m\hat{m} + \frac{n-1}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x^*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(\frac{\hat{q}}{2}\sum_{\alpha} x_{\alpha}^2 - \hat{m}\sum_{\alpha} x_* x_{\alpha}\right)\right] \mathcal{D}Z \exp\left(-iz\sqrt{\hat{q}}\sum_{\alpha} x_{\alpha}\right)\right] \right\}_{x_*}^{N}$$

Hubbard-Stratonovich Transformation

$$\exp\left(-\frac{a}{2}x^2\right) = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2a} - ixz\right) dz, \quad \forall a > 0$$

$$\begin{split} & \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] \\ & = \int \prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \int \prod_{\alpha < \beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2} \left[\sum_{\alpha} m_{\alpha}^2 + \sum_{\alpha < \beta} q_{\alpha\beta}^2\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha < \beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right) \left\{ \mathbb{E}_{x^*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\sum_{\alpha} \hat{m}_{\alpha}x_*x_{\alpha} - \sum_{\alpha < \beta} \hat{q}_{\alpha\beta}x_{\alpha}x_{\beta}\right)\right]\right\}^N \end{split}$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \int \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(\frac{\lambda N}{2} \left[nm^2 + \frac{n^2 - n}{2}q^2\right] + N \left[nm\hat{m} + \frac{n^2 - n}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\hat{m}\sum_{\alpha} x_* x_{\alpha} - \hat{q}\sum_{\alpha < \beta} x_{\alpha} x_{\beta}\right) \right] \right\}_{\mathcal{D}Z}^{N}$$

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \ \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(nN \left[\frac{\lambda}{2}m^2 + \frac{\lambda}{4}(n-1)q^2 + m\hat{m} + \frac{n-1}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(\frac{\hat{q}}{2}\sum_{\alpha} x_{\alpha}^2 - \hat{m}\sum_{\alpha} x_* x_{\alpha}\right) \int \mathcal{D}Z \exp\left(-iz\sqrt{\hat{q}}\sum_{\alpha} x_{\alpha}\right) \right] \right\}_{x_*}^{N}$$

$$\begin{split} & \mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] \\ & = \int\!\prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \!\int\!\prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left(\frac{\lambda N}{2} \left[\sum_{\alpha} m_{\alpha}^2 + \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] + N \left[\sum_{\alpha} m_{\alpha}\hat{m}_{\alpha} + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right]\right) \! \left\{ \mathbb{E}_{\boldsymbol{x}^*} \left[\int\!\prod_{\alpha} P_{\boldsymbol{X}}\left(\boldsymbol{x}_{\alpha}\right) \mathrm{d}\boldsymbol{x}_{\alpha} \exp\left(-\sum_{\alpha} \hat{m}_{\alpha}\boldsymbol{x}_*\boldsymbol{x}_{\alpha} - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta}\boldsymbol{x}_{\alpha}\boldsymbol{x}_{\beta}\right)\right]\right\}^N \end{split}$$

Assuming replica symmetric ansatz $m_{\alpha}\equiv m, \hat{m}_{\alpha}\equiv \hat{m}, q_{\alpha\beta}\equiv q, \hat{q}_{\alpha\beta}\equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \int \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(\frac{\lambda N}{2} \left[nm^2 + \frac{n^2 - n}{2}q^2\right] + N \left[nm\hat{m} + \frac{n^2 - n}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\hat{m} \sum_{\alpha} x_* x_{\alpha} - \hat{q} \sum_{\alpha < \beta} x_{\alpha} x_{\beta}\right) \right] \right\}_{\mathcal{D}Z}^{N} = \int \mathrm{d}\hat{m} \ \mathrm{d}m \ \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(nN \left[\frac{\lambda}{2}m^2 + \frac{\lambda}{4}(n-1)q^2 + m\hat{m} + \frac{n-1}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x^*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(\frac{\hat{q}}{2} \sum_{\alpha} x_{\alpha}^2 - \hat{m} \sum_{\alpha} x_* x_{\alpha}\right) \int \mathcal{D}Z \exp\left(-iz\sqrt{\hat{q}} \sum_{\alpha} x_{\alpha}\right) \right] \right\}_{x_*}^{N}$$

$$\left\{ \mathbb{E}_{x_*} \left[\int \mathcal{D}z \prod_{\alpha} \left\{ \int P_X \left(x_{\alpha} \right) dx_{\alpha} \exp \left(\frac{\hat{q}}{2} x_{\alpha}^2 - \hat{m} x_* x_{\alpha} - iz \sqrt{\hat{q}} x_{\alpha} \right) \right\} \right] \right\}^N$$

$$\begin{split} & \mathbb{E}_{\mathbf{x}^*, \mathbf{\xi}} \left[Z^n \right] \\ & = \int \prod_{\alpha} \mathrm{d}\hat{m}_{\alpha} \mathrm{d}m_{\alpha} \int \prod_{\alpha < \beta} \mathrm{d}\hat{q}_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp \left(\frac{\lambda N}{2} \left[\sum_{\alpha} m_{\alpha}^2 + \sum_{\alpha < \beta} q_{\alpha\beta}^2 \right] + N \left[\sum_{\alpha} m_{\alpha} \hat{m}_{\alpha} + \sum_{\alpha < \beta} q_{\alpha\beta} \hat{q}_{\alpha\beta} \right] \right) \left\{ \mathbb{E}_{x^*} \left[\int \prod_{\alpha} P_X \left(x_{\alpha} \right) \mathrm{d}x_{\alpha} \exp \left(- \sum_{\alpha} \hat{m}_{\alpha} x_* x_{\alpha} - \sum_{\alpha < \beta} \hat{q}_{\alpha\beta} x_{\alpha} x_{\beta} \right) \right] \right\}^N \\ \end{split}$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \int \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(\frac{\lambda N}{2} \left[nm^2 + \frac{n^2 - n}{2}q^2\right] + N \left[nm\hat{m} + \frac{n^2 - n}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\hat{m}\sum_{\alpha} x_* x_{\alpha} - \hat{q}\sum_{\alpha < \beta} x_{\alpha} x_{\beta}\right) \right] \right\}_{\mathcal{D}Z}^{N} = \int \mathrm{d}\hat{m} \ \mathrm{d}m \ \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(nN \left[\frac{\lambda}{2}m^2 + \frac{\lambda}{4}(n-1)q^2 + m\hat{m} + \frac{n-1}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x^*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(\frac{\hat{q}}{2}\sum_{\alpha} x_{\alpha}^2 - \hat{m}\sum_{\alpha} x_* x_{\alpha}\right) \int \mathcal{D}Z \exp\left(-iz\sqrt{\hat{q}}\sum_{\alpha} x_{\alpha}\right) \right] \right\}_{N}^{N}$$

$$\left\{ \mathbb{E}_{x_*} \left[\int \mathcal{D}z \prod_{\alpha} \left\{ \int P_X \left(x_{\alpha} \right) dx_{\alpha} \exp \left(\frac{\hat{q}}{2} x_{\alpha}^2 - \hat{m} x_* x_{\alpha} - iz \sqrt{\hat{q}} x_{\alpha} \right) \right\} \right] \right\}^N$$

Now take the limit $n \to 0$ $\mathbb{E}[X^n] = \mathbb{E}[e^{n\log(X)}] \approx \mathbb{E}[1 + n\log(X)] = 1 + n\mathbb{E}[\log(X)] = \exp(\log(1 + n\mathbb{E}[\log(X)])) \approx \exp(n\mathbb{E}[\log(X)])$

$$= \int \! \mathrm{d}\hat{m} \ \mathrm{d}m \ \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(nN\left[\frac{\lambda}{2}m^2 - \frac{\lambda}{4}q^2 + m\hat{m} - \frac{1}{2}q\hat{q}\right]\right) \exp\left\{nN\mathbb{E}_{x_*}\left[\int \mathcal{D}z \log\left(\int P_X(x)\mathrm{d}x \exp\left(\frac{\hat{q}}{2}x^2 - \hat{m}x_*x - \mathrm{i}z\sqrt{q}x\right)\right)\right]\right\}$$

$$\begin{split} &\mathbb{E}_{\mathbf{x}^*,\boldsymbol{\xi}}\left[Z^n\right] \\ &= \int\!\prod_{\boldsymbol{\alpha}} \mathrm{d}\hat{m}_{\boldsymbol{\alpha}} \mathrm{d}m_{\boldsymbol{\alpha}} \!\int\!\prod_{\boldsymbol{\alpha}<\boldsymbol{\beta}} \mathrm{d}\hat{q}_{\boldsymbol{\alpha}\boldsymbol{\beta}} \mathrm{d}q_{\boldsymbol{\alpha}\boldsymbol{\beta}} \exp\left(\frac{\lambda N}{2} \left[\sum_{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}}^2 + \sum_{\boldsymbol{\alpha}<\boldsymbol{\beta}} q_{\boldsymbol{\alpha}\boldsymbol{\beta}}^2 \right] + N \left[\sum_{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}}\hat{m}_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\alpha}<\boldsymbol{\beta}} q_{\boldsymbol{\alpha}\boldsymbol{\beta}}\hat{q}_{\boldsymbol{\alpha}\boldsymbol{\beta}} \right] \right) \left\{ \mathbb{E}_{\boldsymbol{x}^*} \left[\int\!\!\prod_{\boldsymbol{\alpha}} P_{\boldsymbol{X}}\left(\boldsymbol{x}_{\boldsymbol{\alpha}}\right) \mathrm{d}\boldsymbol{x}_{\boldsymbol{\alpha}} \exp\left(-\sum_{\boldsymbol{\alpha}} \hat{m}_{\boldsymbol{\alpha}}\boldsymbol{x}_*\boldsymbol{x}_{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\alpha}<\boldsymbol{\beta}} \hat{q}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\boldsymbol{x}_{\boldsymbol{\alpha}}\boldsymbol{x}_{\boldsymbol{\beta}} \right) \right] \right\}^N \end{split}$$

Assuming replica symmetric ansatz $m_\alpha \equiv m, \hat{m}_\alpha \equiv \hat{m}, q_{\alpha\beta} \equiv q, \hat{q}_{\alpha\beta} \equiv \hat{q}$.

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \int \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(\frac{\lambda N}{2} \left[nm^2 + \frac{n^2 - n}{2}q^2\right] + N \left[nm\hat{m} + \frac{n^2 - n}{2}q\hat{q}\right]\right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(-\hat{m}\sum_{\alpha} x_* x_{\alpha} - \hat{q}\sum_{\alpha < \beta} x_{\alpha} x_{\beta}\right)\right] \right\}_{\mathcal{D}Z} = \int \mathrm{d}z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left\{ \mathbb{E}_{x_*} \left[\int \prod_{\alpha} P_X\left(x_{\alpha}\right) \mathrm{d}x_{\alpha} \exp\left(\frac{\hat{q}}{2}\sum_{\alpha} x_{\alpha}^2 - \hat{m}\sum_{\alpha} x_* x_{\alpha}\right) \int \mathcal{D}z \exp\left(-iz\sqrt{\hat{q}}\sum_{\alpha} x_{\alpha}\right)\right] \right\}_{x_*}^{N}$$

$$\left\{ \mathbb{E}_{x_*} \left[\int \mathcal{D}z \prod_{\alpha} \left\{ \int P_X \left(x_{\alpha} \right) dx_{\alpha} \exp \left(\frac{\hat{q}}{2} x_{\alpha}^2 - \hat{m} x_* x_{\alpha} - iz \sqrt{\hat{q}} x_{\alpha} \right) \right\} \right] \right\}^N$$

Now take the limit $n \to 0$ $\mathbb{E}[X^n] = \mathbb{E}[e^{n \log(X)}] \approx \mathbb{E}[1 + n \log(X)] = 1 + n\mathbb{E}[\log(X)] = \exp(\log(1 + n\mathbb{E}[\log(X)])) \approx \exp(n\mathbb{E}[\log(X)])$

$$= \int \mathrm{d}\hat{m} \ \mathrm{d}m \ \mathrm{d}\hat{q} \ \mathrm{d}q \exp\left(nN\left[\frac{\lambda}{2}m^2 - \frac{\lambda}{4}q^2 + m\hat{m} - \frac{1}{2}q\hat{q}\right]\right) \exp\left\{nN\mathbb{E}_{x_*}\left[\int \mathcal{D}z \log\left(\int P_X(x)\mathrm{d}x \exp\left(\frac{\hat{q}}{2}x^2 - \hat{m}x_*x - \mathrm{i}z\sqrt{q}x\right)\right)\right]\right\}$$

 $= \int d\hat{m} dm d\hat{q} dq \exp(nN\Phi(m, q, \hat{m}, \hat{q}))$

$$\Phi(m,q,\hat{m},\hat{q}) = \frac{\lambda}{4} \left(2m^2 - q^2 \right) + m\hat{m} - \frac{1}{2}q\hat{q} + \mathbb{E}_{x^*} \left[\int \mathcal{D}z \log \left(\int P_X(x) dx \exp \left[\frac{\hat{q}}{2} x^2 + \left(\sqrt{\hat{q}} z - \hat{m} x_* \right) x \right] \right) \right]$$

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$$m_{\alpha} = \frac{1}{N} \sum_{i} x_{i}^{(\alpha)} x_{i}^{*}$$
 "overlap with the truth"

$$q_{\alpha\beta} = \frac{1}{N} \sum_{i} x_{i}^{(\alpha)} x_{i}^{(\beta)} \quad \text{"replicas overlap"}$$

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Nishimori Identity (for two Replicas)

Consider a model $y = f(x^*) + \eta$ where f is a continuous bounded function, and η is the additive noise. Denote $\langle ... \rangle_{x^*,\eta}$ the expectation with respect to $P(X = ... | Y = f(x^*) + \eta)$, then

$$\mathbb{E}_{X^*,\eta}\left[\left\langle g\left(X,X^*\right)\right\rangle_{X^*,\eta}\right] = \mathbb{E}_{X^*,\eta}\left[\left\langle g\left(X^{(1)},X^{(2)}\right)\right\rangle_{X^*,\eta}\right]$$

where $X^{(1)}, X^{(2)}$ are two independent replicas distributed as $P(X = . \mid Y = f(x^*) + \eta)$

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Nishimori identity implies m = q, $\hat{m} = \hat{q}$

$$\Phi_{\mathsf{Nishi}}(m,\hat{m}) = \Phi(m,q,\hat{m},\hat{q}) \left|_{q=m,\hat{q}=\hat{m}} = \frac{\lambda}{4} m^2 + \frac{1}{2} m \hat{m} + \mathbb{E}_{x^*} \left[\int \mathcal{D}z \log \left(\int P_X(x) \mathrm{d}x e^{\frac{\hat{m}}{2}x^2 - \left(\mathrm{i}\sqrt{\hat{m}}z + \hat{m}x_* \right)x} \right) \right]$$

$$\Phi(m,q,\hat{m},\hat{q}) = \frac{\lambda}{4} \left(2m^2 - q^2 \right) + m\hat{m} - \frac{1}{2}q\hat{q} + \mathbb{E}_{x^*} \left[\int \mathcal{D}z \log \left(\int P_X(x) dx \exp \left[\frac{\hat{q}}{2} x^2 + \left(\sqrt{\hat{q}} z - \hat{m} x_* \right) x \right] \right) \right]$$

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Extremization implies $\hat{m} = -\lambda m$

$$\Phi_{\text{RS}}(m) \triangleq \Phi_{\text{Nishi}}(m, \hat{m}) \Big|_{\hat{m} = -\lambda m} = -\frac{\lambda}{4} m^2 + \mathbb{E}_{x_*, z} \left[\log \left(\int P_X(x) dx e^{\frac{-\lambda m}{2} x^2 + \left(\lambda m x_* + \sqrt{\lambda m} z\right) x} \right) \right]$$

Recall on what we care about

$$\mathrm{MMSE}(\lambda) = \mathbb{E}_{\mathbf{Y}, x^*} \left[\left(\langle x \rangle_{\mathbf{Y}} - x^* \right)^2 \right] = \mathbb{E}_{\mathbf{Y}, x^*} \left[\langle x \rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^* \langle x \rangle_{\mathbf{Y}} \right]$$

Recall on what we care about

$$\begin{aligned} \text{MMSE}(\lambda) &= \mathbb{E}_{\mathbf{Y},x^*} \left[\left(\langle x \rangle_{\mathbf{Y}} - x^* \right)^2 \right] = \mathbb{E}_{\mathbf{Y},x^*} \left[\langle x \rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^* \langle x \rangle_{\mathbf{Y}} \right] \\ &= \mathbb{E}_{x^*} \left[(x^*)^2 \right] - \mathbb{E}_{\mathbf{x}^*,\xi} \left[\left\langle \frac{1}{N} \sum_i x_i^* x_i \right\rangle_{\mathbf{x}^*,\xi} \right] \\ & N \to \infty \\ &\approx \mathbb{E}_{x^*} \left[(x^*)^2 \right] - m^* (\lambda) \end{aligned}$$

Recall on what we care about

$$\mathsf{MMSE}(\lambda) = \mathbb{E}_{\mathbf{Y},x^*} \left[\left\langle x \right\rangle_{\mathbf{Y}}^2 - x^* \right]^2 \right] = \mathbb{E}_{\mathbf{Y},x^*} \left[\left\langle x \right\rangle_{\mathbf{Y}}^2 + (x^*)^2 - 2x^* \left\langle x \right\rangle_{\mathbf{Y}} \right]$$

$$= \mathbb{E}_{x^*} \left[(x^*)^2 \right] - \mathbb{E}_{\mathbf{x}^*,\xi} \left[\left\langle \frac{1}{N} \sum_{i} x_i^* x_i \right\rangle_{\mathbf{x}^*,\xi} \right]$$

$$\approx \mathbb{E}_{x^*} \left[(x^*)^2 \right] - m^* (\lambda)$$

A Simple Solvable Case

Assume now that each $x_i^* = \pm 1$ are distributed with equal probabilities, i.e. $P_X(x) = \frac{1}{2} [\delta(x-1) + \delta(x+1)]$ (Radamacher's Distributed), in homework 3, you'll show that

$$\Phi_{\rm RS}(m,\lambda) = -\frac{\lambda m^2}{4} - \frac{\lambda m}{2} + \frac{1}{2} \mathbb{E}_z \left[\log \left\{ \frac{1}{2} (\cosh(2\lambda m) + \cosh(2z\sqrt{\lambda m})) \right\} \right]$$

and that

$$m^*(\lambda) = \mathbb{E}_z \begin{vmatrix} 2\sinh(2\lambda m) + \frac{z}{\sqrt{\lambda m}} \sinh(2z\sqrt{\lambda m}) \\ -1 \\ \cosh(2\lambda m) + \cosh(2z\sqrt{\lambda m}) \end{vmatrix} - 1$$

Example of Computational Phase Transition (for Statistical Estimation)

