
Maximizing nongaussianity

Exercise T6.1: Solving the ICA problem by maximizing nongaussianity (tutorial)

- (a) What are the ambiguities and the limitations of the solutions found by ICA?
- (b) What role does *whitening/sphering* play in the context of ICA?
- (c) Why are Gaussians bad for ICA?
- (d) How do we find independent components by maximizing nongaussianity?
- (e) What measures do we have for nongaussianity and how do we use each for solving the ICA problem?

Exercise H6.1: Kurtosis of Toy Data (homework, 6 points)

The file `distrib.mat` contains three toy datasets (`uniform`, `normal`, `laplacian`)¹. Each is made up of 10,000 samples with 2 sources (i.e. $N = 2, p = 10,000$). You are asked to do the following for each dataset:

- (a) Apply the following mixing matrix $\underline{\mathbf{A}}$ to the original sources $\underline{\mathbf{s}}$:

$$\begin{aligned}\underline{\mathbf{A}} &= \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \\ \underline{\mathbf{x}} &= \underline{\mathbf{A}} \underline{\mathbf{s}}.\end{aligned}$$

- (b) Center the mixtures $\underline{\mathbf{x}}$ to zero mean.
- (c) Decorrelate the mixtures from (b) by applying principal component analysis (PCA) on them and project them onto the PCs.
- (d) Scale the decorrelated mixtures from (c) to unit variance in each PC direction. The mixtures are now *whitened* (*sphered*).
- (e) Rotate the whitened mixtures by different angles θ

$$\begin{aligned}\underline{\mathbf{x}}_{\theta} &= \underline{\mathbf{R}}_{\theta} \underline{\mathbf{x}} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \underline{\mathbf{x}} \\ \theta &= 0, \frac{\pi}{50}, \dots, 2\pi\end{aligned}$$

and calculate the (excess) kurtosis² empirically for each dimension in $\underline{\mathbf{x}}$:

$$\text{kurt}(x_{\theta}) = \langle x_{\theta}^4 \rangle - 3 \underbrace{\langle x_{\theta}^2 \rangle}_{=1}^2.$$

¹Python users can load the content of `.mat` files into a `dict` using the function `loadmat` from `scipy.io`

²Here and in the lecture notes the so-called *excess* Kurtosis is used which yields a value of 0 for normally distributed random variables. Additionally, this definition does not explicitly normalize by the standard deviation, because the standard deviation of each dimension is 1 after whitening.

- (f) Find the minimum and maximum kurtosis value for the first dimension and rotate the data accordingly.
- Plot the original dataset (sources) and the mixtures after the steps (a), (b), (c), (d), and (f) as a scatter plot and display the respective marginal histograms.
 - For step (e) plot the kurtosis value $\text{kurt}(x_\theta)$ of each dimension in $\underline{\mathbf{x}}$ as a function of the rotation angle θ for each dimension.
 - Compare the histograms after rotation by θ_{min} and θ_{max} for the different distributions.

Exercise H6.2: Negentropy is scale-invariant**(homework, 4 points)**

The differential entropy of an N -dimensional random vector \underline{X} with probability density $p(\underline{\mathbf{x}})$ is defined as

$$H(\underline{X}) = - \int_{\mathbb{R}^N} p(\underline{\mathbf{x}}) \log p(\underline{\mathbf{x}}) d\underline{\mathbf{x}}$$

The negentropy is defined as

$$J(\underline{X}) = H(\underline{X}_{Gauss}) - H(\underline{X})$$

where \underline{X}_{Gauss} is an N -dimensional multivariate Gaussian random vector with the same covariance matrix as \underline{X} .

Show that the negentropy is invariant w.r.t. to an invertible ($\det \underline{\mathbf{A}} \neq 0$) linear transformations $\underline{\mathbf{y}} = \underline{\mathbf{A}} \underline{\mathbf{x}}$, i.e.

$$J(\underline{\mathbf{A}} \underline{X}) = J(\underline{X})$$

from which it follows that the negentropy is scale-invariant.

Use that the differential entropy of a multivariate N -dimensional Gaussian random vector \underline{X} with covariance matrix $\underline{\Sigma}$ has the form

$$H(\underline{X}_{Gauss}) = \frac{1}{2} \log |\det \underline{\Sigma}| + \frac{N}{2} (1 + \log 2\pi)$$

Remark: Differential entropy itself is not scale-invariant.